Mathematics of Image and Data Analysis
Math 5467

Lecture 2: Linear Algebra & Python

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Announcements

- HW1 due Friday. Submission will be via Google drive shared folder. Details to come on Wednesday.
  - Code as Google Colab Notebook (or .py or .ipynb files)
  - Math can be included in Google Colab Notebook (it supports LaTeX), or typed up in LaTeX (submit a PDF), or handwritten and scanned (high quality) with scanner or smartphone app.
- 3 choices for project 1 are up on the website. Choose 1 to complete.

  http://www-users.math.umn.edu/~jwcalder/5467S21/homework.html
  - Project descriptions are in the course notes, and Python notebooks on course website.
- Glad to see students using Piazza. I will try to answer questions nightly from now on.

Last time: Linear algebra review

- Capital letters A, B, C for matrices (entries are $A(i,j)$)
- Lower case letteers $x, y, z, x_1, x_2, x_3, x_4, \ldots$ for (column) vectors.
- $e_1, e_2, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
- Matrix multiplication: A is $m \times n$ and B is $n \times p$ then $C = AB$ is the $m \times p$ matrix with entries
$$C(i,j) = \sum_{k=1}^{n} A(i,k)B(k,j).$$
- A^T denotes the transpose of A.
- Dot product $x^T y = \sum_{i=1}^{n} x(i)y(i)$.
- Norm: $\|x\| = \sqrt{x^T x} = \sqrt{x(1)^2 + x(2)^2 + \cdots + x(n)^2}$.
- Algebra: $\|x \pm y\|^2 = \|x\|^2 \pm 2x^T y + \|y\|^2$.

Rank-one matrix

For vectors x, y of length n, the rank-one matrix $A = xy^T$ is the $n \times n$ matrix with entries

$$A(i, j) = x(i)y(j).$$

It is called rank-one since the range of A is one dimensional and spanned by the vector x. Indeed,

$$Az = xy^T z = (y^T z)x$$

for any vector z.

Exercise

Let $x_1, x_2, x_3, \ldots, x_m$ be a collection of vectors of length n. Define the $m \times n$ matrix

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{bmatrix}.$$

Show that

$$\sum_{i=1}^{m} x_i x_i^T = X^T X.$$

$$X^T X z = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{bmatrix} z$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} x_1^T z \\ x_2^T z \\ \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x_m^T z \end{bmatrix}$$

$$= x_1 x_1^T z + x_2 x_2^T z + \cdots + x_m x_m^T z$$

$$= \left(\sum_{i=1}^{m} x_i x_i^T \right) z \qquad \blacksquare$$

Today

- Projection
- Introduction to Numpy

Projection

Let $L \subset \mathbb{R}^n$ be a linear subspace spanned by the orthonormal vectors $v_1, v_2, \ldots, v_p$, where $p \leq n$. That is

$$L = \left\{ \sum_{i=1}^{p} a_i v_i \; : \; a_i \in \mathbb{R} \right\}.$$

$v_i v_i^T = \|v_i\|^2 = 1$

In this case, L is p-dimensional.

- Orthonormal means that $\|v_i\| = 1$ and $\cancel{v_i^T v_i} = 0$ for $i \neq j$.

$v_i^T v_j = 0 \quad i \neq j$

Definition 1. The *projection* of a point $x \in \mathbb{R}^n$ onto L, denoted $\text{Proj}_L x$, is the closest point in the subspace L to x. That is, $\text{Proj}_L x \in L$ satisfies

$$\|\text{Proj}_L x - x\| \leq \|y - x\| \quad \text{for all } y \in L.$$

Projection

$$\|x-y\|^2 = \|x\|^2 - 2x^Ty + \|y\|^2$$

We claim that

$$\text{Proj}_L x = \sum_{i=1}^{p} (x^T v_i) v_i$$

Proof: $\left\| x - \sum_{i=1}^{p} a_i v_i \right\|^2$

$$= \|x\|^2 - 2x^T \sum_{i=1}^{p} a_i v_i + \left\| \sum_{i=1}^{p} a_i v_i \right\|^2$$

