Mathematics of Image and Data Analysis
Math 5467

Lecture 15: Multi-dimensional DFT and Image Denoising

Instructor: Jeff Calder
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/5467S21

Last time

- Total Variation (TV) regularized denoising

Today

- Multi-dimensional DFT
- Image denoising

Higher dimensions

Let

$$\mathbb{Z}_n^d = \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{d \text{ times},}$$

where $d \geq 1$. Each $k \in \mathbb{Z}_n^d$ has d components

$$k = (k(1), k(2), \ldots, k(d)).$$

We denote the dot product of $k, \ell \in \mathbb{Z}_n^d$ by

$$k \cdot \ell = \sum_{j=1}^{d} k(j)\ell(j).$$

We denote by $L^2(\mathbb{Z}_n^d)$ the space of function $f : \mathbb{Z}_n^d \to \mathbb{C}$ equipped with the inner product

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}_n^d} f(k)\overline{g(k)}.$$

We also have the induced norm $\|f\|^2 = \langle f, f \rangle$.

Convolution

The *discrete cyclic convolution* of $f, g \in L^2(\mathbb{Z}_n^d)$ is given by

$$(f * g)(k) = \sum_{\ell \in \mathbb{Z}_n^d} f(\ell) g(k - \ell).$$

Note that the sums above are short-hand for d sums, and we have

$$\sum_{k \in \mathbb{Z}_n^d} = \underbrace{\sum_{k(1)=0}^{n-1} \sum_{k(2)=0}^{n-1} \cdots \sum_{k(d)=1}^{n-1}}_{d \text{ sums}} .$$

Multi-dimensional DFT

Definition 1. The *(multi-dimensional) Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n^d) \to L^2(\mathbb{Z}_n^d)$ given by

$$\mathcal{D}f(k) = \sum_{\ell \in \mathbb{Z}_n^d} f(\ell) e^{-2\pi i k \cdot \ell / n}. \tag{1}$$

The *(mult-dimensional) Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n^d) \to L^2(\mathbb{Z}_n^d)$ given by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n^d} \sum_{k \in \mathbb{Z}_n^d} f(k) e^{2\pi i k \cdot \ell / n}. \tag{2}$$

Reduction to one dimension

It is important to point out that the multi-dimensional DFT can be viewed as applying d one dimensional DFTs to the individual coordinates. Indeed, we consider the case of $d = 2$ where we can write

$$
\begin{aligned}
\mathcal{D}f(k) &= \mathcal{D}f(k(1), k(2)) \\
&= \sum_{\ell(1)=0}^{n-1} \sum_{\ell(2)=0}^{n-1} f(\ell(1), \ell(2)) e^{-2\pi i \overbrace{(k(1)\ell(1)+k(2)\ell(2))}^{k\cdot\ell}/n} \\
&= \underbrace{\sum_{\ell(1)=0}^{n-1} e^{-2\pi i k(1)\ell(1)/n} \underbrace{\left(\sum_{\ell(2)=0}^{n-1} f(\ell(1), \ell(2)) e^{-2\pi i k(2)\ell(2)/n} \right)}_{\text{One dimensional DFT in } \ell(2)}}_{\text{One dimensional DFT in } \ell(1)} .
\end{aligned}
$$

In terms of images, we can think that the two dimensional DFT is just taking the one dimensional DFT of the rows, and then the one dimensional DFT of the columns (or vice versa).

Basic properties

Theorem 2. *For every $f \in L^2(\mathbb{Z}_n^d)$ we have $f = \mathcal{D}\mathcal{D}^{-1}f = \mathcal{D}^{-1}\mathcal{D}f$. Furthermore, the following properties hold for each $f, g \in L^2(\mathbb{Z}_n^d)$.*

(i) $\langle f, g \rangle = \frac{1}{n^d}\langle \mathcal{D}f, \mathcal{D}g \rangle$,

(ii) $\|f\|^2 = \frac{1}{n^d}\|\mathcal{D}f\|^2$,

Parseval

(iii) $\mathcal{D}(f * g) = \mathcal{D}f \cdot \mathcal{D}g$. → Convolution property.

Exercise 3. Prove Theorem 2. △

Discrete derivatives

Let $e_1, e_2, \ldots, e_d \in \mathbb{R}^d$ be the standard basis vectors in $\mathbb{R}^d$. We define the forward difference in the j^{th} direction, $\nabla_j^+ : L^2(\mathbb{Z}_n^d) \to L^2(\mathbb{Z}_n^d)$, by

$$\nabla_j^+ u(k) = u(k + e_j) - u(k).$$

$e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$

j^{th} coord.

Similarly, the backward difference ∇_j^- by

$$\nabla_j^- u(k) = u(k) - u(k - e_j).$$

The discrete Laplacian Δ is defined by

$$\Delta u = \sum_{j=1}^{d} \nabla_j^+ \nabla_j^- u.$$

Integration (summation) by parts formulas

Proposition 4. *For all $u, v \in L^2(\mathbb{Z}_n^d)$ and $j = 1, 2, \ldots, d$, the following hold.*

(i) $\langle \nabla_j^- u, v \rangle = -\langle u, \nabla_j^+ v \rangle$

(ii) $\langle \nabla_j^+ u, v \rangle = -\langle u, \nabla_j^- v \rangle$

(iii) $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$ Δ self-adjoint.

