Mathematics of Image and Data Analysis
Math 5467

Lecture 12: Signal Denoising

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Last time

- Parseval and Convolution

Today

- Signal Denoising with Tikhonov regularization

Recall

Definition 1. The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \sum_{k=0}^{n-1} f(k)\omega^{-k\ell} = \sum_{k=0}^{n-1} f(k)e^{-2\pi i k\ell/n},$$

where $\omega = e^{2\pi i/n}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$.

Definition 2 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{2\pi i k\ell/n}.$$

Lemma 3 (Convolution and the DFT)**.** *For $f, g \in L^2(\mathbb{Z}_n)$ we have*

(1)
$$\mathcal{D}(f * g) = \mathcal{D}f \cdot \mathcal{D}g.$$

$$(f * g)(k) = \sum_{j=0}^{n-1} f(j)g(k-j)$$

Exercise on discrete derivatives

Exercise 4. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

(i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

$$\nabla^- f(k) = f(k) - f(k-1).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = g * f$ and use this with Lemma 3 to show that $\mathcal{D}(\nabla^- f)(k) = (1 - \omega^{-k})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

(ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

$$\nabla^+ f(k) = f(k+1) - f(k).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^+ f = g * f$ and use this with Lemma 3 to show that $\mathcal{D}(\nabla^+ f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

(iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

$$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i\sin(2\pi k/n)\mathcal{D}f(k).$$

$$\nabla^\dagger f - \nabla^- f$$

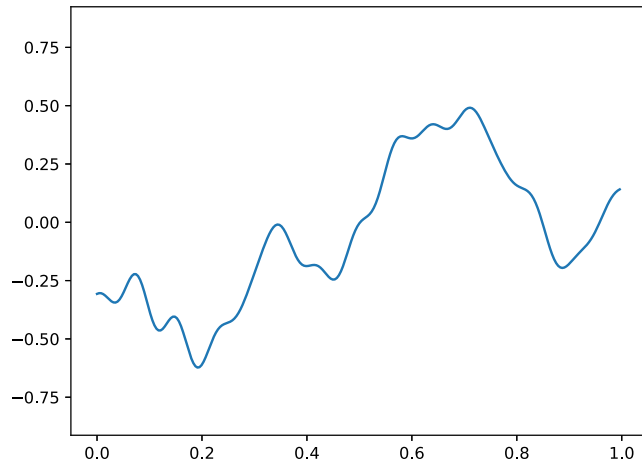(iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

$$\Delta f(k) = \nabla^+ \nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$
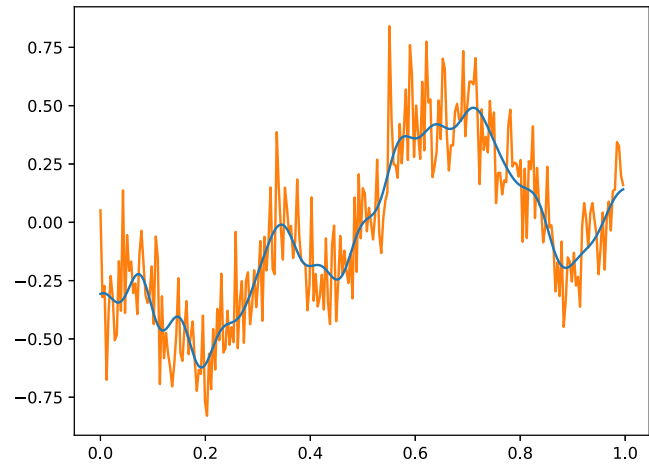
Use parts (i) and (ii) to show that

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k). \qquad \triangle$$

Signal denoising

(a) Clean signal   (b) Noisy signal

Figure 1: Example of a clean signal, and a noisy version, corrupted with additive Gaussian noise. The clean signal is superimposed over the noisy one in (b) for reference.

Sources of noise

A brief (and not exhaustive) list of sources of signal noise.

- Signal acquisition (microphone diaphragm, camera sensor).

- Thermal noise in electronic circuitry.

- Noise from signal compression artifacts.

- Corruption of wireless signals.

