Mathematics of Image and Data Analysis
Math 5467

Lecture 1: Introduction

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http://www-users.math.umn.edu/~jwcalder/5467S21

Course Information

- Main course website: http://www-users.math.umn.edu/~jwcalder/5467S21
- Canvas for Zoom links, grades: https://canvas.umn.edu/courses/218893
- Piazza for Q/A: https://piazza.com/umn/spring2021/math5467
  - Sign up for Piazza and make a post in the first week (see HW1).
- Zoom links available on Canvas and sent by email.
- Office hours will be on https://gather.town (link on Canvas)
- 4 homework assignments and 3 projects.
  - Please collaborate with other students!
- Take home exam over 2 days.

Expectations for Zoom classes

- Attend class and keep your video on.
- Lectures will be interactive, and involve working together in groups within Zoom breakout rooms.
- Lectures will not be recorded, since students are expected to attend and engage.
- Ask questions and interact with your peers in breakout rooms.

Audio signals

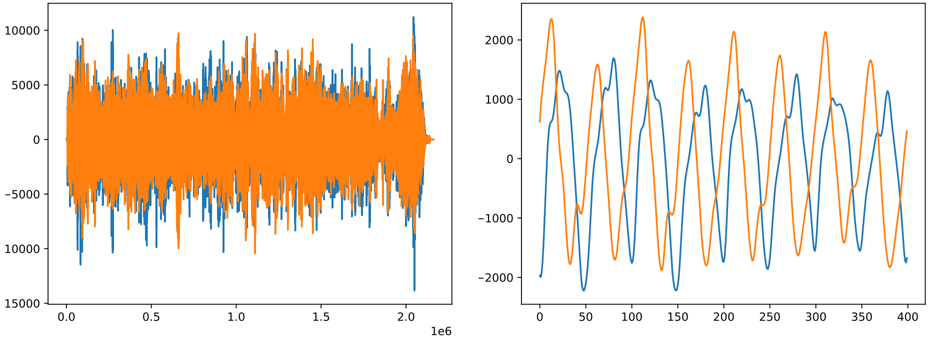


Figure 1: Example of a stereo audio signal from a piece of classical music. The left figure shows both channels over the whole song, while the right figure shows a short clip.

Audio signals

- CD audio has 44,100 samples per second (Hz), with 2 channels and 16 bits per sample

$$\underbrace{2}_{\text{Channels}} \times \underbrace{44,100}_{\text{Samples per Second}} \times \underbrace{16}_{\text{Bits per Sample}} = 1,411,200 \text{ bits/second.}$$

- In terms of kilobits (kbit), 1,411 kbit/sec.
- In terms of megabits (Mbit), 1.4 Mbit/sec.
- How many Mbits or MB of space would a 4 minute song take up?
  - 1 MB = 8 Mbit

Audio signals

Questions:

- How to compress audio without destroying sound quality?
- How to determine what is said in an audio sample? (or determine which song is playing)
- How to demix or remove noise?

Images

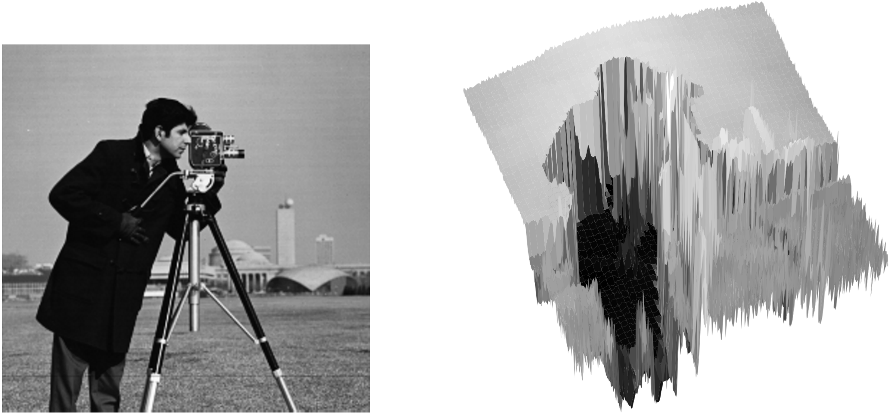


Figure 2: Example of a grayscale digital image.

