Mathematics of Image and Data Analysis
Math 5467

Lecture 13: TV Denoising

Instructor: Jeff Calder
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/5467S21

Last time

- Tikhonov regularized denoising

Today

- Total Variation (TV) regularized denoising

Tikhonov regularization

Let $f \in L^2(\mathbb{Z}_n)$ be the noisy signal. Tikhonov regularized denoising minimizes the energy $E : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$E(u) = \underbrace{\sum_{k=0}^{n-1} |u(k) - f(k)|^2}_{\text{Data Fidelity}} + \lambda \underbrace{\sum_{k=0}^{n-1} |u(k) - u(k-1)|^2}_{\text{Regularizer}}, \tag{1}$$

where $\lambda \geq 0$ is a parameter.

Main ideas:

- Data fidelity keeps the denoised signal close to the noisy signal f.
- Regularizer removes the noise.

Tikhonov regularization

We recall the backward difference $\nabla^- : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ is defined by

$$\nabla^- u(k) = u(k) - u(k-1),$$

while the forward difference is $\nabla^+ u(k) = u(k+1) - u(k)$. The discrete Laplacian is

$$\Delta u = \nabla^+ \nabla^- u = \nabla^- \nabla^+ u.$$

In terms of this notation, the Tikhonov regularized denoising problem is

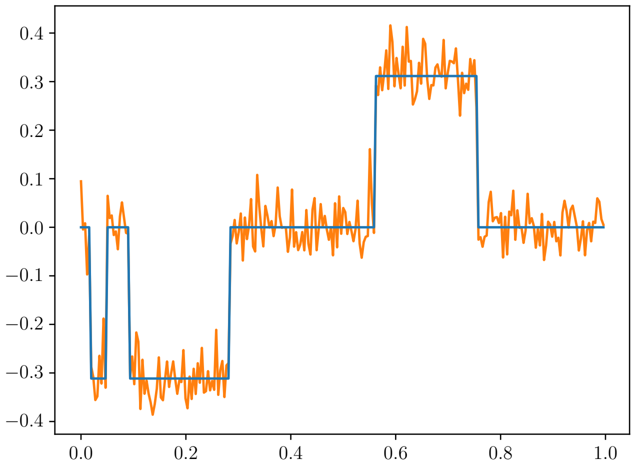
$$\min_{u \in L^2(\mathbb{Z}_n)} E(u) = \|u - f\|^2 + \lambda \|\nabla^- u\|^2. \tag{2}$$

Theorem 1. *Let $\lambda \geq 0$ and $f \in L^2(\mathbb{Z}_n)$. Then there exists a unique solution $u \in L^2(\mathbb{Z}_n)$ of the optimization problem (2). Furthermore, the minimizer u is also characterized as the unique solution of the Euler-Lagrange equation*

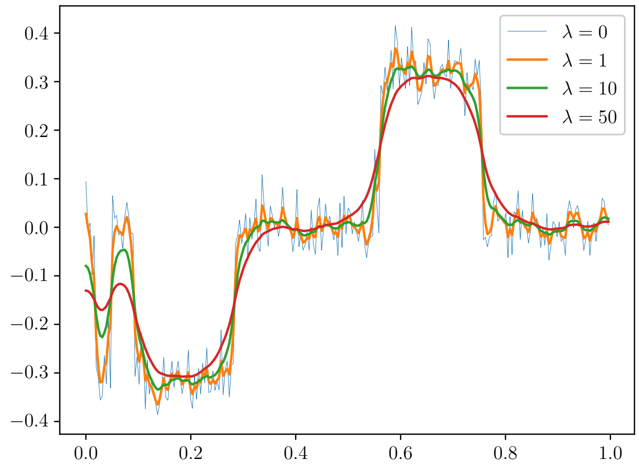
$$u - \lambda \Delta u = f. \tag{3}$$

$\nabla E = 0$

Tikhonov regularization



(a) Noisy signal



(b) Tikhonov denoising

Total Variation Regularization

Total Variation (TV) regularization replaces the squared difference by the absolute differences in the regularizer.

$$E(u) = \frac{1}{2}\sum_{k=0}^{n-1} |u(k) - f(k)|^2 + \lambda \underbrace{\sum_{k=0}^{n-1} |u(k) - u(k-1)|}_{\text{Total variation of } u.} . \tag{4}$$

- TV regularization is better at preserving edges (sharp changes) in the signal.
- The analysis is more involved, since the denosing equation is *nonlinear*.

