Math 8590: Viscosity Solutions
The Perron Method

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The Perron method

Definition 1. We define the upper semicontinuous envelope of a function $u : \mathcal{O} \to \mathbb{R}$ by

$$u^*(x) := \limsup_{\mathcal{O} \ni y \to x} u(y).$$

The lower semicontinuous envelope of u is defined by

$$u_*(x) := \liminf_{\mathcal{O} \ni y \to x} u(y).$$

- Note $u_* = -(-u)^*$.
- The function u^* is the smallest upper semicontinuous function that is pointwise greater than or equal to u.
- The function u_* is the greatest lower semicontinuous function that is less than u.
- Note that $u_* \leq u \leq u^*$, and $u^* = u_* = u$ if and only if u is continuous.

The Perron method

Consider the second order nonlinear equation

$$H(D^2u, Du, u, x) = 0 \quad \text{in } U, \tag{1}$$

where H is continuous and $U \subset \mathbb{R}^n$ is open. Let $w \in \mathrm{LSC}(\overline{U})$ be a viscosity supersolution of (1) and define

$$\mathcal{F} := \Big\{ v \in \mathrm{USC}(\overline{U}) \,:\, v \text{ is a subsolution of (1) and } v \leq w \text{ in } \overline{U} \Big\},$$

and

$$u(x) := \sup\{v(x) \,:\, v \in \mathcal{F}\}. \tag{2}$$

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Two key lemmas:

Lemma 1. *Suppose* $\mathcal{F}$ *is nonempty. Then the upper semicontinuous function* u^* *is a viscosity subsolution of* (1)

Lemma 2. *Let* $u \in \mathcal{F}$*. If* u_* *is not a viscosity supersolution of* (1)*, then there exists* $v \in \mathcal{F}$ *such that* $v(x) > u(x)$ *for some* $x \in U$*.*

$$\mathcal{F} := \Big\{ v \in \mathrm{USC}(\overline{U}) \,:\, v \text{ is a subsolution of (1) and } v \le w \text{ in } \overline{U} \Big\},$$

$$u(x) := \sup\{v(x) \,:\, v \in \mathcal{F}\}.$$

The Perron method

Assume H is continuous and satisfies all monotonicity and regularity requirements of previous theorems.

Theorem 1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and Lipschitz continuous, and suppose that*

$$K := \sup \left\{ |H(p,x)| \ : \ |p| \leq Lip(g) \ \textit{and} \ x \in \mathbb{R}^n \right\} < \infty.$$

Then for every $T > 0$ there exists a unique bounded viscosity solution $u \in C(\mathbb{R}^n \times [0,T])$ of

$$\begin{cases} u_t + H(Du, x) = 0 & \textit{in } \mathbb{R}^n \times (0,T) \\ u = g & \textit{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{3}$$