Tikhonov regularization

Let $f \in L^2(\mathbb{Z}_n)$ be the noisy signal. Tikhonov regularized denoising minimizes the energy $E : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

(2) $$E(u) = \underbrace{\sum_{k=0}^{n-1} |u(k) - f(k)|^2}_{\text{Data Fidelity}} + \lambda \underbrace{\sum_{k=0}^{n-1} |u(k) - u(k-1)|^2}_{\text{Regularizer}},$$

$\nabla u(k)$

where $\lambda \geq 0$ is a parameter.

Main ideas:

- Data fidelity keeps the denoised signal close to the noisy signal f.

- Regularizer removes the noise.

Tikhonov regularization

We recall the backward difference $\nabla^- : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ is defined by

$$\nabla^- u(k) = u(k) - u(k-1),$$

while the forward difference is $\nabla^+ u(k) = u(k+1) - u(k)$. The discrete Laplacian is

$$\Delta u = \nabla^+ \nabla^- u = \nabla^- \nabla^+ u.$$

In terms of this notation, the Tikhonov regularized denoising problem is

(3)
$$\min_{u \in L^2(\mathbb{Z}_n)} E(u) = \|u - f\|^2 + \lambda \|\nabla^- u\|^2.$$

fidelity   regularizer

Theorem 5. *Let $\lambda \geq 0$ and $f \in L^2(\mathbb{Z}_n)$. Then there exists a unique solution $u \in L^2(\mathbb{Z}_n)$ of the optimization problem* (3). *Furthermore, the minimizer u is also characterized as the unique solution of the Euler-Lagrange equation*

(4) $\nabla E = 0$ $\longleftrightarrow$ $\underline{u - \lambda \Delta u = f.}$ on $\mathbb{Z}_n$.

Properties of discrete derivatives

Before proving the theorem, we need to establish some basic properties of discrete derivatives. These are the discrete analogs of integration by parts formulas.

Proposition 6. *For all $u, v \in L^2(\mathbb{Z}_n)$ the following hold.*

(i) $\langle \nabla^- u, v \rangle = -\langle u, \nabla^+ v \rangle$

(ii) $\langle \nabla^+ u, v \rangle = -\langle u, \nabla^- v \rangle$

(iii) $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$

$$\int_0^1 u' v \, dx = uv \Big|_0^1 - \int u v' \, dx$$

on periodic domain

$u(0) = u(1)$
$v(0) = v(1)$.

Remark 7. We can take $u = v$ to obtain

$$\|\nabla^+ u\|^2 = \|\nabla^- u\|^2 = -\langle \Delta u, u \rangle.$$

$$\|\nabla^- u\|^2 = \langle \nabla^- u, \nabla^- u \rangle = -\langle \nabla^+ \nabla^- u, u \rangle$$

$\uparrow$ Regularizer

(ii)

(i) Proof

$$\langle \nabla^- u, v \rangle = \sum_{k=0}^{n-1} \underbrace{(u(k) - u(k-1))}_{\nabla^- u(k)} \overline{v(k)}$$

$$\Rightarrow = \sum_{k=0}^{n-1} u(k) \overline{v(k)} - \sum_{k=0}^{n-1} u(k-1) \overline{v(k)}.$$

$$j = k-1$$

$$= \sum_{k=0}^{n-1} u(k) \overline{v(k)} - \sum_{j=-1}^{n-2} u(j) \overline{v(j+1)}$$

$$\boxed{-1 = n-1 \quad \text{mod } n}$$

$$= \sum_{k=0}^{n-1} u(k) \overline{v(k)} - \sum_{j=0}^{n-1} u(j) \overline{v(j+1)}$$

$$= \sum_{k=0}^{n-1} u(k) \overline{\left(v(k) - v(k+1) \right)}$$

$$\underbrace{}_{-D^+ v(k)}$$

$$= - \langle u, D^+ v \rangle.$$

(ii) Similar proof, or by (i).