Exercise 5. Prove Proposition 4. △

Exercise 6. Complete the following exercises.

(i) Show that $\mathcal{D}(\nabla_j^- f)(k) = (1 - \omega^{-k(j)})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

(ii) Show that $\mathcal{D}(\nabla_j^+ f)(k) = (\omega^{k(j)-1)})\mathcal{D}f(k)$.

(iii) Show that

$$\mathcal{D}(\Delta f)(k) = 2\mathcal{D}f(k) \sum_{j=1}^{d} (\cos(2\pi k(j)/n) - 1).$$

△

Image denoising

We consider gradient regularized image denoising:

$$\min_{u\in L^2(\mathbb{Z}_n^d)} E_\Phi(u) = \frac{1}{2}\|u-f\|^2 + \lambda \sum_{j=1}^{d} \|\Phi(\nabla_j^- u)\|_1, \tag{3}$$

where $f \in L^2(\mathbb{Z}_n^d)$ is the noisy image and the denoised image is the minimizer u. The function $\Phi : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, convex, and satisfies $\Phi(0) = 0$. We also recall that since these conditions imply Φ is nonnegative, we have

$$\|\Phi(\nabla_j^- u)\|_1 = \sum_{k\in\mathbb{Z}_n^d} \Phi(\nabla_j^- u(k))).$$

The choice of $\Phi(t) = \frac{1}{2}t^2$ leads to Tikhonov image denoising, while $\Phi(t) = |t|$ (or the regularized $\Phi(t) = \sqrt{t^2+\varepsilon^2}$) leads to Total Variation (TV) regularization.

The Euler-Lagrange equation

As before, we compute

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E_\Phi(u+tv) &= \frac{d}{dt}\Big|_{t=0}\left[\frac{1}{2}\|u+tv-f\|^2 + \lambda\sum_{j=1}^{d}\|\Phi(\nabla_j^- u + t\nabla_j^- v)\|_1\right] \\
&= \langle u-f, v\rangle + \lambda\sum_{j=1}^{d}\langle \Phi'(\nabla_j^- u), \nabla_j^- v\rangle \\
&= \langle u-f, v\rangle - \lambda\sum_{j=1}^{d}\langle \nabla_j^+\Phi'(\nabla_j^- u), v\rangle \\
&= \langle \nabla E(u), v\rangle,
\end{aligned}
$$

integration by parts.

where

$$
\nabla E_\Phi(u) = u - f - \lambda\sum_{j=1}^{d}\nabla_j^+\Phi'(\nabla_j^- u).
$$

Tikhonov regularization

$$D_j^+ \Phi'(D_j^- u) = D_j^+ D_j^- u$$

For Tikhonov regularization, $\Phi(t) = \frac{1}{2}t^2$, $\Phi'(t) = t$ and the Euler-Lagrange equation is

$$u - \lambda \Delta u = f.$$

$$\Delta = \sum_{j=1}^{d} D_j^+ D_j^-$$

Solution via DFT:

$$D(\Delta u)(k) = 2\, Du(k) \sum_{j=1}^{d} \left(\cos(2\pi k(j)/n) - 1 \right)$$

$$\left(1 - 2\lambda \sum_{j=1}^{d} \left(\cos(2\pi k(j)/n) - 1 \right) \right) Du(k) = Df(k)$$

Set $$G_\lambda(k) = \left(1 - 2\lambda \sum_{j=1}^{d} \left(\cos(2\pi k(j)/n) - 1 \right) \right)^{-1}$$

Hence $\quad u = D^{-1}(G_\lambda \cdot Df)$.

$$= g_\lambda * f \qquad , \; g_\lambda = D^{-1} G_\lambda .$$

Total Variation (or general) Regularization

For general nonlinear Φ, we use gradient descent, which iterates

$$\begin{aligned} u_{j+1} &= u_j - dt \nabla E(u_j) \\ &= u_j - dt \left(u_j - \lambda \sum_{m=1}^{d} \nabla_m^+ \Phi'(\nabla_m^- u_j) - f \right). \end{aligned}$$

$f = 0$

Von Neumann stability analysis:

$$D_m^+ \Phi'(D_m^- u_j) \simeq C_\Phi D_m^+ D_m^- u_j \,, \quad C_\Phi = \max \Phi''$$

$$u_{j+1} = u_j - dt\left(u_j - \lambda C_\Phi \sum_{m=1}^{d} D_m^+ D_m^- u_j\right)$$

$$= (1-dt)\, u_j + (\lambda C_\Phi \Delta u_j)\, dt$$

Take DFT on both sides

$$D u_{j+1}(k) = \underbrace{\left(1 - dt + dt\, 2\lambda c_{\Phi} \sum_{m=1}^{d} \left(\cos\left(2\pi k(m)/n\right) - 1\right)\right)}_{\lambda_k} D u_j$$

Need $|\lambda_k| \le 1$ for stability

$$1 - dt \ge \lambda_k \ge 1 - dt\left(1 + 2\lambda c_{\Phi} \sum_{m=1}^{d} 2\right)$$

$$= 1 - dt\left(1 + 4d\lambda c_{\Phi}\right) \overset{\text{want}}{\ge} -1$$

So $\lambda_k \geq -1 \iff \boxed{dt \leq \frac{2}{1+4d\lambda c_{\Phi}}}$

Image denoising (.ipynb)