Math 8590: Viscosity Solutions
Method of Vanishing Viscosity

Instructor: Jeff Calder
Office: 538 Vincent
Email: jcalder@umn.edu
Office Hours: TBD

http://www-users.math.umn.edu/~jwcalder/8590F18

Vanishing viscosity

Consider the viscous Hamilton-Jacobi equation

$$\begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{1}$$

We now examine convergence of the solution u_ε of (1) to the unique viscosity solution of

$$\begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2}$$

Assumptions

$U \subset \mathbb{R}^n$ is open and bounded, $H(p,x)$ is continuous.

Coercivity:

$$\liminf_{|p|\to\infty} H(p,x) > 0 \quad \text{uniformly in } x \in U, \tag{3}$$

Nonnegativity:

$$-H(0,x) \geq 0 \quad \text{for all } x \in U. \tag{4}$$

Exterior sphere condition: There exists $r > 0$ such that for every $x_0 \in \partial U$ there is a point $x_0^* \in \mathbb{R}^n \setminus \overline{U}$ for which

$$B(x_0^*, r) \cap \overline{U} = \{x_0\}. \tag{5}$$

Basic estimates

$$
\begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{6}
$$

Lemma 1. *Let $\varepsilon > 0$ and let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ be a solution of (6). Then*

$$
0 \le u_\varepsilon \le \sup_{x \in U} |H(0,x)| \quad \textit{in } U. \tag{7}
$$

Weak upper and lower limits

Definition 1. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of real-valued functions on $\overline{U}$.

The *upper weak limit* $\overline{u} : \overline{U} \to \mathbb{R}$ of the family $\{u_\varepsilon\}_{\varepsilon>0}$ is defined by

$$\overline{u}(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y). \tag{8}$$

Similarly, the *lower weak limit* $\underline{u} : \overline{U} \to \mathbb{R}$ is defined by

$$\underline{u}(x) = \liminf_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y). \tag{9}$$

Lemma 2. *Suppose the family* $\{u_\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded. Then* $\overline{u} \in USC(\overline{U})$ *and* $\underline{u} \in LSC(\overline{U})$.

Convergence of vanishing viscosity

$$\begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{10}$$

$$\begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{11}$$

Theorem 1. *For each $\varepsilon > 0$ let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ solve (10). Then $u_\varepsilon \to u$ uniformly on $\overline{U}$ as $\varepsilon \to 0^+$, where u is the unique viscosity solution of (11).*

Convergence rate

Lemma 3. *Let $u \in USC(\overline{U})$ be a nonnegative viscosity subsolution of (12). Then there exists C depending only on H such that*

$$|u(x) - u(y)| \leq C|x - y| \quad \text{for all } x, y \in \overline{U}.$$

$$\begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{12}$$

Proof. Fix $x \in U$ and define

$$w(y) := u(y) - C|y - x|.$$

Then w attains its maximum at some $y_0 \in \overline{U}$... □

Convergence rate

We assume that for every $R > 0$ there exists C_R such that

$$H(p, y) - H(p, x) \leq C_R|x - y| \quad \text{for all } x, y \in U \text{ and } |p| \leq R. \tag{13}$$

Theorem 2. *For each $\varepsilon > 0$, let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ solve (14), and let u be the unique viscosity solution of (15). Then there exists C depending only on H such that*

$$|u - u_\varepsilon| \leq C\sqrt{\varepsilon}.$$

$$\begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{14}$$

$$\begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{15}$$

Convergence rate

Proof. We show that $u - u_\varepsilon \le C\sqrt{\varepsilon}$. Define

$$\Phi(x, y) = u(x) - u_\varepsilon(y) - \frac{\alpha}{2}|x - y|^2,$$

where α is to be determined. Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\max_{\overline{U}\times\overline{U}} \Phi = \Phi(x_\alpha, y_\alpha).$$

It follows from Lipschitzness of u that

$$|x_\alpha - y_\alpha| \le \frac{C}{\alpha}.$$

Claim:

$$u(x_\alpha) - u_\varepsilon(y_\alpha) \le C\left(\frac{1}{\alpha} + \alpha\varepsilon\right).$$