Variational Regularized Denoising

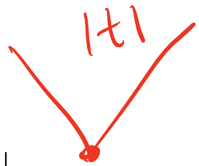
$\|x\|_1 = \sum_{k=0}^{n-1} |x(k)|$

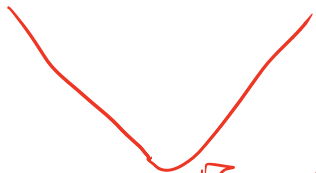
We will proceed in generality, studying regularizers of the form

$$\sum_{k=0}^{n-1} \Phi(u(k) - u(k-1)) = \sum_{k=0}^{n-1} \Phi(\nabla^- u(k)) = \|\Phi(\nabla^- u)\|_1, \tag{5}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable, convex, and even function satisfying $\Phi(0) = 0$.

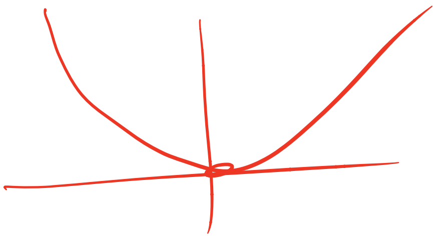
- Tikhonov is $\Phi(t) = t^2$
- Total Variation (TV) is $\Phi(t) = |t|$.
- We will approximate TV by $\Phi(t) = \sqrt{t^2 + \varepsilon^2}$.

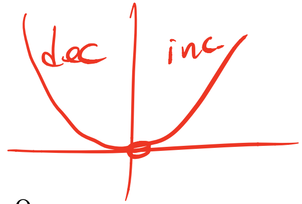




Rounding off corner.

Φ





Convexity

We say Φ convex if $\Phi'' \geq 0$. We also assumed Φ is even and $\Phi(0) = 0$.

The following properties hold:

$$\Phi'(0) = \lim_{h \to 0} \frac{\Phi(h) - \Phi(-h)}{2h} = 0$$

(i) Φ' is increasing.

(ii) Since Φ is even and $\Phi(0) = 0$ we have $\Phi'(0) = 0$.

(iii) $\Phi'(t) \leq 0$ for $t < 0$ and $\Phi'(t) \geq 0$ for $t > 0$. $\Longrightarrow \Phi \geq 0$

(iv) For any $t, s \in \mathbb{R}$ we have

$$(\Phi'(t) - \Phi'(s))(t - s) \geq 0.$$

(monotonicity)

Proof (iv): If $t \geq s$ then $\Phi'(t) - \Phi'(s) \geq 0$ and $t - s \geq 0$

If $t \leq s$ then $\Phi'(t) - \Phi'(s) \leq 0$ and $t - s \leq 0$

$\blacksquare$

Total Variation Denoising

The Total Variation (TV) regularized denoising function is

$$E_\Phi(u) = \frac{1}{2}\|u - f\|^2 + \lambda\|\Phi(\nabla^- u)\|_1. \tag{6}$$

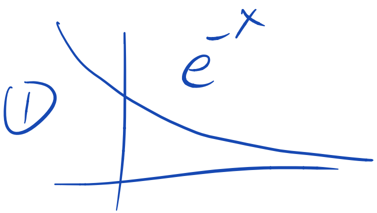
The denoised signal u is found by minimizing E_Φ.

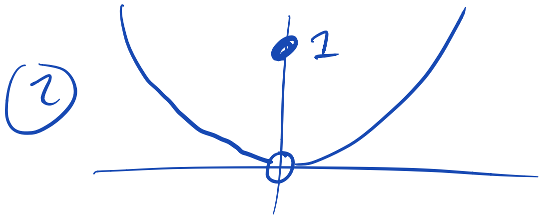
Note: We will work with real-value signals in this lecture, for simplicity. We denote by $L^2(\mathbb{Z}_n;\mathbb{R})$ the subspace of $L^2(\mathbb{Z}_n)$ consisting of $f : \mathbb{Z}_n \to \mathbb{R}$.

