# Mathematics of Image and Data Analysis
# Math 5467

## Lecture 9: Discrete Fourier Transform

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# Last time

- PageRank

# Today

- Discrete Fourier Transform (DFT)

# Audio compression basis

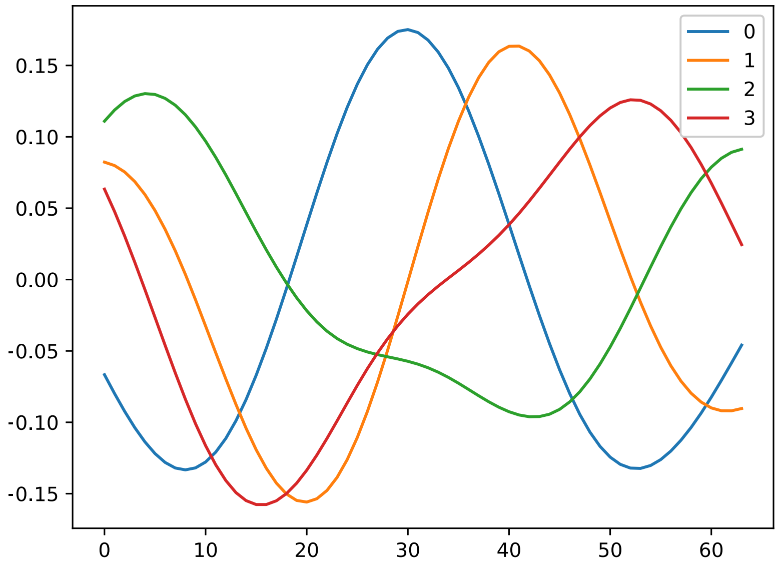



Figure 1: The first 4 principal components computed during PCA-based audio compression. Two of the basis functions strongly resemble the trigonometric functions sin and cos.

# A role for a hand-crafted change of basis

- PCA finds the best change of basis that represents your data with as few basis vectors as possible.
- In some setting PCA is too expensive (embedded environments, cell phones, digital cameras, video surveillance, etc.).
- A hand-crafted change of basis can be computed very efficiently and studied much more deeply mathematically.

# Complex numbers

We recall that a complex number has the form $z = a + ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. The set of all complex numbers is denoted $\mathbb{C}$. For a complex number $z = a + ib$, the *complex conjugate*, denoted $\overline{z}$, is given by

$$\overline{z} = a - ib.$$

The *modulus* of $z$, denoted $|z|$, is given by

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}.$$

$i^2 = -1$

$$\underbrace{(a+ib)}_{z}\underbrace{(a-ib)}_{\overline{z}} = a^2 + b^2$$

# Complex exponential and Euler's formula

The complex exponential of $z \in \mathbb{C}$ is defined by the Taylor series expansion

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

$e^{i\pi} = -1$

The Taylor series is absolutely convergent in the whole complex plane. A very important identity involving the complex exponential is Euler's identity

formula

(1) $$e^{it} = \cos t + i \sin t$$

for all real numbers $t \in \mathbb{R}$.

Short proof: $f(t) = \cos t + i \sin t$

$$f'(t) = -\sin t + i \cos t$$

$i^2 = -1$

$$= i(i \sin t + \cos t)$$

## Proof of Euler's formula

So $\boxed{f'(t) = i f(t)}$

$$\frac{d}{dt}\left(\frac{e^{it}}{f(t)}\right) = \frac{f(t)\, i e^{it} - e^{it} f'(t)}{f(t)^2}$$

$f' = if$

$$= \frac{f(t)\, i e^{it} - e^{it}\, i f(t)}{f(t)^2}$$

$$= 0$$

So $\dfrac{e^{it}}{f(t)} = C$ constant.

Set $t=0$, $e^0 = 1$

$f(0) = \cos(0) + i\sin(0)$

$= 1$

$\Longrightarrow C = 1$ $\blacksquare$

# The Discrete Fourier Transform (DFT)

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$ (i.e., integers $p, q \in \mathbb{Z}_n$ are added, subtracted, or multiplied, the result is interpreted modulo $n$).