(iii) $\langle \Delta u, v \rangle = \langle D^+ \overset{D^-}{\overbrace{D^-}} u, v \rangle$

$$\overset{(ii)}{=} - \langle D^- u, \overset{D^+}{\overbrace{D^-}} v \rangle$$

$$\overset{(i)}{=} \langle u, D^+ D^- v \rangle = \langle u, \Delta v \rangle.$$

Proof of Theorem : First show that

$$(\divideontimes) \quad u - \lambda \Delta u = f$$

has a unique solution u for each $\lambda \geq 0$, $f \in L^2(\mathbb{R}_n)$

Uniqueness ? Let u, v be solutions of $(\divideontimes)$. Set $w = u - v$.

Note
$$w - \lambda \Delta w = 0$$

Take inner product on both sides with w :

$$\langle w - \lambda \Delta w, w \rangle = 0$$

$$\|w\|^2 - \lambda \langle \Delta w, w \rangle = 0$$

$$\|w\|^2 + \lambda \|\nabla w\|^2 = 0 \implies w = 0$$

$\implies u = v$. Existence of a solutions follows

by Rank-Nullity Theorem.

Now, we show that the unique solution u of ($\clubsuit$) minimizes $E(u) = \|u - f\|^2 + \lambda \|\nabla u\|^2$.

Let $w \in L^2(\mathbb{Z}_n)$. Want to show $E(u) \leq E(w)$.

Set $v = w - u$ and for $t \in \mathbb{R}$. set

$$e(t) = E(u + tv).$$

Then

$$E(w) - E(u) = e(1) - e(0) = \int_0^1 e'(t) \, dt$$

for $t \geq 0$

We claim $e'(t) \geq 0$ and so $E(\omega) - E(u) \geq 0$.

$$\|g + h\|^2 = \|g\|^2 + 2\,\mathrm{Re}\,\langle g, h \rangle + \|h\|^2$$

$$e(t) = \|u + tv - f\|^2 + \lambda \|\bar{\nabla} u + t\bar{\nabla} v\|^2$$

$$= \|u - f\|^2 + 2t\,\mathrm{Re}\,\langle u - f, v \rangle + t^2 \|v\|^2$$

$$+ \lambda \|\bar{\nabla} u\|^2 + 2\lambda t\,\mathrm{Re}\,\langle \bar{\nabla} u, \bar{\nabla} v \rangle + \lambda t^2 \|\bar{\nabla} v\|^2$$

$$\langle \bar{\nabla} u, \bar{\nabla} v \rangle = -\langle \bar{\nabla}^\dagger \bar{\nabla} u, v \rangle = -\langle \Delta u, v \rangle$$

$$e(t) = E(u) + 2t\,\mathrm{Re}\,\langle u - f - \lambda \Delta u, v \rangle$$
$$+ t^2 \left(\|v\|^2 + \lambda \|\bar{\nabla} v\|^2 \right).$$

$(**)\ e'(t) = 2\,\mathrm{Re}\langle \underbrace{u - \lambda \Delta u - f}_{0}, v\rangle$

Since u solves $(*)$ $+\ 2t\left(\|v\|^2 + \lambda\|\bar{\nabla} v\|^2\right)$

$$= 2t\left(\|v\|^2 + \lambda\|\bar{\nabla} v\|^2\right) \geq 0 \quad \text{for } t \geq 0.\ \checkmark$$

$\Rightarrow\ u$ minimizes E. To show that minimizer is unique, let u be any minimizer. Let $v \in L^2(\mathbb{Z}_n)$ and $e(t) = E(u+tv)$. Note $e'(0) = 0$, since $e(0) = E(u) \leq E(u+tv) = e(t)$.

By $(**)$ $\quad 0 = e'(0) = 2\,\mathrm{Re}\langle u - \lambda\Delta u - f, v\rangle$

Take $v = u - \lambda\Delta u - f$ to get $\|u - \lambda\Delta u - f\|^2 = 0$.

$\Rightarrow\ u - \lambda\Delta u = f \Rightarrow$ minimizers are unique $\boxed{}$

Maximum principle

Proposition 8 (Maximum Principle). *Let $\lambda \geq 0$ and let $f \in L^2(\mathbb{Z}_n)$ be real-valued. Let $u \in L^2(\mathbb{Z}_n)$ be the denoised signal, i.e., the solution of*

$$u - \lambda \Delta u = f,$$

which is also real-valued. Then we have

(5)
$$\min_{\mathbb{Z}_n} f \leq u \leq \max_{\mathbb{Z}_n} f.$$

How to solve the Euler-Lagrange equation?