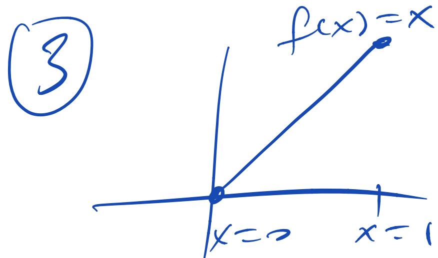
Existence of a minimizer

Lemma 2. *For any $f \in L^2(\mathbb{Z}_n;\mathbb{R})$ and $\lambda \geq 0$, there exists $u \in L^2(\mathbb{Z}_n;\mathbb{R})$ minimizing E_Φ, i.e., $E_\Phi(u) \leq E_\Phi(w)$ for all $w \in L^2(\mathbb{Z}_n;\mathbb{R})$. Furthermore, u satisfies*

$$\min_{\mathbb{Z}_n} f \leq u \leq \max_{\mathbb{Z}_n} f. \tag{7}$$







The proof is based on a simple fact: A continuous function on a closed and bounded subset of $\mathbb{R}^n$ attains its minimum value.

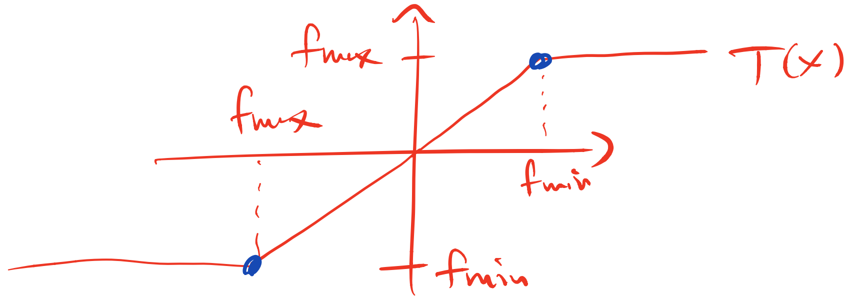
- $f(x) = e^{-x}$ does not have a minimum value on $\mathbb{R}$ (unbounded set).
- $f(x) = x^2$ for $x \neq 0$ and $f(0) = 1$ does not have a minimum value (discontinuous function).
- $f(x) = x$ does not have a minimum value on $(0, 1)$ (open set).

Proof: Define

$$T(x) = \begin{cases} f_{min}, & \text{if } x < f_{min} \\ x, & \text{if } f_{min} \le x \le f_{max} \\ f_{max}, & \text{if } x > f_{max} \end{cases}$$

$$f_{min} = \min_{\mathbb{R}^n} f$$

$$f_{max} = \max_{\mathbb{R}^n} f$$



Claim: $\boxed{|T(x) - T(y)| \le |x - y|}$

Assume $x \ge y$ and compute

$$T(x) - T(y) = \int_y^x T'(t)\,dt \le \int_y^x \left(\max_{\mathbb{R}} T'\right) dt$$

$$= \left(\max_{\mathbb{R}} T'\right) \int_y^x 1 \, dt$$

$$= \underbrace{\left(\max_{\mathbb{R}} T'\right)}_{=1} |x-y|$$

Let $u \in \ell^2(\mathbb{Z}_n;\mathbb{R})$ and define $w(k) = T(u(k))$

Claim: $E_\Phi(w) \leq E_\Phi(u)$.

$$E_\Phi(w) = \frac{1}{2} \sum_{k=0}^{n-1} |w(k)-f(k)|^2 + \lambda \sum_{k=0}^{n-1} |w(k)-w(k-1)|$$

$$|w(k)-f(k)|^2 = |T(u(k)) - T(f(k))|^2 \leq |u(k)-f(k)|^2$$

Works for TV

$T(f) = f.$

Lipschitzness of T

$$\boxed{\begin{aligned} |w(k) - w(k-1)| &= |T(u(k)) - T(u(k-1))| \\ \text{T Lipschitz} \Rightarrow \quad &\le |u(k) - u(k-1)| \quad \checkmark \end{aligned}} \quad (*)$$