Images

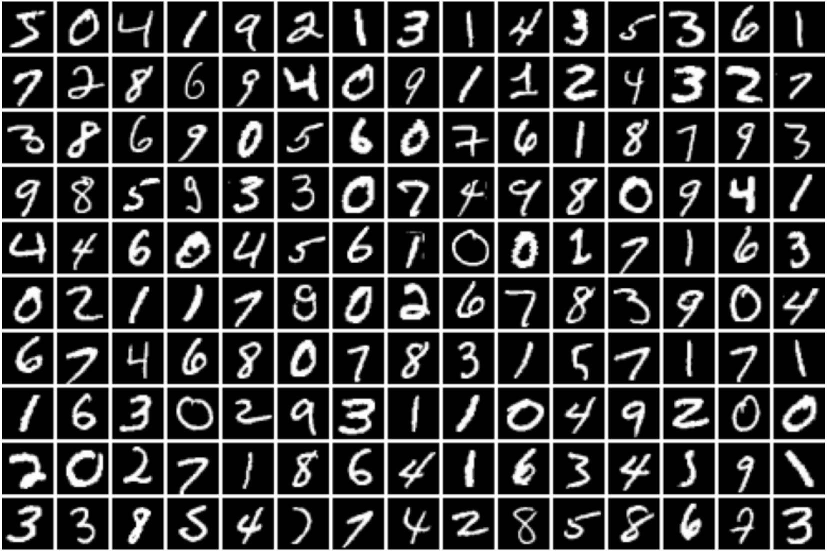
- Modern smartphone can have 12 million pixels (MP).
- For a color image this means storing 36 million numbers.
- With 8-bits per sample, this takes 36 MB of space.
- How much space would a color image from a 46 MP camera take up?

Images

Problems in image analysis:

- Compression
- Segmentation
- Inpainting
- Denoising, deblurring
- Classification
- Recognition

Data analysis



MNIST dataset

Data analysis

Problems in data analysis:

- Classification
- Clustering
- Ranking
- Dimension reduction...

Overview of course

- Data analysis
  - Principal component analysis (PCA)
  - k-means clustering
  - Spectral clustering
  - Google's PageRank
- Fourier Analysis
- Wavelet Analysis
- Variational methods
- Machine learning
  - Basic algorithms
  - Graph-based learning
  - Neural networks
  - Convolutional neural networks

Python

- We will use Python for computational examples during class, and students will use Python on homework assignments and for projects.
- Course website has information for how to get access to Python.
- We will cover an introduction to Python in the first 2 weeks.
- To start today: Introduction to Python

Linear algebra review

- Capital letters A, B, C for matrices (entries are $A(i, j)$)
- Lower case letteers $x, y, z, x_1, x_2, x_3, x_4, \ldots$ for (column) vectors.
- $e_1, e_2, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
- Matrix multiplication: A is $m \times n$ and B is $n \times p$ then $C = AB$ is the $m \times p$ matrix with entries
$$C(i, j) = \sum_{k=1}^{n} A(i, k)B(k, j).$$
- A^T denotes the transpose of A.
- Dot product $x^T y = \sum_{i=1}^{n} x(i)y(i)$.
- Norm: $\|x\| = \sqrt{x^T x} = \sqrt{x(1)^2 + x(2)^2 + \cdots + x(n)^2}$.
- Algebra: $\|x \pm y\|^2 = \|x\|^2 \pm 2x^T y + \|y\|^2$.

Rank-one matrix

For vectors x, y of length n, the rank-one matrix $A = xy^T$ is the $n \times n$ matrix with entries

$$A(i, j) = x(i)y(j).$$

It is called rank-one since the range of A is one dimensional and spanned by the vector x. Indeed,

$$Az = xy^T z = (y^T z)x$$

for any vector z.

Exercise

Let $x_1, x_2, x_3, \dots, x_m$ be a collection of vectors of length n. Define the $m \times n$ matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{bmatrix}.$$

Show that

$$\sum_{i=1}^{m} x_i x_i^T = X^T X.$$