# Lecture notes on viscosity solutions

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# Contents

# Chapter 1

# Introduction

These notes are concerned with viscosity solutions for fully nonlinear equations. A majority of the notes are concerned with Hamilton-Jacobi equations of the form

$$H(Du, u, x) = 0.$$

First order equations generally do not admit classical solutions, due to the possibility of crossing characteristics. On the other hand, there are infinitely many Lipschitz continuous functions that satisfy the equation almost everywhere. Since the equation is nonlinear, we cannot define weak solutions via integration by parts. In this setting, the correct notion of weak solution is the *viscosity solution*, discovered by by Crandall, Evans and Lions [5,7]. At a high level, the notion of viscosity solution selects, from among the infinitely many Lipschitz continuous solutions, the one that is 'physically correct' for a *very* wide range of applications.

Viscosity solutions have proven to be extremely useful, and this is largely because very strong comparison and stability results are available via the maximum (or comparison) principle. As we shall see, these results come almost directly from the definitions. As such, viscosity solutions could easily have been called "comparison solutions" or "$L^\infty$-stable solutions". The term "viscosity" comes from the original motivation for the definitions via the *method of vanishing viscosity* (see Section 1.3 and Chapter 5). Viscosity solutions have a wide range of applications, including problems in optimal control theory. A good reference for the first order theory is the book by Bardi and Capuzzo-Dolcetta [1], and Evans [11, Chapter 10].

Since viscosity solutions are defined by, and based upon, the maximum principle, it is natural that they extend to fully nonlinear second order equations of the form

$$F(D^2u, Du, u, x) = 0,$$

provided $F$ satisfies some form of ellipticity. However, in the early days of the theory, it was not clear that uniqueness would hold for second order equations, since the standard proof of uniqueness for first order equations does not directly extend. The first uniqueness result for second order equations is due to Jensen [13], and his role in the theory is immortalized in Jensen's Lemma (see Lemma 12.1), which is a crucial technical tool in the second order theory. Good references for second order theory include the User's Guide [6], Crandall's introductory paper [4], and the book by Katzourakis [14].

These notes were designed to illustrate the theory and applications of viscosity solutions. They are written in a lecture style and are not meant to be a thorough reference. We do prove the comparison principle for first and second order equations in full generality for semi-continuous sub- and supersolutions. When considering applications, we take simple settings where the main ideas are present, but the proofs are particularly simple. Almost all of the applications (e.g., convergence rates, homogenization, etc.) can be stated and proved in far more generality. However, the ideas in these notes contain the essence of the key tools for many of these problems.

The organization of these notes is as follows. In Sections 1.1, 1.2, 1.3, and 1.4 we give several different motivational examples leading to the definition of viscosity solution. In Chapter 2 we give the main definitions of viscosity solutions, and provide a number of interesting exercises. In Chapter 3 we prove the comparison principle for viscosity solutions of first order equations. In Chapter 4 we discuss the Hamilton-Jacobi-Bellman equation from optimal control theory in the special case of shortest path problems (i.e., distance functions). Chapter 5 treats the method of vanishing viscosity, proving convergence via the weak upper and lower limits, the $O(\sqrt{\varepsilon})$ convergence rate, and a one-sided $O(\varepsilon)$ rate when the solution is semiconcave. In Chapter 6 we briefly discuss boundary conditions in the viscosity sense. Chapter 7 covers the Perron method for establishing existence of viscosity solutions. In Chapter 8 we discuss the inf- and sup-convolutions and their role in constructing semiconvex and semiconcave approximate viscosity sub- and supersolutions. In Chapter 9 we construct convergent finite difference schemes for viscosity solutions, and we prove $O(\sqrt{h})$ and one-sided $O(h)$ convergence rates. In Chapter 10 we give a brief introduction to homogenization, and illustrate the perturbed test function method. Chapter 11 establishes comparison principles for first order equations with discontinuous coefficients. Finally, in Chapter 12 we prove the comparison principle for viscosity solutions of second order equations, and discuss some applications.

While most of the notes address first order Hamilton-Jacobi equations, I have extended results to second order equations when the proofs are similar. In particular, Chapter 7 (the Perron method), Chapter 8 (inf- and sup-convolutions), and Chapter 12 address general second order equations. Let me also mention that the references are lacking; in future versions of these notes I plan to extend the bibliography considerably.

## 1.1 An example

We begin with a simple example. Let $\Gamma$ be a closed subset of $\mathbb{R}^n$ and let $u : \mathbb{R}^n \to [0,\infty)$ be the *distance function to* $\Gamma$, defined by

$$u(x) = \operatorname{dist}(x,\Gamma) := \min_{y\in\Gamma} |x-y|. \tag{1.1}$$

**Exercise 1.1.** Verify that $u$ is 1-Lipschitz, that is, $|u(x)-u(y)| \le |x-y|$ for all $x, y \in \mathbb{R}^n$.

Let $U = \mathbb{R}^n \setminus \Gamma$ and fix any ball $B(x,r) \subset U$. We claim that

$$u(x) = r + \min_{z\in\partial B(x,r)} u(z). \tag{1.2}$$

To see this, fix $z \in \partial B(x,r)$ minimizing the right hand side of (1.2). Select $y \in \Gamma$ such that $u(z) = |z-y|$ and compute

$$u(x) \le |x-y| \le |x-z| + |z-y| = r + u(z).$$

For the other direction, fix $y \in \Gamma$ such that $u(x) = |x-y|$. Let $z \in \partial B(x,r)$ lie on the line segment between $x$ and $y$. Then

$$u(x) = |x-y| = |x-z| + |z-y| \ge r + u(z).$$

Equation (1.2) is called a *dynamic programming principle* (DPP), and it gives a *local* characterization of the distance function $u$ as the solution of a Hamilton-Jacobi equation. Indeed, suppose for the moment that $u \in C^1(U)$. Notice we can rewrite the DPP as

$$\max_{z\in\partial B(x,r)} \{u(x) - u(z) - r\} = 0. \tag{1.3}$$

Since $u \in C^1(U)$ the Taylor expansion

$$u(x) - u(z) = Du(x)\cdot(x-z) + o(|x-z|)$$

holds as $z \to x$. Dividing both sides of (1.3) by $r$ and using the Taylor expansion yields

$$\max_{z \in \partial B(x,r)} \left\{ Du(x) \cdot \left( \frac{x-z}{r} \right) - 1 \right\} = o\,(1)\,.$$

Setting $a = \frac{x-z}{r}$ and sending $r \to 0^+$ we deduce

$$\max_{|a|=1} \{Du(x) \cdot a - 1\} = 0. \tag{1.4}$$

The partial differential differential equation (1.4) is called a *Hamilton-Jacobi-Bellman equation*, and is a direct consequence of the DPP (1.2). Notice that (1.4) implies that $Du(x) \neq 0$, and the maximum occurs at $a = Du(x)/|Du(x)|$. We therefore find that

$$\left.\begin{aligned} |Du| &= 1 \quad \text{in } U \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}\right\} \tag{1.5}$$

Equation (1.5) is a special case of the *eikonal equation*, which has applications in geometric optics, wave propagation, level set methods for partial differential equations, and computer vision.

In general $u \notin C^1(U)$, so this argument is only a heuristic.

**Exercise 1.2.** Compute the distance function $u$ to $\Gamma = \{0\}$, $\Gamma = \{x_n = 0\}$, $\Gamma = \{x_1 \cdots x_n = 0\}$ and $\Gamma = \partial B(0,1)$. In which cases does it hold that $u \in C^1(U)$?

**Exercise 1.3.** Show that when $U \subset \mathbb{R}^n$ is open and bounded, and $\Gamma = \partial U$, there does not exist a classical solution $u \in C^1(U) \cap C(\overline{U})$ of (1.5).

In light of these facts, the natural question is how to rescue this argument so that it holds when $u \notin C^1(U)$? Since the argument above relies only on a first order Taylor expansion of $u$ around $x$, the argument is valid at all points of differentiability of $u$. As $u$ is Lipschitz continuous, $u$ is differentiable almost everywhere. Thus $u$ satisfies (1.5) almost everywhere, and is called a *Lipschitz almost everywhere solution*. Unfortunately, Lipschitz almost everywhere solutions are in general not unique.

**Exercise 1.4.** Consider the following one dimensional version of (1.5):

$$|u'(x)| = 1 \quad \text{for } x \in (0,1), \qquad \text{and } u(0) = u(1) = 0. \tag{1.6}$$

Show that there are infinitely many Lipschitz almost everywhere solutions $u$ of (1.6).

Since our goal is to uniquely characterize $u$ as a solution of (1.5), we need a notion of solution of Hamilton-Jacobi equations that is weaker than classical solutions, yet more restrictive than Lipschitz almost everywhere solutions. Equivalently, we need to discover some additional condition that selects among the infinitely many Lipschitz almost everywhere solutions the one that is 'physically correct'.

## 1.2 Motivation via dynamic programming

Before giving the definitions, let us proceed further with the distance function example. As with all notions of weak solutions to partial differential equations, we will push derivatives off of $u$ and onto a class of smooth test functions. Since (1.5) is not in divergence form, the classical trick of integration by parts does not work. Instead, consider the following: Fix a point $x \in U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$. Then we have

$$u(x) - u(z) \geq \varphi(x) - \varphi(z) \quad \text{for all } z \text{ near } x.$$

Substituting this into the dynamic programming principle (1.3) we deduce

$$\max_{z \in \partial B(x,r)} \{\varphi(x) - \varphi(z) - r\} \leq 0,$$

for $r > 0$ sufficiently small. Since $\varphi$ is smooth, the argument in Section 1.1 can be used to conclude that

$$|D\varphi(x)| - 1 = \max_{|a|=1} \{D\varphi(x) \cdot a - 1\} \leq 0. \tag{1.7}$$

A similar argument can be used to show that for every $x \in U$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$

$$|D\varphi(x)| - 1 = \max_{|a|=1} \{D\varphi(x) \cdot a - 1\} \geq 0. \tag{1.8}$$

It is worthwhile taking a moment to observe what is going on here. If $u - \varphi$ has a local maximum at $x$, we can replace $\varphi$ by $\varphi + C$ so that $u(x) = \varphi(x)$ and $u(y) \leq \varphi(y)$ for all $y$ near $x$. Thus, the graph of $\varphi$ touches the graph of $u$ from above at the point $x$. We have shown above that any such test function $\varphi$ must be a subsolution of the PDE at $x$. Similarly, whenever the graph of $\varphi$ touches $u$ from below at $x$, $\varphi$ must be a supersolution of the PDE. See Figure 1.1 for a visual illustration. These ideas should remind you of maximum principle arguments. Indeed, in the case that $u$ is differentiable at

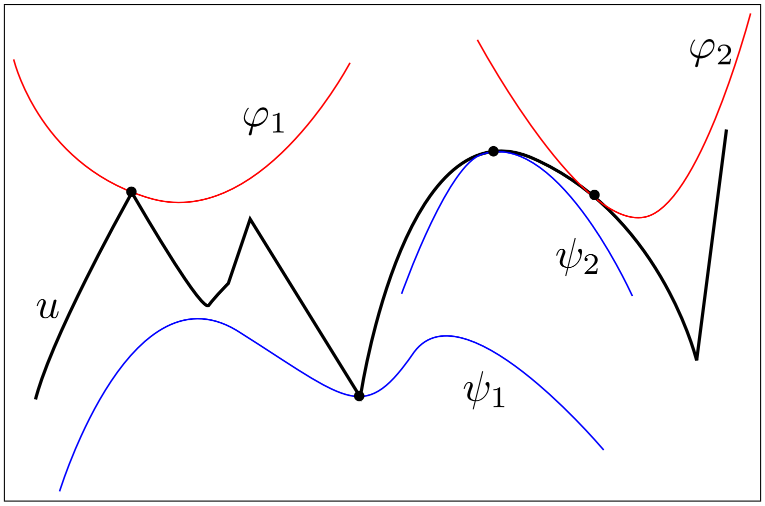


Figure 1.1: An illustration of test functions touching a nonsmooth function $u$ from above and below. The functions $\varphi_1$ and $\varphi_2$, drawn in red, touch $u$ from above, while $\psi_1$ and $\psi_2$, drawn in blue, touch $u$ from below.

$x$, we have $Du(x) = D\varphi(x)$ whenever $\varphi$ touches $u$ from above or below at $x$. These observations say that the function $u$ is not just any Lipschitz solution of (1.5); it is a Lipschitz solution that respects the maximum principle in a certain way. We will see shortly that $u$ is the only Lipschitz solution of (1.5) with these properties.

## 1.3 Motivation via vanishing viscosity

A general principle of PDEs is that equations are governed primarily by their highest order terms. With this in mind, consider the semilinear viscous Hamilton-Jacobi equation

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon\Delta u_\varepsilon = 0 \quad \text{in } U. \tag{1.9}$$

Since the highest order term in (1.9) is $-\varepsilon\Delta u_\varepsilon$, which is uniformly elliptic, we can in very general settings prove existence and uniqueness of smooth solutions $u_\varepsilon$ of (1.9) subject to, say, Dirichlet boundary conditions $u = g$ on $\partial U$. In fact, we did this for a special case of (1.9) using Schaefer's Fixed Point Theorem earlier in the course. As a remark, the additional second order term $\varepsilon\Delta u_\varepsilon$ is called a *viscosity term*, since for the Navier-Stokes equations such a term models the viscosity of the fluid.

Now suppose that $u_\varepsilon$ converges uniformly to a continuous function $u$ as $\varepsilon \to 0^+$. Let $x \in U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$. By replacing $\varphi(y)$ with $\varphi(y) + |x-y|^2$, we may assume that $u - \varphi$ has a strict local maximum at $x$. It follows that there exists a sequence $\varepsilon_k \to 0^+$ and $x_k \to x$ such that $u_{\varepsilon_k} - \varphi$ has a local maximum at $x_k$.

**Exercise 1.5.** Prove the preceding statement.

Therefore $Du_{\varepsilon_k}(x_k) = D\varphi(x_k)$, $\Delta u_{\varepsilon_k}(x_k) \le \Delta\varphi(x_k)$, and

$$\begin{aligned} &H(D\varphi(x_k), u_{\varepsilon_k}(x_k), x_k) - \varepsilon_k \Delta\varphi(x_k) \\ &\qquad\qquad \le H(Du_{\varepsilon_k}(x_k), u_{\varepsilon_k}(x_k), x_k) - \varepsilon_k \Delta u_{\varepsilon_k}(x_k) \le 0. \end{aligned}$$

Sending $\varepsilon_k \to 0^+$ we find that

$$H(D\varphi(x), u(x), x) \le 0.$$

We can similarly argue that whenever $u - \varphi$ has a local minimum at $x$

$$H(D\varphi(x), u(x), x) \ge 0.$$

Notice we have recovered the same conditions on $u$ as in Section 1.2. The technique of sending $\varepsilon \to 0$ in (1.9) is called the *method of vanishing viscosity*, and served as the original motivation for the definition of viscosity solution.

## 1.4 Motivation via the maximum principle

A less common motivation for the definition of viscosity solution comes from the maximum principle. Since the well-posedness theory for viscosity solutions is based on the maximum principle, it is arguably a more important way of thinking about viscosity solutions compared to the method of vanishing viscosity.

Suppose that $u \in C^1(U) \cap C(\overline{U})$ is a solution of

$$H(Du, x) = 0 \quad \text{in } U. \tag{1.10}$$

If $\varphi \in C^\infty(\mathbb{R}^n)$ is any function satisfying

$$H(D\varphi, x) > 0 \quad \text{in } U,$$

then we immediately have

$$\max_{\overline{U}} (u - \varphi) = \max_{\partial U} (u - \varphi),$$

that is, the maximum principle holds when comparing $u$ against strict super solutions. In fact, we can say a bit more. Since we know that $Du(x) \neq D\varphi(x)$ for all $x \in U$, the maximum of $u - \varphi$ cannot be attained in $U$. This implies that

$$u \leq \varphi \text{ on } \partial U \implies u < \varphi \text{ in } U.$$

The observations above hold equally well for any $V \subset\subset U$. That is, if $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies

$$H(D\varphi, x) > 0 \quad \text{in } V, \tag{1.11}$$

then we have

$$u \leq \varphi \text{ on } \partial V \implies u < \varphi \text{ in } V. \tag{1.12}$$

Similarly, if $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies

$$H(D\varphi, x) < 0 \quad \text{in } V, \tag{1.13}$$

then we have

$$u \geq \varphi \text{ on } \partial V \implies u > \varphi \text{ in } V. \tag{1.14}$$

Now suppose we have a continuous function $u \in C(\overline{U})$ that satisfies the maximum (or rather, comparison) principle against smooth strict super and subsolutions, as above. What can we say about $u$? Does $u$ solve (1.10) in any reasonable sense? To answer these questions, we need to formulate what it means for a continuous function to satisfy the maximum principles stated above.

For every $V \subset\subset U$ we define

$$S^+(V) = \{\varphi \in C^\infty(\mathbb{R}^n) \,:\, H(D\varphi(x), x) > 0 \text{ for all } x \in V\}, \tag{1.15}$$

and

$$S^-(V) = \{\varphi \in C^\infty(\mathbb{R}^n) \,:\, H(D\varphi(x), x) < 0 \text{ for all } x \in V\}. \tag{1.16}$$

Let $u \in C(\overline{U})$. Suppose that for every $V \subset\subset U$, $u$ satisfies (1.12) for all $\varphi \in S^+(V)$ and (1.14) for all $\varphi \in S^-(V)$. Such a function $u$ could be called a *comparison* solution of (1.10), since it is defined precisely to satisfy the comparison or maximum principle.

We now derive a much simpler property that is satisfied by $u$. Let $\psi \in C^\infty(\mathbb{R}^n)$ and $x \in U$ such that $u - \psi$ has a local maximum at $x$. This means that for some $r > 0$

$$u(y) - \psi(y) \leq u(x) - \psi(x) \text{ for all } y \in B(x, r).$$

Define

$$\varphi(y) := \psi(y) + u(x) - \psi(x).$$

Then $u \leq \varphi$ on the ball $B(x,r)$ and $u(x) = \varphi(x)$. Therefore, $u - \varphi$ attains its maximum over the ball $B(x,r)$ at the interior point $x$. It follows from our definition of $u$ that $\varphi \notin S^+(B^0(x,r))$, and hence

$$H(D\varphi(x_r), x_r) \leq 0 \text{ for some } x_r \in B^0(x,r).$$

Sending $r \to 0$ we have $x_r \to x$ which yields

$$H(D\psi(x), x) \leq 0. \tag{1.17}$$

That is, for any $x \in U$ and $\psi \in C^\infty(\mathbb{R}^n)$ such that $u - \psi$ has a local maximum at $x$ we deduce (1.17). It is left as an exercise to the reader to show that whenever $u - \psi$ has a local minimum at $x$ we have

$$H(D\psi(x), x) \geq 0.$$

# Chapter 2

# Definitions

Let us now consider a general second order nonlinear partial differential equation

$$H(D^2u, Du, u, x) = 0 \quad \text{in } \mathcal{O}, \tag{2.1}$$

where $H$ is continuous and $\mathcal{O} \subset \mathbb{R}^n$. We recall that a function $u : \mathcal{O} \subset \mathbb{R}^n \to \mathbb{R}$ is *upper* (resp. *lower*) *semicontinuous* at $x \in \mathcal{O}$ provided

$$\limsup_{\mathcal{O} \ni y \to x} u(y) \leq u(x) \quad (\text{resp. } \liminf_{\mathcal{O} \ni y \to x} u(y) \geq u(x)).$$

To be precise, we recall the definitions

$$\limsup_{\mathcal{O} \ni y \to x} u(y) := \inf_{r>0} \sup\{u(y) \,:\, y \in \mathcal{O} \cap B(x,r)\},$$

and

$$\liminf_{\mathcal{O} \ni y \to x} u(y) := \sup_{r>0} \inf\{u(y) \,:\, y \in \mathcal{O} \cap B(x,r)\}.$$

Let USC($\mathcal{O}$) (resp. LSC($\mathcal{O}$)) denote the collection of functions that are upper (resp. lower) semicontinuous at all points in $\mathcal{O}$. We make the following definitions.

**Definition 2.1** (Viscosity solution)**.** We say that $u \in \text{USC}(\mathcal{O})$ is a *viscosity subsolution* of (2.1) if for every $x \in \mathcal{O}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$ with respect to $\mathcal{O}$

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \leq 0.$$

We will often say that $u \in \text{USC}(\mathcal{O})$ is a viscosity solution of $H \leq 0$ in $\mathcal{O}$ when $u$ is a viscosity subsolution of (2.1).

Similarly, we say that $u \in \mathrm{LSC}(\mathcal{O})$ is a *viscosity supersolution* of (2.1) if for every $x \in \mathcal{O}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$ with respect to $\mathcal{O}$

$$H(D^2\varphi(x), D\varphi(x), u(x), x) \geq 0.$$

We also say that $u \in \mathrm{LSC}(\mathcal{O})$ is a viscosity solution of $H \geq 0$ in $\mathcal{O}$ when $u$ is a viscosity supersolution of (2.1).

Finally, we say $u$ is *viscosity solution* of (2.1) if $u$ is both a viscosity subsolution and a viscosity supersolution.

We immediately have comparison against strict super and subsolutions.

**Theorem 2.2.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose that $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies*

$$H(D^2\varphi, D\varphi, x) > 0 \text{ in } U \tag{2.2}$$

*and let $u \in USC(\overline{U})$ be a viscosity solution of $H \leq 0$ in $U$. Then we have that*

$$\max_{\overline{U}}(u - \varphi) = \max_{\partial U}(u - \varphi). \tag{2.3}$$

*Similarly, if*

$$H(D^2\varphi, D\varphi, x) < 0 \text{ in } U \tag{2.4}$$

*and $u \in LSC(\overline{U})$ is a viscosity solution of $H \geq 0$ in $U$ then*

$$\min_{\overline{U}}(u - \varphi) = \min_{\partial U}(u - \varphi). \tag{2.5}$$

*Proof.* Since $u - \varphi$ is upper semicontinuous, its maximum is attained at some point $x \in \overline{U}$. If $x \in U$ then by the definition of viscosity subsolution we would have

$$H(D^2\varphi(x), D\varphi(x), x) \leq 0,$$

which contradicts (2.2) and completes the proof of (2.3). The proof of (2.5) is similar. □

Although we made the definitions for second order equations, we will mostly study first order Hamilton-Jacobi equations of the form

$$H(Du, u, x) = 0 \quad \text{in } \mathcal{O}. \tag{2.6}$$

Some remarks are in order.

**Remark 2.3.** The argument given in Section 1.2 shows that the distance function $u$ defined by (1.1) is a viscosity solution of (1.5).

**Remark 2.4.** In light of Figure 1.1, when $u - \varphi$ has a local maximum at $x$, we will say that $\varphi$ touches $u$ from above at $x$. Likewise, when $u - \varphi$ has a local minimum at $x$, we will say that $\varphi$ touches $u$ from below at $x$.

**Remark 2.5.** It is possible that for some $x \in \mathcal{O}$, there are no admissible test functions $\varphi$ in the definition of viscosity sub- or supersolution. For example, if $n = 1$ and $u(x) := |x|$, there does not exist $\varphi \in C^\infty(\mathbb{R})$ touching $u$ from above at $x = 0$ (why?). Of course, it is possible to touch $u(x) = |x|$ from below at $x = 0$ (e.g., take $\varphi \equiv 0$). A more intricate example is the function

$$v(x) = \begin{cases} x\sin(\log(|x|)), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Even though $v$ is Lipschitz continuous, there are no smooth functions $\varphi$ touching from above or below at $x = 0$.

It turns out the set of points at which there are admissible test functions is dense (see Exercise 2.18).

**Remark 2.6.** A viscosity solution $u$ is necessarily continuous, being both upper and lower semicontinuous. It is very useful to allow viscosity sub- and supersolutions to be merely semicontinuous. We will see applications of this later.

**Remark 2.7.** The condition that $\varphi \in C^\infty(\mathbb{R}^n)$ can be replaced by $\varphi \in C(\mathbb{R}^n)$ and $\varphi$ is differentiable at $x$ (see [11, Section 10.1.2]). It follows that if $u$ is a viscosity solution of (2.6), $\mathcal{O}$ is open, and $u$ is differentiable at $x \in U$, then

$$H(Du(x), u(x), x) = 0.$$

Indeed, we can simply take $\varphi = u$ in the sub and supersolution properties. Therefore, a Lipschitz viscosity solution is also an almost everywhere solution. The converse is not true. A similar remark holds for second order equations, provided $u$ is twice differentiable at $x$.

**Remark 2.8.** The set $\mathcal{O} \subset \mathbb{R}^n$ need not be open. In some settings, we may take $\mathcal{O} = U \cup \Gamma$, where $U \subset \mathbb{R}^n$ is open and $\Gamma \subset \partial U$. The reader should note that $u - \varphi$ is assumed to have a local max or min at $x \in \mathcal{O}$ *with respect to* the set $\mathcal{O}$. This allows a very wide class of test function when $x \in \partial\mathcal{O}$, and as a consequence, classical solutions on non-open sets need not be viscosity solutions at boundary points (see Exercise 2.12).

**Remark 2.9.** If $u$ is a viscosity solution of (2.6), then $u$ is generally *not* a viscosity solution of

$$-H(Du, u, x) = 0 \quad \text{in } \mathcal{O}.$$

Although this is counterintuitive, it is a necessary peculiarity in the theory of viscosity solutions, and is important to keep in mind. See Exercise 2.13.

**Exercise 2.10.** Let $U \subset \mathbb{R}^n$ be open, and suppose $u \in C^1(U)$ is a classical solution of $H(Du, u, x) = 0$ in $U$. Show that $u$ is a viscosity solution of $H(Du, u, x) = 0$ in $U$.

**Exercise 2.11.** Show that the distance function $u$ defined by (1.1) is the *unique* viscosity solution of (1.5). [Hint: Use Theorem 2.2 and compare $u$ against a suitable family of strict super and subsolutions of (1.1).]

**Exercise 2.12.** Show that $u(x) := x$ is a viscosity solution of $u' - 1 = 0$ on $(0, 1]$, but is *not* a viscosity solution on $[0, 1)$. [Hint: Examine the subsolution condition at $x = 0$. This exercise shows that smooth solutions need not be viscosity solutions at boundary points.]

**Exercise 2.13.** Verify that $u(x) = -|x|$ is a viscosity solution of $|u'(x)|-1 = 0$ on $\mathbb{R}$, but is *not* a viscosity solution of $-|u'(x)| + 1 = 0$ on $\mathbb{R}$. What is the viscosity solution of the second PDE?

Exercise 2.13 shows that, roughly speaking, viscosity solutions allow 'corners' or 'kinks' in only one direction. Changing the sign of the equation reverses the orientation of the allowable corners.

**Exercise 2.14.** Let $u : (0, 1) \to \mathbb{R}$ be continuous. Show that the following are equivalent.

(i) $u$ is nondecreasing.

(ii) $u$ is a viscosity solution of $u' \geq 0$ on $(0, 1)$.

(iii) $u$ is a viscosity solution of $-u' \leq 0$ on $(0, 1)$.

[Hint: For the hard direction, suppose that $u' \geq 0$ in the viscosity sense on $(0, 1)$, but $u$ is not nondecreasing on $(0, 1)$. Show that there exists $0 < x_1 < x_2 < x_3 < 1$ such that $u(x_3) < u(x_2) < u(x_1)$. Construct a test function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' < 0$ such that $\varphi$ touches $u$ from below somewhere in the interval $(x_1, x_3)$. Drawing a picture might help.]

**Exercise 2.15.** Let $u : (0,1) \to \mathbb{R}$ be continuous. Show that $u$ is convex on $(0,1)$ if and only if $u$ is a viscosity solution of $-u'' \leq 0$ on $(0,1)$. Show that in general, convex functions are not viscosity solutions of $u'' \geq 0$ on $(0,1)$. **Note:** The PDE $u'' \geq 0$ is not even degenerate elliptic in the sense of (3.4).

[Hint: The hint for the hard direction is similar to Exercise 2.14. Suppose that $-u'' \leq 0$ on $(0,1)$ but $u$ is not convex on $(0,1)$. Then there exists $0 < x_1 < x_2 < 1$ and $\lambda \in (0,1)$ such that

$$u(\lambda x_1 + (1-\lambda)x_2) > \lambda u(x_1) + (1-\lambda)u(x_2).$$

Construct a test function $\varphi \in C^\infty(\mathbb{R})$ with $\varphi'' < 0$ such that $\varphi$ touches $u$ from above somewhere in the interval $(x_1, x_2)$.]

**Exercise 2.16.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. Show that $u$ is a viscosity solution of $|Du| \leq \mathrm{Lip}(u)$ and $-|Du| \geq -\mathrm{Lip}(u)$ on $\mathbb{R}^n$.

**Exercise 2.17.** We define the superdifferential of $u$ at $x$ to be

$$D^+u(x) := \Big\{ p \in \mathbb{R}^n \,:\, u(y) \leq u(x) + p \cdot (y-x) + o(|y-x|) \text{ as } y \to x \Big\}.$$

Similarly, we define the subdifferential

$$D^-u(x) := \Big\{ p \in \mathbb{R}^n \,:\, u(y) \geq u(x) + p \cdot (y-x) + o(|y-x|) \text{ as } y \to x \Big\}.$$

Suppose $U \subset \mathbb{R}^n$ is open. Show that $u \in \mathrm{USC}(\overline{U})$ is a viscosity subsolution of $H(Du, u, x) = 0$ on $U$ if and only if

$$H(p, u(x), x) \leq 0 \quad \text{for all } x \in U \text{ and } p \in D^+u(x).$$

Similarly, show that $u \in \mathrm{LSC}(\overline{U})$ is a viscosity supersolution of $H(Du, u, x) = 0$ in $U$ if and only if

$$H(p, u(x), x) \geq 0 \quad \text{for all } x \in U \text{ and } p \in D^-u(x).$$

[Hint: Show that $p \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\mathbb{R}^n)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at $x$. A similar statement holds for the subdifferential.]

**Exercise 2.18.** Let $u \in \mathrm{USC}(\mathbb{R}^n)$. Show that the set

$$A := \{x \in \mathbb{R}^n \,:\, D^+u(x) \neq \varnothing\}$$

is dense in $\mathbb{R}^n$. [Hint: Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, and consider the maximum of $u - \varphi_\varepsilon$ over $B(x_0, 1)$, where $\varphi_\varepsilon(x) := \frac{|x-x_0|^2}{\varepsilon}$. Send $\varepsilon \to 0$.]

**Exercise 2.19.** Let $U \subset \mathbb{R}^n$ be open. Suppose that $u \in C(U)$ satisfies

$$u(x) = \fint_{B(x,\varepsilon)} u\, dy + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+$$

for every $x \in U$. Note this is an asymptotic version of the mean value property. Show that $u$ is a viscosity solution of

$$-\Delta u = 0 \quad \text{in} \;\; U.$$

[Hint: Show that for every $\varphi \in C^\infty(\mathbb{R}^n)$

$$-\Delta\varphi(x) = 2(n+2)\fint_{B(x,\varepsilon)} \frac{\varphi(x) - \varphi(y)}{\varepsilon^2}\, dy + o(1) \quad \text{as } \varepsilon \to 0^+.$$

Then verify the viscosity sub- and supersolution properties directly from the definitions.]

**Exercise 2.20.**

(a) Let $u, v \in \mathrm{USC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2w, Dw, w, x) \leq 0 \;\; \text{in U.} \tag{2.7}$$

Show that $w(x) := \max\{u(x), v(x)\}$ is a viscosity solution of (2.7) (i.e., the pointwise maximum of two subsolutions is again a subsolution).

(b) Let $u, v \in \mathrm{LSC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2w, Dw, w, x) \geq 0 \;\; \text{in U.} \tag{2.8}$$

Show that $w(x) := \min\{u(x), v(x)\}$ is a viscosity solution of (2.8).

**Exercise 2.21.** For each $k \in \mathbb{N}$, let $u_k \in C(U)$ be a viscosity solution of

$$H(D^2u_k, Du_k, u_k, x) = 0 \;\; \text{in } U.$$

Suppose that $u_k \to u$ locally uniformly on $U$ (this means $u_k \to u$ uniformly on every $V \subset\subset U$). Show that $u$ is a viscosity solution of

$$H(D^2u, Du, u, x) = 0 \;\; \text{in } U.$$

Thus, viscosity solutions are stable under uniform convergence.

**Exercise 2.22.** Suppose that $p \mapsto H(p,x)$ is *convex* for any fixed $x$. Let $u \in C^{0,1}_{loc}(U)$ satisfy

$$\lambda u(x) + H(Du(x), x) \leq 0 \quad \text{for a.e. } x \in U,$$

where $\lambda \geq 0$. Show that $u$ is a viscosity solution of

$$\lambda u + H(Du, x) \leq 0 \ \text{ in } U.$$

Give an example to show that the same result does not hold for supersolutions. [Hint: Mollify $u$: $u_\varepsilon := \eta_\varepsilon * u$. For $V \subset\subset U$, use Jensen's inequality to show that

$$\lambda u_\varepsilon(x) + H(Du_\varepsilon(x), x) \leq h_\varepsilon(x) \ \text{ for all } x \in V$$

and $\varepsilon > 0$ sufficiently small, where $h_\varepsilon \to 0$ uniformly on $V$. Then apply an argument similar to Exercise 2.21.]

# Chapter 3

# A comparison principle

The utility of viscosity solutions comes from the fact that we can prove existence and uniqueness under very broad assumptions on the Hamiltonian $H$. Uniqueness of viscosity solutions is based on the maximum principle. In this setting, the maximum principle gives a *comparison principle*, which states that subsolutions must lie below supersolutions, provided their boundary conditions do as well.