This means we can minimize E_{Φ} over the closed and bounded set

$$M = \left\{ u \in \ell^2(\mathbb{Z}_n; \mathbb{R}) : \begin{array}{l} f_{min} \le u(k) \le f_{max} \\ \text{for all } k \end{array} \right\}.$$

$$\hat{=} \; [f_{min}, f_{max}]^n \subseteq \mathbb{R}^n.$$

So $\exists$ minimizer $u \in M$ of E_Φ, minimal over M, so

$$E_\Phi(u) \leq E_\Phi(w) \quad \forall w \in M.$$

If $v \in L^2(\mathbb{Z}_n; \mathbb{R})$, then

$$E_\Phi(u) \leq E_\Phi(T(v)) \leq E_\Phi(v).$$

$\blacksquare$

(*) For general Φ: (Recall $w(k) = T(u(k))$)

$$\Phi(\nabla^- w(k)) = \Phi(w(k) - w(k-1))$$

Φ even

$$= \Phi(|w(k) - w(k-1)|)$$

$\Phi(t)$ is increasing for $t>0$

$$\leq \Phi(|u(k) - u(k-1)|)$$

$$= \Phi(u(k) - u(k-1)) = \Phi(\nabla^- u(k)).$$

Euler-Lagrange equation

Lemma 3. *Let $f \in L^2(\mathbb{Z}_n;\mathbb{R})$ and $\lambda \geq 0$. Then the minimizer $u \in L^2(\mathbb{Z}_n;\mathbb{R})$ of E_Φ is unique and is characterized as the unique solution of the Euler-Lagrange equation*

$$u - \lambda\nabla^+\Phi'(\nabla^- u) = f. \tag{8}$$

Tikhonov: $u - \lambda \Delta u = f$ $\quad \left(\Phi(t) = \frac{1}{2}t^2\right)$

$\Phi'(t) = t$

Recall

Proposition 4. *For all $u, v \in L^2(\mathbb{Z}_n)$ the following hold.*

(i) $\langle \nabla^- u, v\rangle = -\langle u, \nabla^+ v\rangle$

(ii) $\langle \nabla^+ u, v\rangle = -\langle u, \nabla^- v\rangle$

(iii) $\langle \Delta u, v\rangle = \langle u, \Delta v\rangle$

Proof: let u be a minimizer of E_{Φ}.
Let $v \in L^2(\mathbb{Z}_n; \mathbb{R})$ and set

$$e(t) = E_{\Phi}(u+tv).$$

Since $e(0) = E_{\Phi}(u) \leq E_{\Phi}(u+tv) = e(t)$
e has a minimizer at $t=0$. Hence

$$0 = e'(0) = \frac{d}{dt}\Big|_{t=0} E_{\Phi}(u+tv).$$

$$e(t) = E_{\Phi}(u+tv)$$

$$= \underbrace{\frac{1}{2}\|u+tv-f\|^2}_{= \frac{1}{2}\left(\|u-f\|^2 + 2t\langle u-f, v\rangle + t^2\|v\|^2\right)} + \lambda \sum_{k=0}^{n-1} \Phi\left(\nabla^- u(k) + t\nabla^- v(k)\right)$$

$$e'(t) = \langle u-f, v\rangle + \cancelto{0 \text{ when } t=0.}{t\|v\|^2}$$

$$+ \lambda \sum_{k=0}^{n-1} \Phi'\left(\nabla^- u(k) + \cancelto{0}{t}\,\nabla^- v(k)\right) \nabla^- v(k)$$

$$0 = e'(0) = \langle u-f, v\rangle + \lambda\langle \Phi'(\nabla^- u), \nabla^- v\rangle$$

$$= \langle u-f, v\rangle - \lambda\langle \nabla^+ \Phi'(\nabla^- u), v\rangle$$

Gateaux Derivative $= \langle u - \lambda D^+ \Phi'(D^- u) - f, v \rangle.$

$$\frac{d}{dt}\Big|_{t=0} E_\Phi(u+tv) = \langle \underbrace{u - \lambda D^+ \Phi'(D^- u) - f}_{DE_\Phi(u) := \text{Gradient of } E.}, v \rangle = 0$$

Directional Derivative

Since this holds for all v, choose

$v = DE_\Phi(u)$ to get $\| DE_\Phi(u) \|^2 = 0$

or $\quad DE_\Phi(u) = 0.$

Aside: If $f:\mathbb{R}^n \to \mathbb{R}$, $\left.\frac{d}{dt}\right|_{t=0} f(x+tv) = \nabla f(x)\cdot v$