We can use the DFT (and FFT) to solve the Tikhonov regularized denoising equation

$$u - \lambda \Delta u = f.$$

Indeed, the solution is given by

(6) $$u = \mathcal{D}^{-1}(G_\lambda \cdot \mathcal{D}f).$$

where

$$G_\lambda(k) = \frac{1}{1 + 2\lambda - 2\lambda \cos(2\pi k/n)}.$$

Take DFT on both sides of $u - \lambda \Delta u = f$

$$Du - \lambda D(\Delta u) = Df$$

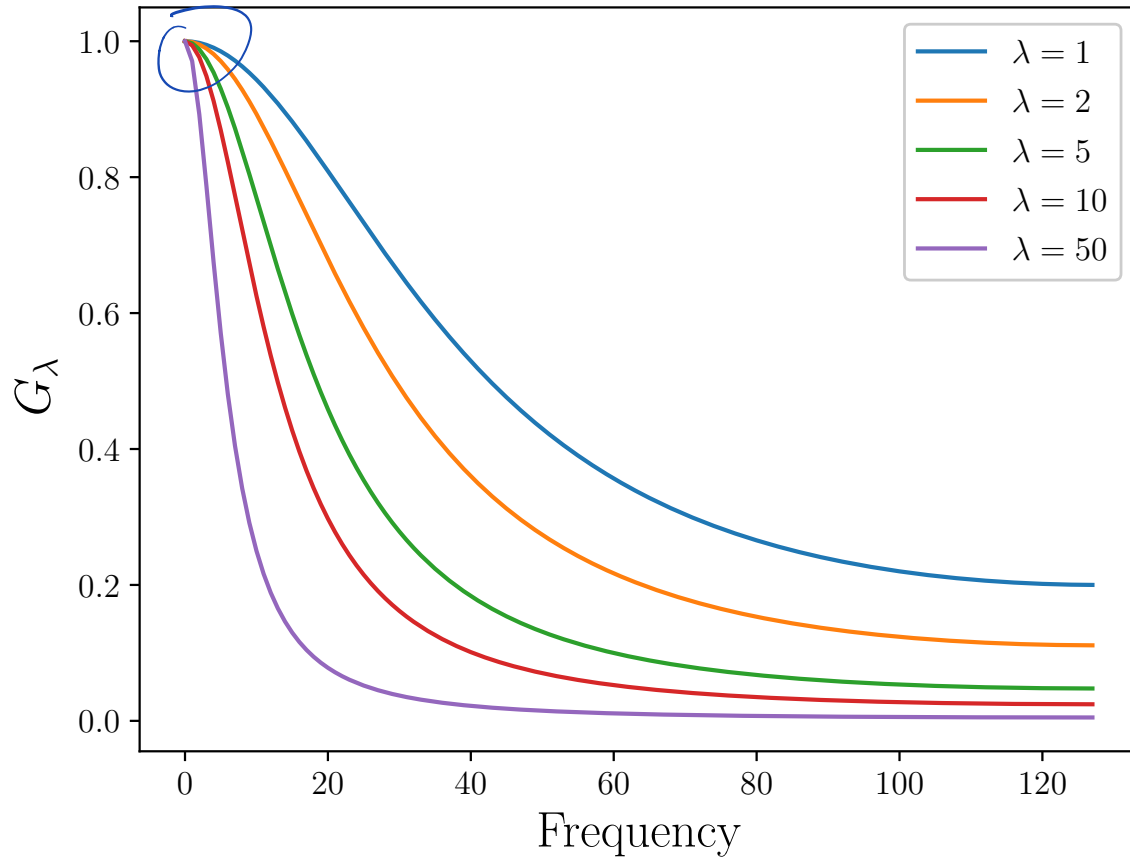$$Du(k) - \lambda 2 (\cos(2\pi k/n) - 1) Du(k) = Df(k)$$