As motivation, let us give the formal comparison principle argument for smooth sub- and super solutions. Let $u, v \in C^2(U) \cap C(\overline{U})$ such that

$$\left.\begin{aligned} H(D^2u, Du, u, x) < H(D^2v, Dv, v, x) &\quad \text{in } U \\ u \le v &\quad \text{on } \partial U. \end{aligned}\right\} \tag{3.1}$$

We would like to find conditions on $H$ for which (3.1) implies that $u \le v$ in $U$ as well. The classical maximum principle argument examines the maximum of $u - v$ over $U$. We may as well assume $\max_{\overline{U}}(u - v) > 0$, or else we are done. Therefore $u - v$ attains a positive maximum at some $x \in U$. At this maximum we have

$$u(x) > v(x), \quad Du(x) = Dv(x), \quad \text{and} \quad D^2u(x) \le D^2v(x). \tag{3.2}$$

Here, the notation $X \le Y$ for symmetric matrices $X$ and $Y$ means that $Y - X$ is non-negative definite. Writing $p = Du(x) = Dv(x)$ and recalling (3.1) we deduce

$$0 < H(D^2v(x), p, v(x), x) - H(D^2u(x), p, u(x), x).$$

We obtain a contradiction provided

$$H(X, p, r, x) \le H(X, p, s, x) \quad \text{whenever } r \le s, \tag{3.3}$$

and

$$H(X,p,z,x) \geq H(Y,p,z,x) \quad \text{whenever } X \leq Y. \tag{3.4}$$

The condition (3.4) is called *ellipticity*, or sometimes *degenerate ellipticity*. The condition (3.3) is the familiar monotonicity we encountered when studying linear elliptic equations

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu, \tag{3.5}$$

where $c \geq 0$ was necessary for maximum principle arguments to hold.

When $u$ and $v$ are semicontinuous viscosity solutions, it is possible that at a maximum of $u - v$ there may be no admissible test functions for $u$ or $v$ (or both) in the definition of viscosity solution. It is quite extraordinary that the basic structure of the argument can be rescued in this general setting. The proof is based on the unusual (and clever) idea of doubling the variables. We present the argument here in the simplest setting of a bounded domain $U \subset \mathbb{R}^n$ in order to convey the main ideas, and present more general cases later. We also restrict our study to first order equations, where the proofs are considerably simpler. We refer the reader to the user's guide [6] for more details on the comparison principle for second order equations.

We require the additional regularity hypothesis

$$H(p,z,y) - H(p,z,x) \leq \omega(|x-y|(1+|p|)) \tag{3.6}$$

for all $x, y \in U$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$, where $\omega$ is a modulus of continuity (i.e., $\omega$ is nonnegative, $\omega(0) = 0$ and $\omega$ is continuous at 0).

**Theorem 3.1** (Comparison with strict subsolution)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, suppose $H = H(p,z,x)$ satisfies (3.3) and (3.6), and fix $\varepsilon > 0$. Let $u \in USC(\overline{U})$ be a viscosity solution of $H \leq -\varepsilon$ in $U$ and let $v \in LSC(\overline{U})$ be a viscosity solution of $H \geq 0$ in $U$. If $u \leq v$ on $\partial U$ then $u \leq v$ on $U$.*

*Proof.* Assume to the contrary that $\sup_{\overline{U}}(u - v) > 0$. For $\alpha > 0$ consider the auxiliary function

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha}{2}|x-y|^2 \quad (x, y \in \overline{U}). \tag{3.7}$$

Since $u \in \mathrm{USC}(\overline{U})$ and $v \in \mathrm{LSC}(\overline{U})$, $\Phi \in \mathrm{USC}(\overline{U} \times \overline{U})$. As $U$ is bounded, $\Phi$ assumes its maximum value at some $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$. Note that

$$u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 = \Phi(x_\alpha, y_\alpha) \geq \sup_{\overline{U}}(u - v) > 0.$$

As $u$ and $-v$ are bounded above on $\overline{U}$ we have

$$|x_\alpha - y_\alpha|^2 \le \frac{C}{\alpha}.$$

By passing to a subsequence, if necessary, we may assume that $x_\alpha \to x_0$ and $y_\alpha \to x_0$ as $\alpha \to \infty$ for some $x_0 \in \overline{U}$. By the upper semicontinuity of $\Phi$

$$\limsup_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) \le u(x_0) - v(x_0) \le \liminf_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha).$$

Therefore $\lim_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) = u(x_0) - v(x_0)$ and it follows that

$$\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0. \tag{3.8}$$

Furthermore, since $u \le v$ on $\partial U$ and $u(x_0) > v(x_0)$, we must have $x_0 \in U$. Therefore $(x_\alpha, y_\alpha) \in U \times U$ for large enough $\alpha$.

Consider $\varphi(x) = \frac{\alpha}{2}|x - y_\alpha|^2$. By the definition of $(x_\alpha, y_\alpha)$, $u - \varphi$ has a local maximum at the point $x_\alpha \in U$. Therefore

$$H(\alpha(x_\alpha - y_\alpha), u(x_\alpha), x_\alpha) \le -\varepsilon. \tag{3.9}$$

Likewise, $v - \psi$ has a local minimum at $y_\alpha$, where $\psi(y) = -\frac{\alpha}{2}|x_\alpha - y|^2$. Therefore

$$H(\alpha(x_\alpha - y_\alpha), v(y_\alpha), y_\alpha) \ge 0. \tag{3.10}$$

Since $\Phi(x_\alpha, y_\alpha) \ge \sup_{\overline{U}}(u - v) > 0$ we must have $u(x_\alpha) > v(y_\alpha)$. Combining the monotonicity of $H$ (3.3) with (3.10) we have

$$H(\alpha(x_\alpha - y_\alpha), u(x_\alpha), y_\alpha) \ge 0. \tag{3.11}$$

Subtract (3.9) from (3.11) to find

$$\begin{aligned} \varepsilon &\le H(\alpha(x_\alpha - y_\alpha), u(x_\alpha), y_\alpha) - H(\alpha(x_\alpha - y_\alpha), u(x_\alpha), x_\alpha) \\ &\overset{(3.6)}{\le} \omega(\alpha|x_\alpha - y_\alpha|^2 + |x_\alpha - y_\alpha|). \end{aligned}$$

Sending $\alpha \to \infty$ and recalling (3.8) contradicts the positivity of $\varepsilon$. □

It is a good idea to master the proof of Theorem 3.1. The main idea is doubling the number of variables and defining the auxiliary function $\Phi$ (3.7). The rest of the proof boils down to using the boundary conditions to show that $\Phi$ assumes its maximum value at an interior point, and then obtaining a contradiction from the viscosity sub- and supersolution conditions. The proof can be extended to unbounded domains, other boundary conditions, and discontinuous Hamiltonians $H$ by adding appropriate terms to the auxiliary function.

The comparison result in Theorem 3.1 requires the subsolution to be strict, i.e. $H(Du, u, x) \le -\varepsilon < 0$. This condition can be relaxed provided we can perturb a viscosity solution $u$ of $H \le 0$ so that it is a strict subsolution. This requires some sort of strict monotonicity in the Hamiltonian $H$. We first record a general result, and then discuss the type of monotonicity conditions one can place on $H$.

**Corollary 3.2.** *Let $U \subset \mathbb{R}^n$ be open and bounded and suppose $H$ satisfies (3.3) and (3.6). Let $u \in USC(\overline{U})$ be a viscosity solution of $H \le 0$ in $U$ and let $v \in LSC(\overline{U})$ be a viscosity solution of $H \ge 0$ in $U$. Suppose there exists a sequence $u_k \in USC(\overline{U})$ such that $u_k \to u$ pointwise on $U$, $u_k \le v$ on $\partial U$, and each $u_k$ is a viscosity solution of $H \le -\frac{1}{k}$ in $U$. Then $u \le v$ on $U$.*

*Proof.* By comparison with strict subsolution (Theorem 3.1), we have $u_k \le v$ on $U$ for all $k$. Since $u_k \to u$ pointwise on $U$, $u \le v$ on $U$. □

The hypotheses of Corollary 3.2 do not hold in general, and require some further conditions on $H$. We record some important cases here.

1. Suppose there exists $\gamma > 0$ such that
$$H(p, z+h, x) - H(p, z, x) \ge \gamma h \quad (h > 0). \tag{3.12}$$
This is strict form of the monotonicity condition (3.3). Then the hypotheses of Corollary 3.2 hold with $u_k = u - \frac{1}{\gamma k}$. Notice that the monotonicity condition (3.3) allows $H$ to have no dependence on $u$, whereas the strict monotonicity condition (3.12) requires such a dependence. A special case of (3.12) is a Hamilton-Jacobi equation with zeroth order term
$$u + H(Du, x) = 0 \quad \text{in } U.$$

2. Suppose there exists $\gamma > 0$ and $i \in \{1, \dots, n\}$ such that
$$H(p + he_i, z, x) - H(p, z, x) \ge \gamma h \quad (h > 0). \tag{3.13}$$
Here, $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. The hypotheses of Corollary 3.2 hold with $u_k = u - \frac{x_i - a}{\gamma k}$, where $a = \min_{\overline{U}} x_i$. A special case of (3.13) is the time-dependent Hamilton-Jacobi equation
$$u_t + H(Du, x) = 0 \quad \text{in } U \times (0, T).$$

3. Consider the case where $H$ has no dependence on $u$, i.e., $H(p, z, x) = H(p, x)$. Suppose $p \mapsto H(p, x)$ is convex, and there exists $\varphi \in C^\infty(\overline{U})$ such that
$$H(D\varphi(x), x) + \gamma \leq 0 \quad \text{in } U$$
where $\gamma > 0$. Then the hypotheses of Corollary 3.2 hold with
$$u_k = \varepsilon_k \varphi + (1 - \varepsilon_k)u,$$
where $\varepsilon_k = \frac{1}{\gamma k}$. Note that we can assume that $\varphi \leq 0$, due to the fact that $H$ has no dependence on $u$. A special case is the eikonal equation (1.5), in which case we can take $\varphi \equiv 0$.

**Exercise 3.3.** For each of the cases listed above, verify that $u_k$ is a viscosity solution of $H \leq -\frac{1}{k}$ in $U$, $u_k \leq u$ for all $k$, and $u_k \to u$ uniformly on $\overline{U}$.

The comparison principle from Corollary 3.2 shows that if $H$ satisfies (3.3) and (3.6), and any one of the conditions listed above holds, then there exists at most one viscosity solution $u \in C(\overline{U})$ of the Dirichlet problem

$$\left.\begin{aligned} H(Du, u, x) &= 0 \quad \text{in } U \\ u &= g \quad \text{on } \partial U \end{aligned}\right\} \tag{3.14}$$

where $g : \partial U \to \mathbb{R}$ is continuous and $U \subset \mathbb{R}^n$ is open and bounded. Indeed, suppose $u, v \in C(\overline{U})$ are viscosity solutions of $H = 0$ in $U$. Then by (3.3) $u - C$ is a viscosity subsolution of $H = 0$ in $U$ for all $C > 0$. Setting $C = \max_{\partial U} |u - v|$ we have
$$u - C = u - \max_{\partial U} |u - v| \leq v \quad \text{on } \partial U.$$
By Corollary (3.2), $u - C \leq v$ in $U$. Swapping the roles of $u$ and $v$ we have $v - C \leq u$ in $U$. Therefore
$$\max_{\overline{U}} |u - v| \leq \max_{\partial U} |u - v|.$$
Hence, if $u = v = g$ on $\partial U$, then $u = v$ in $U$.

**Exercise 3.4.** Show that the following PDEs are degenerate elliptic.

(i) The linear elliptic operator (3.5), provided $\sum_{i,j=1}^n a^{ij}\eta_i\eta_j \geq 0$.

(ii) The Monge-Ampère equation
$$-\det(D^2 u) + f = 0,$$
provided $u$ is convex.

(iii) The $\infty$-Laplace equation $-\Delta_\infty u + f = 0$, where

$$\Delta_\infty u := \frac{1}{|Du|^2} \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}.$$

(iv) The level-set equation for motion by mean curvature

$$u_t - \Delta u + \frac{1}{|Du|^2} \sum_{i,j=1}^n u_{x_i x_j} u_{x_i} u_{x_j} = 0.$$

[Hint: For (ii), use the linear algebra identity

$$\det(A) = \min_{v_1,\ldots,v_n} \prod_{i=1}^n v_i^T A v_i,$$

where $A \geq 0$ is symmetric, and the minimum is over all orthonormal bases of $\mathbb{R}^n$. This is closely related to Hadamard's determinant inequality. For (iv), write the equation as

$$u_t - \sum_{i,j=1}^n a^{ij}(Du) u_{x_i x_j} = 0,$$

where

$$a^{ij}(Du) = \delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}.$$

Verify the ellipticity when $Du \neq 0$. (Remark: To handle $Du = 0$, we redefine viscosity solutions by taking the upper and lower semicontinuous envelopes of

$$F(X,p) = \sum_{i,j=1}^n a^{ij}(p) X_{ij},$$

in the sub- and supersolution properties, respectively.)]

**Exercise 3.5.** Consider the Hamilton-Jacobi equation

$$\text{(H)} \quad \begin{cases} |Du| - \sqrt{u} = 0 & \text{in } B^0(0,1) \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

(a) Show that there are infinitely many nonnegative viscosity solutions $u \in C(B(0,1))$ of (H). [Hint: For every $0 \leq \lambda \leq 1$, show that

$$u_\lambda(x) := \frac{1}{4} \max\{0, \lambda - |x|\}^2$$

is a viscosity solution of (H).]

(b) Explain why uniqueness fails for (H), i.e., which hypothesis from Theorem 3.1 is not satisfied.

(c) Show that $u_1$ (i.e., $u_\lambda$ with $\lambda := 1$) is the unique viscosity solution of (H) that is positive on $B^0(0,1)$. [Hint: Show that if $\widetilde{u}$ is another viscosity solution of (H) that is positive on $B^0(0,1)$, then $w := 2\sqrt{u}$ and $\widetilde{w} := 2\sqrt{\widetilde{u}}$ are both viscosity solutions of the eikonal equation in $B^0(0,1)$.]

# Chapter 4

# The Hamilton-Jacobi-Bellman equation

We now aim to generalize the distance function example in Section 1.1. Consider the following calculus of variations problem:

$$T(x,y) = \inf\Big\{ I[\mathbf{w}] \,:\, \mathbf{w} \in C^1([0,1];\overline{U}),\ \mathbf{w}(0) = x, \ \text{and } \mathbf{w}(1) = y \Big\}, \quad (4.1)$$

where

$$I[\mathbf{w}] := \int_0^1 L(\mathbf{w}'(t), \mathbf{w}(t))\, dt. \quad (4.2)$$

Here, $U \subset \mathbb{R}^n$ is open, bounded, and path connected with a Lipschitz boundary, and $x, y \in \overline{U}$. We assume that $L : \mathbb{R}^n \times \overline{U} \to \mathbb{R}$ is continuous,

$$p \mapsto L(p,x) \text{ is positively 1-homogeneous,} \quad (4.3)$$

and

$$L(p,x) > 0 \text{ for all } p \neq 0,\ x \in \overline{U}. \quad (4.4)$$

Recall that positively 1-homogeneous means means that $L(\alpha p, x) = \alpha L(p,x)$ for all $\alpha > 0$ and $x \in \overline{U}$.

Let us note a few consequences of these assumptions. First, the 1-homogeneity requires that $L(0,x) = 0$ for all $x$. Since $L$ is continuous on the compact set $\{|p| = 1\} \times \overline{U}$, and $L(p,x) > 0$ for $p \neq 0$, we have

$$\gamma := \inf_{\substack{|p|=1 \\ x \in \overline{U}}} \{L(p,x)\} > 0.$$

Since $L$ is 1-homogeneous, we conclude that

$$L(p,x) \geq \gamma |p| \quad \text{for all } p \in \mathbb{R}^n,\ x \in \overline{U}. \quad (4.5)$$

Therefore, for any $\mathbf{w} \in C^1([0,1];\overline{U})$

$$\gamma\,\ell(\mathbf{w}) = \gamma \int_0^1 |\mathbf{w}'(t)|\,dt \leq \int_0^1 L(\mathbf{w}'(t),\mathbf{w}(t))\,dt = I[\mathbf{w}], \tag{4.6}$$

where $\ell(\mathbf{w})$ denotes the length of $\mathbf{w}$. Hence, curves that minimize, or nearly minimize $I$ must have bounded length.

Instead of looking for minimizing curves $\mathbf{w}$ via the Euler-Lagrange equations, we consider the *value function*

$$u(x) = \inf_{y\in\partial U} \left\{g(y) + T(x,y)\right\}, \tag{4.7}$$

where $g : \partial U \to \mathbb{R}$. In the case where $U$ is convex, $L(p,x) = |p|$, and $g \equiv 0$, $u$ is the distance function to the boundary $\partial U$. We assume throughout this section that the *compatibility condition*

$$g(x) - g(y) \leq T(x,y) \quad \text{for all } x,y \in \partial U \tag{4.8}$$

holds. The compatibility condition ensures that $u$ assumes its boundary values $u = g$ on $\partial U$ continuously.

**Proposition 4.1.** *For any $x,y \in \overline{U}$ such that the line segment between $x$ and $y$ belongs to $\overline{U}$ we have*

$$T(x,y) \leq K|x-y|, \tag{4.9}$$

*where $K = \sup_{x\in\overline{U},|p|=1} L(p,x)$.*

*Proof.* Take $\mathbf{w}(t) = x + t(y-x)$. Then

$$T(x,y) \leq \int_0^1 L(y-x, x+t(y-x))\,dt = |x-y| \int_0^1 L(p, x+t(y-x))\,dt.$$

where $p = (y-x)/|y-x|$ and we used the fact that $L$ is positively 1-homogeneous. The result immediately follows. $\square$

**Lemma 4.2.** *For all $x,y,z \in \overline{U}$ we have*

$$T(x,z) \leq T(x,y) + T(y,z). \tag{4.10}$$

*Proof.* Let $\varepsilon > 0$. For $i = 1,2$, let $\mathbf{w}_i \in C^1([0,1];\overline{U})$ such that $\mathbf{w}_1(0) = x$, $\mathbf{w}_1(1) = y$, $\mathbf{w}_2(0) = y$, $\mathbf{w}_2(1) = z$ and

$$T(x,y) + T(y,z) + \varepsilon \geq I[\mathbf{w}_1] + I[\mathbf{w}_2].$$

Define

$$\mathbf{w}(t) = \begin{cases} \mathbf{w}_1(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ \mathbf{w}_2(2t-1), & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Note we can reparameterize $\mathbf{w}$, if necessary, so that $\mathbf{w} \in C^1([0,1]; \overline{U})$, and we easily compute that $I[\mathbf{w}_1] + I[\mathbf{w}_2] = I[\mathbf{w}]$. Since $\mathbf{w}(0) = x$ and $\mathbf{w}(1) = z$ we have

$$T(x,z) \le I[\mathbf{w}] \le T(x,y) + T(y,z) + \varepsilon.$$

Sending $\varepsilon \to 0$ completes the proof. □

We now establish the important dynamic programming principle for the value function $u$.

**Lemma 4.3.** *For every $B(x,r) \subset U$ we have*

$$u(x) = \inf_{y \in \partial B(x,r)} \{u(y) + T(x,y)\}. \tag{4.11}$$

*Proof.* Fix $x \in U$ with $B(x,r) \subset U$, and let $v(x)$ denote the right hand side of (4.11).

We first show that $u(x) \ge v(x)$. Let $\varepsilon > 0$. Then there exists $z \in \partial U$ and $\mathbf{w} \in C^1([0,1]; \overline{U})$ such that $\mathbf{w}(0) = x$, $\mathbf{w}(1) = z$ and

$$g(z) + I[\mathbf{w}] \le u(x) + \varepsilon. \tag{4.12}$$

Let $y \in \partial B(x,r)$ and $s \in (0,1)$ such that $\mathbf{w}(s) = y$. Define

$$\mathbf{w}_1(t) = \mathbf{w}(st) \quad \text{and} \quad \mathbf{w}_2(t) = \mathbf{w}(s + t(1-s)).$$

Then we have $I[\mathbf{w}_1] + I[\mathbf{w}_2] = I[\mathbf{w}]$. Furthermore, $\mathbf{w}_1(0) = x$, $\mathbf{w}_1(1) = \mathbf{w}_2(0) = y$ and $\mathbf{w}_2(1) = z$. Combining these observations with (4.12) and (4.1) we have

$$u(x) + \varepsilon \ge g(z) + I[\mathbf{w}_1] + I[\mathbf{w}_2] \ge u(y) + T(x,y) \ge v(x).$$

Since $\varepsilon > 0$ is arbitrary, $u(x) \ge v(x)$.

To show that $u(x) \le v(x)$, note that by Lemma 4.2 we have

$$g(z) + T(x,z) \le g(z) + T(y,z) + T(x,y),$$

for any $y \in U$ and $z \in \partial U$. Therefore

$$\begin{aligned} u(x) &= \inf_{z \in \partial U} \{g(z) + T(x,z)\} \\ &\le \inf_{z \in \partial U} \{g(z) + T(y,z)\} + T(x,y) \\ &= u(y) + T(x,y), \end{aligned} \tag{4.13}$$

for any $y \in U$, and the result easily follows. □

We now establish regularity of the value function $u$.

**Lemma 4.4.** *The value function $u$ is locally Lipschitz continuous in $U$ and assumes the boundary values $u = g$ on $\partial U$, in the sense that for all $x \in \partial U$*

$$\lim_{\substack{y\to x\\ y\in U}} u(y) = g(x).$$

*Proof.* Let $x, y \in U$ such that the line segment between $x$ and $y$ is contained in $U$. By (4.13) and Proposition 4.1 we have

$$u(x) \leq u(y) + T(x,y) \leq u(y) + K|x-y|.$$

Therefore $u$ is Lipschitz on any convex subset of $U$, and hence $u$ is locally Lipschitz.

To show that $u$ assumes the boundary values $g$, we need to use the compatibility condition (4.8) and the Lipschitzness of the boundary $\partial U$. Fix $x_0 \in \partial U$. Up to orthogonal transformation, we may assume that $x_0 = 0$ and

$$U \cap B(0,r) = \{x \in B(0,r) \, : \, x_n \geq h(\widetilde{x})\},$$

for $r > 0$ sufficiently small, where $x = (\widetilde{x}, x_n) \in \mathbb{R}^n$, $h : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz continuous, and $h(0) = 0$. Let $x \in U \cap B(0,r)$ and define

$$y = (x_1, \ldots, x_{n-1}, \mathrm{Lip}(h)|\widetilde{x}|).$$

Then $|x - y| \leq C|x|$ and $y_n = \mathrm{Lip}(h)|\widetilde{y}|$. It follows that the line segment from $y$ to $0$ is contained in $U$, as well as the segment from $x$ to $y$. In light of Proposition 4.1 and Lemma 4.2 we have

$$\begin{aligned} u(x) &= \inf_{z\in\partial U}\{g(z) + T(x,z)\} \\ &\leq g(0) + T(x,0) \\ &\leq g(0) + T(x,y) + T(y,0) \\ &\leq g(0) + C|x-y| + C|y| \\ &\leq g(0) + C|x|. \end{aligned}$$

Now let $\varepsilon > 0$ and $z \in \partial U$ such that

$$u(x) + \varepsilon \geq g(z) + T(x,z).$$

Invoking the compatibility condition (4.8) we have

$$\begin{aligned} u(x) + \varepsilon &\geq g(0) - T(0,z) + T(x,z) \\ &\geq g(0) - T(0,x) \geq g(0) - C|x|. \end{aligned}$$

Therefore $|u(x) - g(0)| \leq C|x|$ for all $x \in U \cap B(0,r)$, and the result immediately follows. ∎

We can now characterize $u$ as the viscosity solution of a Hamilton-Jacobi equation. We define

$$H(p,x) = \sup_{|a|=1} \{-p \cdot a - L(a,x)\}. \tag{4.14}$$

**Lemma 4.5.** *$H$ is convex in $p$ and satisfies (3.6).*

*Proof.* Let $p, q \in \mathbb{R}^n$, $x \in U$, and $\lambda \in (0,1)$. Then there exists $a \in \mathbb{R}^n$ with $|a| = 1$ such that

$$H(\lambda p + (1-\lambda)q, x) = -(\lambda p + (1-\lambda)q) \cdot a - L(a,x).$$

We compute

$$\begin{aligned} H(\lambda p + (1-\lambda)q, x) &= \lambda(-p \cdot a - L(a,x)) + (1-\lambda)(-q \cdot a - L(a,x)) \\ &\leq \lambda H(p,x) + (1-\lambda)H(q,x). \end{aligned}$$

Therefore $p \mapsto H(p,x)$ is convex.

Now fix $p \in \mathbb{R}^n$ and $x, y \in U$. There exists $a \in \mathbb{R}^n$ with $|a| = 1$ such that

$$H(p,x) = -p \cdot a - L(a,x).$$

Therefore we have

$$\begin{aligned} H(p,x) - H(p,y) &\leq -p \cdot a - L(a,x) - (-p \cdot a - L(a,y)) \\ &= L(a,y) - L(a,x) \leq \omega(|x-y|), \end{aligned}$$

where $\omega$ is the modulus of continuity of $L$. □

**Theorem 4.6.** *The value function $u$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$\left.\begin{aligned} H(Du,x) &= 0 \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}\right\} \tag{4.15}$$

*Proof.* We first verify that $u$ is a viscosity subsolution of (4.15). Let $x \in U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$. Choose $r > 0$ sufficiently small so that

$$u(y) - u(x) \leq \varphi(y) - \varphi(x) \quad \text{for all } y \in B(x,r) \subset U.$$

By the dynamic programming principle (4.11)

$$0 = \inf_{y \in \partial B(x,r)} \{u(y) - u(x) + T(x,y)\} \leq \inf_{y \in \partial B(x,r)} \{\varphi(y) - \varphi(x) + T(x,y)\}.$$

Since $\varphi \in C^\infty(\mathbb{R}^n)$ there exists $C > 0$ such that

$$\varphi(y) - \varphi(x) \le D\varphi(x) \cdot (y-x) + C|x-y|^2 \quad \text{for all } y \in B(x,r).$$

Therefore we have

$$0 \le \inf_{y \in \partial B(x,r)} \{D\varphi(x) \cdot (y-x) + T(x,y)\} + Cr^2. \tag{4.16}$$

Notice now that

$$\begin{aligned} T(x,y) &\le r \int_0^1 L\left(\tfrac{y-x}{r}, x + t(y-x)\right)\, dt \\ &\le r \int_0^1 L\left(\tfrac{y-x}{r}, x\right) + o(1)\, dt \\ &= rL\left(\tfrac{y-x}{r}, x\right) + o(r) \end{aligned}$$

as $r \to 0^+$, due to the continuity of $L$. Inserting this into (4.16) and dividing by $r$ we have

$$0 \le \inf_{y \in \partial B(x,r)} \left\{D\varphi(x) \cdot \tfrac{y-x}{r} + L\left(\tfrac{y-x}{r}, x\right)\right\} + o(1),$$

as $r \to 0^+$. Setting $a = (y-x)/r$ and sending $r \to 0^+$ we find that

$$H(D\varphi(x), x) = -\inf_{|a|=1} \{D\varphi(x) \cdot a + L(a,x)\} \le 0.$$

Therefore $u$ is a viscosity subsolution of (4.15).

We now show that $u$ is a viscosity supersolution of (4.15). Let $x \in U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$. We must show that

$$H(D\varphi(x), x) = -\inf_{|a|=1} \{D\varphi(x) \cdot a + L(a,x)\} \ge 0.$$

Suppose to the contrary that there exists $\theta > 0$ such that

$$\inf_{|a|=1} \{D\varphi(x) \cdot a + L(a,x)\} \ge \theta.$$

Then there exists $r_0 > 0$ such that $B(x, r_0) \subset U$ and

$$D\varphi(y) \cdot a + L(a,y) \ge |a|\frac{\theta}{2} \tag{4.17}$$

for all $a \in \mathbb{R}^n$ and $y \in B(x, r_0)$. Notice we used the 1-homogeneity of $L$ above. Since $u - \varphi$ has a local minimum at $x$, we may as well also assume that

$$u(y) - u(x) \ge \varphi(y) - \varphi(x) \quad \text{for all } y \in B(x, r_0).$$

Let $0 < r < r_0$. By the dynamic programming principle (4.11) there exist $y \in \partial B(x,r)$ and $\mathbf{w} \in C^1([0,1];\overline{U})$ with $\mathbf{w}(0) = x$ and $\mathbf{w}(1) = y$ such that

$$u(x) \geq u(y) + I[\mathbf{w}] - \frac{\theta}{4}r. \tag{4.18}$$

By (4.6) and Lemma 4.4 we have

$$\gamma\,\ell(\mathbf{w}) \leq I[\mathbf{w}] \leq u(x) - u(y) + \frac{\theta}{4}r \leq Cr.$$

Fix $0 < r < r_0$ small enough so that $\ell(\mathbf{w}) < r_0$. Then $\mathbf{w}(t) \in B^0(x, r_0)$ for all $t \in [0,1]$. We can now invoke (4.17) to find that

$$\begin{aligned}
u(y) - u(x) &\geq \varphi(y) - \varphi(x) \\
&= \int_0^1 \frac{d}{dt}\varphi(\mathbf{w}(t))\,dt \\
&= \int_0^1 D\varphi(\mathbf{w}(t)) \cdot \mathbf{w}'(t)\,dt \\
\text{by (4.17)} \qquad &\geq \frac{\theta}{2}\int_0^1 |\mathbf{w}'(t)|\,dt - \int_0^1 L(\mathbf{w}'(t), \mathbf{w}(t))\,dt \\
&\geq \frac{\theta}{2}r - \int_0^1 L(\mathbf{w}'(t), \mathbf{w}(t))\,dt.
\end{aligned}$$

Combining this with (4.18) we have

$$\frac{\theta}{4}r \geq u(y) - u(x) + \int_0^1 L(\mathbf{w}'(t), \mathbf{w}(t))\,dt \geq \frac{\theta}{2}r,$$

which is a contradiction.

Note that $\varphi = 0$ is smooth strict subsolution, since

$$H(D\varphi(x), x) = -\inf_{|a|=1}\{L(a,x)\} \leq -\inf_{\substack{|a|=1 \\ y \in \overline{U}}}\{L(a,y)\} = -\gamma < 0.$$

Therefore $H(D\varphi(x), x) + \gamma \leq 0$ in $U$. Since $p \mapsto H(p,x)$ is convex, Corollary 3.2 and the remarks thereafter guarantee that $u$ is the unique viscosity solution of (4.15). □

**Remark 4.7.** We remark that Theorem 4.6 establishes existence of a viscosity solution of (4.15) when $H$ is given by (4.14). In Chapter 5 we show how to establish existence with the method of vanishing viscosity. Other techniques for establishing existence include the Perron method [6] and convergence of finite difference approximations.

**Exercise 4.8.** Let $1 < p < \infty$ and define

$$|x|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Assume $U \subset \mathbb{R}^n$ is open, bounded, and path connected with Lipschitz boundary $\partial U$, and let $f : \overline{U} \to \mathbb{R}$ be continuous and positive. Show that there exists a unique viscosity solution $u \in C(\overline{U})$ of the p-eikonal equation

$$\text{(P)} \quad \begin{cases} |Du|_p = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

[Hint: Construct $u$ as the value function

$$u(x) = \inf\{T(x, y) \,:\, y \in \partial U\},$$

where

$$T(x, y) = \inf \left\{ I[\mathbf{w}] \,:\, \mathbf{w} \in C^1([0, 1]; \overline{U}),\ \mathbf{w}(0) = x,\ \mathbf{w}(1) = y \right\},$$

$$I[\mathbf{w}] = \int_0^1 f(\mathbf{w}(t)) |\mathbf{w}'(t)|_q \, dt,$$

and $q$ is the Hölder conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.]

# Chapter 5

# Convergence of vanishing viscosity

Consider the viscous Hamilton-Jacobi equation

$$\left.\begin{aligned} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon\Delta u_\varepsilon = 0 \quad &\text{in } U \\ u_\varepsilon = 0 \quad &\text{on } \partial U. \end{aligned}\right\} \tag{5.1}$$

In this section we show that solutions $u_\varepsilon$ of (5.1) converge as $\varepsilon \to 0$ to the unique viscosity solution of

$$\left.\begin{aligned} u + H(Du, x) = 0 \quad &\text{in } U \\ u = 0 \quad &\text{on } \partial U. \end{aligned}\right\} \tag{5.2}$$

The main structural assumptions we place on $H$ are *coercivity*:

$$\liminf_{|p|\to\infty} H(p,x) > 0 \quad \text{uniformly in } x \in U, \tag{5.3}$$

and *nonnegativity*

$$-H(0,x) \geq 0 \quad \text{for all } x \in U. \tag{5.4}$$

The reason we call (5.4) *nonnegativity* is that when $H(p,x) = G(p) - f(x)$ and $G(0) \geq 0$, (5.4) implies that $f \geq 0$.

The main structural condition we place on $U$ is the following *exterior sphere condition*: There exists $r > 0$ such that for every $x_0 \in \partial U$ there is a point $x_0^* \in \mathbb{R}^n \setminus \overline{U}$ for which

$$B(x_0^*, r) \cap \overline{U} = \{x_0\}. \tag{5.5}$$

Throughout this section we assume that $U \subset \mathbb{R}^n$ is open, bounded, and satisfies the exterior sphere condition, and $H$ satisfies (3.6) and is continuous, coercive, and nonnegative. As a result, Corollary 3.2 guarantees that a comparison principle holds for (5.2). We note the results in this section hold under more general assumptions than these, but the proofs are particularly simple and illustrative in this special case.

## 5.1 Weak upper and lower limits

Our first technique for proving convergence of the vanishing viscosity method will be the method of weak upper and lower limits. These techniques are very general and apply to a wide range of problems.

We first need some basic estimates on solutions of (5.1).