We now show that solutions of

$$(1) \quad u - \lambda D^+ \Phi'(D^- u) = f$$

are unique. Let v be another solution

$$(2) \quad v - \lambda D^+ \Phi'(D^- v) = f$$

Diff of (1) − (2)

$$u - v - \lambda D^+\left(\Phi'(D^- u) - \Phi'(D^- v)\right) = 0$$

Take inner product with $u-v$ on both sides:

$$\langle u-v-\lambda D^{+}(\Phi'(D^{-}u)-\Phi'(D^{-}v)),\ u-v\rangle = 0$$

$$\|u-v\|^2 - \lambda\langle D^{+}(\Phi'(D^{-}u)-\Phi'(D^{-}v)),\ u-v\rangle = 0$$

prop. $-D^{-}$

$$\|u-v\|^2 + \lambda\langle \Phi'(D^{-}u)-\Phi'(D^{-}v),\ D^{-}u-D^{-}v\rangle = 0$$

$$(\Phi'(t)-\Phi'(s))(t-s) \geq 0$$

due to convexity of Φ

$\Rightarrow$ both terms above are ≥ 0

and they sum to zero, so

both vanish $\Rightarrow u = v$. ▨

The gradient of E_Φ

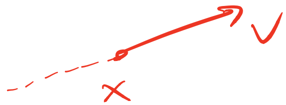
The gradient of E_Φ can be interpreted as

$$\nabla E_\Phi(u) = u - \lambda \nabla^+ \Phi'(\nabla^- u) - f.$$

Definition of ∇E_Φ is

$$\frac{d}{dt}\Big|_{t=0} E_\Phi(u+tv) = \langle \nabla E_\Phi(u), v \rangle.$$

If $f: \mathbb{R}^n \to \mathbb{R}$ then



$x, v \in \mathbb{R}^n$

$$\frac{d}{dt}\Big|_{t=0} f(x+tv) = \nabla f(x) \cdot v = \langle \nabla f, v \rangle.$$

Gradient Descent

We can minimize E_Φ by gradient descent

$$(GD) \qquad u_{j+1} = u_j - dt\nabla E_\Phi(u_j) = u_j - dt\,\big(u_j - \underbrace{\lambda\nabla^+\Phi'(\nabla^- u_j)}_{C\lambda\Delta u} - f\big)$$

Linear approx.

Time step restriction: For stability and convergence of the gradient descent iteration, we have a time step restriction

$$dt \leq \frac{2}{1 + 4C_\Phi\lambda},$$

where $C_\Phi = \max_{t\in\mathbb{R}} \Phi''(t)$. This follows from a Von Nuemann analysis using the DFT.

Von Neumann Analysis:

① Linearize the GD equation.

$$\Phi'(t) - \Phi'(s) = \int_s^t \Phi''(\tau)\,d\tau$$

$$\approx \Phi''(t) \cdot (t-s)$$

Set $\quad C_\Phi = \max_{t \in \mathbb{R}} \Phi''(t)$

and replace

$$\nabla^+ \Phi'(\nabla^- u(k)) = \Phi'(\overbrace{\nabla^- u(k+1)}^{t}) - \Phi'(\overbrace{\nabla^- u(k)}^{s})$$

$$\simeq C_\Phi (\nabla^- u(k+1) - \nabla^- u(k)).$$

$$= C_\Phi \nabla^+ \nabla^- u(k) = C_\Phi \Delta u$$

Linearized eq:

$$u_{j+1} = u_j - dt\left(u_j - \lambda C_\Phi \Delta u_j - f\right).$$

Take DFT on both sides

$$D u_{j+1} = (1-dt) D u_j + \lambda C_{\Phi} dt\, D(\Delta u_j) + \overbrace{\cancel{dt\, Df}}^{\text{assume } 0}$$

$$D(\Delta u_j)(k) = 2(\cos(2\pi k/n) - 1) D u_j(k).$$

$$D u_{j+1} = \overbrace{\left(1 - dt + 2\lambda C_{\Phi} dt (\cos(2\pi k/n) - 1)\right)}^{\lambda_k} D u_j$$