□

Convergence rate

Exercise 1. Show that the solution u_ε of

$$|u_\varepsilon'(x)| - \varepsilon u_\varepsilon''(x) = 1 \quad \text{for } x \in (-1,1)$$

satisfying $u_\varepsilon(-1) = u_\varepsilon(1) = 0$ is

$$u_\varepsilon(x) = 1 - |x| - \varepsilon \left(e^{-\frac{1}{\varepsilon}|x|} - e^{-\frac{1}{\varepsilon}} \right).$$

In this case, $|u - u_\varepsilon| \leq C\varepsilon$, where $u(x) = 1 - |x|$ is the viscosity solution of $|u'(x)| = 1$ on $(-1,1)$ with $u(-1) = u(1) = 0$.

Convergence rate

Exercise 2. Show that if $u \in C^2(\overline{U})$, then

$$|u - u_\varepsilon| \leq C\varepsilon.$$

$$\begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U. \end{cases} \tag{16}$$

$$\begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{17}$$

C^2-type estimates

Let $G(p)$ be convex, and suppose $u \in C_c^\infty(\mathbb{R}^n)$ is a solution of

$$u + G(Du) = f \quad \text{in } \mathbb{R}^n.$$

Exercise 3. Show that $D^2u \leq cI$, where $c = \max_{\substack{x\in\mathbb{R}^n \\ |\xi|=1}} f_{\xi\xi}$.

Semiconcavity

Definition 2. We say $u \in C(\overline{U})$ is *semiconcave* with constant c if u is a viscosity solution of

$$-D^2 u \geq -cI \quad \text{in } U. \tag{18}$$

Semiconcavity

Definition 3. We say $u \in C(\overline{U})$ is *semiconcave* with constant c if u is a viscosity solution of

$$-D^2u \geq -cI \quad \text{in } U. \tag{19}$$

- We say u is a viscosity solution of (19) provided $D^2\varphi(x) \leq cI$ whenever $\varphi \in C^\infty(\mathbb{R}^n)$ and $u - \varphi$ has a local minimum at x. Equivalently

$$-\max_{|\xi|=1} u_{\xi\xi} \geq -c \quad \text{in } \mathbb{R}^n.$$

- Notice that $v := u - \frac{1}{2}c|x|^2$ is a viscosity solution of $-D^2v \geq 0$, hence v is concave (due to a generalization of a homework Exercise).

- We also note that (19) is equivalent to

$$u(x+h) - 2u(x) + u(x-h) \leq c|h|^2 \quad \text{for all } x, h \in \mathbb{R}^n.$$

- A function u is called *semiconvex* if $-u$ is semiconcave.

Semiconcavity

Theorem 3. *Assume* $p \mapsto G(p)$ *is convex,* $G(0) = 0$*, and* $f \in C_c^2(\mathbb{R}^n)$*. Let* $u \in C(\mathbb{R}^n)$ *be a compactly supported viscosity solution of*

$$u + G(Du) = f \quad \text{in } \mathbb{R}^n. \tag{20}$$

Then u *is a viscosity solution of*

$$-D^2 u \geq -cI \quad \text{in } \mathbb{R}^n, \tag{21}$$

where $c = \max_{\substack{x \in \mathbb{R}^n \\ |\xi|=1}} f_{\xi\xi}$*. That is,* u *is semiconcave with constant* c*.*

One-sided rate

Theorem 4. *Assume* $p \mapsto G(p)$ *is convex and nonnegative with* $G(0) = 0$, *and* $f \in C_c^2(U)$ *is nonnegative. Let* $u \in C(\overline{U})$ *be the viscosity solution of*

$$\left.\begin{aligned} u + G(Du) &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{aligned}\right\} \tag{22}$$

and let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ *solve*

$$\left.\begin{aligned} u_\varepsilon + G(Du_\varepsilon) - \varepsilon \Delta u_\varepsilon &= f \quad \text{in } U \\ u_\varepsilon &= 0 \quad \text{on } \partial U, \end{aligned}\right\} \tag{23}$$

Then there exists a constant C *such that*

$$u_\varepsilon - u \leq C\varepsilon.$$