**Lemma 5.1.** *Let $\varepsilon > 0$ and let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ be a solution of (5.1). Then*

$$0 \le u_\varepsilon \le \sup_{x \in U} |H(0,x)| \quad \text{in } U. \tag{5.6}$$

*Proof.* The argument is based on the maximum principle. Due to the compactness of $\overline{U}$, $u_\varepsilon$ must attain its maximum value at some $x_0 \in \overline{U}$. If $x_0 \in \partial U$ then $u(x_0) = 0$. If $x_0 \in U$ then $Du_\varepsilon(x_0) = 0$ and $\Delta u_\varepsilon(x_0) \le 0$. Therefore

$$u_\varepsilon(x_0) = \varepsilon \Delta u_\varepsilon(x_0) - H(Du_\varepsilon(x_0), x_0) \le \sup_{x \in U} |H(0,x)|.$$

Likewise, $u_\varepsilon$ attains its minimum value at some $y_0 \in \overline{U}$. If $y_0 \in \partial U$ then $u_\varepsilon(y_0) = 0$. If $y_0 \in U$ then $Du_\varepsilon(x_0) = 0$ and $\Delta u_\varepsilon(x_0) \ge 0$. Since $H$ is nonnegative (5.4)

$$u_\varepsilon(y_0) = \varepsilon \Delta u_\varepsilon(y_0) - H(0, y_0) \ge 0.$$

Therefore $u_\varepsilon \ge 0$ throughout $U$. □

**Definition 5.2.** Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of real-valued functions on $\overline{U}$.

The *upper weak limit* $\overline{u} : \overline{U} \to \mathbb{R}$ of the family $\{u_\varepsilon\}_{\varepsilon>0}$ is defined by

$$\overline{u}(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y). \tag{5.7}$$

Similarly, the *lower weak limit* $\underline{u} : \overline{U} \to \mathbb{R}$ is defined by

$$\underline{u}(x) = \liminf_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y). \tag{5.8}$$

The limits above are taken with $y \in \overline{U}$.

The upper and lower weak limits are fundamental objects in the theory of viscosity solutions and allow *passage to the limit* in a wide variety of applications.

**Lemma 5.3.** *Suppose the family $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded. Then $\overline{u} \in USC(\overline{U})$ and $\underline{u} \in LSC(\overline{U})$.*

*Proof.* By the uniform boundedness assumption, $\overline{u}$ and $\underline{u}$ are bounded real-valued functions on $\overline{U}$. We will show that $\overline{u} \in \mathrm{USC}(\overline{U})$; the proof that $\underline{u} \in \mathrm{LSC}(\overline{U})$ is very similar.

We assume by way of contradiction that $x_k \to x$ and $\overline{u}(x_k) \geq \overline{u}(x) + \delta$ for some $\delta > 0$ and all $k$ large enough, where $x_k, x \in \overline{U}$. By the definition of $\overline{u}$, for each $k$ there exists $y_k$ and $\varepsilon_k$ such that $|x_k - y_k| < 1/k$, $\varepsilon_k < 1/k$ and

$$u_{\varepsilon_k}(y_k) \geq \overline{u}(x_k) - \frac{\delta}{2} \geq \overline{u}(x) + \frac{\delta}{2}$$

for sufficiently large $k$. Therefore

$$\liminf_{k\to\infty} u_{\varepsilon_k}(y_k) > \overline{u}(x),$$

which is a contradiction to the definition of $\overline{u}$, since $y_k \to x$ and $\varepsilon_k \to 0^+$. $\square$

**Theorem 5.4.** *For each $\varepsilon > 0$ let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ solve (5.1). Then $u_\varepsilon \to u$ uniformly on $\overline{U}$ as $\varepsilon \to 0^+$, where $u$ is the unique viscosity solution of (5.2).*

The idea of the proof is to show that $\overline{u}$ is a viscosity subsolution of (5.2), and $\underline{u}$ is a viscosity supersolution of (5.2). Then provided we can show that $\overline{u} \leq \underline{u}$ on $\partial U$, the comparison principle will show that $\overline{u} \equiv \underline{u}$, and uniform convergence follows.

*Proof.* By Lemma 5.1, the family $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded, hence by Lemma 5.3, $\overline{u} \in \mathrm{USC}(\overline{U})$ and $\underline{u} \in \mathrm{LSC}(\overline{U})$.

We claim that $\overline{u}$ is a viscosity solution of $\overline{u} + H(D\overline{u}, x) \leq 0$ in $U$. To establish the claim, let $x \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a local maximum at $x$. By replacing $\varphi(y)$ with $\varphi(y) + |x-y|^2$, there is no loss in assuming that $\overline{u} - \varphi$ has a strict local maximum at $x$. It follows that there exists $x_k \to x$ and $\varepsilon_k \to 0^+$ such that $u_{\varepsilon_k}(x_k) \to \overline{u}(x)$ and $u_{\varepsilon_k} - \varphi$ has a local maximum at $x_k$[1]. Since $u_\varepsilon$ is twice continuously differentiable and $\varphi$ is smooth, we have

$$Du_{\varepsilon_k}(x_k) = D\varphi(x_k) \quad \text{and} \quad \Delta u_{\varepsilon_k}(x_k) \leq \Delta\varphi(x_k).$$

It follows that

$$\begin{aligned}\overline{u}(x) + H(D\varphi(x), x) &= \lim_{k\to\infty} u_{\varepsilon_k}(x_k) + H(D\varphi(x_k), x_k) - \varepsilon_k \Delta\varphi(x_k) \\ &\leq \lim_{k\to\infty} u_{\varepsilon_k}(x_k) + H(Du_{\varepsilon_k}(x_k), x_k) - \varepsilon_k \Delta u_{\varepsilon_k}(x_k) = 0.\end{aligned}$$

[1]See Exercise 5.5

Therefore $\overline{u}$ is a viscosity solution of $\overline{u} + H(D\overline{u}, x) \le 0$ in $U$. A similar argument shows that $\underline{u}$ is a viscosity solution of $\underline{u} + H(D\underline{u}, x) \ge 0$ in $U$.

By Lemma 5.1, $u_\varepsilon \ge 0$ for all $\varepsilon > 0$, thus $\underline{u} \ge 0$ on $\partial U$. We claim that $\overline{u} \le 0$ on $\partial U$. To see this, fix $x_0 \in \partial U$. By coercivity of $H$ (5.3), select $C > 0$ and $\delta > 0$ so that

$$H(Cp, x) \ge \delta \quad \text{for all } x \in U \text{ and } |p| = 1.$$

By the exterior sphere condition there exists $r > 0$ and $x_0^* \in \mathbb{R}^n \setminus \overline{U}$ such that $|x_0 - x_0^*| = r$ and

$$\psi(x) := C(|x - x_0^*| - r) \ge 0 \quad \text{for all } x \in \overline{U}.$$

We note that

$$|D\psi(x)| = C \quad \text{and} \quad \Delta\psi(x) = \frac{C(n-1)}{|x - x_0^*|} \le \frac{C(n-1)}{r}$$

for all $x \in U$. It follows that

$$\psi(x) + H(D\psi(x), x) - \varepsilon\Delta\psi(x) \ge \delta - C(n-1)\varepsilon r^{-1}$$

for all $x \in U$. Since $\psi \ge 0$ on $\partial U$, we can for sufficiently small $\varepsilon > 0$ use a maximum principle argument similar to Lemma 5.1 to show that $u_\varepsilon \le \psi$ on $\overline{U}$. It follows that

$$\overline{u}(x_0) \le \limsup_{x \to x_0} \psi(x) = 0.$$

This establishes the claim.

We have shown that $\overline{u} + H(D\overline{u}, x) \le 0$ and $\underline{u} + H(D\underline{u}, x) \ge 0$ in the viscosity sense, and $\overline{u} \le \underline{u}$ on $\partial U$. By the comparison principle from Corollary 3.2 and the remarks thereafter, $\overline{u} \le \underline{u}$ on $U$. Since $\underline{u} \le \overline{u}$ by definition, we have $\overline{u} \equiv \underline{u}$. It follows that $u_\varepsilon \to u$ uniformly on $\overline{u}$ as $\varepsilon \to 0^+$[2]. □

**Exercise 5.5.** Suppose $\overline{u} - \varphi$ has a strict local maximum at $x \in U$. Show that there exists $x_k \to x$ and $\varepsilon_k \to 0$ such that $u_{\varepsilon_k}(x_k) \to \overline{u}(x)$ and $u_{\varepsilon_k} - \varphi$ has a local maximum at $x_k$.

**Exercise 5.6.** Show that if $\overline{u} \equiv \underline{u}$ then $u_\varepsilon \to u$ uniformly on $\overline{U}$.

[2] See Exercise 5.6

## 5.2 The $O(\sqrt{\varepsilon})$ rate

With a bit more work, and a few more hypotheses on $H$, we can actually say something about the rate of convergence $u_\varepsilon \to u$. This also gives us a second approach for proving convergence of the vanishing viscosity method.

The main ingredient is the Lipschitz continuity of the viscosity solution of (5.2). The following lemma gives a standard argument for proving that viscosity subsolutions are Lipschitz continuous.

**Lemma 5.7.** *Let $u \in USC(\overline{U})$ be a nonnegative viscosity subsolution of (5.2). Then there exists $C$ depending only on $H$ such that*

$$|u(x) - u(y)| \le C|x - y| \quad \text{for all } x, y \in \overline{U}.$$

*Proof.* By coercivity of $H$ (5.3) select $C > 0$ and $\delta > 0$ so that

$$H(Cp, x) \ge \delta \quad \text{for all } x \in U \text{ and } |p| = 1. \tag{5.9}$$

Fix $x \in U$ and define

$$w(y) = u(y) - C|y - x|.$$

Then $w$ attains its maximum at some $y_0 \in \overline{U}$. If $y_0 \in \partial U$ then since $u \le 0$ on $\partial U$ we have

$$u(x) = w(x) \le w(y_0) = u(y_0) - C|y_0 - x| < 0,$$

which contradicts the nonnegativity of $u$. If $y_0 \in U$ and $y_0 \ne x$, then the viscosity subsolution property implies that

$$u(y_0) + H\left(C\frac{y_0 - x}{|y_0 - x|}, y_0\right) \le 0, \tag{5.10}$$

which contradicts (5.9). Therefore $w$ must attain its maximum at $y_0 = x$ and we have

$$u(y) - C|x - y| = w(y) \le w(x) = u(x)$$

for all $y \in \overline{U}$. Since $x \in U$ was arbitrary, the result follows. □

To prove a rate of convergence $u_\varepsilon \to u$, we need to assume Lipschitz regularity of $H$. In particular, instead of (3.6), we assume that for every $R > 0$ there exists $C_R$ such that

$$H(p, y) - H(p, x) \le C_R|x - y| \quad \text{for all } x, y \in U \text{ and } |p| \le R. \tag{5.11}$$

**Theorem 5.8.** *For each $\varepsilon > 0$, let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ solve (5.1), and let $u$ be the unique viscosity solution of (5.2). Then there exists $C$ depending only on $H$ such that*

$$|u - u_\varepsilon| \leq C\sqrt{\varepsilon}.$$

*Proof.* We first show that $u - u_\varepsilon \leq C\sqrt{\varepsilon}$. Define

$$\Phi(x, y) = u(x) - u_\varepsilon(y) - \frac{\alpha}{2}|x - y|^2,$$

where $\alpha$ is to be determined. Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\max_{\overline{U}\times\overline{U}} \Phi = \Phi(x_\alpha, y_\alpha).$$

Since $\Phi(x_\alpha, y_\alpha) \geq \Phi(y_\alpha, y_\alpha)$ we have

$$\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq C|x_\alpha - y_\alpha|,$$

owing to the Lipschitz estimate from Lemma 5.7. Therefore

$$|x_\alpha - y_\alpha| \leq \frac{C}{\alpha}. \tag{5.12}$$

We claim that

$$u(x_\alpha) - u_\varepsilon(y_\alpha) \leq C\left(\tfrac{1}{\alpha} + \alpha\varepsilon\right). \tag{5.13}$$

To see this: If $x_\alpha \in \partial U$ then

$$u(x_\alpha) - u_\varepsilon(y_\alpha) \leq 0,$$

due to Lemma 5.1 and the boundary condition $u = 0$ on $\partial U$. If $y_\alpha \in \partial U$ then

$$u(x_\alpha) - u_\varepsilon(y_\alpha) \leq u(x_\alpha) - u(y_\alpha) \leq C|x_\alpha - y_\alpha| \leq \frac{C}{\alpha}.$$

If $(x_\alpha, y_\alpha) \in U \times U$ then $x \mapsto u(x) - \frac{\alpha}{2}|x - y_\alpha|^2$ has a maximum at $x_\alpha$ and hence

$$u(x_\alpha) + H(p_\alpha, x_\alpha) \leq 0, \tag{5.14}$$

where $p_\alpha = \alpha(x_\alpha - y_\alpha)$. Similarly, $y \mapsto u_\varepsilon(y) + \frac{\alpha}{2}|x_\alpha - y|^2$ has a minimum at $y_\alpha$ and hence $Du_\varepsilon(y_\alpha) = p_\alpha$ and $-\Delta u_\varepsilon(y_\alpha) \leq \alpha n$. Therefore

$$0 = u_\varepsilon(y_\alpha) + H(p_\alpha, y_\alpha) - \varepsilon\Delta u_\varepsilon(y_\alpha) \leq u_\varepsilon(y_\alpha) + H(p_\alpha, y_\alpha) + \alpha n\varepsilon.$$

Subtracting this from (5.14) we have

$$u(x_\alpha) - u_\varepsilon(y_\alpha) \le H(p_\alpha, y_\alpha) - H(p_\alpha, x_\alpha) + \alpha n\varepsilon \le \frac{C}{\alpha} + \alpha n\varepsilon,$$

due to (5.11), (5.12) and the inequality $|p_\alpha| = \alpha|x_\alpha - y_\alpha| \le C$. This establishes the claim.

By (5.13) and the definition of $\Phi$

$$\max_{\overline{U}}(u - u_\varepsilon) \le \Phi(x_\alpha, y_\alpha) \le u(x_\alpha) - u_\varepsilon(y_\alpha) \le C\left(\tfrac{1}{\alpha} + \alpha\varepsilon\right).$$

Selecting $\alpha = 1/\sqrt{\varepsilon}$ completes the proof.

The proof that $u_\varepsilon - u \le C\sqrt{\varepsilon}$ is similar, and is left to Exercise 5.9. □

**Exercise 5.9.** Complete the proof of Theorem 5.8 by showing that

$$u_\varepsilon - u \le C\sqrt{\varepsilon}.$$

[Hint: Define the auxilliary function

$$\Phi(x, y) = u_\varepsilon(x) - u(y) - \frac{\alpha}{2}|x - y|^2.$$

Then proceed as in the proof of Theorem 5.8. You will need to use the exterior sphere condition and the barrier function method from the proof of Theorem 5.4 to handle the case when $y_\alpha \in \partial U$.]

**Exercise 5.10.** Show that the solution $u_\varepsilon$ of

$$|u'(x)| - \varepsilon u''(x) = 1 \quad \text{for } x \in (-1, 1)$$

satisfying $u(-1) = u(1) = 0$ is

$$u_\varepsilon(x) = 1 - |x| - \varepsilon\left(e^{-\frac{1}{\varepsilon}|x|} - e^{-\frac{1}{\varepsilon}}\right).$$

In this case, $|u - u_\varepsilon| \le C\varepsilon$, where $u(x) = 1 - |x|$ is the viscosity solution of $|u'(x)| = 1$ on $(-1, 1)$ with $u(-1) = u(1) = 0$.

## 5.3 Semiconcavity and an $O(\varepsilon)$ one-sided rate

Exercise 5.10 suggests that in some situations we might expect to see the better rate $|u - u_\varepsilon| \le C\varepsilon$. To see when this might hold, let us proceed formally with maximum principle arguments. Let $x \in U$ be a maximum of $u_\varepsilon - u$. Then $u - u_\varepsilon$ has a minimum at $x$, and the definition of viscosity solution yields

$$u(x) + H(Du_\varepsilon(x), x) \geq 0,$$

since $u_\varepsilon \in C^2(U)$. Since $u_\varepsilon$ solves (5.1), we find that

$$u_\varepsilon(x) - u(x) \leq \varepsilon \Delta u_\varepsilon(x).$$

In the case that $u \in C^2(\overline{U})$, we have $\Delta u_\varepsilon(x) \leq \Delta u(x)$ and thus

$$\max_{\overline{U}}(u_\varepsilon - u) \leq \varepsilon \sup_{y \in \overline{U}} \Delta u(y) = C\varepsilon.$$

A similar argument shows that

$$\max_{\overline{U}}(u - u_\varepsilon) \leq -\varepsilon \inf_{y \in \overline{U}} \Delta u(y) = C\varepsilon.$$

In general, we do not expect $u \in C^2(\overline{U})$. However, there are situations where the second derivatives of $u$ are bounded from above or from below, and we can obtain one-sided rates $u_\varepsilon - u \leq C\varepsilon$ or $u - u_\varepsilon \leq C\varepsilon$. This is possible, for example, when $H(p, x) = G(p) - f(x)$, where $G$ is convex and $f$ has bounded second derivatives. To see why, we again proceed formally, and assume $u \in C_c^\infty(\mathbb{R}^n)$ is a solution of

$$u + G(Du) = f \quad \text{in } \mathbb{R}^n.$$

Differentiate the PDE above twice in an arbitrary direction $\xi \in \mathbb{R}^n$ with $|\xi| = 1$ to obtain

$$u_{\xi\xi} + \sum_{i,j=1}^n G_{p_i p_j}(Du) u_{x_i \xi} u_{x_j \xi} + \sum_{i=1}^n G_{p_i}(Du) u_{x_i \xi\xi} = f_{\xi\xi}.$$

Since $G$ is convex, the second term above is nonnegative. Setting $v = u_{\xi\xi}$ we find that

$$v + \sum_{i=1}^n G_{p_i}(Du) v_{x_i} \leq f_{\xi\xi} \ \text{ in } \mathbb{R}^n.$$

Since $v$ is compactly supported, $v$ attains its maximum over $\mathbb{R}^n$ at some $x \in \mathbb{R}^n$ and $Dv(x) = 0$. Therefore

$$\sup_{y \in \mathbb{R}^n} u_{\xi\xi} = v(x) \leq f_{\xi\xi}(x).$$

It follows that $D^2u \le cI$ at all points in $\mathbb{R}^n$, where

$$c := \max_{\substack{x\in\mathbb{R}^n \\ |\xi|=1}} f_{\xi\xi}(x) \ge 0. \tag{5.15}$$

Since $u$ is not generally smooth, these arguments are only a heuristic. The following theorem makes the arguments rigorous in the viscosity sense.

**Theorem 5.11.** *Assume $p \mapsto G(p)$ is convex, $G(0) = 0$, and $f \in C_c^2(\mathbb{R}^n)$. Let $u \in C(\mathbb{R}^n)$ be a compactly supported viscosity solution of*

$$u + G(Du) = f \quad \text{in } \mathbb{R}^n. \tag{5.16}$$

*Then $u$ is a viscosity solution of*

$$-D^2u \ge -cI \quad \text{in } \mathbb{R}^n, \tag{5.17}$$

*where $c$ is given by (5.15).*

**Remark 5.12.** We say $u$ is a viscosity solution of (5.17) provided $D^2\varphi(x) \le cI$ whenever $\varphi \in C^\infty(\mathbb{R}^n)$ and $u - \varphi$ has a local minimum at $x$. This is equivalent to the condition that $u$ is a viscosity solution of

$$-\max_{|\xi|=1} u_{\xi\xi} \ge -c \quad \text{in } \mathbb{R}^n.$$

A function $u$ satisfying (5.17) is called *semiconcave*, with *semiconcavity constant $c$*. Notice that $v := u - \frac{1}{2}c|x|^2$ is a viscosity solution of $-D^2v \ge 0$, hence $v$ is concave (due to a generalization of Exercise 2.15).

We also note that (5.17) is equivalent to

$$u(x+h) - 2u(x) + u(x-h) \le c|h|^2 \quad \text{for all } x, h \in \mathbb{R}^n,$$

which is often the definition of semiconcavity. A function $u$ is called *semiconvex* if $-u$ is semiconcave.

*Proof.* Consider the auxiliary function

$$\Phi(x,y,z) = u(x) - 2u(y) + u(z) - \frac{\alpha}{2}|x-2y+z|^2 - \frac{c}{2}|x-y|^2 - \frac{c}{2}|z-y|^2. \tag{5.18}$$

Let $(x_\alpha, y_\alpha, z_\alpha) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$\max_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} \Phi = \Phi(x_\alpha, y_\alpha, z_\alpha).$$

By the definition of viscosity solution we have

$$u(x_\alpha) + G(p_\alpha + c(x_\alpha - y_\alpha)) \le f(x_\alpha),$$

$$u(z_\alpha) + G(p_\alpha + c(z_\alpha - y_\alpha)) \le f(z_\alpha),$$

and

$$u(y_\alpha) + G(p_\alpha + \tfrac{c}{2}(x_\alpha - y_\alpha) + \tfrac{c}{2}(z_\alpha - y_\alpha)) \ge f(y_\alpha),$$

where $p_\alpha = \alpha(x_\alpha - 2y_\alpha + z_\alpha)$. Since $G$ is convex

$$\begin{aligned} &2G(p_\alpha + \tfrac{c}{2}(x_\alpha - y_\alpha) + \tfrac{c}{2}(z_\alpha - y_\alpha)) \\ &\qquad\qquad \le G(p_\alpha + c(x_\alpha - y_\alpha)) + G(p_\alpha + c(z_\alpha - y_\alpha)). \end{aligned}$$

It follows that

$$u(x_\alpha) - 2u(y_\alpha) + u(z_\alpha) \le f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha). \tag{5.19}$$

Since $\Phi(y+h, y, y-h) \le \Phi(x_\alpha, y_\alpha, z_\alpha)$ for any $y, h \in \mathbb{R}^n$ we find that

$$\begin{aligned} &u(y+h) - 2u(y) + u(y-h) - c|h|^2 \\ &\qquad \le f(x_\alpha) - 2f(y_\alpha) + f(z_\alpha) - \frac{\alpha}{2}|x_\alpha - 2y_\alpha + x_\alpha|^2 \\ &\qquad\qquad - \frac{c}{2}|x_\alpha - y_\alpha|^2 - \frac{c}{2}|z_\alpha - y_\alpha|^2. \end{aligned} \tag{5.20}$$

We now aim to bound the terms on the right hand side. Since $\Phi(x_\alpha, y_\alpha, z_\alpha) \ge \Phi(0,0,0) = 0$, we have

$$\frac{\alpha}{2}|x_\alpha - 2y_\alpha + z_\alpha|^2 + \frac{c}{2}|x_\alpha - y_\alpha|^2 + \frac{c}{2}|z_\alpha - y_\alpha|^2 \le C.$$

Therefore, by passing to a subsequence, there exists $y_0, h_0 \in \mathbb{R}^n$ such that

$$y_\alpha \to y_0,\ x_\alpha - y_\alpha \to h_0,\ \text{ and }\ y_\alpha - z_\alpha \to h_0,$$

as $\alpha \to \infty$. Therefore

$$\limsup_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha, z_\alpha) \le u(y_0 + h_0) - 2u(y_0) + u(y_0 - h_0) - c|h_0|^2.$$

For each $\alpha$ we have

$$\Phi(x_\alpha, y_\alpha, z_\alpha) \ge \Phi(y_0 + h_0, y_0, y_0 - h_0) = u(y_0 + h_0) - 2u(y_0) + u(y_0 - h_0) - c|h_0|^2,$$

and so we deduce

$$\lim_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha, z_\alpha) = u(y_0 + h_0) - 2u(y_0) + u(y_0 - h_0) - c|h_0|^2.$$

Therefore
$$\alpha|x_\alpha - 2y_\alpha + z_\alpha|^2 \to 0 \text{ as } \alpha \to \infty.$$

Passing to limits in (5.20) we have

$$\begin{aligned} u(y+h) - 2u(y) + u(y-h) - c|h|^2 \\ &\le f(y_0+h_0) - 2f(y_0) + f(y_0-h_0) - c|h_0|^2 \\ &\le c|h_0|^2 - c|h_0|^2 \le 0, \end{aligned}$$

for all $y, h \in \mathbb{R}^n$. Now let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $y \in \mathbb{R}^n$. Then

$$u(y+h) - u(y) \ge \varphi(y+h) - \varphi(y) \quad \text{for small } |h|.$$

Therefore

$$\varphi(y+h) - 2\varphi(y) + \varphi(y-h) \le u(y+h) - 2u(y) + u(y-h) \le c|h|^2$$

for small $|h|$. It follows that $\varphi_{\xi\xi}(y) \le c$ for all $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and so $D^2\varphi(y) \le cI$. □

The second derivative estimate from Theorem 5.11 allows us to prove a better one-sided rate in the method of vanishing viscosity.

**Theorem 5.13.** *Assume $p \mapsto G(p)$ is convex and nonnegative with $G(0) = 0$, and $f \in C_c^2(U)$ is nonnegative. Let $u \in C(\overline{U})$ be the viscosity solution of*

$$\left.\begin{aligned} u + G(Du) &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{aligned}\right\} \tag{5.21}$$

*and let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ solve*

$$\left.\begin{aligned} u_\varepsilon + G(Du_\varepsilon) - \varepsilon\Delta u_\varepsilon &= f \quad \text{in } U \\ u_\varepsilon &= 0 \quad \text{on } \partial U, \end{aligned}\right\} \tag{5.22}$$

*Then there exists a constant $C$ such that*

$$u_\varepsilon - u \le C\varepsilon.$$

*Proof.* Define

$$v(x) = \begin{cases} u(x), & \text{if } x \in U \\ 0, & \text{otherwise.} \end{cases}$$

We claim that $v \in C(\mathbb{R}^n)$ is a viscosity solution of

$$v + G(Dv) = f \quad \text{in } \mathbb{R}^n.$$

Since $v(x) = f(x) = G(0) = 0$ for $x \notin \overline{U}$, we just need to check that $v$ is a viscosity solution at boundary points $x \in \partial U$. Furthermore, since $G \geq 0$ and $u(x) = f(x) = 0$ for $x \in \partial U$, the viscosity supersolution property holds trivially. For the subsolution property, let $x \in \partial U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $v - \varphi$ has a local maximum at $x$. Since $v$ is nonnegative (Lemma 5.1) and $v(x) = 0$ we have

$$\varphi(y) - \varphi(x) \geq v(y) - v(x) \geq 0$$

for all $y$ near $x$. Therefore $\varphi$ has a local minimum at $x$, and so $D\varphi(x) = 0$. It follows that

$$v(x) + G(D\varphi(x)) = 0 + G(0) = 0 \leq f(x).$$

This establishes the claim.

By Theorem 5.11, $v$ is a viscosity solution of $-D^2 v \geq -cI$ on $\mathbb{R}^n$, where $c$ is given by (5.15). Since $u = v$ on the open set $U$, it follows that $u$ is a viscosity solution of $-D^2 u \geq -cI$ on $U$. Note that $u = u_\varepsilon$ on $\partial U$. Suppose that $\max_{\overline{U}}(u_\varepsilon - u) > 0$ and let $x \in U$ be a point at which $u_\varepsilon - u$ assumes its positive maximum. Then $u - u_\varepsilon$ has a minimum at $x$ and hence

$$u(x) + G(Du_\varepsilon(x)) \geq f(x),$$

and $D^2 u_\varepsilon(x) \leq cI$. It follows that $\Delta u_\varepsilon(x) \leq nc$ and therefore

$$\max_{\overline{U}}(u_\varepsilon - u) = u_\varepsilon(x) - u(x) \leq \varepsilon \Delta u_\varepsilon(x) \leq cn\varepsilon. \qquad \square$$

**Exercise 5.14.**

(a) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du, u, x) = 0 \quad \text{in } U.$$

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\Psi' > 0$. Show that $v := \Psi \circ u$ is a viscosity solution of

$$H((\Phi' \circ v)Dv, \Phi \circ v, x) = 0 \quad \text{in } U,$$

where $\Phi := \Psi^{-1}$.

(b) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du) = f \quad \text{in } U,$$

and suppose that $H$ is positively 1-homogeneous. Define the Kružkov Transform of $u$ by $v := -e^{-u}$. Use part (a) to show that $v$ is a viscosity solution of

$$fv + H(Dv) = 0 \quad \text{in } U. \tag{5.23}$$

[Remark: The Kružkov Transform is a standard technique for introducing a zeroth order term. When $f > 0$, this term has the correct sign for a comparison principle to hold for (5.23). This also shows that we do not lose much in the way of generality by studying equations with zeroth order terms.]

# Chapter 6

# Boundary conditions in the viscosity sense

We say that $u$ satisfies the boundary condition from (3.14) in the *strong sense* provided $u = g$ on $\partial U$. This is the usual sense, and is how we have been interpreting boundary conditions thus far. However, depending on the geometry of the projected characteristics, the Dirichlet problem (3.14) with boundary conditions in the strong sense is in general overdetermined. For example, the solution $u$ of

$$u_{x_1} + u_{x_2} = 0 \quad \text{in } B(0,1) \subset \mathbb{R}^2$$

is constant along the projected characteristics

$$x(s) = (x_0 + s, s) \quad (x_0 \in \mathbb{R}).$$

Since each projected characteristic intersects $\partial B(0,1)$ at two points, we cannot specify arbitrary Dirichlet conditions on $\partial B(0,1)$. The solution $u$ is in fact uniquely determined by its values on $\partial B(0,1) \cap \{x_1 + x_2 \leq 0\}$. Clearly we need some weaker notion of boundary conditions if we expect to get existence.

**Exercise 6.1.** Consider the ordinary differential equation

$$u_\varepsilon'(x) - \varepsilon u_\varepsilon''(x) = 1, \quad u_\varepsilon(0) = u_\varepsilon(1) = 0.$$

Find explicitly the solution $u_\varepsilon$ and sketch its graph. Show that $u_\varepsilon(x) \to x$ pointwise on $[0,1)$ as $\varepsilon \to 0$.

The previous exercise suggests that $u(x) = x$ should be the viscosity solution of

$$u'(x) = 1, \quad u(0) = u(1) = 0,$$

even though $u(1) \neq 0$. The issue is that the problem above is overdetermined, so we lose one of the boundary conditions in the vanishing viscosity limit. The same thing happens in a more complicated manner in higher dimensions.

In order to make sense of this, we should consider carefully how boundary conditions behave in the vanishing viscosity limit. Let $u_\varepsilon$ be a smooth solution of

$$H(Du_\varepsilon, u_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 \quad \text{in } U, \tag{6.1}$$

and assume that $u_\varepsilon \leq g$ on $\partial U$, where $g : \partial U \to \mathbb{R}$ is continuous. Exercise (6.1) shows that we cannot expect $u_\varepsilon$ to converge uniformly on $\overline{U}$. Instead, let us consider the weak upper limit

$$\overline{u}(x) = \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y),$$

where we assume that $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded, and $y \in \overline{U}$. Select a point $x \in \partial U$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a strict local maximum at $x$ over the set $\overline{U}$. Then there exists $\varepsilon_k \to 0^+$ and $x_k \to x$ such that $x_k \in \overline{U}$, $u_{\varepsilon_k}(x_k) \to \overline{u}(x)$ and $u_{\varepsilon_k} - \varphi$ has a local max at $x_k$ over $\overline{U}$. By passing to a further subsequence, if necessary, we may assume that either (A) $x_k \in U$ for all $k$, or (B) $x_k \in \partial U$ for all $k$.

If (A) holds, then we conclude, as in Section 1.3, that

$$H(D\varphi(x), \overline{u}(x), x) \leq 0.$$

If (B) holds, then

$$\overline{u}(x) = \lim_{k\to\infty} u_{\varepsilon_k}(x_{\varepsilon_k}) = g(x).$$

Hence we find that either $\overline{u}(x) = g(x)$ or $H(D\varphi(x), \overline{u}(x), x) \leq 0$. If we weaken the boundary condition by assuming merely that $u_\varepsilon \leq g$ on $\partial U$, then we would find that for each $x \in \partial U$, either $\overline{u}(x) \leq g(x)$ or $H(D\varphi(x), \overline{u}(x), x) \leq 0$. This can be compactly written as

$$\min\{H(D\varphi(x), \overline{u}(x), x), \overline{u}(x) - g(x)\} \leq 0.$$

We can make the same argument with the weak lower limit $\underline{u}$ to find that when $\underline{u} - \varphi$ has a local minimum at $x \in \partial U$ we have

$$\max\{H(D\varphi(x), \underline{u}(x), x), \underline{u}(x) - g(x)\} \geq 0,$$

provided $u_\varepsilon \geq g$ on $\partial U$.

This motivates the following definitions.

**Definition 6.2.** We say $u \in \mathrm{USC}(\overline{U})$ is a *viscosity subsolution* of (3.14) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$ with respect to $\overline{U}$

$$\begin{cases} H(D\varphi(x), u(x), x) \le 0, & \text{if } x \in U \\ \min\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \le 0 & \text{if } x \in \partial U. \end{cases}$$

Likewise, we say that $u \in \mathrm{LSC}(\overline{U})$ is a *viscosity supersolution* of (3.14) if for all $x \in \overline{U}$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$ with respect to $\overline{U}$

$$\begin{cases} H(D\varphi(x), u(x), x) \ge 0, & \text{if } x \in U \\ \max\{H(D\varphi(x), u(x), x), u(x) - g(x)\} \ge 0 & \text{if } x \in \partial U. \end{cases}$$

Finally, we say that $u$ is a *viscosity solution* of (3.14) if $u$ is both a viscosity sub- and supersolution. In this case, we say that the boundary conditions in (3.14) hold in the *viscosity sense*

**Exercise 6.3.** Show that $u(x) = x$ is a viscosity solution of

$$u'(x) = 1, \quad u(0) = u(1) = 0,$$

on the interval $U = (0, 1)$ in the sense of Definition 6.2.