$\frac{k}{n}$ = frequency

$$D u_{j+1}(k) = \lambda_k D u_j(k)$$

<u>induction</u>

$$D u_j(k) = \lambda_k^j D u_0(k).$$

For the scheme to be stable need

$$\forall k, \quad -1 \le \lambda_k \le 1$$

$$1 \ge \lambda_k = 1 - dt + 2\lambda C_{\Phi}\, dt \left(\overbrace{\underbrace{\cos(2\pi k/n) - 1}_{\le 0}}^{\ge -2} \right)$$

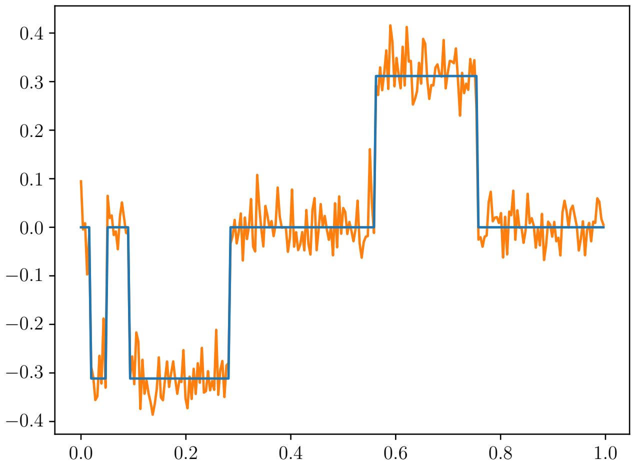
$$\ge 1 - dt - 4\lambda C_{\Phi}\, dt$$

want ≥ -1

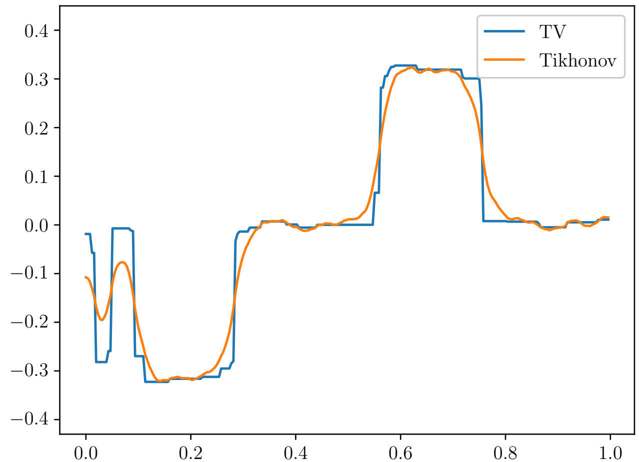
$$1 - dt(1 + 4\lambda C_{\Phi}) \ge -1$$

$$\iff \boxed{dt \le \frac{2}{1 + 4\lambda C_{\Phi}}}$$

Total Variation Denoising



(c) Noisy signal



(d) Denoised

Convergence of Gradient Descent

Theorem 5. *Let $f \in L^2(\mathbb{Z}_n;\mathbb{R})$ and $\lambda \geq 0$. Let u_j be the iterations of the gradient descent scheme for minimizing E_Φ and let u be the solution of (8) (the minimizer of E_Φ). Assume that the time step dt satisfies*

$$dt < \frac{2}{1 + 16C_\Phi^2\lambda^2}. \tag{9}$$

Then u_j converges to u as $j \to \infty$, and the difference $u_j - u$ satisifes

$$\underbrace{\|u_{j+1} - u\|^2}_{\|e_{j+1}\|^2} \leq \mu \|\underbrace{u_j - u}_{\|e_j\|^2}\|^2 \tag{10}$$

Linear convergence rate μ.

where

$$\mu := (1-dt)^2 + 16C_\Phi^2 dt^2\lambda^2 < 1. \tag{11}$$

(8) $\qquad u - \lambda D^+ \Phi'(D^- u) = f$

Proof: Write GD as

① $$u_{j+1} = (1-dt)u_j + dt\lambda D^+ \Phi'(D^- u_j) + dtf$$

and write (8) as

② $$u = (1-dt)u + dt\lambda D^+ \Phi'(D^- u) + dtf$$

Write $e_j = u_j - u$. Subtract ② from ①:

$$e_{j+1} = (1-dt)e_j + dt\lambda D^+\left(\Phi'(D^- u_j) - \Phi'(D^- u)\right).$$