$$Du(k)\left(1 - 2\lambda\left(\cos(2\pi k/n) - 1\right)\right) = Df(k)$$

$$Du(k) = G_\lambda(k)\,Df(k)$$

$$u = D^{-1}\left(G_\lambda \cdot Df\right).$$

The low-pass filter G_λ

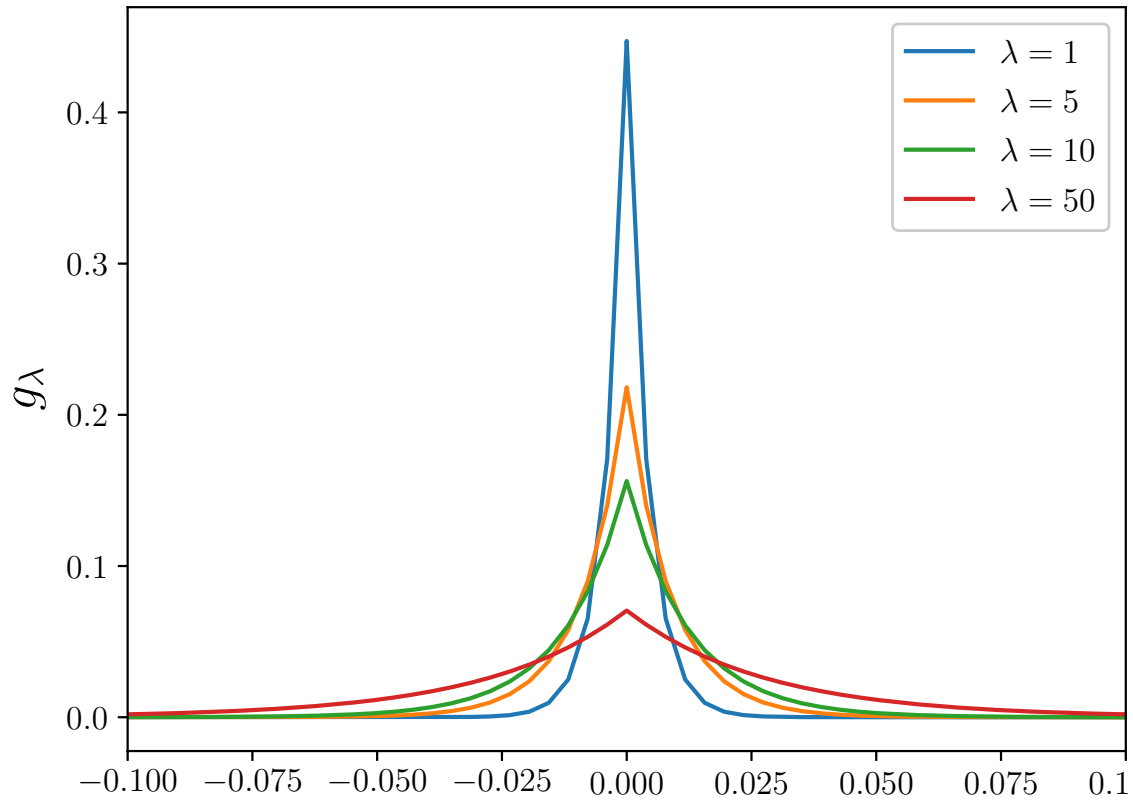# The filtering perspective

$$D(f * g) = Df \cdot Dg.$$

Defining $g_\lambda = \mathcal{D}^{-1}G_\lambda$ we have

$$u = \mathcal{D}^{-1}(\mathcal{D}g_\lambda \cdot \mathcal{D}f) = g_\lambda * f.$$

The function g_λ is often called a *kernel*. The convolution to compute $u(k)$ should be thought of as a weighted average, weighted by g_λ, of the values of f nearby k. Indeed, we have

$$u(k) = (g_\lambda * f)(k) = \sum_{j=0}^{n-1} g_\lambda(j)f(k - j).$$

The convolutional kernel g_λ (fundamental solution)

Tikhonov denoising (.ipynb)