It is possible to prove a comparison principle for viscosity sub- and supersolutions in the sense of Definition 6.2, provided the semicontinuous solutions attain their boundary values continuously. We will not give the proof in full generality (see [1]). In some special cases, it is also possible to recover strong boundary conditions from boundary conditions in the viscosity sense. The typical approach is to select a sequence of test functions at a boundary point for which the sub- or supersolution property is violated. This is often possible because the class of admissible test functions at boundary points is very large (since the admissibility condition is "one-sided").

With the exception of Theorem 9.8, we will generally not use Definition 6.2 in these notes. Hence, unless otherwise stated, all viscosity solutions should be interpreted in the sense of the definitions in Chapter 2.

We give here a comparison principle in the special case where we have additional information concerning at which boundary points the Dirichlet condition holds, and at which points the PDE should hold. We also relax the assumption that $U$ is bounded, and instead assume that the sub- and supersolutions are bounded.

We assume the usual monotonicity (3.3) and regularity (3.6) conditions on $H$ hold. In addition we assume

$$|H(p, z, x) - H(q, z, x)| \le \omega_1(|p - q|), \tag{6.2}$$

where $\omega_1$ is a modulus of continuity. In this case, we can prove the following comparison principle.

**Theorem 6.4.** *Let $U \subset \mathbb{R}^n$ be open and suppose $\partial U = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ is relatively open and $\Gamma_1 \cap \Gamma_2 = \varnothing$. Let $u \in USC(\overline{U})$ be a bounded viscosity solution of $H \le -\varepsilon < 0$ on $U \cup \Gamma_1$, and let $v \in LSC(\overline{U})$ be a bounded viscosity solution of $H \ge 0$ on $U \cup \Gamma_1$. If $u \le v$ on $\Gamma_2$ then $u \le v$ on $U$.*

The proof is very similar to that of Theorem 3.1, so we will briefly outline the details. The main difficulty is to ensure that the auxiliary function assumes its maximum on the *unbounded* domain $U$. We also remark that the theorem holds when $U = \mathbb{R}^n$ and $\Gamma_1 = \Gamma_2 = \varnothing$.

*Proof.* Let $\lambda > 0$ and define

$$u_\lambda(x) = u(x) - \frac{\lambda}{2}\log(1 + |x|^2).$$

Let $x \in U \cup \Gamma_1$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u_\lambda - \varphi$ has a local maximum at $x$. Then $u - \frac{\lambda}{2}\log(1 + |x|^2) - \varphi$ has a local max at $x$, and therefore

$$H(\lambda(1 + |x|^2)^{-1}x + D\varphi(x), u(x), x) \le -\varepsilon.$$

By (6.2) and (3.3) we have

$$\begin{aligned} H(D\varphi(x), u_\lambda(x), x) &\le H(D\varphi(x), u(x), x) \\ &\le H(\lambda(1 + |x|^2)^{-1}x + D\varphi(x), u(x), x) + \omega_1(\lambda) \\ &\le -\varepsilon + \omega_1(\lambda). \end{aligned}$$

Therefore, there exists $\Lambda > 0$ such that for all $0 < \lambda < \Lambda$, $u_\lambda$ is a viscosity solution of

$$H(Du_\lambda, u_\lambda, x) \le -\frac{\varepsilon}{2} \quad \text{in } U \cup \Gamma_1. \tag{6.3}$$

We will prove that $u_\lambda \le v$ on $\overline{U}$ for all $0 < \lambda < \Lambda$. To see this, assume to the contrary that $\sup_{\overline{U}}(u_\lambda - v) > 0$. For $\alpha > 0$ define the auxiliary function

$$\Phi(x, y) = u_\lambda(x) - v(y) - \frac{\alpha}{2}|x - y|^2. \tag{6.4}$$

Since $u$ and $v$ are bounded

$$0 \leq \frac{\alpha}{2}|x-y|^2 \leq u_\lambda(x) - v(y) \leq C - \frac{\lambda}{2}\log(1+|x|^2),$$

for any $(x,y) \in \overline{U} \times \overline{U}$ such that $\Phi(x,y) \geq 0$. Since $\sup_{\overline{U}\times\overline{U}} \Phi > 0$, we find that $\Phi$ attains its maximum at some $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ satisfying

$$\frac{\lambda}{2}\log(1+|x_\alpha|^2) + \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq C.$$

It follows that there exists $x_0 \in \overline{U}$ such that, up to a subsequence, $x_\alpha \to x_0$ and $y_\alpha \to x_0$. Therefore

$$\limsup_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) \leq u_\lambda(x_0) - v(x_0) \leq \liminf_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha),$$

due to the upper semicontinuity of $\Phi$. Therefore

$$\lim_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) = u_\lambda(x_0) - v(x_0) \quad \text{and} \quad \lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0. \tag{6.5}$$

Since $\Phi(x_\alpha, y_\alpha) \geq \sup_{\overline{U}}(u_\lambda - v) > 0$, we have

$$u_\lambda(x_0) > v(x_0) \text{ and } u_\lambda(x_\alpha) > v(y_\alpha).$$

Since $u_\lambda \leq v$ on $\Gamma_2$, we must have $x_0 \in U \cup \Gamma_1$. Since $\Gamma_1$ is relatively open, $x_\alpha, y_\alpha \in U \cup \Gamma_1$ for sufficiently large $\alpha$.

By the viscosity sub- and supersolution properties, (3.3) and (3.6) we have

$$\begin{aligned}
\frac{\varepsilon}{2} &\leq H(p_\alpha, v(y_\alpha), y_\alpha) - H(p_\alpha, u_\lambda(x_\alpha), x_\alpha) \\
&\leq H(p_\alpha, u_\lambda(x_\alpha), y_\alpha) - H(p_\alpha, u_\lambda(x_\alpha), x_\alpha) \\
&\leq \omega((1+|p_\alpha|)|x_\alpha - y_\alpha|),
\end{aligned}$$

where $p_\alpha = \alpha(x_\alpha - y_\alpha)$. By (6.5),

$$(1+|p_\alpha|)|x_\alpha - y_\alpha| \to 0 \text{ as } \alpha \to \infty.$$

Sending $\alpha \to \infty$ contradicts the positivity of $\varepsilon$.

Therefore $u_\lambda \leq v$ for all $0 < \lambda < \Lambda$. It follows that $u \leq v$ on $\overline{U}$. □

**Exercise 6.5.** Let $\Gamma \subset \mathbb{R}^n$ be closed and bounded. Consider eikonal equation

$$\text{(H)} \quad \begin{cases} |Du| = 1 & \text{in } \mathbb{R}^n \setminus \Gamma \\ u = 0 & \text{on } \Gamma. \end{cases}$$

(a) Show that there is at most one viscosity solution $u \in C(\mathbb{R}^n)$ of (H) satisfying the boundary condition at infinity

$$\lim_{|x|\to\infty} u(x) = \infty. \tag{6.6}$$

[Hint: Theorem 6.4 does not apply, since $u$ and $v$ are unbounded. To prove uniqueness, let $u, v \in C(\mathbb{R}^n)$ be two viscosity solutions of (H) satisfying (6.6). Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} \Psi(s) = s, & \text{if } s \le 1, \\ \Psi(s) \le 2, & \text{for all } s \in \mathbb{R}, \\ 0 < \Psi'(s) \le 1, & \text{for all } s \in \mathbb{R}. \end{cases}$$

For $R > 1$ define

$$w(x) := (R-1)\,\Psi(R^{-1}u(x)).$$

Show that $w \le 2R$ is a viscosity solution of

$$|Dw| + \frac{1}{R} \le 1 \quad \text{in } \mathbb{R}^n \setminus \Gamma.$$

Use the doubling of the variables argument to show that $w \le v$ on $\mathbb{R}^n \setminus \Gamma$. Complete the argument from here.]

(b) Show that the solution is not unique without (6.6).

## 6.1 Time-dependent Hamilton-Jacobi equations

As an application of Theorem 6.4, we will prove a comparison principle for the time-dependent Hamilton-Jacobi equation

$$\left.\begin{aligned} u_t + H(Du, x) &= 0 \quad \text{in } \mathbb{R}^n \times (0, T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{aligned}\right\} \tag{6.7}$$

We assume, as before, that $H$ is continuous and satisfies (3.6) and (6.2).

**Theorem 6.6.** *Let $u \in USC(\mathbb{R}^n \times [0, T])$ be a bounded viscosity subsolution of (6.7), and let $v \in LSC(\mathbb{R}^n \times [0, T])$ be a bounded viscosity supersolution of (6.7). Then $u \le v$ on $\mathbb{R}^n \times [0, T]$.*

Perhaps there is a bit of abuse of notation here. When we say $u$ is a subsolution of (6.7), we mean that $u$ is a solution of $u_t + H \le 0$ in $\mathbb{R}^n \times (0, T)$ *and* $u \le g$ at $t = 0$. Likewise, a supersolution is assumed to satisfy $v \ge g$ at $t = 0$, hence $u \le v$ at $t = 0$.

*Proof.* Set $U = \mathbb{R}^n \times (0,T)$, $\Gamma_1 = \mathbb{R}^n \times \{t = T\}$, and $\Gamma_2 = \mathbb{R}^n \times \{t = 0\}$. Our aim is to apply Theorem 6.4. For this, we only need to show that $u$ and $v$ are viscosity sub- and supersolutions on the extended set

$$U \cup \Gamma_1 = \mathbb{R}^n \times (0,T].$$

That is, we need to allow $t = T$ in the sub- and supersolution properties.

Let $(x_0,T) \in \Gamma_1$, and let $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $u - \varphi$ has a local maximum at $(x_0,T)$. As before, we may assume the local maximum is strict. For $x \in \mathbb{R}^n$ and $0 < t < T$, define

$$\varphi^\varepsilon(x,t) := \varphi(x,t) + \frac{\varepsilon}{T-t}.$$

Then there exist sequences $\varepsilon_k \to 0^+$ and $(x_k,t_k) \to (x_0,T)$ such that $0 < t_k < T$ and $u - \varphi^{\varepsilon_k}$ has a local maximum at $(x_k,t_k)$. Therefore

$$\varphi_t^{\varepsilon_k}(x_k,t_k) + H(D\varphi^{\varepsilon_k}(x_k,t_k),x_k) \le 0,$$

and hence

$$\varphi_t(x_k,t_k) + \frac{\varepsilon_k}{(T-t_k)^2} + H(D\varphi(x_k,t_k),x_k) \le 0.$$

Letting $k \to \infty$ we find that

$$\varphi_t(x_0,T) + H(D\varphi(x_0,T),x_0) \le 0.$$

We can similarly verify that $v$ is a viscosity supersolution of $u_t + H = 0$ on $\Gamma_1$. We can therefore invoke Theorem 6.4, Corollary 3.2, and the remarks thereafter to obtain that $u \le v$ on $\mathbb{R}^n \times [0,T]$. □

We can also easily prove continuous dependence on the initial data.

**Corollary 6.7.** *Let $u,v \in C(\mathbb{R}^n \times [0,T])$ be bounded, and assume that $w := u$ and $w := v$ are viscosity solutions of*

$$w_t + H(Dw,x) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

*Then*

$$\sup_{\mathbb{R}^n \times [0,T]} |u - v| \le \sup_{x \in \mathbb{R}^n} |u(x,0) - v(x,0)|.$$

*Proof.* Let $C := \sup_{x \in \mathbb{R}^n} |u(x,0) - v(x,0)|$. Then $u - C \le v$ at $t = 0$, and by Theorem 6.6 we have $u - v \le C$ on $\mathbb{R}^n \times [0,T]$. The inequality $v - u \le C$ follows by swapping the roles of $u$ and $v$. □

## 6.2 The Hopf-Lax Formula

In the case that $H = H(p)$ and $H$ is convex and superlinear, i.e.,

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty,$$

we have the Hopf-Lax formula

$$u(x,t) = \min_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\},$$

where

$$L(v) = \sup_{p\in\mathbb{R}^n} \{p \cdot v - H(p)\}$$

is the Legendre transform of $H$. In this case, $H$ and $L$ are Legendre duals, and we also have

$$H(p) = \sup_{v\in\mathbb{R}^n} \{p \cdot v - L(v)\}.$$

Under the assumption that $g$ is Lipschitz continuous and bounded, we showed last semester that $u$ is a Lipschitz continuous almost everywhere solution of (6.7). It turns out that the Hopf-Lax formula gives the unique viscosity solution of (6.7). We'll sketch the argument here, for a complete proof see [11, Chapter 10].

Let $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$ and $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ such that $u - \varphi$ has a local maximum at $(x_0, t_0)$. We may assume that for some $r > 0$

$$u(x_0,t_0) = \varphi(x_0,t_0) \quad \text{and} \quad u(x,t) \le \varphi(x,t) \quad \text{for } |x-x_0|^2 + |t-t_0|^2 < r^2.$$

Since $g$ is bounded, $u$ is bounded, and we can also assume that $u \le \varphi$ on $\mathbb{R}^n \times (0,T)$. To see why, we note first that we can multiply $\varphi$ by a bump function to ensure $\varphi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$. Then we can set

$$C := \sup_{\mathbb{R}^n\times[0,T]} u + \sup_{\mathbb{R}^n\times\mathbb{R}} |\varphi|,$$

and

$$\overline{\varphi}(x,t) := \varphi(x,t) + \frac{C}{r^2}(|x-x_0|^2 + |t-t_0|^2).$$

Then $\overline{\varphi}(x_0,t_0) = u(x_0,t_0)$, $\overline{\varphi} \ge u$ for $(x,t) \in B((x_0,t_0), r)$, and

$$\overline{\varphi}(x,t) \ge \varphi(x,t) + C \ge \sup_{\mathbb{R}^n\times[0,T]} u \ge u,$$

for $(x, t) \notin B((x_0, t_0), r)$. Since $\overline{\varphi}_t(x_0, t_0) = \varphi_t(x_0, t_0)$ and $D\overline{\varphi}(x_0, t_0) = D\varphi(x_0, t_0)$, we can without loss of generality replace $\varphi$ by $\overline{\varphi}$.

We recall that the function $u$ defined by the Hopf-Lax formula satisfies the property

$$u(x_0, t_0) = \min_{x \in \mathbb{R}^n} \left\{ (t - t_0) L \left( \frac{x_0 - x}{t_0 - t} \right) + u(x, t) \right\}$$

for all $0 < t < t_0$. Since $u(x_0, t_0) = \varphi(x_0, t_0)$ and $u \leq \varphi$ we have

$$\varphi(x_0, t_0) \leq \min_{x \in \mathbb{R}^n} \left\{ (t - t_0) L \left( \frac{x_0 - x}{t_0 - t} \right) + \varphi(x, t) \right\},$$

for all $0 < t < t_0$. The reader should notice the similarity with the proof of Theorem 4.6. Since $\varphi$ is smooth, the same arguments that showed that $u$ is a Lipschitz almost everywhere solution (see [11, Section 3.3]) prove that

$$\varphi_t + H(D\varphi) \leq 0 \quad \text{at } (x_0, t_0).$$

The proof that $u$ is a viscosity supersolution is similar.

# Chapter 7

# The Perron method

We define the upper semicontinuous envelope of a function $u : \mathcal{O} \to \mathbb{R}$ by

$$u^*(x) := \limsup_{\mathcal{O} \ni y \to x} u(y).$$

The function $u^*$ is the smallest upper semicontinuous function that is pointwise greater than or equal to $u$. The lower semicontinuous envelope of $u$, defined by $u_* := -(-u)^*$, is correspondingly the greatest lower semicontinuous function that is less than $u$. Note that $u_* \leq u \leq u^*$, and $u^* = u_* = u$ if and only if $u$ is continuous.

The Perron method is a powerful technique for proving existence of viscosity solutions. The idea is to construct a solution as an upper envelope of subsolutions. Consider the second order nonlinear equation

$$H(D^2u, Du, u, x) = 0 \quad \text{in } U, \tag{7.1}$$

where $H$ is continuous and $U \subset \mathbb{R}^n$ is open. Let $w \in \text{LSC}(\overline{U})$ be a viscosity supersolution of (7.1) and define

$$\mathcal{F} := \Big\{ v \in \text{USC}(\overline{U}) \, : \, v \text{ is a subsolution of (7.1) and } v \leq w \text{ in } \overline{U} \Big\},$$

and

$$u(x) := \sup\{v(x) \, : \, v \in \mathcal{F}\}. \tag{7.2}$$

The function $u$ is presumably a prime candidate for a viscosity solution of (7.1).

We now establish two lemmas that are fundamental to the Perron method.

**Lemma 7.1.** *Suppose $\mathcal{F}$ is nonempty. Then the upper semicontinuous function $u^*$ is a viscosity subsolution of* (7.1)

*Proof.* Let $x_0 \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^* - \varphi$ has a local maximum at $x_0$. Replacing $\varphi$ by $\varphi + u^*(x_0) - \varphi(x_0) + |x - x_0|^4$, there exists $B(x_0, r) \subset U$ such that

$$u^*(x_0) = \varphi(x_0) \quad \text{and} \quad u^*(x) - \varphi(x) \leq -|x - x_0|^4 \quad \text{for } x \in B(x_0, r). \tag{7.3}$$

By definition of $u^*$ there exists a sequence $x_k \to x_0$ such that $u(x_k) \to u^*(x_0)$. For each $k$ there exists $u_k \in \mathcal{F}$ such that $u_k(x_k) \geq u(x_k) - \frac{1}{k}$. Since $u_k \in \mathrm{USC}(\overline{U})$, $u_k - \varphi$ attains its maximum over $B(x_0, r)$ at some $y_k \in B(x_0, r)$. Furthermore, since $u_k(y_k) \leq u(y_k) \leq u^*(y_k)$ we have by (7.3) that

$$\begin{aligned} |y_k - x_0|^4 &\leq \varphi(y_k) - u^*(y_k) \\ &\leq \varphi(y_k) - u_k(y_k) \\ &\leq \varphi(x_k) - u_k(x_k) \\ &\leq \varphi(x_k) - u(x_k) + \frac{1}{k}. \end{aligned}$$

It follows that $y_k \to x_0$ as $k \to \infty$. Thus for large enough $k$, $u_k - \varphi$ has a local maximum at an interior point $y_k$ of the ball $B(x_0, r)$. By the viscosity subsolution property

$$H(D^2\varphi(y_k), D\varphi(y_k), u_k(y_k), y_k) \leq 0.$$

Since $u(x_k) \to u^*(x_0)$ and

$$u_k(y_k) - \varphi(y_k) \geq u_k(x_k) - \varphi(x_k) \geq u(x_k) - \varphi(x_k) - \frac{1}{k},$$

we have $\liminf_{k\to\infty} u_k(y_k) \geq u^*(x_0)$. Since $u_k(y_k) \leq u^*(y_k)$ and $u^*$ is upper semicontinuous, we find that $u_k(y_k) \to u^*(x_0)$ as $k \to \infty$. Sending $k \to \infty$ and using the continuity of $H$ we have

$$H(D^2\varphi(x_0), D\varphi(x_0), u^*(x_0), x_0) \leq 0. \qquad \square$$

**Lemma 7.2.** *Let $u \in \mathcal{F}$. If $u_*$ is not a viscosity supersolution of (7.1), then there exists $v \in \mathcal{F}$ such that $v(x) > u(x)$ for some $x \in U$.*

*Proof.* Let $u \in \mathcal{F}$ and assume $u_*$ is not a viscosity supersolution of (7.1). Then there exists $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u_* - \varphi$ has a local minimum at $x_0$ and

$$H(D^2\varphi(x_0), D\varphi(x_0), u_*(x_0), x_0) < 0. \tag{7.4}$$

We may assume that $\varphi(x_0) = u_*(x_0)$. If $\varphi(x_0) = w(x_0)$ then $w - \varphi$ has a local minimum at $x_0$, which contradicts (7.4), as $w$ is a supersolution. Therefore

$\varphi(x_0) < w(x_0)$. Hence there exists $\varepsilon > 0$ and a ball $B(x_0, r) \subset U$ such that $\varphi \le u_*$ and $\varphi + \varepsilon \le w$ on $B(x_0, r)$ and

$$H(D^2\varphi(x), D\varphi(x), \varphi(x), x) + \varepsilon \le 0 \quad \text{for } x \in B(x_0, r). \tag{7.5}$$

Set

$$\psi(x) := \varphi(x) + \delta\left(\frac{r^4}{2^4} - |x - x_0|^4\right),$$

and choose $\delta > 0$ small enough so that $\psi \le w$ on $B(x_0, r)$ and

$$H(D^2\psi(x), D\psi(x), \psi(x), x) \le 0 \quad \text{for } x \in B(x_0, r).$$

Define

$$v(x) := \begin{cases} \max\{u(x), \psi(x)\}, & \text{if } x \in B(x_0, r) \\ u(x), & \text{otherwise.} \end{cases}$$

Since $u$ and $\psi$ are subsolutions of $H = 0$ in $B(x_0, r)$, $v$ is a subsolution in $B(x_0, r)$. Furthermore, since

$$\psi(x) \le \varphi(x) \le u(x) \quad \text{for } x \in B(x_0, r) \setminus B(x_0, \tfrac{r}{2}),$$

we have $u = v$ on the annulus $B(x_0, r) \setminus B(x_0, \frac{r}{2})$. Therefore $v$ is a subsolution of (7.1) and $v \le w$ on $U$. Therefore $v \in \mathcal{F}$.

By definition of the lower semicontinuous envelope $u_*$, there exists a sequence $x_k \to x_0$ such that $u(x_k) \to u_*(x_0)$. Since $v \ge \psi$ on $B(x_0, r)$, we have

$$\liminf_{k\to\infty} v(x_k) \ge \lim_{k\to\infty} \psi(x_k) = u_*(x_0) + \delta',$$

where $\delta' = \delta r^4/2^4$. Therefore, for $k$ large enough

$$v(x_k) \ge u(x_k) + \frac{\delta'}{2},$$

or $v(x_k) > u(x_k)$. This completes the proof. □

The remaining ingredient for Perron's method is a comparison principle for (7.1). Let us illustrate the technique on the time-dependent Hamilton-Jacobi equation (6.7). As usual, we assume $H$ is continuous and satisfies (3.3), (3.6), and (6.2).

**Theorem 7.3.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and Lipschitz continuous, and suppose that*

$$K := \sup\left\{|H(p, x)| \ : \ |p| \le \mathrm{Lip}(g) \text{ and } x \in \mathbb{R}^n\right\} < \infty.$$

*Then for every $T > 0$ there exists a unique bounded viscosity solution $u \in C(\mathbb{R}^n \times [0, T])$ of (6.7).*

*Proof.* Define

$$w(x,t) := g(x) + Kt.$$

If $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and $w - \varphi$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$, then $|D\varphi(x_0,t_0)| \leq \mathrm{Lip}(g)$ and $\varphi_t(x_0,t_0) = K$. Therefore

$$\varphi_t(x_0,t_0) + H(D\varphi(x_0,t_0), x_0) \geq K - K = 0.$$

Therefore $w$ is a bounded supersolution of (6.7).

Define

$$\mathcal{F} := \Big\{ v \in \mathrm{USC}(\mathbb{R}^n \times [0,T]) \,:\, v \text{ is a subsolution of (6.7) and } v \leq w \Big\},$$

and

$$u(x,t) := \sup\{v(x,t) \,:\, v \in \mathcal{F}\}.$$

We can verify, as before, that $\widetilde{w}(x,t) := g(x) - Kt$ is a subsolution of (6.7). Therefore $\mathcal{F}$ is nonempty. Since $u \leq w$ and $w$ is continuous, $u^* \leq w$ and so $u^*(x,0) \leq w(x,0) = g(x)$. By Lemma 7.1, $u^*$ is a viscosity subsolution of (6.7). Therefore $u^* \in \mathcal{F}$, and so $u = u^*$.

Since $\widetilde{w} \leq w$, $\widetilde{w} \in \mathcal{F}$ and hence $u \geq \widetilde{w}$. Since $\widetilde{w}$ is continuous, $u_*(x,0) \geq w(x,0) = g(x)$. By Lemma 7.2, $u_*$ is a viscosity supersolution of (6.7). Since $u_*(x,0) = u^*(x,0)$, we can use the comparison principle (Theorem 6.6) to show that $u^* \leq u_*$ on $\mathbb{R}^n \times [0,T]$. Since the opposite inequality is true by definition, we have $u^* = u_* = u$. Therefore $u \in C(\mathbb{R}^n \times [0,T])$ is a bounded viscosity solution of (6.7). Uniqueness follows from Theorem 6.6. □

**Exercise 7.4.** Consider the Hamilton-Jacobi equation

$$u + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n.$$

What (non-trivial) conditions can you place on $H$ to guarantee the existence of a bounded viscosity solution $u \in C(\mathbb{R}^n)$? [Hint: Use the Perron method.]

# Chapter 8

# Smoothing viscosity solutions

Since viscosity solutions are in general only continuous functions, it is useful to be able to construct smoother approximations of viscosity solutions. That is, given a viscosity solution $u$ of $F(D^2u, Du, u, x) = 0$, we would like to construct a sequence of smooth (or just smoother) functions $u_k$ such that $F(D^2u_k, Du_k, u_k, x) \to F(D^2u, Du, u, x)$ and $u_k \to u$ as $k \to \infty$, in some appropriate sense. For linear constant coefficient PDEs, we can construct smooth approximate solutions by mollification, that is, $u_\varepsilon = \eta_\varepsilon * u$, where $\eta_\varepsilon$ is the standard mollifier [11]. The mollified function $u_\varepsilon$ is infinitely differentiable and $u_\varepsilon \to u$ locally uniformly as $\varepsilon \to 0$ provided $u$ is continuous. Since mollification is a linear operation, it commutes with linear constant coefficient PDEs and so $u_\varepsilon$ is also a solution of the linear equation.

Unfortunately mollification is not useful for viscosity solutions, as the following exercise illustrates.

**Exercise 8.1.** Recall from Exercise 2.13 that $u(x) = 1 - |x|$ is a viscosity solution of $|u'(x)| - 1 = 0$. In fact, this is the unique solution with boundary conditions $u(-1) = 0 = u(1)$ on the interval $(-1, 1)$ (why?). Show that there does not exist a sequence $u_k \in C^1([-1, 1])$ such that $u_k \to u$ and $|u_k'| \to 1$ uniformly as $k \to \infty$. This shows that it is impossible, in general, to uniformly approximate a viscosity solution by a classical solution.

Since we cannot smoothly approximate viscosity solutions, we are left to consider approximations that are smoother than the continuous or Lipschitz viscosity solutions, but less regular than classical solutions. Such approximations are provided by the inf- and sup-convolutions, defined below.

**Definition 8.2.** Let $U \subset \mathbb{R}^n$, $u : U \to \mathbb{R}^n$, and $\varepsilon > 0$. We define the *sup-*

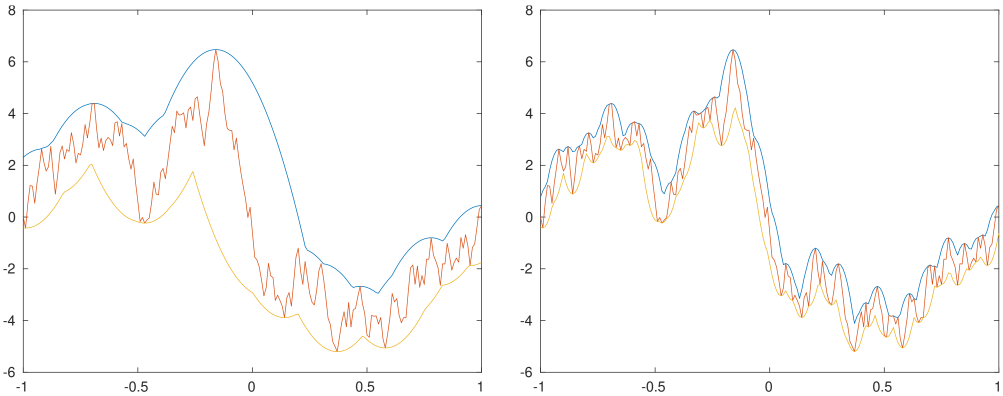


Figure 8.1: Examples of inf- and sup-convolutions of a Brownian motion sample path for $\varepsilon = 0.1$ (left) and $\varepsilon = 0.01$ (right)

*convolution* of $u$, denoted $u^\varepsilon$, to be

$$u^\varepsilon(x) = \sup_{y\in U}\left\{u(y) - \frac{1}{2\varepsilon}|x-y|^2\right\}. \tag{8.1}$$

Similarly, the *inf-convolution* of $u$, denoted $u_\varepsilon$, is defined by

$$u_\varepsilon(x) = \inf_{y\in U}\left\{u(y) + \frac{1}{2\varepsilon}|x-y|^2\right\}. \tag{8.2}$$

We remark that whenever the set $U$ is not specified, it is taken to be the domain of $u$.

The inf- and sup-convolutions are tools that originally appeared in convex analysis—the inf-convolution is called the *Moreau envelop* in optimization [3]—and have been appropriated in the viscosity solution literature due to their useful approximation properties. As we show below, the inf- and sup-convolutions of a viscosity solution $u$ are nearly $C^2$ functions, and are approximate viscosity super- and subsolutions, respectively.

We first establish some basic properties of inf- and sup-convolutions

**Proposition 8.3.** *Suppose $u : U \to \mathbb{R}$ is open and let $u : \overline{U} \to \mathbb{R}$. Then*

*(i) we have $u_\varepsilon \le u \le u^\varepsilon$,*

*(ii) the function $u^\varepsilon + \frac{1}{2\varepsilon}|x|^2$ is convex, and $u_\varepsilon - \frac{1}{2\varepsilon}|x|^2$ is concave,*

*(iii) if $y^\varepsilon \in \arg\max_{y\in\overline{U}}\left\{u(y) - \frac{1}{2\varepsilon}|x-y|^2\right\}$ then $|x-y^\varepsilon|^2 \le 4\|u\|_{L^\infty(U)}\varepsilon$,*

*(iv) if* $y_\varepsilon \in \arg\min_{y\in U}\left\{u(y)+\frac{1}{2\varepsilon}|x-y|^2\right\}$ *then* $|x-y_\varepsilon|^2 \le 4\|u\|_{L^\infty(U)}\varepsilon$,

*(v) both* $u^\varepsilon$ *and* $u_\varepsilon$ *are twice differentiable almost everywhere in* $U$ *and*

$$|u_\varepsilon(x)-u_\varepsilon(y)|, |u^\varepsilon(x)-u^\varepsilon(y)| \le \frac{1}{2\varepsilon}\left(|x-y| + 4\|u\|^{1/2}_{L^\infty(\overline{U})}\varepsilon^{1/2}\right)|x-y|. \tag{8.3}$$

*Proof.* (i) is obvious. For (ii) Note that

$$u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2 = \sup_{y\in\overline{U}}\left\{u(y) + \frac{1}{\varepsilon}x\cdot y - \frac{1}{2\varepsilon}|y|^2\right\},$$

and recall that the supremum of a family of affine functions is convex. The proof that $u_\varepsilon - \frac{1}{2\varepsilon}|x|^2$ is concave is similar.

For (iii), since

$$u^\varepsilon(x) = u(y^\varepsilon) - \frac{1}{2\varepsilon}|x-y^\varepsilon|^2$$

and $u^\varepsilon(x) \ge u(x)$ we have

$$\frac{1}{2\varepsilon}|x-y^\varepsilon|^2 = u(y^\varepsilon) - u^\varepsilon(x) \le u(y^\varepsilon) - u(x) \le 2\|u\|_{L^\infty(\overline{U})}.$$

The proof of (iv) is similar.

For (v), by the Alexandrov Theorem any convex function is twice differentiable almost everywhere. Thus, it follows from (ii) that $u^\varepsilon + \frac{1}{2\varepsilon}|x|^2$ and $u_\varepsilon - \frac{1}{2\varepsilon}|x|^2$ are twice differentiable almost everywhere in $U$, and hence so are $u^\varepsilon$ and $u_\varepsilon$.

To prove the Lipschitz estimate (8.3), let $x, y \in \overline{U}$ and $\delta > 0$. Let $y^\varepsilon \in \overline{U}$ such that

$$u^\varepsilon(x) \le u(y^\varepsilon) - \frac{1}{2\varepsilon}|x-y^\varepsilon|^2 + \delta.$$

Then we have

$$|x-y^\varepsilon|^2 \le 2(2\|u\|_{L^\infty(\overline{U})} + \delta)\varepsilon. \tag{8.4}$$

Since $u^\varepsilon(y) \ge u(y^\varepsilon) - \frac{1}{2\varepsilon}|y-y^\varepsilon|^2$ we have

$$\begin{aligned}
u^\varepsilon(x) - u^\varepsilon(y) &\le \frac{1}{2\varepsilon}\left(|y-y^\varepsilon|^2 - |x-y^\varepsilon|^2\right) + \delta \\
&\le \frac{1}{2\varepsilon}\left((|x-y| + |x-y^\varepsilon|)^2 - |x-y^\varepsilon|^2\right) + \delta \\
&= \frac{1}{2\varepsilon}\left(|x-y|^2 + 2|x-y||x-y^\varepsilon|\right) + \delta \\
&= \frac{1}{2\varepsilon}\left(|x-y| + 2|x-y^\varepsilon|\right)|x-y| + \delta.
\end{aligned}$$

Recalling (8.4) and sending $\delta \to 0$ we have

$$u^\varepsilon(x) - u^\varepsilon(y) \le \frac{1}{2\varepsilon}(|x-y| + 4\|u\|_\infty^{1/2}\varepsilon^{1/2})|x-y|.$$

Reversing the roles of $x$ and $y$ completes the proof. The Lipschitz estimate for $u_\varepsilon$ is similar. $\square$

**Remark 8.4.** Recalling Remark 5.12, it follows from Proposition 8.3(i) that $u^\varepsilon$ is semiconvex and $u_\varepsilon$ is semiconcave. In particular, $-D^2u_\varepsilon \ge -\frac{1}{\varepsilon}I$ and $-D^2u^\varepsilon \le \frac{1}{\varepsilon}I$ in the viscosity sense. Roughly speaking, this means the second derivatives of $u_\varepsilon$ are bounded above by $\frac{1}{\varepsilon}$, while the second derivatives of $u^\varepsilon$ are bounded below by $-\frac{1}{\varepsilon}$.