Take $\| \|^2$ on both sides

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2\langle f, g\rangle$$

$$\|e_{j+1}\|^2 = \|(1-dt)e_j + dt\,\lambda D^+(\Phi'(D^-u_j) - \Phi'(D^-u))\|^2$$

$$= (1-dt)^2\|e_j\|^2 + dt^2\lambda^2\|D^+(\Phi'(D^-u_j) - \Phi'(D^-u))\|^2$$

$$+ 2\,dt(1-dt)\,\lambda\underbrace{\langle D^+(\Phi'(D^-u_j) - \Phi'(D^-u)), e_j\rangle}_{A}$$

$\langle D^+f, g\rangle = -\langle f, D^-g\rangle$

$$A = -\langle \Phi'(D^-u_j) - \Phi'(D^-u), D^-e_j\rangle \qquad e_j = u_j - u$$

$$= -\langle \Phi'(D^-u_j) - \Phi'(D^-u), D^-u_j - D^-u\rangle$$

≤ 0 by monotonicity property (iv).

If $dt \le 1$, drop 3^{rd} term (since ≤ 0) to get

$$\|e_{j+1}\|^2 \le (1-dt)^2 \|e_j\|^2 + dt^2 \lambda^2 \|D^+(\Phi'(D^-u_j) - \Phi'(D^-u))\|^2$$

Aside: $\|D^+ f\|^2 = \sum_{k=0}^{n-1} |f(k) - f(k-1)|^2$

$$= \sum_{k=0}^{n-1} \left(f(k)^2 + f(k-1)^2 - 2f(k)f(k-1)\right)$$

$$= 2\|f\|^2 - 2\sum_{k=0}^{n-1} f(k)f(k-1).$$

Claim: $2ab \le a^2 + b^2$ for any $a, b \in \mathbb{R}$.

True since $(a-b)^2 = a^2 + b^2 - 2ab \geq 0$.

$\Rightarrow \quad -2f(k)f(k-1) \leq f(k)^2 + f(k-1)^2$

by taking $a = -f(k)$, $b = f(k-1)$.

$$\Rightarrow \boxed{\begin{aligned} \|D^+ f\|^2 &\leq 4\|f\|^2 \\ \|D^- f\|^2 &\leq 4\|f\|^2 \end{aligned}}$$

Bound on operator norm of D^+

Here

$$\|D^+(\Phi'(D^-u_j) - \Phi'(D^-u))\|^2 \leq 4\|\Phi'(D^-u_j) - \Phi'(D^-u)\|^2$$

$$\leq 4C_\Phi^2 \|D^-u_j - D^-u\|^2 \quad (*)$$

since $t>s$ $\quad \Phi'(t) - \Phi'(s) = \int_s^t \Phi''(\tau)\,d\tau \leq C_\Phi(t-s)$

"$C_\Phi = \max \Phi''$"

$$(*) = 4 C_\Phi^2 \|\nabla e_j\|^2 \le 16 C_\Phi^2 \|e_j\|^2$$

Hence

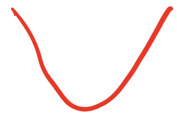
$$\|e_{j+1}\|^2 \le \underbrace{\left[(1-dt)^2 + 16\, dt^2 \lambda^2 C_\Phi^2\right]}_{=\mu} \|e_j\|^2$$