$$\left\| \sum_{i=1}^{p} a_i v_i \right\|^2 = \left(\sum_{i=1}^{p} a_i v_i \right)^T \left(\sum_{j=1}^{p} a_j v_j \right)$$

$$= \sum_{i=1}^{p} a_i v_i^T \sum_{j=1}^{p} a_j v_j$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} a_i a_j \underbrace{v_i^T v_j}_{= \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}}$$

$$= \sum_{i=1}^{p} a_i^2$$

$$\boxed{\left\| \sum_{i=1}^{p} a_i v_i \right\|^2 = \sum_{i=1}^{p} a_i^2}$$

$$\left\| x - \sum_{i=1}^{p} a_i v_i \right\|^2 = \|x\|^2 - 2x^T \sum_{i=1}^{p} a_i v_i + \sum_{i=1}^{p} a_i^2$$

$$= \|x\|^2 + \sum_{i=1}^{p} \left(a_i^2 - 2(x^T v_i) a_i \right)$$

To minimize, differentiate in a_i

$$2a_i - 2(x^T v_i) = 0$$

$$\boxed{a_i = x^T v_i}$$

Projection

Since the v_i are orthonormal, we have by

$$\|\mathrm{Proj}_L x\|^2 = \sum_{j=1}^{p} (x^T v_i)^2 \tag{1}$$

Follows from

$$\left\| \sum_{i=1}^{p} a_i v_i \right\|^2 = \sum_{i=1}^{p} a_i^2$$

with $a_i = x^T v_i$.

$$\mathrm{proj}_L x = \sum_{i=1}^{p} (x^T v_i)\, v_i$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}}_{V} \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_p^T x \end{bmatrix}$$

$$= V \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_p^T \end{bmatrix} x = V V^T x$$

Projection

It can be useful to write the projection in matrix form. Let

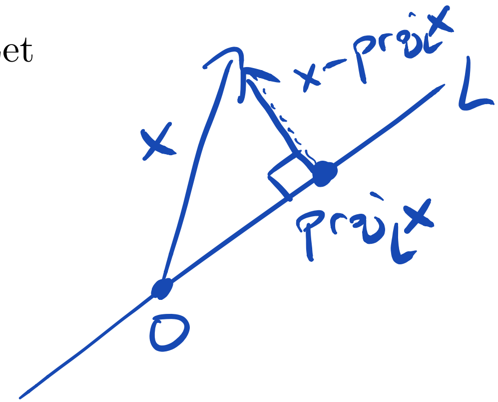
$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}.$$

Then we have

$$\mathrm{Proj}_L x = VV^T x.$$

The *residual* is

$$x - \mathrm{Proj}_L x = (I - VV^T)x.$$



Exercise 2. Show that the projection is orthogonal, that is

$$(x - \mathrm{Proj}_L x)^T v_i = 0, \quad i = 1, \dots, p.$$

Use this to show that

$$\|x\|^2 = \|\mathrm{Proj}_L x\|^2 + \|x - \mathrm{Proj}_L x\|^2.$$

$$\left(x - \sum_{j=1}^{p} (x^T v_j) v_j\right)^T v_i = x^T v_i - \sum_{j=1}^{p} (x^T v_j) v_j^T v_i$$

$$= \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

$$= x^T v_i - x^T v_i = 0$$

$$\|x - \text{proj}_L x\|^2 = \|x\|^2 - 2x^T \text{proj}_L x + \|\text{proj}_L x\|^2$$

$$= \|x\|^2 - 2x^T \sum_{i=1}^{p} (x^T v_i) v_i + \sum_{i=1}^{p} (x^T v_i)^2$$

$$= \|x\|^2 - 2\sum_{i=1}^{p} (x^T v_i)^2 + \sum_{i=1}^{p} (x^T v_i)^2$$

$$= \|x\|^2 - \|\text{proj}_L x\|^2$$

<u>Pythagorean Theorem</u>

$m \times n$ matrix $\quad A = \begin{bmatrix} A(1,1) & A(1,2) & \cdots \\ A(2,1) & A(2,2) & \cdots \\ \vdots & & \ddots \end{bmatrix}$

$$z = Ax, \qquad z(i) = \sum_{j=1}^{n} A(i,j)\, x(j)$$

$$\begin{aligned} A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax \end{aligned}$$

$$\begin{aligned} y^T : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto y^T x \end{aligned}$$