To establish further properties of the sup- and inf-convolutions, we need to assume $u$ has additional regularity.

**Lemma 8.5.** *Suppose $U \subset \mathbb{R}^n$ is open and bounded, and let $u \in C(\overline{U})$. Then $u_\varepsilon, u^\varepsilon \to u$ uniformly on $\overline{U}$. Furthermore, if $u \in C^{0,\alpha}(\overline{U})$ for $0 < \alpha \le 1$ then*

*(i)* $|x - y^\varepsilon|^{2-\alpha} \le 2\varepsilon[u]_{0,\alpha;\overline{U}}$ *for any* $y^\varepsilon \in \arg\max_{y\in\overline{U}}\left\{u(y) - \frac{1}{2\varepsilon}|x-y|^2\right\}$,

*(ii)* $|x - y_\varepsilon|^{2-\alpha} \le 2\varepsilon[u]_{0,\alpha;\overline{U}}$ *for any* $y_\varepsilon \in \arg\min_{y\in\overline{U}}\left\{u(y) + \frac{1}{2\varepsilon}|x-y|^2\right\}$,

*(iii)* $\|u^\varepsilon - u\|_{L^\infty(U)}, \|u_\varepsilon - u\|_{L^\infty(U)} \le (2[u]_{0,\alpha;\overline{U}})^{2/(2-\alpha)}\varepsilon^{\alpha/(2-\alpha)}$, *and*

*(iv)* $u^\varepsilon, u_\varepsilon \in C^{0,\alpha}(\overline{U})$ *and* $[u^\varepsilon]_{0,\alpha;\overline{U}}, [u_\varepsilon]_{0,\alpha;\overline{U}} \le C[u]_{0,\alpha;\overline{U}}$, *with $C$ independent of $\varepsilon > 0$.*

*Proof.* We first prove uniform convergence. Let

$$y^\varepsilon \in \arg\max_{y\in\overline{U}}\left\{u(y) - \frac{1}{2\varepsilon}|x-y|^2\right\}$$

and note that

$$\frac{1}{2\varepsilon}|x-y^\varepsilon|^2 = u(y^\varepsilon) - u^\varepsilon(x) \le u(y^\varepsilon) - u(x). \tag{8.5}$$

Therefore

$$|u^\varepsilon(x) - u(x)| = \left|u(y^\varepsilon) - u(x) - \tfrac{1}{2\varepsilon}|x-y^\varepsilon|^2\right| \le 2|u(y^\varepsilon) - u(x)|. \tag{8.6}$$

By Proposition 8.3(i), $|x - y^\varepsilon| \le 2\|u\|_{L^\varepsilon(\overline{U})}^{1/2}\varepsilon^{1/2}$, and so it follows from uniform continuity of $u$ that $u^\varepsilon \to u$ uniformly on $\overline{U}$ as $\varepsilon \to 0$. The proof for $u_\varepsilon$ is similar.

For (i), we can use (8.5) to deduce

$$\frac{1}{2\varepsilon}|x-y^\varepsilon|^2 \leq [u]_{0,\alpha;\overline{U}}|x-y^\varepsilon|^\alpha,$$

and so

$$|x-y^\varepsilon|^{2-\alpha} \leq 2\varepsilon[u]_{0,\alpha;\overline{U}}.$$

The proof of (ii) is similar.

For (iii), we use (8.6) and (i) to obtain

$$|u^\varepsilon(x)-u(x)| \leq 2[u]_{0,\alpha;\overline{U}}|x-y^\varepsilon|^\alpha \leq 2[u]_{0,\alpha;\overline{U}}(2\varepsilon[u]_{0,\alpha;\overline{U}})^{\alpha/(2-\alpha)}.$$

Finally, for (iv) let $x, y \in \overline{U}$ and let $x^\varepsilon$ such that

$$u^\varepsilon(x) = u(x^\varepsilon) - \frac{1}{2\varepsilon}|x-x^\varepsilon|^2.$$

Using (iii) we obtain

$$\begin{aligned}|u^\varepsilon(x)-u^\varepsilon(y)| &\leq |u^\varepsilon(x)-u(x)| + |u(x)-u(y)| + |u^\varepsilon(y)-u(y)|\\ &\leq 2(2[u]_{0,\alpha;\overline{U}})^{2/(2-\alpha)}\varepsilon^{\alpha/(2-\alpha)} + [u]_{0,\alpha;\overline{U}}|x-y|^\alpha.\end{aligned}$$

If $|x-y|^{2-\alpha} \geq \varepsilon[u]_{0,\alpha;\overline{U}}$ then

$$|u^\varepsilon(x)-u^\varepsilon(y)| \leq C[u]_{0,\alpha;\overline{U}}|x-y|^\alpha.$$

On the other hand, if $|x-y|^{2-\alpha} < \varepsilon[u]_{0,\alpha;\overline{U}}$ then

$$\begin{aligned}u^\varepsilon(x)-u^\varepsilon(y) &\leq \frac{1}{2\varepsilon}\left(|y-x^\varepsilon|^2 - |x-x^\varepsilon|^2\right)\\ &\leq \frac{1}{2\varepsilon}\left(|x-y|^2 + 2|y-x||x-x^\varepsilon|\right)\\ &= \frac{1}{2\varepsilon}(|x-y|^{2-\alpha} + 2|x-y|^{1-\alpha}|x-x^\varepsilon|)|x-y|^\alpha\\ &\leq C[u]_{0,\alpha;\overline{U}}|x-y|^\alpha,\end{aligned}$$

which completes the proof. □

Having established that the sup- and inf-convolutions are smooth(er) approximations of $u$, we turn to the problem of showing that sup- and inf-convolutions preserve the viscosity sub- and super-solution properties, respectively. For this, we introduce additional notation. Given $u \in \mathrm{USC}(\overline{U})$ we define

$$M^\varepsilon(u) = \left\{ x \in U \ : \ \operatorname*{arg\,max}_{y\in\overline{U}} \left\{u(y) - \tfrac{1}{2\varepsilon}|x-y|^2\right\} \cap U \neq \varnothing \right\}, \tag{8.7}$$

and for $u \in \mathrm{LSC}(\overline{U})$ we define

$$M_\varepsilon(u) = \left\{ x \in U \ : \ \arg\min_{y \in \overline{U}} \left\{ u(y) + \tfrac{1}{2\varepsilon}|x-y|^2 \right\} \cap U \neq \varnothing \right\}. \tag{8.8}$$

For the moment, we consider equations of the form

$$F(D^2u, Du) = 0 \ \text{ in } U. \tag{8.9}$$

The following proposition is useful to state independently.

**Proposition 8.6.** *Let $\varepsilon > 0$, $u \in USC(\mathbb{R}^n)$, and $x_0 \in \mathbb{R}^n$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^\varepsilon - \varphi$ has a local max at $x_0$, and let $x_\varepsilon \in \mathbb{R}^n$ such that*

$$u^\varepsilon(x_0) = u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_0 - x_\varepsilon|^2. \tag{8.10}$$

*Define $\psi(x) = \varphi(x + x_0 - x_\varepsilon)$. Then $u - \psi$ has a local max at $x_\varepsilon$ and*

$$D\psi(x_\varepsilon) = D\varphi(x_0) = \frac{1}{\varepsilon}(x_\varepsilon - x_0). \tag{8.11}$$

*Proof.* Let $r > 0$ such that

$$u^\varepsilon(x_0) - \varphi(x_0) \geq u^\varepsilon(x) - \varphi(x) \tag{8.12}$$

whenever $|x - x_0| < r$. It follows from (8.10) and (8.12) that

$$\begin{aligned} u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_0 - x_\varepsilon|^2 - \varphi(x_0) &\geq u^\varepsilon(x) - \varphi(x) \\ &\geq u(y) - \frac{1}{2\varepsilon}|x-y|^2 - \varphi(x), \end{aligned} \tag{8.13}$$

for any $y \in \mathbb{R}^n$ and $|x - x_0| < r$. Set $y = x_\varepsilon$ to obtain

$$\frac{1}{2\varepsilon}|x - x_\varepsilon|^2 + \varphi(x) \geq \frac{1}{2\varepsilon}|x_0 - x_\varepsilon|^2 + \varphi(x_0)$$

for $|x - x_0| < r$. Therefore

$$x \mapsto \frac{1}{2\varepsilon}|x - x_\varepsilon|^2 + \varphi(x)$$

has a local minimum at $x = x_0$ and so

$$D\varphi(x_0) = \frac{1}{\varepsilon}(x_\varepsilon - x_0),$$

which verifies (8.11).

Now in (8.13) set $x_0 - x_\varepsilon = x - y$, that is, choose $y = x - x_0 + x_\varepsilon$. Then

$$u(x_\varepsilon) - \varphi(x_0) \geq u(y) - \varphi(y + x_0 - x_\varepsilon),$$

for $|y - x_\varepsilon| = |x - x_0| < r$. Since $\psi(y) = \varphi(y + x_0 - x_\varepsilon)$ we find that

$$u(x_\varepsilon) - \psi(x_\varepsilon) \geq u(y) - \psi(y)$$

whenever $|y - x_\varepsilon| < r$; that is, $u - \psi$ has a local maximum at $x_\varepsilon$. □

We now show that sup- and inf-convolutions preserve viscosity sub- and super-solution properties.

**Theorem 8.7.** *Let $U \subset \mathbb{R}^n$ be open and bounded. If $u \in USC(\overline{U})$ is a viscosity subsolution of (8.9) then the sup-convolution $u^\varepsilon$ is a viscosity solution of*

$$F(D^2u^\varepsilon, Du^\varepsilon) \leq 0 \quad \text{in } M^\varepsilon(u) \subset U. \tag{8.14}$$

*Similarly, if $u \in LSC(\overline{U})$ is viscosity supersolution of (8.9) then the inf-convolution $u_\varepsilon$ is a viscosity solution of*

$$F(D^2u_\varepsilon, Du_\varepsilon) \geq 0 \quad \text{in } M_\varepsilon(u) \subset U. \tag{8.15}$$

*Proof.* Let $u \in \mathrm{USC}(\overline{U})$ be a viscosity subsolution of (8.9) and define the sup-convolution $u^\varepsilon$. Let $x_0 \in M^\varepsilon(u)$. Then there exists $y_0 \in U$ such that

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2.$$

By Proposition 8.6 $u - \psi$ has a local maximum at $y_0$, where $\psi(x) = \varphi(x + x_0 - y_0)$. Since $u$ is a viscosity subsolution of (8.9) we have

$$F(D^2\varphi(x_0), D\varphi(x_0)) = F(D^2\psi(y_0), D\psi(y_0)) \leq 0.$$

The proof that $u_\varepsilon$ is a viscosity solution of (8.15) is similar. □

**Remark 8.8.** It can be the case that $M^\varepsilon(u) = U$ or $M_\varepsilon(u) = U$. Indeed, if $u = 0$ on $\partial U$ and $u > 0$ in $U$, then we immediately have $M^\varepsilon(u) = U$, and if $u < 0$ in $U$ then $M_\varepsilon(u) = U$. In general, it follows from Lemma 8.5 that

$$M^\varepsilon(u) \cap M_\varepsilon(u) \supset \left\{x \in U \,:\, \mathrm{dist}(x, \partial U) \geq C\varepsilon^{1/(2-\alpha)}\right\},$$

where $\alpha = 0$ when $u$ is bounded, and $\alpha > 0$ if $u \in C^{0,\alpha}(\overline{U})$.

We can immediately state a corollary for equations of the form

$$F(D^2u, Du) = f \quad \text{in } U, \tag{8.16}$$

where $f : \overline{U} \to \mathbb{R}$.

**Corollary 8.9.** *Let $U \subset \mathbb{R}^n$ be open and bounded and suppose $f \in C(\overline{U})$ with modulus of continuity $\omega$. If $u \in USC(\overline{U})$ is a bounded viscosity subsolution of (8.16) then the sup-convolution $u^\varepsilon$ is a viscosity solution of*

$$F(D^2u^\varepsilon, Du^\varepsilon) \le f + \omega(C\varepsilon^{1/2}) \quad \text{in } M^\varepsilon(u) \subset U \tag{8.17}$$

*Similarly, if $u \in LSC(\overline{U})$ is bounded viscosity supersolution of (8.16) then the inf-convolution $u_\varepsilon$ is a viscosity solution of*

$$F(D^2u_\varepsilon, Du_\varepsilon) \ge f - \omega(C\varepsilon^{1/2}) \quad \text{in } M_\varepsilon(u) \subset U. \tag{8.18}$$

*Proof.* Let $x_0 \in M^\varepsilon(u)$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^\varepsilon - \varphi$ has a local maximum at $x_0$. As in the proof of Theorem 8.7 we define $\psi(y) := \varphi(y+x_0-y_0)$ where $y_0 \in U$ is a point for which $u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2$. Then $u - \psi$ has a local maximum at $y_0$ and so

$$F(D^2\varphi(x_0), D\varphi(x_0)) = F(D^2\psi(y_0), D\psi(y_0)) \le f(y_0).$$

By Proposition 8.3 we have

$$f(y_0) \le f(x_0) + \omega(|x_0 - y_0|) \le f(x_0) + \omega(C\varepsilon^{1/2}).$$

The proof of (8.17) is similar. □

**Remark 8.10.** In Corollary 8.9, if $u \in C^{0,\alpha}(\overline{U})$ for $0 < \alpha \le 1$, then we can use Lemma 8.5 to obtain

$$F(D^2u^\varepsilon, Du^\varepsilon) \le f + \omega\left(C\varepsilon^{1/(2-\alpha)}\right) \quad \text{in } M^\varepsilon(u).$$

In particular, if $u$ and $f$ are Lipschitz continuous then

$$F(D^2u^\varepsilon, Du^\varepsilon) \le f + C\varepsilon \quad \text{in } M^\varepsilon(u).$$

A similar remark holds for $u_\varepsilon$.

Our final approximation result is for general first order equations.

$$H(Du, u, x) = 0 \text{ in } U. \tag{8.19}$$

**Theorem 8.11.** *Let $U \subset \mathbb{R}^n$ be open and bounded, and assume $H \in C^{0,1}_{loc}(\mathbb{R}^n \times \mathbb{R} \times \overline{U})$. If $u \in C^{0,1}(\overline{U})$ is a viscosity subsolution of (8.19) then the sup-convolution $u^\varepsilon$ is a viscosity solution of*

$$H(Du^\varepsilon, u^\varepsilon, x) \leq C\varepsilon \quad \text{in } M^\varepsilon(u) \subset U. \tag{8.20}$$

*Similarly, if $u \in C^{0,1}(\overline{U})$ is a viscosity supersolution of (8.19) then the inf-convolution $u_\varepsilon$ is a viscosity solution of*

$$H(Du_\varepsilon, u_\varepsilon, x) \geq -C\varepsilon \quad \text{in } M_\varepsilon(u) \subset U. \tag{8.21}$$

*In both cases, the constant $C$ depends only on $H$ and $\|u\|_{C^{0,1}(\overline{U})}$.*

*Proof.* Since $u \in C^{0,1}(\overline{U})$, there exists $K > 0$ such that $\|u\|_{L^\infty(U)} \leq K$, and $|Du| \leq K$ in $U$ in the viscosity sense (see Exercise 2.16). Since $H$ is locally Lipschitz, there exists $C > 0$ such that

$$|H(p, z, x) - H(p, r, y)| \leq C(|z - r| + |x - y|) \tag{8.22}$$

for $|z|, |r|, |p| \leq K$ and $x, y \in \overline{U}$.

Let $x_0 \in M^\varepsilon(u)$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u^\varepsilon - \varphi$ has a local maximum at $x_0$. As in the proof of Theorem 8.7 we define $\psi(y) := \varphi(y + x_0 - y_0)$ where $y_0 \in U$ is a point for which $u^\varepsilon(x_0) = u(y_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2$. Then $u - \psi$ has a local maximum at $y_0$ and so

$$H(D\psi(y_0), u(y_0), y_0)) \leq 0.$$

Since $|Du| \leq K$ in the viscosity sense, we also have $|D\psi(y_0)| \leq K$. Since $D\psi(y_0) = D\varphi(x_0)$, we have by (8.22) and Lemma 8.5 that

$$\begin{aligned}|H(D\psi(y_0), u(y_0), y_0)) - H(D\varphi(x_0), u(x_0), x_0)|\\ &\leq C(|u(x_0) - u(y_0)| + |x_0 - y_0|)\\ &\leq C|x_0 - y_0|\\ &\leq C\varepsilon,\end{aligned}$$

which completes the proof of (8.20). The proof of (8.21) is similar. □

To conclude this section, we use the inf- and sup-convolution tools developed here to give an alternative proof of the $O(\sqrt{\varepsilon})$ rate in the method of vanishing viscosity.

*Alternative proof of Theorem 5.8.* Recall that $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ is the classical solution of

$$u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon\Delta u_\varepsilon = 0 \text{ in } U \tag{8.23}$$

with boundary condition $u_\varepsilon = 0$ on $\partial U$ (and not the inf-convolution), and $u \in C^{0,1}(\overline{U})$ is the unique viscosity solution of

$$u + H(Du, x) = 0 \text{ in } U$$

with Dirichlet condition $u = 0$ on $\partial U$. Our goal is to show that

$$|u - u_\varepsilon| \leq C\sqrt{\varepsilon}.$$

For $\delta > 0$ let $u^\delta$ be the sup-convolution

$$u^\delta(x) = \sup_{y \in \overline{U}} \left\{ u(y) - \frac{1}{2\delta}|x-y|^2 \right\}.$$

Then $u^\delta$ is semiconvex with constant $-\frac{1}{\delta}$, i.e., $-D^2u^\delta \leq \frac{1}{\delta}I$ on $U$ in the viscosity sense, and by Theorem 8.11 there exists $C > 0$ such that $u^\delta$ is a viscosity solution of

$$u^\delta + H(Du^\delta, x) \leq C\delta \quad \text{in} \quad U_{C\delta},$$

where $U_{C\delta} = \{x \in U : \operatorname{dist}(x, \partial U) \geq C\delta\}$. Let $x_0 \in \overline{U}$ such that

$$\max_{\overline{U}}(u^\delta - u_\varepsilon) = u^\delta(x_0) - u_\varepsilon(x_0).$$

If $x_0 \notin U_{C\delta}$ then $\operatorname{dist}(x_0, \partial U) < C\delta$ and hence by Lemma 8.5 and the nonnegativity of $u_\varepsilon$ we have

$$\max_{\overline{U}}(u^\delta - u_\varepsilon) \leq u^\delta(x_0) - u_\varepsilon(x_0) \leq C\delta.$$

If $x_0 \in U_{C\delta}$ then $u^\delta - u_\varepsilon$ has a local maximum at $x_0$ and so $-D^2u_\varepsilon(x_0) \leq \frac{1}{\delta}I$ and

$$u^\delta(x_0) + H(Du_\varepsilon(x_0), x_0) \leq C\delta.$$

Subtracting (8.23) we have

$$u^\delta(x_0) - u_\varepsilon(x_0) \leq -\varepsilon\Delta u_\varepsilon(x_0) + C\delta \leq C\left(\tfrac{\varepsilon}{\delta} + \delta\right).$$

Optimizing over $\delta$ yields $\delta = \sqrt{\varepsilon}$ and hence

$$u - u_\varepsilon \leq u^\delta - u_\varepsilon \leq C\sqrt{\varepsilon}.$$

We leave the proof of the other direction to Exercise 8.12. □

**Exercise 8.12.** Complete the alternative proof of Theorem 5.8 by showing the $u_\varepsilon - u \leq C\sqrt{\varepsilon}$. Use the inf-convolution instead of the sup-convolution, and recall the barrier function argument from Theorem 5.4.

# Chapter 9

# Finite difference schemes

We briefly consider here the problem of approximating viscosity solutions with finite difference schemes. For simplicity, we restrict our attention to the unit box $[0,1]^n$. Thus we consider the Hamilton-Jacobi equation

$$\left.\begin{aligned} H(Du,u,x) &= 0 \quad \text{in } (0,1)^n \\ u &= g \quad \text{on } \partial(0,1)^n. \end{aligned}\right\} \tag{9.1}$$

Our goal is to design finite difference schemes for (9.1) that converge to the viscosity solution of (9.1) as the grid resolution tends to zero.

We first introduce some notation. For $h>0$ let $\mathbb{Z}_h = \{hz \,:\, z\in\mathbb{Z}\}$ and $\mathbb{Z}_h^n = (\mathbb{Z}_h)^n$. For a set $\mathcal{O}\subset\mathbb{R}^n$ we define $\mathcal{O}_h := \mathcal{O}\cap\mathbb{Z}_h^n$, and $\partial O_h := (\partial O)\cap\mathbb{Z}_h^n$. We will always assume that $1/h$ is an integer. Given a function $u:[0,1]_n^h\to\mathbb{R}$, we define the forward and backward difference quotients by

$$\nabla_i^{\pm}u(x) := \pm\frac{u(x\pm he_i)-u(x)}{h}, \tag{9.2}$$

and we set

$$\nabla^{\pm}u(x) = (\nabla_1^{\pm}u(x),\dots,\nabla_n^{\pm}u(x)).$$

When $u$ is a smooth function restricted to the grid, the forward and backward difference quotients (9.2) offer $O(h)$ (or first order) accurate approximations of $u_{x_i}$. This can be immediately verified by expanding $u$ via its Taylor series.

The idea is to restrict (9.1) to the grid $[0,1]_h^n$, and replace each partial derivative by a corresponding finite difference. However, some care must be taken in how this is done.

**Exercise 9.1.** Consider the following finite difference scheme for the one dimensional eikonal equation (1.6) from Exercise 1.4:

$$|\nabla_1^+ u_h(x)| = 1 \quad \text{for } x\in[0,1)_h, \quad \text{and} \quad u_h(0)=u_h(1)=0. \tag{9.3}$$

Show that the scheme is not well-posed, that is, depending on whether $1/h$ is even or odd, there is either no solution, or there is more than one solution.

## 9.1 Monotone schemes

In light Exercise 9.1, we cannot arbitrarily select forward or backward differences and hope to get a convergent scheme. To see what we should do, consider the special case that (9.1) is the Hamilton-Jacobi-Bellman equation (4.15), and $H$ is given by (4.14), which we recall here:

$$H(p,x) = \sup_{|a|=1} \left\{ -p \cdot a - L(a,x) \right\}.$$

In this case, the solution $u$ satisfies the dynamic programming principle (4.11)

$$u(x) = \inf_{y \in \partial B(x,r)} \{u(y) + T(x,y)\}.$$

The infimum on the right is attained at some $y \in \partial B(x,r)$ so we have

$$u(x) = u(y) + T(x,y).$$

Since $T(x,y) > 0$, this expresses two things. First, there must exist $y \in \partial B(x,r)$ such that $u(y) < u(x)$, and second, $u(x)$ depends only on the neighboring values $u(y)$ that are smaller than $u(x)$. Keeping these ideas in mind, we define the *monotone* finite differences

$$\nabla_i^{\mathbf{m}} u = \mathbf{m}(\nabla_i^+ u, \nabla_i^- u), \tag{9.4}$$

where

$$\mathbf{m}(a,b) = \begin{cases} a, & \text{if } a+b<0 \text{ and } a \le 0 \\ b, & \text{if } a+b \ge 0 \text{ and } b \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

We also define the *monotone gradient* by

$$\nabla^{\mathbf{m}} u = (\nabla_1^{\mathbf{m}} u, \ldots, \nabla_n^{\mathbf{m}} u).$$

The monotone finite difference $\nabla_i^{\mathbf{m}} u$ selects the forward difference when

$$u(x+he_i) = \min\{u(x+he_i), u(x-he_i)\} \le u(x),$$

the backward difference when

$$u(x-he_i) = \min\{u(x+he_i), u(x-he_i)\} \le u(x),$$

and returns zero when

$$u(x) < \min\{u(x + he_i), u(x - he_i)\}.$$

This is consistent with our observations above that there must exist a neighboring grid point where $u$ is smaller, and $u$ locally depends only on such a grid point.

**Exercise 9.2.** Consider the following monotone finite difference scheme for the one dimensional eikonal equation:

$$|\nabla_1^{\mathbf{m}} u_h(x)| = 1 \quad \text{for } x \in (0,1)_h, \quad \text{and } u_h(0) = u_h(1) = 0.$$

Find the solution $u_h$ explicitly, and show that $u_h \to \frac{1}{2} - |x|$ as $h \to 0^+$.

An important property of the monotone difference is the following anti-monotonicity.

**Proposition 9.3.** *If $u(x) = v(x)$ and $u \le v$ then*

$$|\nabla_i^{\mathbf{m}} u(x)| \ge |\nabla_i^{\mathbf{m}} v(x)| \quad \textit{for all } i.$$

*Proof.* The proof follows immediately from the observation that

$$|\nabla_i^{\mathbf{m}} u(x)| = \frac{1}{h} \max\left\{(u(x) - u(x + he_i))_+, (u(x) - u(x - he_i))_+\right\},$$

where $t_+ := \max\{0, t\}$. □

We should note that the condition $u \le v$ from Proposition 9.3 only needs to hold at neighboring grid points to $x$, i.e., $x - he_i$ and $x + he_i$.

**Lemma 9.4.** *Suppose $H$ is given by (4.14) and $L$ satisfies*

$$L(a_1, \ldots, a_n, x) = L(|a_1|, \ldots, |a_n|, x) \quad \textit{for all } x. \tag{9.5}$$

*If $u(x) = v(x)$ and $u \le v$ then*

$$H(\nabla^{\mathbf{m}} u(x), x) \ge H(\nabla^{\mathbf{m}} v(x), x). \tag{9.6}$$

*Proof.* By (9.5) we have

$$\begin{aligned} H(p_1, \ldots, p_n, x) &= \sup_{|a|=1} \{-p \cdot a - L(a, x)\} \\ &= \sup_{|a|=1} \{a_1|p_1| + \cdots + a_n|p_n| - L(a, x)\}. \end{aligned}$$

Therefore $H(p, x) \le H(q, x)$ whenever $|p_i| \le |q_i|$ for all $i$. Combining this with Proposition 9.3 completes the proof. □

Let us suppose now that $H$ satisfies the hypotheses of Lemma 9.4 and suppose $u_h : [0,1]^n_h \to \mathbb{R}$ is a solution of the numerical scheme

$$H(\nabla^{\mathbf{m}} u_h(x), x) = 0 \quad \text{for } x \in (0,1)^n_h.$$

Suppose we can also show that $u_h \to u \in C(\overline{U})$ uniformly on $[0,1]^n$ as $h \to 0^+$. Let $x \in (0,1)^n$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a strict maximum at $x$. Then there exists $h_k \to 0^+$ and $x_k \to x$ such that $x_k \in (0,1)^n_h$ and $u_{h_k} - \varphi$ has a maximum at $x_k$ over the grid $[0,1]^n_{h_k}$. By shifting $\varphi$, if necessary, we may assume that $u_{h_k}(x_k) = \varphi(x_k)$ and $u_{h_k} \leq \varphi$. By Lemma 9.4 we have

$$0 = H(\nabla^{\mathbf{m}} u_{h_k}(x_k), x_k) \geq H(\nabla^{\mathbf{m}} \varphi(x_k), x_k) \longrightarrow H(D\varphi(x), x)$$

as $h_k \to 0^+$. Therefore $u$ is a viscosity subsolution of $H = 0$. We can argue that $u$ is a supersolution similarly.

We note that the key part of the argument above was using (9.6) to replace the numerical solution $u_{h_k}$ by a smooth test function $\varphi$. In more general situations, when $H$ may not be given by (4.14), we can still look for a scheme satisfying a condition like (9.6). Such schemes are called *monotone*, and sometimes *upwind*.

A general finite difference scheme for (9.1) is of the form

$$\left.\begin{aligned} S_h(u_h, u_h(x), x) &= 0 \quad \text{in } (0,1)^n_h \\ u_h &= g \quad \text{on } \partial(0,1)^n_h, \end{aligned}\right\} \tag{9.7}$$

where

$$S_h : X_h \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

and $X_h$ denotes the collection of real-valued functions on $[0,1]^n_h$. We remark that the first argument of $S_h$ represents the dependence of $S_h$ on neighboring grid points, while the second argument represents the dependence of $S_h$ on the grid point $x$.

**Definition 9.5.** We say the scheme $S_h$ is *monotone* if

$$u \leq v \implies S_h(u, t, x) \geq S_h(v, t, x) \tag{9.8}$$

for all $u, v \in X_h$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

When $H$ is given by (4.14) and $L$ satisfies (9.5), the scheme

$$S_h(u, u(x), x) := H(\nabla^{\mathbf{m}} u(x), x)$$

is monotone by Lemma 9.4.

## 9.2 Convergence of monotone schemes

We show here that monotone schemes are convergent provided they are stable and consistent, and the limit PDE is well-posed. These results can be found in [2] in the more general context of degenerate elliptic equations.

We now give the definitions of consistency and stability.

**Definition 9.6.** We say the scheme $S_h$ is *consistent* if

$$\lim_{\substack{y\to x\\ h\to 0^+\\ \gamma\to 0}} S_h(\varphi+\gamma, \varphi(y)+\gamma, y) = H(D\varphi(x), \varphi(x), x) \tag{9.9}$$

for all $\varphi \in C^\infty(\mathbb{R}^n)$.

We note that even though viscosity solutions are not in general smooth, consistency need only be verified for smooth test functions $\varphi \in C^\infty(\mathbb{R}^n)$.

**Definition 9.7.** We say the scheme $S_h$ is *stable* if the solutions $u_h$ are uniformly bounded as $h \to 0^+$, that is, there exists $C > 0$ such that

$$\sup_{h>0} \sup_{x\in[0,1]^n_h} |u_h(x)| \le C.$$

Finally, we need to assume that a comparison principle holds for (9.1). In particular we assume that (9.1) enjoys *strong uniqueness*. This means that whenever $u \in \mathrm{USC}([0,1]^n)$ and $v \in \mathrm{LSC}([0,1]^n)$ are viscosity sub- and super-solutions of (9.1) in the sense of Definition 6.2, we have $u \le v$ on $[0,1]^n$.

**Theorem 9.8.** *Suppose* (9.1) *enjoys strong uniqueness, and $S_h$ is monotone, consistent, and stable. Then $u_h \to u$ uniformly on $[0,1]^n$ as $h \to 0^+$, where $u$ is the unique viscosity solution of* (9.1).

The proof is similar to Theorem 5.4. We sketch the details below. We note that in the context of the following proof, all viscosity solutions are interpreted in the sense of Definition 6.2.

*Proof.* We define the upper and lower weak limits by

$$\overline{u}(x) = \limsup_{(y,h)\to(x,0^+)} u_h(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{(y,h)\to(x,0^+)} u_h(y).$$

The limits above are taken with $y \in [0,1]^n_h$. Since the scheme is stable, both $\overline{u}$ and $\underline{u}$ are bounded real valued functions.

We claim that $\overline{u}$ is a viscosity subsolution of (9.1) in the sense of Definition 6.2. To see this, let $x_0 \in (0,1)^n$ and let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a strict maximum at $x_0$. As in the proof of Theorem 5.4, there exists $h_k \to 0$ and $x_k \to x_0$ such that $u_{h_k}(x_k) \to \overline{u}(x_0)$ and $u_{h_k} - \varphi$ has a maximum at $x_k$, relative to the grid $[0,1]^n_h$. Let us write

$$\varphi_k(x) = \varphi(x) + \gamma_k,$$

where $\gamma_k = u_{h_k}(x_k) - \varphi(x_k)$. Then $\varphi_k(x_k) = u_{h_k}(x_k)$ and $u_{h_k} \leq \varphi_k$. Since the scheme $S_h$ is monotone we have

$$0 = S_h(u_{h_k}, u_{h_k}(x_k), x_k) \geq S_h(\varphi + \gamma_k, \varphi(x_k) + \gamma_k, x_k).$$

Since $S_h$ is consistent, we can send $k \to \infty$ to find that

$$H(D\varphi(x_0), \varphi(x_0), x_0) \leq 0. \tag{9.10}$$

If $x_0 \in \partial[0,1]^n$, then we can arrange it so that $x_k \in \partial[0,1]^n_{h_k}$ for all $k$, or $x_k \in (0,1)^n_{h_k}$ for all $k$. In the first case, we have

$$\overline{u}(x_0) = \lim_{h_k \to 0^+} u_{h_k}(x_k) \leq g(x_0),$$

due to the continuity of $g$. The second case proceeds as above and we find that (9.10) holds. Therefore $\overline{u}$ is a viscosity subsolution of (9.1).

That $\underline{u}$ is a viscosity supersolution of (9.1) is verified similarly. By strong uniqueness we have $\overline{u} = \underline{u}$. Therefore $u_h \to u$ uniformly, where $u$ is the unique viscosity solution of (9.1). □

**Exercise 9.9.** Suppose the numerical solutions $u_h$ are uniformly Lipschitz continuous, i.e., there exists $C > 0$ such that

$$|u_h(x) - u_h(y)| \leq C|x - y| \quad \text{for all } x, y \in [0,1]^n_h \text{ and } h > 0.$$

This is a stronger form of stability. Prove Theorem (9.1) without the strong uniqueness hypothesis. You can assume that ordinary uniqueness holds, that is, there is at most one viscosity solution of (9.1) satisfying the boundary conditions in the usual sense. [Hint: Use the Arzelà-Ascoli Theorem to extract a subsequence $u_{h_k}$ converging uniformly to a continuous function $u \in C([0,1]^n)$. Show that $u$ is the unique viscosity solution of (9.1), and conclude that the entire sequence must converge uniformly to $u$.]

**Exercise 9.10.** Suppose that $S_h$ depends only on the forward and backward neighboring grid points in each direction, so that we can write

$$S_h(u, u(x), x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \dots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x).$$

Let us set $F = F(a_1, \dots, a_{2n}, z, x)$. You may assume that $H$ and $F$ are smooth.