$e_j = u_j - u.$

If $\mu < 1$ then GD converge at rate μ, and $dt \le \ldots$ follows from this ▨

Note: (TV) $\Phi(t) = |t|, \quad C_\Phi = \infty$

(TV_ε) $\Phi_\varepsilon(t) = \sqrt{t^2+\varepsilon^2}$



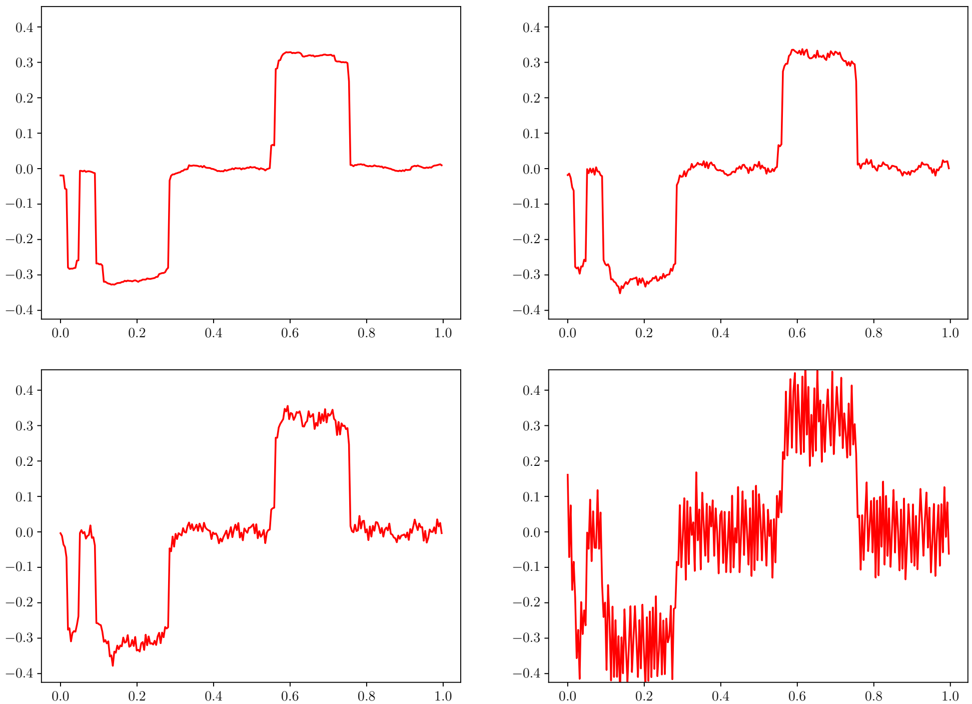
$$\Phi_\varepsilon'(t) = \frac{t}{\sqrt{t^2+\varepsilon^2}}$$

$$\Phi_\varepsilon''(t) = \frac{\sqrt{t^2+\varepsilon^2} - t\,\frac{t}{\sqrt{t^2+\varepsilon^2}}}{t^2+\varepsilon^2}$$

$$= \frac{\varepsilon^2}{(t^2+\varepsilon^2)^{3/2}} \le \frac{\varepsilon^2}{\varepsilon^3} = \frac{1}{\varepsilon} = C_\Phi$$

Nonlinear stability at larger time steps

We set $\varepsilon = 10^{-10}$ and the CFL condition is $dt \sim 5 \times 10^{-10}$.
Figures are $dt = 0.01, 0.05, 0.1, 0.5$.



Local nonlinear stability

A heuristic local version of the Von Neumann analysis for ε-regularzied TV shows that the scheme is stable wherever the gradient of u satisfies

$$|\nabla^- u|^3 \geq \frac{4\lambda\varepsilon^2 dt}{2 - dt}.$$

Thus, oscillations cannot grow infinitely large, since the scheme is stable for larger gradients.

(CFL) $\quad dt \leq \dfrac{2}{1 + 4 C_\Phi \lambda}$ $\quad$ Von Neumann analysis.

Instead of $\quad C_\Phi = \max \Phi''$

$$\nabla^+ \Phi'(\nabla^- u) = \Phi'(\nabla^- u(k+1)) - \Phi'(\nabla^- u(k))$$

$$\simeq \Phi''(\nabla^- u(k))\,(\nabla^- u(k+1) - \nabla^- u(k))$$

$$= \Phi''(\nabla^- u)\,\nabla^+ \nabla^- u$$

$$= \Phi''(\nabla^- u)\,\Delta u.$$

leave as Φ'':

$$dt \leq \frac{2}{1 + 4\,\Phi''(\nabla^- u)\,\lambda}$$

Flip around

$$(\#) \quad \boxed{\Phi''(\nabla^- u) \leq \frac{1}{4\lambda}\left(\frac{2}{dt} - 1\right)}$$

$$\Phi(t) = \sqrt{t^2+\varepsilon^2}$$

Check $$\Phi''(t) = \frac{\varepsilon^2}{(t^2+\cancel{\varepsilon^2})^{3/2}} \le \frac{\varepsilon^2}{t^3}$$

(*) holds if $$\frac{\varepsilon^2}{|\nabla u|^3} \le \frac{1}{4\lambda}\left(\frac{2}{dt}-1\right)$$

Rearrange $$\boxed{|\nabla u|^3 \ge \frac{4\lambda\varepsilon^2 dt}{2-dt}}$$

Total Variation denoising (.ipynb)