Exercises

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$$

Exercise 3. Show that

(i) $V^T V = I$.

$$V^T V = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_p^T \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}$$

(ii) $(VV^T)^2 = VV^T$

(iii) $(I - VV^T)^2 = I - VV^T$.

$$(VV^T)^2 = V\cancel{V^T V}V^T = VV^T$$

$$= \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_p \\ v_2^T v_1 & v_2^T v_2 & \cdots & v_2^T v_p \\ \vdots & & \ddots & \\ v_p^T v_1 & \cdots & \cdots & v_p^T v_p \end{bmatrix} = I \quad \triangle$$

Exercise 4. Let L be a linear subspace of $\mathbb{R}^n$.

(i) Show that $\|\mathrm{Proj}_L x\| \leq \|x\|$.

(ii) Show that $\mathrm{Proj}_L x = x$ if and only if $x \in L$.

(iii) Show that if $\mathrm{Proj}_L x = x$ for all $x \in \mathbb{R}^n$, then $L = \mathbb{R}^n$.

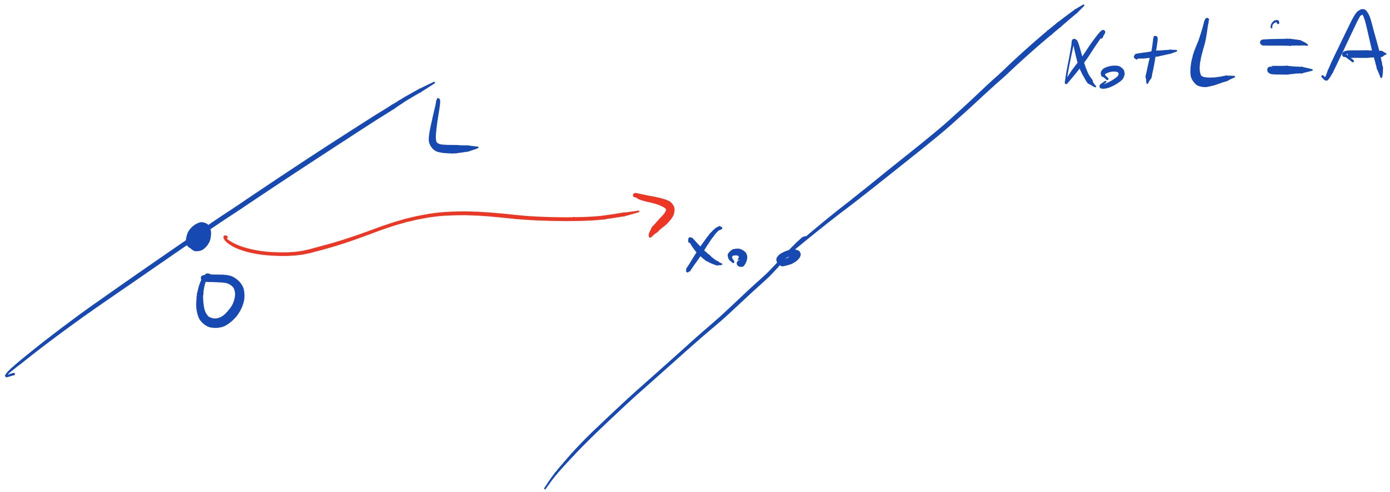
Affine projection

An affine space has the form

$$A = x_0 + L = \{x_0 + y \; : \; y \in L\}$$

where L is a linear subspace of $\mathbb{R}^n$. Projection onto an affine space is given by

$$\mathrm{Proj}_A x = x_0 + \mathrm{Proj}_L(x - x_0). \tag{2}$$



Introduction to Numpy(.ipynb)