(a) Show that $S_h$ is monotone if and only if $F_{a_i} \geq 0$ for all $i$.

(b) Show that $S_h$ is consistent if and only if

$$F(p_1, -p_2, \dots, p_n, -p_n, z, x) = H(p, z, x)$$

for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in [0,1]_h^n$.

(c) Find a monotone and consistent scheme for the linear PDE

$$a_1 u_{x_1} + \cdots + a_n u_{x_n} = f(x),$$

where $a_1, \dots, a_n$ are real numbers. Compare your scheme with the direction of the projected characteristics. [Hint: Your solution should depend on the signs of the $a_i$.]

(d) Suppose that $H$ is Lipschitz continuous and define

$$a := \sup \{ |D_p H(p, z, x)| \,:\, p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in [0,1]^n \}.$$

The Lax-Friedrichs scheme is defined by

$$S_h(u, u(x), x) := H\left(\nabla_h u(x), u(x), x\right) - \frac{ah}{2} \Delta_h u(x),$$

where

$$\nabla_h u(x) := \left( \frac{u(x + he_1) - u(x - he_1)}{2h}, \dots, \frac{u(x + he_n) - u(x - he_n)}{2h} \right),$$

and

$$\Delta_h u(x) := \sum_{i=1}^{n} \frac{u(x + he_i) - 2u(x) + u(x - he_i)}{h^2}.$$

Show that the Lax-Friedrichs scheme is monotone and consistent. [Hint: Rewrite the scheme as a function of the forward and backward differences $\nabla_i^\pm u(x)$, as above.]

**Exercise 9.11.** Let $U := B^0(0,1)$ and $\varepsilon > 0$. Consider the nonlocal integral equation

$$(\mathrm{I}_\varepsilon) \quad \begin{cases} (1+c\varepsilon^2)u_\varepsilon(x) - \fint_{B(x,\varepsilon)} u_\varepsilon \, dy = c\varepsilon^2 f(x) & \text{if } x \in U \\ u_\varepsilon(x) = 0 & \text{if } x \in \Gamma_\varepsilon, \end{cases}$$

where $c = \frac{1}{2(n+2)}$, $u_\varepsilon : \Gamma_\varepsilon \cup U \to \mathbb{R}$, $f \in C(\overline{U})$, and

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus U \,:\, \operatorname{dist}(x, \partial U) \leq \varepsilon\}.$$

Follow the steps below to show that as $\varepsilon \to 0^+$, $u_\varepsilon$ converges uniformly to the viscosity solution $u$ of

$$(\mathrm{P}) \quad \begin{cases} u - \Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

The proof is based on recognizing $(\mathrm{I}_\varepsilon)$ as a monotone approximation scheme for (P). Unless otherwise specified, any function $u : U \to \mathbb{R}$ is implicitly extended to be identically zero on $\Gamma_\varepsilon$.

(a) Show that there exists a unique function $u_\varepsilon \in C(\overline{U})$ solving $(\mathrm{I}_\varepsilon)$. [Hint: Show that the mapping $T : C(\overline{U}) \to C(\overline{U})$ defined by

$$T[u](x) := \frac{1}{1+c\varepsilon^2}\fint_{B(x,\varepsilon)} u \, dy + \frac{c\varepsilon^2}{1+c\varepsilon^2} f(x)$$

is a contraction mapping. Use the usual norm $\|u\| := \max_{\overline{U}} |u|$ on $C(\overline{U})$. Then appeal to Banach's fixed point theorem.]

(b) Define $S_\varepsilon : L^\infty(U \cup \Gamma_\varepsilon) \times \mathbb{R} \times U \to \mathbb{R}$ by

$$S_\varepsilon(u,t,x) := (1+c\varepsilon^2)t - \fint_{B(x,\varepsilon)} u \, dy.$$

Show that $S_\varepsilon$ is monotone, i.e., for all $t \in \mathbb{R}$, $x \in U$, and $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$

$$u \leq v \text{ on } B(x,\varepsilon) \implies S_\varepsilon(u,t,x) \geq S_\varepsilon(v,t,x).$$

(c) Show that the following comparison principle holds: Let $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$ such that $u|_{\overline{U}}, v|_{\overline{U}} \in C(\overline{U})$. If $u \leq v$ on $\Gamma_\varepsilon$ and $S_\varepsilon(u, u(x), x) \leq S_\varepsilon(v, v(x), x)$ at all $x \in U$, then $u \leq v$ on $U$.

(d) Use the comparison principle to show that there exists $C > 0$ such that

$$|u_\varepsilon(x)| \le C(1 + 3\varepsilon - |x|^2),$$

for all $x \in U$ and $0 < \varepsilon \le 1$, where $C$ depends only on $\|f\| = \max_{\overline{U}} |f|$. [Hint: Compare against $v(x) := C(1 + 3\varepsilon - |x|^2)$ and $-v$, and adjust the constant $C$ appropriately.]

(e) Use the method of weak upper and lower limits to show that $u_\varepsilon \to u$ uniformly on $\overline{U}$, where $u$ is the viscosity solution of (P). You may assume a comparison principle holds for (P) for semicontinuous viscosity solutions. That is, if $u \in \mathrm{USC}(\overline{U})$ is a viscosity subsolution of (P) and $v \in \mathrm{LSC}(\overline{U})$ is a viscosity supersolution, and $u \le v$ on $\partial U$, then $u \le v$ in $U$. [Hint: You will find the identity in the hint from Exercise 2.19 useful.]

## 9.3 Local truncation error

The monotonicity condition (Definition 9.5) places severe restrictions on the types of schemes available. It turns out that all monotone schemes for first order equations are at best $O(h)$ accurate. By this, we mean that the local truncation error (obtained by substituting smooth functions into the scheme) is no better than $O(h)$. In other words, any scheme with a local truncation error of $O(h^2)$ or better *cannot* be monotone.

There are many higher order schemes for Hamilton-Jacobi equations. For example, essentially non-oscillatory (ENO) schemes [15], which were originally proposed for hyperbolic conservation laws and have been adapted to Hamilton-Jacobi equations, are widely used and quite successful. Since the ENO schemes are non-monotone, there is as of yet no rigorous theory guaranteeing convergence to the viscosity solution.

As in Exercise 9.10, we write our monotone scheme as

$$F[u](x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \ldots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x).$$

Thus, we are assuming that the neighborhood $N(x)$ of $x$ contains just the forward and backward neighbors in each coordinate direction. Let us write

$$F = F(a_1, \ldots, a_{2n}, z, x)$$

for notational simplicity. Recall from Exercise (9.8) that $F$ is monotone if and only if $F$ is nondecreasing in each $a_i$, i.e., $F_{a_i} \ge 0$ for all $i$. In this case, consistency of the scheme states that

$$F(p_1, -p_1, \ldots, p_n, -p_n, z, x) = H(p, z, x). \tag{9.11}$$

Let $M > 0$ and define

$$\mathcal{S}_M := \Big\{\varphi \in C^\infty(\mathbb{R}^n) \,:\, \|\varphi\|_{C^3(\mathbb{R}^n)} \le M\Big\}.$$

We define the local truncation error by

$$\mathrm{err}(M,h) := \sup_{\substack{\varphi\in\mathcal{S}_M\\ x\in[0,1]^n}} |F[\varphi](x) - H(D\varphi(x),\varphi(x),x)|.$$

**Theorem 9.12.** *Let $F$ be monotone and smooth, and assume $H$ is smooth. Suppose that for some $p \in \mathbb{R}^n$, $z\in\mathbb{R}$, $x\in[0,1]^n$, and $i\in\{1,\dots,n\}$*

$$H_{p_i}(p,z,x) \neq 0. \tag{9.12}$$

*Then there exists $M>0$, $C>0$, $c>0$ and $\overline{h}>0$ such that for all $0<h<\overline{h}$*

$$ch \le \mathit{err}(M,h) \le Ch. \tag{9.13}$$

**Remark 9.13.** The condition (9.12) says that $H$ is not a trivial zeroth order PDE, such as $H(p,z,x)=z$.

*Proof.* The basic idea of the proof is that the monotonicity of $F$ ensures that the second order terms in the Taylor expansion for $u(x)-u(x\pm he_i)$ cannot be cancelled out to improve accuracy.

Without loss of generality, let us assume that $i=1$, $z=0$ and $x=0$ in (9.12). If $p_1=0$, then we can find a nearby point $p$ where $p_1\neq 0$ and (9.12) holds, by smoothness of $H$. Hence we may assume that $p_1>0$. By consistency (Eq. (9.11)) we have

$$F_{a_1}(a^0,0,0) - F_{a_2}(a^0,0,0) \neq 0,$$

where $a^0=(p_1,-p_1,\dots,p_n,-p_n)$. Since $F$ is monotone, $F_{a_i}\ge 0$ for all $i$. We may, without loss of generality, assume that $F_{a_1}(a^0,0,0)>0$. Hence, there exists $r>0$ and $\theta>0$ such that

$$F_{a_1}(a,0,0) \ge \theta \quad \text{whenever } |a-a^0|^2 \le r^2. \tag{9.14}$$

Define

$$\varphi(x) = \frac{1}{2}p_1(x_1+1)^2 - \frac{1}{2}p_1 + p_2x_2 + \cdots + p_nx_n.$$

Then $D\varphi(0)=p$ and $\varphi(0)=0$. We also note that

$$\nabla_1^+\varphi(0) = p_1 + \frac{1}{2}p_1h,\ \ \nabla_1^-\varphi(0) = p_1 - \frac{1}{2}p_1h,\ \text{ and } \nabla_i^\pm\varphi(0) = p_i$$

for $i = 2, \dots, n$. Due to the monotonicity of $F$ and (9.14) we have

$$\begin{aligned}F[\varphi](0) &= F(p_1 - \frac{1}{2}p_1 h, -p_1 - \frac{1}{2}p_1 h, p_2, -p_2, \dots, p_n, -p_n, 0, 0)\\ &\le F(p_1, -p_1, \dots, p_n, -p_n, 0, 0) - \frac{1}{2}p_1\theta h\\ &= H(p, 0, 0) - \frac{1}{2}p_1\theta h,\end{aligned}$$

provided $\frac{1}{2}p_1 h < r$. Thus, there exists $\overline{h} > 0$ such that

$$H(D\varphi(0), \varphi(0), 0) - F[\varphi](0) \ge \frac{1}{2}p_1\theta h =: ch$$

for all $0 < h < \overline{h}$. We can multiply $\varphi$ by a bump function to ensure that $\varphi \in C_c^\infty(\mathbb{R}^n)$, hence $\varphi \in \mathcal{S}_M$ for $M$ large enough.

The upper bound in (9.13) follows from the fact that $F$ is smooth, hence Lipschitz on compact sets, and

$$|\nabla^\pm \varphi(x) - D\varphi(x)| \le Ch,$$

for all $\varphi \in \mathcal{S}_M$, where $C$ depends on $M$. □

## 9.4 The $O(\sqrt{h})$ rate

Even though monotone schemes have $O(h)$ local truncation errors, it turns out that the best global errors that can be established rigorously are worse; they are $O(\sqrt{h})$. This should be compared with the $O(\sqrt{\varepsilon})$ convergence rates established in Chapter 5 for the method of vanishing viscosity. Intuitively, the reason for this is that local truncation errors consider how the scheme acts on smooth functions, and viscosity solutions are in general not smooth. Thus, the usual trick of substituting the solution of the PDE into the scheme does not convert local errors into global errors for viscosity solutions. However, see Section 5.3 for situations where the viscosity solution satisfies a one-sided second derivative bound. In this situation, we would expect a one-sided $O(h)$ rate.

Nevertheless, it is commonplace in practice to observe global errors on the order of $O(h)$ in numerical experiments, even when the solutions are not smooth. There is currently no theory that fully explains this difference between the experimental and theoretical convergence rates.

We assume that $L$ is Lipschitz continuous, and satisfies (9.5) as well as all of the assumptions of Chapter 4, and we take $H$ to be given by (4.14).

**Proposition 9.14.** *The Hamiltonian $H$ is Lipschitz continuous.*

*Proof.* It follows from Lemma 4.5 that $H$ is Lipschitz in $x$. Let $x, p, q \in \mathbb{R}^n$ and choose $a \in \mathbb{R}^n$ with $|a| = 1$ such that

$$H(p, x) = -p \cdot a - L(a, x).$$

Then we have

$$H(q, x) \geq -q \cdot a - L(a, x),$$

and so

$$H(p, x) - H(q, x) \leq (q - p) \cdot a \leq |q - p|.$$

Therefore $H$ is Lipschitz continuous. □

Let $u \in C^{0,1}([0, 1]^n)$ be the unique viscosity solution of

$$\left.\begin{aligned} H(Du, x) &= 0 \quad \text{in } (0, 1)^n \\ u &= 0 \quad \text{on } \partial(0, 1)^n, \end{aligned}\right\} \tag{9.15}$$

and consider the monotone finite difference scheme

$$\left.\begin{aligned} H(\nabla^{\mathbf{m}} u_h(x), x) &= 0 \quad \text{in } (0, 1)^n_h \\ u_h &= 0 \quad \text{on } \partial(0, 1)^n_h. \end{aligned}\right\} \tag{9.16}$$

We first aim to establish existence of a unique solution of (9.16). For this, we need a discrete comparison principle. We say that $u_h : [0, 1]^n_h \to \mathbb{R}$ is a *subsolution* of (9.16) if $H(\nabla^{\mathbf{m}} u_h, x) \leq 0$ in $(0, 1)^n_h$ and $u_h \leq 0$ on $\partial(0, 1)^n_h$. We define supersolutions analogously.

The comparison principle for (9.16) is based on the maximum principle. When $u$ and $v$ are smooth functions, the maximum principle is based on the fact that when $u - v$ has a maximum at $x_0$, we have $Du(x_0) = Dv(x_0)$ and hence

$$H(Du(x_0), x_0) = H(Dv(x_0), x_0). \tag{9.17}$$

If $u$ and $v$ are functions on the grid $[0, 1]^n_h$, then at a max of $u - v$ we have

$$\nabla_i^- u(x_0) \geq \nabla_i^- v(x_0) \quad \text{and} \quad \nabla_i^+ u(x_0) \leq \nabla_i^+ v(x_0).$$

Hence we cannot expect equality like (9.17) at the discrete level. Monotone schemes are designed precisely to give the correct inequality so that the maximum principle holds.

To see how this works, recall from Lemma 9.4 that if $u(x_0) = v(x_0)$ and $u \leq v$, then $H(\nabla^{\mathbf{m}} u(x_0), x_0) \geq H(\nabla^{\mathbf{m}} v(x_0), x_0)$. We can rephrase this in the language used above. Suppose that $u - v$ has a maximum at $x_0$. Then $\widetilde{u}(x_0) = v(x_0)$ and $\widetilde{u} \le v$, where $\widetilde{u}(x) := u(x) + v(x_0) - u(x_0)$. Since $\widetilde{u}$ and $u$ differ by a constant, we have

$$H(\nabla^{\mathbf{m}} u(x_0), x_0) = H(\nabla^{\mathbf{m}} \widetilde{u}(x_0), x_0) \ge H(\nabla^{\mathbf{m}} v(x_0), x_0).$$

This is important, so we we repeat

$$\boxed{\text{If } u - v \text{ has a max at } x \text{ then } H(\nabla^{\mathbf{m}} u(x), x) \ge H(\nabla^{\mathbf{m}} v(x), x).} \tag{9.18}$$

This is the discrete analogue of (9.17) and is exactly what allows maximum principle arguments to hold for monotone finite difference schemes.

**Lemma 9.15.** *If $u$ and $v$ are sub- and supersolutions of (9.16), respectively, then $u \le v$ on $[0,1]^n_h$.*

*Proof.* We will show that for every $\theta \in (0,1)$, $\theta u \le v$. Fix $\theta \in (0,1)$ and assume to the contrary that $\max_{[0,1]^n_h} (\theta u - v) > 0$. Let $x \in [0,1]^n_h$ be a point at which $\theta u - v$ attains its positive maximum. Then by (9.18) we have

$$H(\nabla^{\mathbf{m}} v(x), x) \le H(\theta \nabla^{\mathbf{m}} u(x), x).$$

Since $u \le 0 \le v$ on $\partial(0,1)^n_h$, we must have $x \in (0,1)^n_h$. Due to the convexity of $H$ we have

$$\begin{aligned} H(\theta \nabla^{\mathbf{m}} u(x), x) &= H\left(\theta \nabla^{\mathbf{m}} u(x) + (1-\theta) \cdot 0, x\right) \\ &\le \theta H\left(\nabla^{\mathbf{m}} u(x), x\right) + (1-\theta) H(0, x) \le -(1-\theta)\gamma, \end{aligned}$$

where $\gamma$ is a positive constant depending on $L$. We therefore deduce

$$H(\nabla^{\mathbf{m}} v(x), x) \le -(1-\theta)\gamma,$$

which is a contradiction. Therefore $\theta u \le v$ for all $\theta \in (0,1)$, and hence $u \le v$. □

**Lemma 9.16.** *There exists a unique grid function $u_h : [0,1]^n_h \to \mathbb{R}$ satisfying the monotone scheme (9.16). Furthermore, the sequence $u_h$ is nonnegative and uniformly bounded.*

The proof of Lemma 9.16 is based on the Perron method, but is considerably simpler due to the discrete setting.

*Proof.* Define

$$\mathcal{F} = \Big\{ u : [0,1]_h^n \to \mathbb{R} \,:\, u \text{ is a nonnegative subsolution of } (9.16) \Big\}.$$

Since $H(0,x) \leq 0$, $u \equiv 0$ is a subsolution, so $\mathcal{F}$ is nonempty. Furthermore, we claim that there exists a constant $C > 0$ such that for every $u \in \mathcal{F}$

$$u \leq C.$$

To see this, note that there exists $C > 0$ so that $L(a,x) \leq C$ for all $x \in [0,1]^n$ and $|a| = 1$. Therefore

$$H(p,x) \geq \sup_{|a|=1} \{-p \cdot a - C\} \geq |p| - C,$$

and so $v(x) := Cx_1$ is a supersolution of (9.16). By Lemma 9.15, $u \leq v \leq C$, which establishes the claim.

We now define

$$u_h(x) := \sup\{u(x) \,:\, u \in \mathcal{F}\}.$$

We first show that $u_h$ is a subsolution of (9.16). Fix $x_0 \in (0,1)_h^n$ and let $v^k \in \mathcal{F}$ such that $v^k(x_0) \to u_h(x_0)$ as $k \to \infty$. By passing to a subsequence, if necessary, we may assume that $v^k \to v$ on $[0,1]_h^n$ as $k \to \infty$ for some $v : [0,1]_h^n \to \mathbb{R}$. Then clearly $u_h(x_0) = v(x_0)$, $u_h \geq v$ and by continuity of $H$, $v$ is a subsolution of (9.16). It follows from monotonicity that

$$H(\nabla^{\mathbf{m}} u_h(x_0), x_0) \leq H(\nabla^{\mathbf{m}} v(x_0), x_0) \leq 0.$$

This establishes that $u_h$ is a subsolution of (9.16).

We now show that $u_h$ is a supersolution of (9.16). Assume to the contrary that there exists $x_0 \in (0,1)_h^n$ such that

$$H(\nabla^{\mathbf{m}} u_h(x_0), x_0) < 0.$$

For $\varepsilon > 0$ define

$$v(x) := \begin{cases} u_h(x) + \varepsilon, & \text{if } x = x_0 \\ u_h(x), & \text{otherwise.} \end{cases}$$

By continuity, we can choose $\varepsilon > 0$ small enough so that

$$H(\nabla^{\mathbf{m}} u_h(x_0), x_0) \leq 0.$$

By monotonicity, $u_h$ remains a subsolution of (9.16) at other grid points. Therefore $v \in \mathcal{F}$ and $v(x_0) > u_h(x_0)$, which is a contradiction. Therefore $u_h$ is a solution of (9.16).

Uniqueness follows from Lemma 9.15. □

We are now ready to prove the $O(\sqrt{h})$ convergence rate.

**Theorem 9.17.** *There exists a constant $C > 0$ such that*

$$|u - u_h| \leq C\sqrt{h}.$$

*Proof.* For $\theta \in (0,1)$, to be selected later, define the auxiliary function

$$\Phi(x,y) = \theta u(x) - u_h(y) - \frac{1}{\sqrt{h}}|x-y|^2,$$

for $x \in [0,1]^n$ and $y \in [0,1]^n_h$. Let $(x_h, y_h) \in [0,1]^n \times [0,1]^n_h$ such that

$$\Phi(x_h, y_h) = \max_{[0,1]^n \times [0,1]^n_h} \Phi.$$

Since $\Phi(x_h, y_h) \geq \Phi(y_h, y_h)$ we have

$$\theta u(x_h) - u_h(y_h) - \frac{1}{\sqrt{h}}|x_h - y_h|^2 \geq \theta u(y_h) - u_h(y_h).$$

Therefore

$$\frac{1}{\sqrt{h}}|x_h - y_h|^2 \leq \theta(u(x_h) - u(y_h)) \leq C|x_h - y_h|$$

due to the Lipschitzness of $u$. Therefore

$$|x_h - y_h| \leq C\sqrt{h}.$$

The proof of this is split into three cases now.

Case 1. If $x_h \in \partial(0,1)^n$ then

$$\theta u(x_h) - u_h(y_h) \leq 0$$

since $u_h$ is nonnegative and $u(x_h) = 0$.

Case 2. If $y_h \in \partial(0,1)^n_h$ then

$$\theta u(x_h) - u_h(y_h) = \theta u(x_h) - \theta u(y_h) \leq C|x_h - y_h| \leq C\sqrt{h},$$

since $u_h(y_h) = u(y_h) = 0$ and $\theta \in (0,1)$.

Case 3. Suppose $x_h \in (0,1)^n$ and $y_h \in (0,1)^n_h$. Then

$$x \mapsto u(x) - \frac{1}{\theta\sqrt{h}}|x - y_h|^2$$

has a maximum at $x_h$. Letting $p = \frac{2}{\sqrt{h}}(x_h - y_h)$ we have $H\left(\frac{p}{\theta}, x_h\right) \leq 0$. Therefore

$$\begin{aligned} H(p, x_h) &= H\left(\theta \frac{p}{\theta} + (1-\theta)\cdot 0, x_h\right) \\ &\leq \theta H\left(\frac{p}{\theta}, x_h\right) + (1-\theta)H(0, x_h) \leq -(1-\theta)\gamma, \end{aligned} \tag{9.19}$$

for some $\gamma > 0$ depending only on $L$. Note we used the convexity of $H$ with respect to $p$ above.

Notice that

$$y \mapsto u_h(y) + \frac{1}{\sqrt{h}}|x_h - y|^2$$

has a maximum at $y = y_h$. By (9.18) we have

$$0 = H(\nabla^{\mathbf{m}} u_h(y_h), y_h) \leq H(\nabla^{\mathbf{m}}\psi(y_h), y_h),$$

where $\psi(y) := -\frac{1}{\sqrt{h}}|x_h - y|^2$. Since $|D^2\psi| \leq \frac{1}{\sqrt{h}}$, a Taylor expansion gives

$$|\nabla^{\mathbf{m}}\psi(y_h) - D\psi(y_h)| \leq C\sqrt{h}.$$

Since $H$ is Lipschitz in all variables and $D\psi(y_h) = p$ we have

$$H(p, y_h) + C\sqrt{h} \geq 0.$$

Combining this with (9.19) yields

$$\gamma(1-\theta) \leq H(p, y_h) - H(p, x_h) + C\sqrt{h} \leq C\sqrt{h}$$

for all $\theta \in (0,1)$, due to the Lipschitzness of $H$. For $h$ sufficiently small, we set

$$\theta = 1 - \frac{(C+1)}{\gamma}\sqrt{h},$$

to obtain

$$(C+1)\sqrt{h} \leq C\sqrt{h}.$$

Since this is a contradiction, case 3 is impossible.

We have shown that there exists $K > 0$ such that when $\theta := 1 - K\sqrt{h}$ we have

$$\theta u(x_h) - u_h(y_h) \leq C\sqrt{h}$$

for $h > 0$ sufficiently small. Therefore, there exists $\overline{h} > 0$ such that

$$\max_{[0,1]^n_h}(\theta u - u_h) \leq \theta u(x_h) - u_h(y_h) \leq C\sqrt{h},$$

for $0 < h < \overline{h}$. Therefore

$$u - u_h = \theta u - u_h + (1-\theta)u \leq C\sqrt{h} + K\sqrt{h}\max_{[0,1]^n} u = C\sqrt{h}.$$

For $h \geq \overline{h}$ we have

$$\max_{[0,1]^n_h}(u - u_h) \leq \max_{[0,1]^n}|u| + \max_{[0,1]^n_h}|u_h| \leq C \leq \widetilde{C}\sqrt{h},$$

due to Lemma 9.16, where $\widetilde{C} := C/\sqrt{\overline{h}}$. This completes the proof. □

**Exercise 9.18.** Complete the proof of Theorem 9.17 by showing that $u_h - u \leq C\sqrt{h}$.

## 9.5 One-sided $O(h)$ rate

When the solution is semiconcave, we can prove an $O(h)$ one-sided rate. This result is a direct analog of the one-sided rate we obtained for the method of vanishing viscosity in Section 5.3 via semiconcavity.

As in Section 9.4, we assume that $L$ is Lipschitz continuous, and satisfies (9.5) as well as all of the assumptions of Chapter 4, and we take $H$ to be given by (4.14).

**Proposition 9.19.** *Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is semiconcave with semiconcavity constant $c$. Then for almost every $x_0 \in \mathbb{R}^n$*

$$|\nabla_i^{\mathbf{m}} u(x_0)| \geq |u_{x_i}(x_0)| - \frac{c}{2}h. \tag{9.20}$$

*Proof.* Since $u$ is semiconcave with constant $c$, $u - \frac{c}{2}|x - x_0|^2$ is concave. It is a general fact that convex or concave functions are locally Lipschitz continuous, hence differentiable almost everywhere. Hence, for almost every $x_0$ we have

$$u(x) - \frac{c}{2}|x - x_0|^2 \leq u(x_0) + Du(x_0)\cdot(x - x_0) \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore

$$\frac{u(x_0 \pm he_i) - u(x_0)}{h} \leq \pm u_{x_i}(x_0) + \frac{c}{2}h,$$

for all $h > 0$ and almost every $x_0 \in \mathbb{R}^n$. It follows that

$$\begin{aligned}
|\nabla_i^{\mathbf{m}} u(x_0)| &= \max\{(u(x_0) - u(x_0 - he_i))_+, (u(x_0) - u(x_0 + he_i))_+\} \\
&\geq \max\left\{\left(u_{x_i}(x_0) - \frac{c}{2}h\right)_+, \left(-u_{x_i}(x_0) - \frac{c}{2}h\right)_+\right\} \\
&\geq \max\left\{(u_{x_i}(x_0))_+, (-u_{x_i}(x_0))_+\right\} - \frac{c}{2}h \\
&= |u_{x_i}(x_0)| - \frac{c}{2}h
\end{aligned}$$

for almost every $x_0$. □

**Theorem 9.20.** *Suppose the viscosity solution $u \in C([0,1]^n)$ of (9.15) is semiconcave, and let $u_h : [0,1]_h^n \to \mathbb{R}$ be the solution of the monotone finite difference scheme (9.16). Then there exists a constant $C > 0$ such that*

$$u_h - u \leq Ch. \tag{9.21}$$

*Proof.* Let $x_0 \in (0,1)^n$ such that $u$ is differentiable at $x_0$. By Proposition 9.19 we have

$$|\nabla_i^{\mathbf{m}} u(x_0)| \geq |u_{x_i}(x_0)| - \frac{c}{2}h$$

for all $i$ and $h > 0$. As in the proof of Lemma 9.4

$$H(\nabla^{\mathbf{m}} u(x_0), x_0) \geq H\left(Du(x_0) - \frac{c}{2}h\mathbf{1}, x_0\right) \geq -Ch,$$

due to the Lipschitzness of $H$, and the fact that the Lipschitz viscosity solution $u$ satisfies the PDE (9.15) at each point of differentiability; in particular, at $x_0$. In the above, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. By continuity of $H$ and $u$, we conclude that

$$H(\nabla^{\mathbf{m}} u(x), x) + Ch \geq 0 \quad \text{for all } x \in (0,1)_h^n. \tag{9.22}$$

Now set $v_h(x) = \theta u_h(x)$ for $\theta > 0$ to be determined. Then we have

$$\begin{aligned}
H(\nabla^{\mathbf{m}} v_h(x), x) &= H(\theta \nabla^{\mathbf{m}} u_h(x) + (1-\theta)\cdot 0, x) \\
&\leq \theta H(\nabla^{\mathbf{m}} u_h(x), x) + (1-\theta) H(0, x) \\
&\leq -(1-\theta)\gamma,
\end{aligned}$$

for a constant $\gamma > 0$ depending only on $L$. Note we used the convexity of $H$ above. Set $\theta = 1 - \frac{C}{\gamma}h$ to find that

$$H(\nabla^{\mathbf{m}} v_h(x), x) + Ch \leq 0 \quad \text{for all } x \in (0,1)_h^n,$$

for $h > 0$ small enough that $\theta > 0$. By the discrete comparison principle (Lemma 9.15) we have that $v_h \leq u$ on $[0,1]^n_h$. Therefore $(1 - Ch)u_h \leq u$. Since the sequence $u_h$ is uniformly bounded, we conclude that

$$u_h - u \leq Ch$$

for $h > 0$ sufficiently small. □

Notice in the proof of Theorem 9.20, we only used the fact that $u$ is a Lipschitz almost everywhere solution of (9.15). Therefore, we have the following result.

**Corollary 9.21.** *If $w \in C^{0,1}([0,1]^n)$ is a semiconcave Lipschitz almost everywhere solution of* (9.15)*, then $w$ is the unique viscosity solution of* (9.15)*.*

*Proof.* Let $u$ denote the unique viscosity solution of (9.15). By Exercise 2.22, $w$ is a viscosity subsolution of (9.15), and so by comparison, $w \leq u$. By Theorem 9.20, there exists a constant $C$ such that $w \geq u_h - Ch$ for all $h > 0$, where $u_h$ is the solution of the monotone scheme (9.16). Since $u_h \to u$ uniformly as $h \to 0$, we have $w \geq u$, hence $w = u$. □

# Chapter 10

# Homogenization

Let $u_\varepsilon \in C(\overline{U})$ be a viscosity solution of

$$\left.\begin{aligned} u_\varepsilon + H\left(Du_\varepsilon, \frac{x}{\varepsilon}\right) &= 0 \quad \text{in } U \\ u_\varepsilon &= 0 \quad \text{on } \partial U, \end{aligned}\right\} \tag{10.1}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $U \subset \mathbb{R}^n$ is open and bounded. Since (10.1) is highly oscillatory when $\varepsilon > 0$ is small, we expect $u_\varepsilon$ to have an oscillatory component. The goal of homogenization theory is to describe these oscillations and understand the behavior of the sequence $u_\varepsilon$ as $\varepsilon \to 0^+$.

Our primary assumption is

$$\text{(Periodicity)} \quad y \mapsto H(p,y) \ \text{ is } \mathbb{Z}^n\text{-periodic for all } p \in \mathbb{R}^n. \tag{10.2}$$

This means that $H(p, y + z) = H(p, y)$ for all $z \in \mathbb{Z}^n$ and $y \in \mathbb{R}^n$. We also assume that $H$ satisfies (3.6), (6.2), and is

$$\text{(Coercive)} \quad \liminf_{|p|\to\infty} H(p,y) > 0 \quad \text{uniformly in } y \in \mathbb{R}^n, \tag{10.3}$$

and

$$\text{(Nonnegative)} \quad -H(0,y) \geq 0 \quad \text{for all } y \in \mathbb{R}^n. \tag{10.4}$$

We first record a Lipschitz estimate on the solution $u_\varepsilon$.

**Lemma 10.1.** *There exists a constant $C$ such that for all $\varepsilon > 0$*

$$\|u_\varepsilon\|_{C^{0,1}(\overline{U})} \leq C. \tag{10.5}$$

The proof of Lemma 10.1 is very similar to Lemmas 5.1 and 5.7, so we omit it.

By the Arzelà-Ascoli Theorem, we can pass to a subsequence $u_{\varepsilon_j}$ so that $u_{\varepsilon_j} \to u \in C^{0,1}(\overline{U})$ uniformly on $\overline{U}$. The goal is to identify a PDE that is satisfied by $u$. To do this, we need to understand locally the structure of $u_\varepsilon - u$. Let us suppose that near a point $x_0$, $u_\varepsilon$ has the form

$$u_\varepsilon(x) = u(x) + \varepsilon v(\tfrac{x}{\varepsilon}) + O(\varepsilon^2) \ \text{ as } \varepsilon \to 0^+,$$

where the function $v$ is $\mathbb{Z}^n$-periodic and may depend on the choice of $x_0$. Substituting this into (10.1) we formally have

$$u(x) + \varepsilon v(y) + H(Du(x) + Dv(y), y) + o(1) = 0 \ \text{ as } \varepsilon \to 0^+$$

for $x$ near $x_0$, where $y := \frac{x}{\varepsilon}$. Setting $p = Du(x_0)$ and formally sending $\varepsilon \to 0^+$ we find that

$$H(p + Dv(y), y) = \lambda \ \text{ in } \mathbb{R}^n \tag{10.6}$$

for some $\lambda \in \mathbb{R}$ (here, $\lambda = -u(x_0)$). Equation (10.6) is called a *cell problem*, and its solution $v$ is called a *corrector function*. The corrector describes the high frequency oscillations of $u_\varepsilon$ about the limit $u$ near the point $x_0$.

While the above argument is only a heuristic, it is important because it allows us to identify the cell problem (10.6), which we can study rigorously.

**Lemma 10.2.** *For each $p \in \mathbb{R}^n$, there exists a unique real number $\lambda$ such that* (10.6) *has a $\mathbb{Z}^n$-periodic viscosity solution $v \in C^{0,1}(\mathbb{R}^n)$.*

In light of the lemma, we write

$$\overline{H}(p) := \lambda, \tag{10.7}$$

and the heuristics above suggest that $u$ should be the viscosity solution of

$$u + \overline{H}(Du) = 0 \ \text{ in } U,$$

satisfying $u = 0$ on $\partial U$. The function $\overline{H}$ is called the *effective Hamiltonian*.

*Proof.* The proof is split into several steps.

1. Let $\delta > 0$ and consider the approximating PDE

$$\delta w_\delta + H(p + Dw_\delta, y) = 0 \ \text{ in } \mathbb{R}^n. \tag{10.8}$$

The addition of the zeroth order term guarantees that a comparison principle holds for (10.8) (see Theorem 6.4 and Corollary 3.2). We can prove existence of a viscosity solution of (10.8) via the Perron method. Indeed, let $C > 0$ be large enough so that

$$\delta C + H(p, y) \geq 0 \quad \text{for all } y \in \mathbb{R}^n.$$

Then the constant $C$ is a viscosity supersolution of (10.8). Set

$$\mathcal{F} := \Big\{ u \in \mathrm{USC}(\mathbb{R}^n) \,:\, v \text{ is a subsolution of (10.8) and } v \leq C \Big\}.$$

Since $u = -\widetilde{C}$ is a viscosity subsolution for large enough $\widetilde{C} > 0$, the set $\mathcal{F}$ is nonempty. Define

$$w_\delta(x) = \sup\{u(x) \,:\, u \in \mathcal{F}\}.$$

By Lemmas 7.1 and 7.2, $w_\delta = w_\delta^*$ is a bounded viscosity subsolution of (10.8) and $w_{\delta,*}$ is a bounded viscosity supersolution of (10.8). By comparison for (10.8) we have $w_\delta \leq w_{\delta,*}$, therefore $w_\delta \in C(\mathbb{R}^n)$ is the unique bounded viscosity solution of (10.8). By comparison for (10.8) we have $w_\delta \leq w_{\delta,*}$, therefore $w_\delta \in C(\mathbb{R}^n)$ is the unique bounded viscosity solution of (10.8).

2. We now claim that $w_\delta$ is $\mathbb{Z}^n$-periodic. Suppose to the contrary that there exists $y_0 \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$ such that $w_\delta(y_0 + z) > w_\delta(y_0)$. By the periodicity of $H$, $u(y) := w_\delta(y + z)$ is a viscosity solution of (10.8), and $u \leq C$. Therefore $u \in \mathcal{F}$ and by the definition of $w_\delta$

$$w_\delta(y_0) \geq u(y_0) = w_\delta(y_0 + z) > w_\delta(y_0),$$

which is a contradiction.

3. Since $w_\delta$ is $\mathbb{Z}^n$-periodic, similar arguments to the proof of Lemma 10.1 show that there exists $C > 0$ such that

$$\|\delta w_\delta\|_{C(\mathbb{R}^n)} \leq C$$

and

$$|w_\delta(x) - w_\delta(y)| \leq C|x - y| \quad \text{for all } x, y \in \mathbb{R}^n,$$

where $C$ is independent of $\delta$. We now define

$$v_\delta := w_\delta - \min_{\mathbb{R}^n} w_\delta.$$

Then each $v_\delta$ is a viscosity solution of

$$\delta v_\delta + H(p + Dv_\delta, y) = -\min_{\mathbb{R}^n} \delta w_\delta,$$

and the sequence $v_\delta$ satisfies

$$\sup_{\delta>0} \|v_\delta\|_{C^{0,1}(\mathbb{R}^n)} < \infty.$$

Utilizing the above information and the Arzelà-Ascoli Theorem, we can extract a subsequence $\delta_j \to 0$ such that

$$v_{\delta_j} \to v \quad \text{and} \quad \delta_j w_{\delta_j} \to -\lambda \;\; \text{uniformly on } \mathbb{R}^n,$$

where $v \in C^{0,1}(\mathbb{R}^n)$ is $\mathbb{Z}^n$-periodic and $\lambda \in \mathbb{R}$. By the stability of viscosity solutions under uniform convergence we find that $v$ is a viscosity solution of

$$H(p + Dv, y) = \lambda \;\; \text{in } \mathbb{R}^n.$$

4. We now show that $\lambda$ is unique. Suppose, by way of contradiction, that $\hat{v} \in C^{0,1}(\mathbb{R}^n)$ is a $\mathbb{Z}^n$-periodic viscosity solution of

$$H(p + D\hat{v}, y) = \hat{\lambda} \;\; \text{in } \mathbb{R}^n, \tag{10.9}$$

and, say, $\hat{\lambda} > \lambda$. By the comparison principle from Theorem 6.4, we have $v \leq \hat{v}$ in $\mathbb{R}^n$. This contradicts the fact that we can add an arbitrary constant to $\hat{v}$ without changing (10.9). □

**Theorem 10.3.** *The sequence $u_\varepsilon$ converges uniformly on $\overline{U}$ to the unique viscosity solution $u \in C^{0,1}(\overline{U})$ of*

$$\left.\begin{aligned} u + \overline{H}(Du) &= 0 \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U. \end{aligned}\right\} \tag{10.10}$$

The proof of Theorem 10.3 is based on the "perturbed test function" technique, which was pioneered in [9, 10].

*Proof.* By Lemma 10.1 and the Arzelà-Ascoli Theorem, there exists a function $u \in C^{0,1}(\overline{U})$ and a subsequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} \to u$ uniformly on $\overline{U}$. We claim that $u$ is the unique viscosity solution of (10.10). Once this is established, it immediately follows that $u_\varepsilon \to u$ uniformly on $\overline{U}$.

We first verify that $u$ is a viscosity subsolution of (10.10). The proof is split into three steps.

1. Let $x_0 \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a strict local maximum at $x_0$ and $u(x_0) = \varphi(x_0)$. We must show that

$$\varphi(x_0) + \overline{H}(D\varphi(x_0)) \leq 0.$$

Assume, by way of contradiction, that

$$\varphi(x_0) + \overline{H}(D\varphi(x_0)) =: \theta > 0. \tag{10.11}$$

Set $p = D\varphi(x_0)$ and let $v \in C^{0,1}(\overline{U})$ be a $\mathbb{Z}^n$-periodic viscosity solution of

$$H(p + Dv, y) = \overline{H}(p) \ \text{ in } \ \mathbb{R}^n. \tag{10.12}$$

By adding a constant to $v$, we may assume $v \geq 0$. We now define the perturbed test function

$$\varphi_\varepsilon(x) := \varphi(x) + \varepsilon v\left(\frac{x}{\varepsilon}\right).$$

We note that $\varphi_\varepsilon$ is Lipschitz continuous, but not $C^1$ in general. Since $v \geq 0$, $\varphi_\varepsilon \geq \varphi$.

2. We now claim that

$$\varphi_\varepsilon + H\left(D\varphi_\varepsilon, \frac{x}{\varepsilon}\right) \geq \frac{\theta}{2} \quad \text{in} \ \ B^0(x_0, r) \tag{10.13}$$

in the viscosity sense for small enough $r > 0$ (to be selected later). To see this, let $x_1 \in B^0(x_0, r)$ and $\psi \in C^\infty(\mathbb{R}^n)$ such that $\varphi_\varepsilon - \psi$ has a local minimum at $x_1$ and $\psi(x_1) = \varphi_\varepsilon(x_1)$. Then the mapping

$$x \mapsto \varepsilon v\left(\frac{x}{\varepsilon}\right) - (\psi(x) - \varphi(x)) \ \text{ has a minimum at } x = x_1,$$

and hence

$$y \mapsto v(y) - \eta(y) \ \text{ has a minimum at } y_1 := \frac{x_1}{\varepsilon},$$

where

$$\eta(y) := \frac{1}{\varepsilon}(\psi(\varepsilon y) - \varphi(\varepsilon y)).$$

Since $v$ is a viscosity solution of (10.12) we deduce

$$H(p + D\eta(y_1), y_1) \geq \overline{H}(p).$$

Since $\varphi_\varepsilon(x_0) \geq \varphi(x_0)$ we have by (10.11) that

$$\varphi_\varepsilon(x_0) + H\left(D\varphi(x_0) - D\varphi(x_1) + D\psi(x_1), \frac{x_1}{\varepsilon}\right) \geq \theta.$$

By (6.2), there exists a sufficiently small radius $r > 0$ such that

$$\varphi_\varepsilon(x_1) + H\left(D\psi(x_1), \frac{x_1}{\varepsilon}\right) \geq \frac{\theta}{2},$$

which establishes the claim.

3. We can select $r > 0$ smaller, if necessary, so that $u+2\delta \leq \varphi$ on $\partial B(x_0, r)$ for some $0 < \delta < \frac{\theta}{2}$. Then for $\varepsilon_j$ sufficiently small, we have

$$u_{\varepsilon_j} + \delta \leq \varphi \leq \varphi_{\varepsilon_j} \quad \text{on } \partial B(x_0, r).$$

We note that $u := u_{\varepsilon_j} + \delta$ is a viscosity solution of

$$u + H\left(Du, \frac{x}{\varepsilon}\right) = \delta < \frac{\theta}{2} \quad \text{in } B^0(x_0, r).$$

By the comparison principle we have $u_{\varepsilon_j} + \delta \leq \varphi_{\varepsilon_j}$ throughout the ball $B^0(x_0, r)$. Sending $\varepsilon_j \to 0$, we arrive at the contradiction $u(x_0) + \delta \leq \varphi(x_0)$. Therefore $u$ is a viscosity subsolution of (10.10).

The proof that $u$ is a viscosity supersolution of (10.10) is similar. $\square$

# Chapter 11

# Discontinuous coefficients

Here, we briefly consider Hamilton-Jacobi equations with discontinuous coefficients, and illustrate how to define viscosity solutions and extend the comparison principle to this setting. For simplicity, we consider the Hamilton-Jacobi equation

$$\left.\begin{aligned} H(Du) &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}\right\} \tag{11.1}$$

The comparison principles we have established so far, Theorems 3.1 and 6.4, require $f \in C(U)$. There are many applications where $f$ may be discontinuous, as is illustrated by the example below.

**Example 11.1** (Shape from shading)**.** *Let $n = 2$. Suppose we are photographing an object given by the graph of a function $u : U \to \mathbb{R}$ with a camera and light source positioned at $(0, 0, T)$ and pointing in the downward direction $(0, 0, -1)$. The camera's light source illuminates the object, and the camera captures a grayscale image $I : U \to [0, 1]$ proportional to the amount of reflected light returning to the camera. The* shape from shading *problem is to reconstruct the object $u$ from the image $I$.*

*We will show that the shape from shading problem reduces to solving a Hamilton-Jacobi equation. Let*

$$\mathbf{n}(x) = \frac{(u_{x_1}, u_{x_2}, 1)}{\sqrt{|Du|^2 + 1}}$$

*be the upward unit normal to the object. If the light source and camera are far from the object (so $T \gg 1$) then the intensity $I(x)$ of the image can be approximated by*

$$I(x) = \mathbf{n}(x) \cdot (0, 0, 1) = \frac{1}{\sqrt{|Du|^2 + 1}}. \tag{11.2}$$

*Therefore, we find that*

$$|Du| = \sqrt{I(x)^{-2} - 1} =: f(x). \tag{11.3}$$

*Hence, given appropriate boundary conditions, the shape from shading problem reduces to solving the eikonal equation. If the object $u$ is not a smooth graph–it may have corners–then $I$, and hence $f$, may be discontinuous. For more details on shape from shading and connections to viscosity solutions, we refer the reader to [16].*

Example 11.1 motivates the need for a theory of viscosity solutions with discontinuous coefficients. Since our definition of viscosity solution (Definition 2.1) assumed continuity, we first need to revisit definitions.

As motivation, we consider the method of vanishing viscosity for $f$ possibly discontinuous. In the viscous regularization, we replace $f$ with the mollification $f_\varepsilon := \eta^\varepsilon * f$, where $\eta^\varepsilon$ is the standard mollifier, yielding

$$\left.\begin{aligned} H(Du_\varepsilon) - \varepsilon\Delta u_\varepsilon = f_\varepsilon & \quad \text{in } U \\ u_\varepsilon = g_\varepsilon & \quad \text{on } \partial U. \end{aligned}\right\} \tag{11.4}$$

Suppose, as before, that $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$. Let $x_0 \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a strict local max at $x_0$. Then there exists $x_k \to x_0$ and $\varepsilon_k \to 0$ such that $u_{\varepsilon_k} - \varphi$ has a local max at $x_k$. Therefore $Du_{\varepsilon_k}(x_k) = D\varphi(x_k)$ and $\Delta u_{\varepsilon_k}(x_k) \le \Delta\varphi(x_k)$. This yields

$$H(D\varphi(x_k)) - \varepsilon_k\Delta\varphi(x_k) \le H(Du_{\varepsilon_k}(x_k)) - \varepsilon_k\Delta u_{\varepsilon_k}(x_k) = f_{\varepsilon_k}(x_k).$$

Sending $\varepsilon_k \to 0$ we have

$$H(D\varphi(x_0)) \le \liminf_{k\to\infty} f_{\varepsilon_k}(x_k).$$

Noting that

$$f_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta^\varepsilon(x-y) f(y)\, dy \le \sup_{B(x,\varepsilon)} f$$

we find that

$$H(D\varphi(x_0)) \le \liminf_{k\to\infty} \sup_{B(x,\varepsilon_k)} f \le f^*(x_0),$$

where $f^*$ is the upper semicontinuous envelope of $f$, defined in Chapter 7.

The discussion above motivates the following definitions.

**Definition 11.1** (Viscosity solution)**.** Let $f : U \to \mathbb{R}$. We say that $u \in \mathrm{USC}(\overline{U})$ is a *viscosity subsolution* of (11.1) if for every $x \in U$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$ we have

$$H(D\varphi(x)) \leq f^*(x).$$

Similarly, we say that $u \in \mathrm{LSC}(\overline{U})$ is a *viscosity supersolution* of (11.1) if for every $x \in U$ and every $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$ we have

$$H(D\varphi(x)) \geq f_*(x).$$

In general, the standard doubling variables arguments used to prove comparison in Theorems 3.1 and 6.4 does not extend directly to Definition 11.1, since $f$ is not continuous. However, there are special cases where the auxiliary function can be modified to compensate for discontinuous $f$. We first illustrate this first in a special case.

**Theorem 11.2.** *Let $U = B^0(0,1)$ and set $B^+ = U \cap \{x_n > 0\}$, $B^- = U \cap \{x_n < 0\}$, and $\Gamma = U \cap \{x_n = 0\}$. Assume that $f|_{B^+} \in C(\overline{B^+})$, $f|_{B^-} \in C(\overline{B^-})$ and for all $x \in \Gamma$*

$$\lim_{B^- \ni y \to x} f(y) \leq \lim_{B^+ \ni y \to x} f(y). \tag{11.5}$$

*Let $\varepsilon > 0$ and let $u, v \in C^{0,1}(\overline{U})$ such that $H(Du) \leq f$ and $H(Dv) \geq f + \varepsilon$ in $U$ in the viscosity sense of Definition 11.1. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v). \tag{11.6}$$

The proof of Theorem 11.2 uses a modified doubling the variables argument. The proof given below is borrowed in part from [8].

*Proof.* First, we claim that if $x, y \in U$ with $y_n > x_n$ then

$$f^*(x) - f_*(y) \leq 2\omega(|x - y|), \tag{11.7}$$

where $\omega$ is the modulus of continuity of $f|_{B^+}$ and $f|_{B^-}$. To see this, note that if $x_n = 0$ then by (11.5) $f^*(x) = \lim_{B^+ \ni y \to x} f(y)$ and $f$ is continuous at $y \in B^+$ yielding

$$f^*(x) - f_*(y) = f^*(x) - f(y) \leq \omega(|x - y|).$$

If $y_n = 0$ then $f_*(y) = \lim_{B^- \ni y \to x} f(y)$ and $f$ is continuous at $x \in B^-$. Hence

$$f^*(x) - f_*(y) = f(x) - f_*(y) \leq \omega(|x - y|).$$

Finally, assume $x_n \neq 0$ and $y_n \neq 0$. If $x, y \in B^-$ or $x, y \in B^+$ then (11.7) is trivially true, so we can assume $x \in B^-$ and $y \in B^+$. Let $z \in \Gamma$ such that $z$ lies on the line between $x$ and $y$, that is $z = \theta x + (1-\theta)y$ for some $\theta \in (0,1)$ and $z_n = 0$. Then we have

$$\begin{aligned} f^*(x) - f_*(y) &= f(x) - f_*(z) + f_*(z) - f^*(z) + f^*(z) - f(y) \\ &\leq f(x) - \lim_{B^- \ni w \to z} f(w) + \lim_{B^+ \ni w \to z} f(w) - f(y) \\ &\leq \omega(|x-z|) + \omega(|y-z|) \\ &\leq 2\omega(|x-y|), \end{aligned}$$

which establishes the claim.

We now prove (11.6). We can assume that $u \leq v$ on $\partial U$, and assume to the contrary that $\delta := \max_{\overline{U}}(u-v) > 0$. Then there exists $x_0 \in U$ such that $u(x_0) - v(x_0) = \delta$. We define the auxiliary function

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha}{2}\left|x - y + \frac{1}{\sqrt{\alpha}}e_n\right|^2 \tag{11.8}$$

where $e_n = (0, 0, \ldots, 0, 1)$. Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U}\times\overline{U}} \Phi.$$

We claim that

$$\lim_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) = \delta. \tag{11.9}$$

To see this, note that

$$\Phi(x_\alpha, y_\alpha) \geq \Phi(x_0, x_0 + \tfrac{1}{\sqrt{\alpha}}e_n) = u(x_0) - v(x_0 + \tfrac{1}{\sqrt{\alpha}}e_n) \to \delta$$

as $\alpha \to \infty$, and so

$$\liminf_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) \geq \delta > 0.$$

So for $\alpha > 0$ large enough $\Phi(x_\alpha, y_\alpha) > 0$ and so

$$\frac{\alpha}{2}\left|x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}}e_n\right|^2 \leq u(x_\alpha) - v(y_\alpha) \leq C.$$

Therefore

$$\left|x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}}e_n\right|, |x_\alpha - y_\alpha| \leq \frac{C}{\sqrt{\alpha}}. \tag{11.10}$$

It follows that

$$\Phi(x_\alpha, y_\alpha) \le u(x_\alpha) - v(y_\alpha) = u(x_\alpha) - u(y_\alpha) + u(y_\alpha) - v(y_\alpha) \le \frac{C}{\sqrt{\alpha}} + \delta$$

and so $\limsup_{\alpha\to\infty} \Phi(x_\alpha, y_\alpha) \ge \delta$, which establishes the claim.

For large enough $\alpha$, $u(x_\alpha) - v(y_\alpha) \ge \frac{\delta}{2}$, and so $x_\alpha, y_\alpha \in U$. By the viscosity sub- and super-solution properties we have

$$H(p_\alpha) \le f^*(x_\alpha) \text{ and } H(p_\alpha) \ge f_*(y_\alpha) + \varepsilon,$$

where

$$p_\alpha = \alpha\left(x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}} e_n\right).$$

Therefore

$$\varepsilon \le f^*(x_\alpha) - f_*(y_\alpha). \tag{11.11}$$

Setting $w_\alpha = \sqrt{\alpha}\left(x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}} e_n\right)$ we have

$$y_\alpha = x_\alpha + \frac{1}{\sqrt{\alpha}}(e_n - w_\alpha). \tag{11.12}$$

Notice that

$$\begin{aligned} \frac{1}{2}|w_\alpha|^2 &= u(x_\alpha) - v(y_\alpha) - \Phi(x_\alpha, y_\alpha) \\ &\le u(x_\alpha) - u(y_\alpha) + u(y_\alpha) - v(y_\alpha) - \Phi(x_\alpha, y_\alpha) \\ &\le \frac{C}{\sqrt{\alpha}} + \delta - \Phi(x_\alpha, y_\alpha), \end{aligned}$$

and so $w_\alpha \to 0$ as $\alpha \to \infty$. It follows from (11.12) that $y_{\alpha,n} > x_{\alpha,n}$ for $\alpha$ sufficiently large. Thus by (11.7) and (11.11) we have

$$\varepsilon \le f^*(x_\alpha) - f_*(y_\alpha) \le \omega(|x_\alpha - y_\alpha|) \le \omega(C\alpha^{-1/2}).$$

Sending $\alpha \to \infty$ yields a contradiction. □

We can generalize the argument in some ways. We follow [8] and make the assumption that

(D) For all $x_0 \in U$ there exists $\varepsilon_{x_0} > 0$ and $\eta_{x_0} \in \mathbb{S}^{n-1}$ such that

$$f^*(x) - f_*(x + rd) \le \omega(|x - x_0| + r), \tag{11.13}$$

for all $x \in U$, $r > 0$ and $d \in \mathbb{S}^{n-1}$ such that $|d - \eta_{x_0}| < \varepsilon_{x_0}$ and $x + rd \in U$, where $\omega$ is a modulus of continuity.

This models the situation where the domain can be decomposed as the disjoint union $U = U_1 \cup U_2 \cup \Gamma$ where $U_1, U_2$ are open and $\Gamma = \partial U_1 \cap \partial U_2 \cap U$ is the boundary between $U_1$ and $U_2$. Then (D) is satisfied provided $\Gamma$ is a Lipschitz hypersurface, $f|_{U_1} \in C(\overline{U_1})$, $f|_{U_2} \in C(\overline{U_2})$, and

$$\lim_{U_1 \ni y \to x} f(y) \leq \lim_{U_2 \ni y \to x} f(y)$$

for all $x \in \Gamma$.

**Exercise 11.3.** Prove the assertion above.

We now give a more general comparison principle assuming (D) holds, and that $H$ is continuous.

**Theorem 11.4.** *Let $U \subset \mathbb{R}^n$ be open and bounded, assume $f : U \to \mathbb{R}$ satisfies (D) and $H \in C(\mathbb{R}^n)$. Let $\varepsilon > 0$ and let $u, v \in C^{0,1}(\overline{U})$ such that $H(Du) \leq f$ and $H(Dv) \geq f + \varepsilon$ in $U$ in the viscosity sense of Definition 11.1. Then*

$$\max_{\overline{U}}(u - v) = \max_{\partial U}(u - v). \tag{11.14}$$

*Proof.* We sketch the proof, as it is similar to Theorem 11.2. We may assume that $u \leq v$ on $\partial U$, and assume to the contrary that $\delta := \max_{\overline{U}}(u - v) > 0$. Let $x_0 \in U$ such that $u(x_0) - v(x_0) = \delta$. We define the auxiliary function

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}\left|x - y + \frac{1}{\sqrt{\alpha}}\eta_{x_0}\right|^2 - |x - x_0|^2. \tag{11.15}$$

Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U} \times \overline{U}} \Phi.$$

As usual, we have $|x_\alpha - y_\alpha| \leq C/\alpha$, $\Phi(x_\alpha, y_\alpha) \to \delta$, and

$$\frac{\alpha}{2}\left|x - y + \frac{1}{\sqrt{\alpha}}\eta_{x_0}\right|^2 + |x_\alpha - x_0|^2 \to 0 \text{ as } \alpha \to \infty.$$

Therefore $x_\alpha, y_\alpha \to x_0$ as $\alpha \to \infty$ and so $x_\alpha, y_\alpha \in U$ for $\alpha$ sufficiently large.

Write $p_\alpha = \alpha(x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}}\eta_{x_0})$. By the viscosity sub- and supersolution properties

$$H(p_\alpha + 2(x - x_0)) \leq f^*(x_\alpha) \text{ and } H(p_\alpha) \geq f_*(y_\alpha) + \varepsilon.$$

Therefore

$$0 < \varepsilon \leq H(p_\alpha) - H(p_\alpha + x_\alpha - x_0) + f^*(x_\alpha) - f_*(y_\alpha). \tag{11.16}$$

Since $u, v \in C^{0,1}(\overline{U})$, there exists $C > 0$ such that (see Exercise 2.16)

$$|p_\alpha|, |p_\alpha + 2(x_\alpha - x_0)| \leq C$$

for all $\alpha$. Since $H$ is uniformly continuous on compact sets we have

$$\lim_{\alpha\to\infty} H(p_\alpha) - H(p_\alpha + x_\alpha - x_0) = 0.$$

Setting $w_\alpha = \sqrt{\alpha}\left(x_\alpha - y_\alpha + \frac{1}{\sqrt{\alpha}}\eta_{x_0}\right)$ we have

$$y_\alpha = x_\alpha + \frac{1}{\sqrt{\alpha}}\left(\eta_{x_0} - w_\alpha\right). \tag{11.17}$$

Since $w_\alpha \to 0$ as $\alpha \to \infty$ we can invoke (D) to find that

$$f^*(x_\alpha) - f_*(y_\alpha) \leq \omega\left(|x_\alpha - x_0| + \frac{1}{\sqrt{\alpha}}\right)$$

for $\alpha$ sufficiently large. Inserting this into (11.16) and taking $\alpha \to \infty$ yields a contradiction. □

# Chapter 12

# Second order equations

We consider in this section the comparison principle for viscosity solutions of second order equations

$$F(D^2u, Du, u, x) = 0 \quad \text{in } U, \tag{12.1}$$

where $U \subset \mathbb{R}^n$, and $F$ is degenerate elliptic (see (3.4)) and satisfies the usual monotonicity in $u$ (see (3.3)). Our treatment will loosely follow [4], though we prefer to avoid the super/sub-jet terminology. A comprehensive reference on the theory of second order equations with the sharpest results is given the User's Guide [6].

We first examine why the method of proof we used for first order equations (see Theorem 3.1) does not work here. The comparison principle for first order equations is based on doubling the variables and examining the maximum of

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2$$

as $\alpha \to \infty$. The key step was identifying that at a maximum $(x_\alpha, y_\alpha)$ of $\Phi$, the smooth function $\varphi(x) := \frac{\alpha}{2}|x - y_\alpha|^2$ touches $u$ from above at $x_\alpha$, and $\psi(y) = -\frac{\alpha}{2}|x_\alpha - y|^2$ touches $v$ from below at $y_\alpha$. Furthermore, we have the magic property $D\varphi(x_\alpha) = D\psi(y_\alpha)$, which replaces the classical identity $Du(x) = Dv(x)$ at a maximum of $u - v$ when $u, v$ are differentiable. For second order equations, we also need the identity $D^2u(x) \leq D^2v(x)$ at the max of $u - v$. However, $D^2\varphi(x_\alpha) = \alpha I \gg -\alpha I = D^2\psi(y_\alpha)$. So we appear to be at an impasse.

However, we have not used one important piece of information; namely that $(x, y) \mapsto \Phi(x, y)$ is jointly maximal at $(x_\alpha, y_\alpha)$. If we, for the moment, assume $u, v \in C^2$, then the condition that $(x_\alpha, y_\alpha)$ maximize $\Phi$ can be written as

$$\begin{bmatrix} D^2u(x_\alpha) & 0 \\ 0 & -D^2v(y_\alpha) \end{bmatrix} \leq \alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. \tag{12.2}$$

Since the right hand side annihilates vectors of the form $(\eta, \eta)$ for $\eta \in \mathbb{R}^n$, we find that $\eta^T D^2u(x_\alpha)\eta \leq \eta^T D^2v(y_\alpha)\eta$ for all $\eta \in \mathbb{R}^n$—that is $D^2u(x_\alpha) \leq D^2v(y_\alpha)$. So when $u, v$ are sufficiently smooth, the doubling variables argument contains enough information to utilize the maximum principle for second order equations.

This suggests performing some regularization of $u$ and $v$, and then applying the doubling variables argument to the regularizations. The standard regularizers in viscosity solutions are the inf- and sup-convolutions, defined in Chapter 8. We replace the subsolution $u$ with the sup-convolution $u^\varepsilon$, and the supersolution $v$ with the inf-convolution $v_\varepsilon$. Thus, we consider the doubling variables argument in the form

$$\Phi_\varepsilon(x, y) := u^\varepsilon(x) - v_\varepsilon(y) - \frac{\alpha}{2}|x - y|^2.$$

The key is that (see Chapter 8) $u^\varepsilon$ remains a subsolution (approximately) and $v_\varepsilon$ remains a supersolution, so we have not lost much by making this substitution, and we have gained a great deal of regularity. However, to use this additional regularity, we require a more refined understanding of semiconvex functions.

## 12.1 Jensen's Lemma

In this section we prove Jensen's Lemma, which gives necessary conditions for a semiconvex function to attain its maximum value.

**Lemma 12.1** (Jensen's Lemma)**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be semiconvex and let $x_0$ be a strict local maximum of $\varphi$. For $p \in \mathbb{R}^n$ set $\varphi_p(x) = \varphi(x) - p \cdot (x - x_0)$. Then for $r > 0$ sufficiently small and all $\delta > 0$ the set*

$$K = \{y \in B(x_0, r) : \exists p \in B(0, \delta) \text{ such that } \varphi_p(x) \leq \varphi_p(y) \text{ for } x \in B(x_0, r)\}$$

*has positive measure.*

**Remark 12.2.** Note that the condition $\varphi_p(x) \leq \varphi_p(y)$ for $x \in B(x_0, r)$ is simply stating that $x \mapsto \varphi(x) - p \cdot (x - x_0)$ has a local maximum at $y$, and so the linear function $L(x) = p \cdot (x - x_0)$ touches $\varphi$ from above at $y$. In other words, near a strict maximum of a semiconvex function there are a lot of points with very small gradients, so the function looks in some sense "round" near its maximum.

We now turn to the proof of Jensen's Lemma. The proof requires the *area formula*, which is a generalization of the change of variables formula in Lebesgue integration. Note we write $\#A$ to denote the number of points in $A \subset \mathbb{R}^n$ and $|A|$ to denote the Lebesgue measure.

**Theorem 12.3** (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map and set $Jf(x) = |det(Df(x))|$. Then for every Lebesgue measurable $A \subset \mathbb{R}^n$*

$$\int_{f(A)} \#(A \cap f^{-1}(\{x\}))\,dx = \int_A Jf(x)\,dx. \tag{12.3}$$

*In particular, if $f$ is injective then $|f(A)| = \int_A Jf(x)\,dx$.*

We refer the reader to [12] for a proof of the area formula.

**Remark 12.4.** Since $\#(A \cap f^{-1}(\{x\})) \geq 1$ for $x \in f(A)$, it follows from the area formula that

$$|f(A)| = \int_{f(A)} dx \leq \int_A Jf(x)\,dx. \tag{12.4}$$

This form of the area formula is used in the proof of Jensen's Lemma.

*Proof of Jensen's Lemma.* Let $r > 0$ be small enough so that $\varphi(x_0) > \varphi(x)$ for all $x \in B(x_0, r)$ with $x \neq x_0$, and let $a > 0$ such that $\varphi(x) + a \leq \varphi(x_0)$ for all $x \in \partial B(x_0, r)$. Let $\varepsilon > 0$ and define the mollification $\varphi^\varepsilon = \varphi * \eta^\varepsilon$. Then $\varphi_\varepsilon \to \varphi$ uniformly on $B(x_0, r)$ as $\varepsilon \to 0$. Define the corresponding sets

$$K^\varepsilon = \left\{y \in B(x_0, r) : \exists p \in B(0, \delta) \text{ such that } \varphi_p^\varepsilon(x) \leq \varphi_p^\varepsilon(y) \text{ for } x \in B(x_0, r)\right\},$$

where $\varphi_p^\varepsilon(x) = \varphi^\varepsilon(x) - p \cdot (x - x_0)$. Notice that for $y \in \partial B(x_0, r)$ we have

$$\begin{aligned}\varphi_p^\varepsilon(y) - \varphi_p^\varepsilon(x_0) &= \varphi_p^\varepsilon(y) - \varphi(y) + \varphi(y) - \varphi(x_0) + \varphi(x_0) - \varphi_p^\varepsilon(x_0)\\ &\leq 2\|\varphi - \varphi^\varepsilon\|_{L^\infty(B(x_0,r))} + 2|p|r - a.\end{aligned}$$

Therefore, for $\varepsilon$ and $\delta$ sufficiently small, $\varphi_p^\varepsilon(y) < \varphi_p^\varepsilon(x_0)$ for all $p$ with $|p| \leq \delta$ and all $y \in \partial B(x_0, r)$. Thus, every maximum of $\varphi_p^\varepsilon$ with respect to $B(x_0, r)$ lies in the interior $B^0(x_0, r)$ when $|p| \leq \delta$. At a maximum $y \in B^0(x_0, r)$ of $\varphi_p^\varepsilon$ we have $D\varphi^\varepsilon(y) = p$, and so $D\varphi^\varepsilon(K^\varepsilon) \supset B(0, \delta)$. For the rest of the proof we fix $\delta > 0$ sufficiently small, as above.

Now, let $\lambda > 0$ such that $\varphi(x) + \frac{\lambda}{2}|x|^2$ is convex. This yields

$$-\lambda I \leq D^2\varphi^\varepsilon(x) = D^2\varphi_p^\varepsilon(x) \leq 0 \quad \text{for} \quad x \in K^\varepsilon.$$

In particular, it follows that $|\det(D^2\varphi^\varepsilon(x))| \le \lambda^n$ for $x \in K^\varepsilon$ and so by the area formula (Theorem 12.3) we have

$$|B(0,\delta)| \le |D\varphi(K^\varepsilon)| \le \int_{K^\varepsilon} |\det D^2\varphi(x)|\,dx \le |K^\varepsilon|\lambda^n.$$

Therefore

$$|K^\varepsilon| \ge \frac{\alpha(n)\delta^n}{\lambda^n}. \tag{12.5}$$

Since $\varphi^\varepsilon \to \varphi$ uniformly, if $x \in K^{\varepsilon_j}$ for a sequence $\varepsilon_j \to 0$ then $x \in K$. Therefore

$$\chi_K(x) \ge \limsup_{m\to\infty} \chi_{K^{\frac{1}{m}}}(x),$$

where $\chi_A$ is the indicator function of the set $A$. By Fatou's Lemma

$$|K| = \int_K dx \ge \limsup_{m\to\infty} \int_{K^{\frac{1}{m}}} dx \ge \frac{\alpha(n)\delta^n}{\lambda^n},$$

which completes the proof. □

The following proposition illustrates the usefulness of Jensen's Lemma in establishing the maximum principle for semiconvex functions.

**Proposition 12.5.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be semiconvex and let $x_0$ be a local maximum of $\varphi$. Then there exists $x_k \to x_0$ such that $\varphi$ is twice differentiable at $x_k$, $D\varphi(x_k) \to 0$ as $k \to \infty$ and $D^2\varphi(x_k) \le \varepsilon_k I$ for a sequence $\varepsilon_k \to 0$.*

**Remark 12.6.** Proposition 12.5, which is a restatement of Jensen's Lemma, is the semiconvex analog of the condition that $D\varphi = 0$ and $D^2\varphi \le 0$ at the maximum of a $C^2$ function.

*Proof.* Define $\psi(x) = \varphi(x) - |x - x_0|^4$. Then $\psi$ has a strict local max at $x_0$, and $\psi$ is semiconvex. Let $r_k > 0$ be a decreasing sequence of real numbers converging to zero. By Lemma 12.1 (Jensen's Lemma), there is a corresponding decreasing sequence $\delta_k > 0$ such that $\delta_k \to 0$ and

$$\{y \in B(x_0, r_k) : \exists p \in B(0,\delta_k) \text{ such that } \psi_p(x) \le \psi_p(y) \text{ for } x \in B(x_0, r_k)\}$$

has positive measure, where $\psi_p(x) = \psi(x) + p\cdot(x - x_0)$. Since $\psi$ is twice differentiable almost everywhere, there exists $x_k \in B^0(x_0, r_k)$ and $p_k \in B(0,\delta_k)$ such that $\psi_{p_k}$ has a local maximum at $x_k$ and $\psi$ is twice differentiable at $x_k$. Hence $x_k \to x_0$, $p_k \to 0$, $D\psi(x_k) = p_k$ and $D^2\psi(x_k) = D^2\psi_{p_k}(x_k) \le 0$. The proof is completed by noting that $\varphi$ is also twice differentiable along the sequence $x_k$, and that

$$|D\varphi(x_k) - p_k| \le 4|x_k - x_0|^3 \quad\text{and}\quad D^2\varphi(x_k) \le 12|x_k - x_0|^2 I. \qquad \square$$

## 12.2 Comparison for continuous functions

We now use Jensen's lemma to prove comparison principles for second order equations of the form (12.1) for continuous sub- and supersolutions.

We assume throughout this section that $U \subset \mathbb{R}^n$ is open and bounded, and $F$ is continuous, degenerate elliptic (i.e., satisfies (3.4)), and satisfies the monotonicity condition (3.3). We also assume $F$ satisfies

$$F(X, p, z, y) - F(X, p, z, x) \leq \omega(|x - y|(1 + |p|)) \tag{12.6}$$

for all $x, y \in U$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and symmetric matrices $X$, where $\omega$ is a modulus of continuity.

We first prove comparison when the subsolution is semiconvex, and the supersolution is semiconcave.

**Lemma 12.7** (Semiconvex comparison)**.** *Let $u \in C(\overline{U})$ be a semiconvex viscosity subsolution of (12.1), and let $v \in C(\overline{U})$ be a semiconcave viscosity solution of*

$$F(D^2v, Dv, v, x) - \delta \geq 0 \quad \text{in } U,$$

*for some $\delta > 0$. If $u \leq v$ on $\partial U$ then $u \leq v$ in $U$.*

*Proof.* We use doubling the variables and Jensen's lemma. Assume to the contrary that $\max_{\overline{U}}(u - v) > 0$. Define the auxiliary function

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2, \tag{12.7}$$

and let $x_\alpha, y_\alpha \in \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U} \times \overline{U}} \Phi.$$

As in the proof of Theorem 3.1 we have $x_\alpha, y_\alpha \in U$ for $\alpha$ large enough and

$$\alpha|x_\alpha - y_\alpha|^2 \longrightarrow 0. \tag{12.8}$$

Since $u$ and $-v$ are semiconvex, the auxiliary function $\Phi : \overline{U} \times \overline{U} \to \mathbb{R}$ is semiconvex. By Proposition 12.5 there exists a sequence $(x_\alpha^k, y_\alpha^k) \in U \times U$ such that $x_\alpha^k \to x_\alpha$ and $y_\alpha^k \to y_\alpha$ as $k \to \infty$, $\Phi$ is twice differentiable at $(x_\alpha^k, y_\alpha^k)$, $D_{xy}\Phi(x_\alpha^k, y_\alpha^k) \to 0$ as $k \to \infty$, and $D^2_{xy}\Phi(x_\alpha^k, y_\alpha^k) \leq \varepsilon_k I$ for a sequence $\varepsilon_k \to 0$. Here, $D_{xy}$ denotes the gradient jointly in $(x, y)$. It follows that $u$ is twice differentiable at $x_\alpha^k$ and $v$ is twice differentiable at $y_\alpha^k$. Furthermore, for $p_\alpha = \alpha(x_\alpha - y_\alpha)$ we have $Du(x_\alpha^k) \to p_\alpha$ and $Dv(x_\alpha^k) \to p_\alpha$ as $k \to \infty$, and

$$-CI \leq \begin{bmatrix} D^2u(x_\alpha^k) & 0 \\ 0 & -D^2v(y_\alpha^k) \end{bmatrix} \leq \alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \varepsilon_k I. \tag{12.9}$$

The lower bound in (12.9) follows from semiconvexity of $u$ and $-v$, while the upper bound follows from $D^2_{xy}\Phi(x^k_\alpha, y^k_\alpha) \le \varepsilon_k I$. By conjugating both sides with vectors of the form $(\eta, \eta) \in \mathbb{R}^{2n}$ we have

$$\eta^T D^2 u(x^k_\alpha)\eta \le \eta^T D^2 v(x^k_\alpha)\eta + 2\varepsilon_k |\eta|^2,$$

for all $\eta \in \mathbb{R}^n$, and hence $D^2u(x^k_\alpha) \le D^2 v(x^k_\alpha) + 2\varepsilon_k I$. Using (12.9), we can, upon passing to a subsequence, assume that $D^2u(x^k_\alpha) \to X_\alpha$ and $D^2v(y^k_\alpha) \to Y_\alpha$ as $k \to \infty$, where $X_\alpha \le Y_\alpha$.

By the viscosity sub- and supersolution properties, and Remark 2.7, we have

$$F(D^2u(x^k_\alpha), Du(x^k_\alpha), u(x^k_\alpha), x^k_\alpha) \le 0 \tag{12.10}$$

and

$$F(D^2v(y^k_\alpha), Dv(y^k_\alpha), v(y^k_\alpha), y^k_\alpha) \ge \delta. \tag{12.11}$$

Taking $k \to \infty$ and using continuity of $F$, $u$, and $v$ we have

$$F(X_\alpha, p_\alpha, u(x_\alpha), x_\alpha) \le 0$$

and

$$F(Y_\alpha, p_\alpha, v(y_\alpha), y_\alpha) \ge a,$$

where $X_\alpha \le Y_\alpha$. Since $\Phi(x_\alpha, y_\alpha) \ge \max_{\overline{U}}(u - v) > 0$ we have $u(x_\alpha) > v(y_\alpha)$, and so by monotonicity and degenerate ellipticity, we have

$$\delta \le F(Y_\alpha, p_\alpha, v(y_\alpha), y_\alpha) \le F(X_\alpha, p_\alpha, u(x_\alpha), y_\alpha).$$

Applying (12.6) and (12.10) we find that

$$\delta \le \omega((1 + |p_\alpha|)|x_\alpha - y_\alpha|).$$

Sending $\alpha \to \infty$ and recalling (12.8) yields a contradiction. □

Using the inf- and sup-convolutions, we can extend the semiconvex comparison principle to continuous functions. Here, we assume $F$ has the form

$$F(X, p, z, x) = \lambda z + H(X, p) - f(x), \tag{12.12}$$

where $\lambda \ge 0$. Then the regularity condition (12.6) is equivalent to the condition $f \in C(\overline{U})$.

**Theorem 12.8** (Continuous comparison)**.** *Assume $F$ has the form (12.12). Let $u \in C(\overline{U})$ be a viscosity subsolution of (12.1), and let $v \in C(\overline{U})$ be a viscosity solution of*

$$F(D^2v, Dv, v, x) - \delta \ge 0 \quad \text{in } U,$$

*for some $\delta > 0$. If $u \le v$ on $\partial U$ then $u \le v$ in $U$.*

*Proof.* For $\varepsilon > 0$, let $u^\varepsilon$ be the sup-convolution of $u$, and let $v_\varepsilon$ be the inf-convolution of $v$, as defined in Chapter 8. By an argument similar to Corollary 8.9, we have that

$$\lambda u^\varepsilon + H(D^2 u^\varepsilon, Du^\varepsilon) - f \le g(\varepsilon) \ \text{ in } U_\varepsilon$$

and

$$\lambda v_\varepsilon + H(D^2 v_\varepsilon, Dv_\varepsilon) - f \ge \delta - h(\varepsilon) \ \text{ in } U_\varepsilon$$

hold in the viscosity sense, where

$$U_\varepsilon = \{x \in U \, : \, \text{dist}(x, \partial U) \ge C\varepsilon\}$$

for a constant $C > 0$, and $g, h$ are nonnegative continuous functions with $g(0) = h(0) = 0$. Let $m_\varepsilon = \sup_{\overline{U} \setminus U_\varepsilon}(u^\varepsilon - v_\varepsilon)$. Since $u, v \in C(\overline{U})$, $u^\varepsilon \to u$ and $v_\varepsilon \to v$ uniformly, and $u \le v$ on $\partial U$, we have $m_\varepsilon \to 0$ as $\varepsilon \to 0$. Define $w^\varepsilon = u^\varepsilon - m_\varepsilon$. Then $w^\varepsilon$ satisfies

$$\lambda w^\varepsilon + H(D^2 w^\varepsilon, Dw^\varepsilon) - f \le g(\varepsilon) - \lambda m_\varepsilon \ \text{ in } U_\varepsilon$$

in the viscosity sense, and $w^\varepsilon \le v_\varepsilon$ on $\partial U_\varepsilon$. For $\varepsilon > 0$ sufficiently small, we can apply Lemma 12.7 to show that $w^\varepsilon \le v_\varepsilon$ on $U_\varepsilon$, and so

$$u^\varepsilon \le v_\varepsilon + m_\varepsilon \quad \text{on } U_\varepsilon.$$

Sending $\varepsilon \to 0$ we recover $u \le v$ on $U$. □

## 12.3 Superjets and subjets

There is a common alternative definition of viscosity solutions that is worth discussing briefly. We first make a definition.

**Definition 12.9.** Let $\mathcal{O} \subset \mathbb{R}^n$, $u : \mathcal{O} \to \mathbb{R}$, and $x_0 \in \mathcal{O}$. The superjet $J^{2,+}_{\mathcal{O}} u(x_0)$ is defined as the set of all $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ for which

$$u(x) \le u(x_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T X (x - x_0) + o(|x - x_0|^2)$$

as $\mathcal{O} \ni x \to x_0$.

Similarly, the subjet $J^{2,-}_{\mathcal{O}} u(x_0)$ is defined as the set of all $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ for which

$$u(x) \ge u(x_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T X (x - x_0) + o(|x - x_0|^2)$$

as $\mathcal{O} \ni x \to x_0$.

The following proposition is immediate.

**Proposition 12.10.** *Let $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ is open. If $u : \mathcal{O} \to \mathbb{R}$ is twice differentiable at $x \in U$ then*

$$J^{2,+}u(x) \cap J^{2,-}u(x) = (Du(x), D^2u(x)).$$

The relationship between superjets and subjets and viscosity solutions is illuminated by the following result.

**Proposition 12.11.** *Let $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ is open. We have*

$$J^{2,+}u(x_0) = \left\{(D\varphi(x_0), D^2\varphi(x_0)) \,:\, \varphi \in C^2(\mathbb{R}^n) \text{ and } u - \varphi \text{ has a local max at } x_0\right\}.$$

*and*

$$J^{2,-}u(x_0) = \left\{(D\varphi(x_0), D^2\varphi(x_0)) \,:\, \varphi \in C^2(\mathbb{R}^n) \text{ and } u - \varphi \text{ has a local min at } x_0\right\}.$$

*Proof.* If $u - \varphi$ has a local maximum at $x_0$, then clearly $(D\varphi(x_0), D^2\varphi(x_0)) \in J^{2,+}u(x_0)$.

We now prove the converse. We may take $x_0 = 0$ and $u(0) = 0$ for simplicity. Let $(p, X) \in J^{2,+}u(0)$. By definition, for some $r_0 > 0$ we have

$$u(x) \leq p \cdot x + \frac{1}{2}x^T X x + g(x)|x|^2$$

for $|x| \leq r_0$, where $g : U \to \mathbb{R}$ is continuous and $g(0) = 0$. Define

$$\rho(r) = \max_{|x|\leq r} |g(x)|.$$

Then $\rho : [0, r_0] \to [0, \infty)$ is continuous and nondecreasing with $\rho(0) = 0$ and $g(x) \leq \rho(|x|)$. Therefore

$$u(x) \leq p \cdot x + \frac{1}{2}x^T X x + \rho(|x|)|x|^2$$

for $|x| \leq r_0$. Define the $C^2$ function

$$\sigma(r) = \int_0^r \int_0^s \rho(t)\, dt\, ds,$$

and note that

$$\sigma(3r) \geq \int_r^{3r} \int_r^s \rho(t)\, dt\, ds \geq \int_r^{3r} (s - r)\rho(r)\, ds \geq 2r^2\rho(r).$$

Therefore, for $|x| \le r_0$ we have

$$u(x) \le p \cdot x + \frac{1}{2} x^T X x + \sigma(3|x|) =: \varphi(x).$$

Notice that $\varphi \in C^2(B(0, r_0))$, $u - \varphi$ has a local max at $x = 0$, $p = D\varphi(0)$ and $X = D^2\varphi(0)$. We can extend $\varphi$ to a function $\varphi \in C^2(\mathbb{R}^n)$ with a bump function argument. The proof for the superjet is similar. □

We now give an alternative characterization of viscosity solutions of (12.1) in terms of superjets and subjets.

**Theorem 12.12.** *Let $U \subset \mathbb{R}^n$ be open and assume $F$ is continuous in all variables. If $u \in USC(\overline{U})$ is a viscosity subsolution of (12.1) then*

$$F(X, p, u(x), x) \le 0 \quad \textit{for all } x \in U \textit{ and } (p, X) \in J^{2,+}u(x). \tag{12.13}$$

*Similarly, if $v \in LSC(\overline{U})$ is a viscosity supersolution (12.1) then*

$$F(X, p, v(x), x) \ge 0 \quad \textit{for all } x \in U \textit{ and } (p, X) \in J^{2,-}v(x). \tag{12.14}$$

**Remark 12.13.** The conditions (12.13) and (12.14) are sometimes given as the definitions of viscosity solutions (see, e.g., [6]). While this notation may seem convenient and compact, nobody quite likes this "jet" business [4].

*Proof.* Let $u \in \mathrm{USC}(\overline{U})$ be a viscosity subsolution of (12.1) and let $x_0 \in U$ and $(p, X) \in J^{2,+}u(x)$. By Proposition 12.11 there exists $\varphi \in C^2(\mathbb{R}^n)$ such that $u - \varphi$ has a strict local maximum at $x_0$, $p = D\varphi(x_0)$ and $X = D^2\varphi(x_0)$. Define the standard mollification $\varphi^\varepsilon = \eta^\varepsilon * \varphi$. Since $\varphi^\varepsilon \to \varphi$ locally uniformly, there exists $\varepsilon_k \to 0$ and $x_k \to x_0$ such that $u(x_k) \to u(x_0)$ and $u - \varphi^{\varepsilon_k}$ has a local max at $x_k$ for each $k \ge 1$. Since $\varphi \in C^\infty(\mathbb{R}^n)$, the viscosity subsolution property yields

$$F(D^2\varphi^{\varepsilon_k}(x_k), D\varphi^{\varepsilon_k}(x_k), u(x_k), x_k) \le 0 \quad (k \ge 1).$$

Since $\varphi \in C^2(\mathbb{R}^n)$, we have $D\varphi^\varepsilon \to D\varphi$ and $D^2\varphi^\varepsilon \to D^2\varphi$ locally uniformly as $\varepsilon \to 0$. Since $F$ is continuous and $u(x_k) \to u(x_0)$, we can send $k \to \infty$ to obtain

$$F(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \le 0.$$

Since $p = D\varphi(x_0)$ and $X = D^2\varphi(x_0)$ we have

$$F(X, p, u(x_0), x_0) \le 0,$$

which completes the proof. The proof for the superjet is similar. □

As a consequence of Theorem 12.12, we can prove consistency of viscosity solutions with classical solutions under minimal regularity assumptions.

**Corollary 12.14.** *Let $U \subset \mathbb{R}^n$ be open and assume $F$ is continuous in all variables. If $u \in USC(\overline{U})$ is a viscosity subsolution of (12.1) and $u$ is twice differentiable at some $x \in U$ then*

$$F(D^2u(x), Du(x), u(x), x) \le 0.$$

*Similarly, if $v \in LSC(\overline{U})$ is a viscosity supersolution of (12.1) and $v$ is twice differentiable at some $x \in U$ then*

$$F(D^2v(x), Dv(x), v(x), x) \ge 0.$$

*Proof.* The proof follows directly from Proposition 12.10 and Theorem 12.12, since $(Du(x), D^2u(x)) \in J^{2,+}u(x)$ and $(Dv(x), D^2v(x)) \in J^{2,-}v(x)$. □

## 12.4 Semicontinuous comparison

In order to use the Perron method (Chapter 7), or the weak upper and lower limits (Chapters 5 and 9), we require a comparison principle for semicontinuous sub- and supersolutions, which is somewhat more involved than the continuous comparison principle given in Section 12.2. Often, the comparison principle in its full semicontinuous glory is proved using the superjet and subjet definitions of viscosity solutions introduced in Section 12.3. Good references for this include [4,6].

Here, we give a proof that avoids the notion of superjets and subjets, yet is sharp in its generality. The key technical point is that the inf- and sup-convolutions must be done within the doubling variables argument, and in particular must be applied to the penalty term $\frac{\alpha}{2}|x-y|^2$ as well, instead of performing them separately as was done in Theorem 12.8. Furthermore, we do not send $\varepsilon \to 0$ in the sup-convolution, so we avoid the issue that $u^\varepsilon$ does not converge uniformly to $u$ in the semi-continuous case. The proof of Theorem 12.15 is borrowed partly from [4], with appropriate translations to use test functions instead of superjets and subjets.

**Theorem 12.15.** *Assume $U \subset \mathbb{R}^n$ is open and bounded, and that $F$ is continuous, and satisfies (3.3), (3.4), and (12.6). Let $u \in USC(\overline{U})$ be a viscosity subsolution of (12.1), and let $v \in LSC(\overline{U})$ be a viscosity solution of*

$$F(D^2v, Dv, v, x) - \delta \ge 0 \quad \text{in } U,$$

*for some $\delta > 0$. If $u \le v$ on $\partial U$ then $u \le v$ in $U$.*

*Proof.* We assume, by way of contradiction, that $\max_{\overline{U}}(u - v) > 0$. For $\alpha > 0$ define

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2, \tag{12.15}$$

and let $x_\alpha, y_\alpha \in \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U}\times\overline{U}} \Phi.$$

As in the proof of Theorem 3.1 we have $x_\alpha, y_\alpha \in U$ for $\alpha$ large enough and

$$\alpha|x_\alpha - y_\alpha|^2 \longrightarrow 0. \tag{12.16}$$

The proof is split into 3 steps.

1. We first make a reduction. Define

$$f(x) = u(x + x_\alpha) - u(x_\alpha) - \alpha x \cdot (x_\alpha - y_\alpha),$$

and

$$g(y) = v(y + y_\alpha) - v(y_\alpha) - \alpha y \cdot (x_\alpha - y_\alpha).$$

Then we have

$$\begin{aligned}
&f(x) - g(y) - \frac{\alpha}{2}|x - y|^2 \\
&= u(x + x_\alpha) - v(y + y_\alpha) - \frac{\alpha}{2}|x - y|^2 - \alpha(x - y) \cdot (x_\alpha - y_\alpha) + v(y_\alpha) - u(x_\alpha) \\
&= u(x + x_\alpha) - v(y + y_\alpha) - \frac{\alpha}{2}|x - y + x_\alpha - y_\alpha|^2 + \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + v(y_\alpha) - u(x_\alpha) \\
&= \Phi(x + x_\alpha, y + y_\alpha) + \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + v(y_\alpha) - u(x_\alpha).
\end{aligned}$$

Therefore

$$f(x) - g(y) - \frac{\alpha}{2}|x - y|^2$$

attains its maximum at $(x, y) = (0, 0)$, and $f(0) = g(0) = 0$. Thus, we have

$$f(x) - g(y) \leq \frac{\alpha}{2}|x - y|^2 \tag{12.17}$$

for $x, y$ near 0.

2. We now take the sup-convolution on both sides of (12.17) jointly in $(x, y)$ to obtain (see Exercise 12.16)

$$f^\varepsilon(x) - g_\varepsilon(y) \leq (1 - 2\alpha\varepsilon)^{-1}\frac{\alpha}{2}|x - y|^2 \tag{12.18}$$

for $x, y$ near 0. Since $f^\varepsilon \geq f$ and $g_\varepsilon \leq g$ we have

$$f^\varepsilon(0) - g_\varepsilon(0) \geq f(0) - g(0) = 0$$

and so by (12.18) we have $f^\varepsilon(0) = g_\varepsilon(0)$. Since $f^\varepsilon(0) \geq f(0) = 0$ and $g_\varepsilon(0) \leq g(0) = 0$ we have $f^\varepsilon(0) = 0 = g_\varepsilon(0)$. Therefore the function

$$f^\varepsilon(x) - g_\varepsilon(y) - (1-2\alpha\varepsilon)^{-1}\frac{\alpha}{2}|x-y|^2 - |x|^4 - |y|^4$$

attains a strict local maximum at $(x, y) = (0, 0)$. By Lemma 12.1 (Jensen's Lemma), there exists $x^k, y^k \to 0$ and $\xi^k, \zeta^k \to 0$ such that

$$f^\varepsilon(x) - g_\varepsilon(y) - (1-2\alpha\varepsilon)^{-1}\frac{\alpha}{2}|x-y|^2 - |x|^4 - |y|^4 - \xi^k \cdot x - \zeta^k \cdot y$$

has a local maximum at $(x^k, y^k)$, $f^\varepsilon$ is twice differentiable at $x^k$, and $g_\varepsilon$ is twice differentiable at $y^k$. Let $r_k = \max\{|x^k|^2, |y^k|^2\}$. The first order conditions for a maximum yield

$$p^k := Df^\varepsilon(x^k) = (1-2\alpha\varepsilon)^{-1}\alpha(x^k - y^k) + 4|x^k|^2 x^k + \xi^k, \tag{12.19}$$

and

$$q^k := Dg_\varepsilon(y^k) = (1-2\alpha\varepsilon)^{-1}\alpha(x^k - y^k) - 4|y^k|^2 y^k + \zeta^k. \tag{12.20}$$

Note that $p^k, q^k \to 0$ as $k \to \infty$. The second order condition for a maximum, and semiconvexity, yield

$$-\frac{1}{\varepsilon}I \leq \begin{bmatrix} X^k & 0 \\ 0 & -Y^k \end{bmatrix} \leq (1-2\alpha\varepsilon)^{-1}\alpha \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + 12r_k^2 I, \tag{12.21}$$

where $X^k = D^2 f^\varepsilon(x^k)$ and $Y^k = D^2 g_\varepsilon(y^k)$. In particular, for all $\eta \in \mathbb{R}^n$ we have

$$\eta^T X^k \eta \leq \eta^T Y^k \eta + 12r_k^2|\eta|^2,$$

and so $X^k \leq Y^k + 12r_k^2 I$. Passing to a subsequence, if necessary, there exists $X_\alpha, Y_\alpha \in \mathcal{S}(n)$ such that $X^k \to X_\alpha$, $Y^k \to Y_\alpha$, and $X_\alpha \leq Y_\alpha$.

3. By Proposition 12.11, there exists $\varphi \in C^2(\mathbb{R}^n)$ such that $f^\varepsilon - \varphi$ has a local max at $x^k$ and $D^2\varphi(x_k) = X^k$. Let $x_\varepsilon^k \in U$ such that

$$f^\varepsilon(x^k) = f(x_\varepsilon^k) - \frac{1}{2\varepsilon}|x^k - x_\varepsilon^k|^2. \tag{12.22}$$

By Proposition 8.6 $f(x) - \varphi_\varepsilon(x)$ has a local maximum at $x_\varepsilon^k$, where $\varphi_\varepsilon(x) := \varphi(x + x^k - x_\varepsilon^k)$, and we have

$$D\varphi_\varepsilon(x_\varepsilon^k) = p^k = \frac{1}{\varepsilon}(x_\varepsilon^k - x^k),$$

and $D^2\varphi_\varepsilon(x_\varepsilon^k) = X^k$. Since $p^k \to 0$ as $k \to \infty$ we have $x_\varepsilon^k \to 0$ as $k \to \infty$. In particular, from (12.22) and the continuity of $f^\varepsilon$ we have

$$f(x_\varepsilon^k) \longrightarrow f^\varepsilon(0) = f(0) = 0 \tag{12.23}$$

as $k \to \infty$. Since $f - \varphi_\varepsilon$ has a local max at $x_\varepsilon^k$ we see that

$$x \mapsto u(x) - \alpha(x - x_\alpha) \cdot (x_\alpha - y_\alpha) - \varphi_\varepsilon(x - x_\alpha)$$

has a local max at $x_\varepsilon^k + x_\alpha$. Therefore, setting $p_\alpha = \alpha(x_\alpha - y_\alpha)$ we have

$$F(X^k, p_\alpha + p^k, u(x_\varepsilon^k + x_\alpha), x_\varepsilon^k + x_\alpha) \le 0.$$

Since $f(x_\varepsilon^k) \to 0$ as $k \to \infty$ we have $u(x_\varepsilon^k + x_\alpha) \to u(x_\alpha)$. Thus, we can take $k \to \infty$ above and recall $F$ is continuous to obtain

$$F(X_\alpha, p_\alpha, u(x_\alpha), x_\alpha) \le 0. \tag{12.24}$$

We can make a similar argument for $g_\varepsilon$ to obtain

$$F(Y_\alpha, p_\alpha, v(y_\alpha), y_\alpha) \ge \delta. \tag{12.25}$$

Since $\Phi(x_\alpha, y_\alpha) > 0$ we have $u(x_\alpha) > v(y_\alpha)$. Subtracting (12.24) from (12.25) and using (3.3), (3.4) and (12.6) we have

$$\begin{aligned}\delta &\le F(Y_\alpha, p_\alpha, v(y_\alpha), y_\alpha) - F(X_\alpha, p_\alpha, u(x_\alpha), x_\alpha)\\ &\le F(X_\alpha, p_\alpha, u(x_\alpha), y_\alpha) - F(X_\alpha, p_\alpha, u(x_\alpha), x_\alpha)\\ &\le \omega(|x_\alpha - y_\alpha|(1 + |p_\alpha|)).\end{aligned}$$

Sending $\alpha \to \infty$ yields a contradiction, due to (12.16). □

**Exercise 12.16.** Define

$$w(x, y) = \frac{\alpha}{2}|x - y|^2.$$

Show that the sup-convolution

$$w^\varepsilon(x, y) = \sup_{(x', y') \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ \frac{\alpha}{2}|x' - y'|^2 - \frac{1}{2\varepsilon}|x - x'|^2 - \frac{1}{2\varepsilon}|y - y'|^2 \right\}$$

is given by

$$w^\varepsilon(x, y) = (1 - 2\alpha\varepsilon)^{-1}\frac{\alpha}{2}|x - y|^2,$$

provided $1 - 2\alpha\varepsilon \ne 0$.

## 12.5 A problem on an unbounded domain

In this section we give a brief application of the comparison principle from Section 12.4. In particular, we study the equation

$$u + F(D^2u) = f \quad \text{on } \mathbb{R}^n. \tag{12.26}$$

The well-posedness theory is far more general; we study this simple problem to illustrate the main ideas, and to show how to handle the unbounded domain. Throughout this section, we assume $F$ is uniformly continuous and degenerate elliptic, and $f$ is continuous.

We first prove a comparison principle.

**Lemma 12.17.** *Let $u \in USC(\mathbb{R}^n)$ be a viscosity subsolution of (12.26) and let $v \in LSC(\mathbb{R}^n)$ be a viscosity supersolution of (12.26). If*

$$\lim_{|x|\to\infty} \frac{u(x) - v(x)}{|x|^2} = 0 \tag{12.27}$$

*then $u \le v$ on $\mathbb{R}^n$.*

*Proof.* For $\varepsilon > 0$ define

$$u_\varepsilon(x) = u(x) - \frac{1}{2}\varepsilon|x|^2 - c_\varepsilon,$$

for $c_\varepsilon > 0$ to be determined. Then $u_\varepsilon$ is a viscosity solution of

$$u_\varepsilon + F(D^2u_\varepsilon + \varepsilon I) \le f - \frac{1}{2}\varepsilon|x|^2 - c_\varepsilon \;\; \text{on } \mathbb{R}^n.$$

Since $F$ is uniformly continuous, we can choose $c_\varepsilon \to 0$ as $\varepsilon \to 0$ so that

$$u_\varepsilon + F(D^2u_\varepsilon) \le f - \frac{1}{2}\varepsilon|x|^2 - \varepsilon \;\; \text{on } \mathbb{R}^n$$

in the viscosity sense. Note that

$$\frac{u_\varepsilon(x) - v(x)}{|x|^2} \le \frac{u(x) - v(x)}{|x|^2} - \frac{1}{2}\varepsilon.$$

Thus, by (12.27) there exists $r_\varepsilon > 0$ such that $u_\varepsilon \le v$ for $|x| \ge r_\varepsilon$. By Theorem 12.15 we have $u_\varepsilon \le v$ on $B(0, r)$ for all $r > r_\varepsilon$, and thus $u_\varepsilon \le v$ on $\mathbb{R}^n$. Sending $\varepsilon \to 0$ completes the proof. □

By Lemma 12.17 there is at most one viscosity solution of (12.26) satisfying the growth bound

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^2} = 0 \tag{12.28}$$

We now turn to proving existence of a solution, via the Perron method. Here, we look for a solution with at most linear growth at infinity.

**Theorem 12.18.** *Suppose there exists $C_f > 0$ so that*

$$|f(x)| \leq C_f(1 + |x|). \tag{12.29}$$

*Then there exists a unique viscosity solution $u \in C(\mathbb{R}^n)$ of (12.26) satisfying $\lim_{|x|\to\infty} \frac{u(x)}{|x|^2} = 0$. Furthermore, we have*

$$|u(x)| \leq C(1 + |x|) \tag{12.30}$$

*for some $C > 0$.*

*Proof.* Let $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi(x) = |x|$ for $|x| \geq 1$, and $\psi \geq 0$. Define $w(x) = K_1 + K_2\psi(x)$, for $K_1, K_2$ to be determined. Note that

$$D^2\psi(x) = \frac{1}{|x|}I - \frac{x \otimes x}{|x|^3} \quad \text{for } |x| \geq 1.$$

Therefore $|D^2\psi|$ is bounded on $\mathbb{R}^n$ and

$$w + F(D^2w) - f \geq |x|\left(K_2 + \frac{F(K_2D^2\psi)}{|x|} - C_f\left(1 + \frac{1}{|x|}\right)\right)$$

for $|x| \geq 1$. Choose $K_2 = 2C_f$. Then there exists $r > 1$ such that for $|x| > r$ we have

$$w + F(D^2w) - f \geq \frac{1}{2}C_f|x| > 0.$$

Now, for $|x| \leq r$ we have

$$w + F(D^2w) - f \geq K_1 + F(K_2D^2\psi) - f(x) > 0$$

for $K_1 > 0$ sufficiently large. Therefore, for large enough $K_1, K_2$, $w$ is a supersolution of (12.26) on $\mathbb{R}^n$ and

$$|w(x)| \leq C(1 + |x|). \tag{12.31}$$

We also note that the same construction yields that $v := -w$ is a subsolution of (12.26).

Now, define

$$\mathcal{F} = \{v \in \mathrm{USC}(\mathbb{R}^n) \, : \, v \text{ is a subsolution of (12.26) and } v \leq w\}, \tag{12.32}$$

and

$$u(x) := \sup\{v(x) \, : \, v \in \mathcal{F}\}. \tag{12.33}$$

Since $v = -w \in \mathcal{F}$, we have $-w \leq u \leq w$, and so $u$ satisfies (12.30). By Lemma 7.1 $u^*$ is a viscosity subsolution of (12.26). Since $u \leq w$ and $w$ is continuous, we have $u^* \leq w$ and so $u^* \in \mathcal{F}$ and $u = u^*$. By Lemma 7.2, $u_*$ is a viscosity supersolution of (12.26). Since $u$ satisfies (12.30), we can invoke Lemma 12.17 to obtain $u^* \leq u_*$, and so $u = u^* = u_*$ is the unique viscosity solution of (12.26). □

**Exercise 12.19.** Modify the proof of Theorem 12.18 to hold under the condition

$$|f(x)| \leq C_f(1 + |x|^\alpha) \tag{12.34}$$

for some $C_f > 0$ and $\alpha < 2$.

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