**Example 1.** In $\mathbb{Z}_4$ we have $2 + 2 = 4 = 0 \bmod 4$. $\triangle$

$\cong \mathbb{C}^n$

Let $L^2(\mathbb{Z}_n)$ denote the vector space of functions $f : \mathbb{Z}_n \to \mathbb{C}$. We define the inner product on $L^2(\mathbb{Z}_n)$ by

$$\langle f, g \rangle = \sum_{k=0}^{n-1} f(k)\overline{g(k)}.$$

The norm of $f \in L^2(\mathbb{Z}_n)$ is defined by $\|f\| = \sqrt{\langle f, g \rangle}$.

# The Discrete Fourier Transform (DFT)

The DFT is an orthogonal change of basis in $L^2(\mathbb{Z}_n)$ that expresses a function $f :\in L^2(\mathbb{Z}_n) \to \mathbb{C}$ in terms sinusoidal basis functions of different frequencies

$$k \mapsto e^{2\pi i \sigma k} = \cos(2\pi\sigma k) + i\sin(2\pi\sigma k). \tag{2}$$

**Which frequencies?** $\sigma$ = frequency (#cycles/unit)

$T = \frac{1}{\sigma}$ = Period

Period should divide evenly into $n$

So that $e^{2\pi i \sigma k}$ is $n$-periodic.

$n = T \cdot \ell \quad , \quad \ell = 0, \ldots, n-1$

$$n = \frac{\ell}{\sigma} \quad \text{or} \quad \sigma = \frac{\ell}{n}$$

Note if $\ell = n$ then

$$e^{2\pi i k} = \underbrace{\cos(2\pi k)}_{=1} + i \underbrace{\sin(2\pi k)}_{=0}$$

same as $\ell = 0$

# DFT basis functions

We define

$$u_\ell(k) := e^{2\pi i k\ell/n}, \quad \ell = 0, 1, \ldots, n-1. \tag{3}$$

It is often useful to note that we can set $\omega = e^{2\pi i/n}$ and write

$$u_\ell(k) = \omega^{k\ell}.$$

The complex number $\omega$ is an $n^{\text{th}}$ root of unity, meaning that

$$\omega^n = e^{2\pi i} = 1.$$

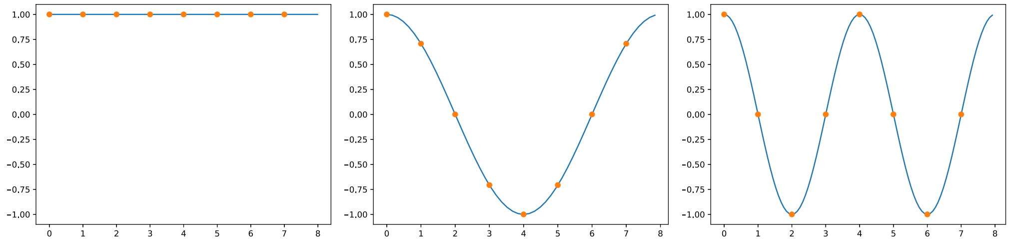
We also have $\overline{\omega} = e^{-2\pi i/n} = \omega^{-1}$.

$$\overline{e^{it}} = \cos t - i \sin t$$

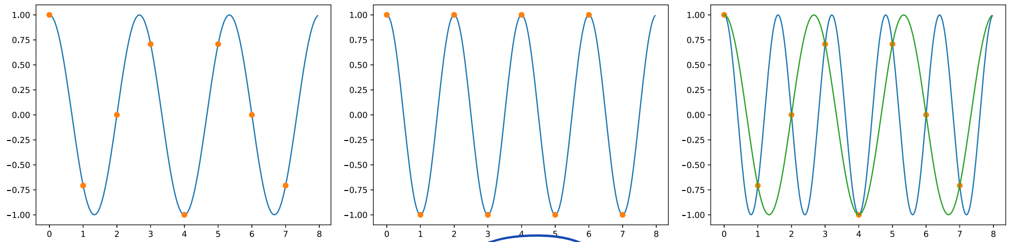
$$= \cos(-t) + i \sin(-t) = e^{-it}$$

$\cos(2\pi k\ell/n)$

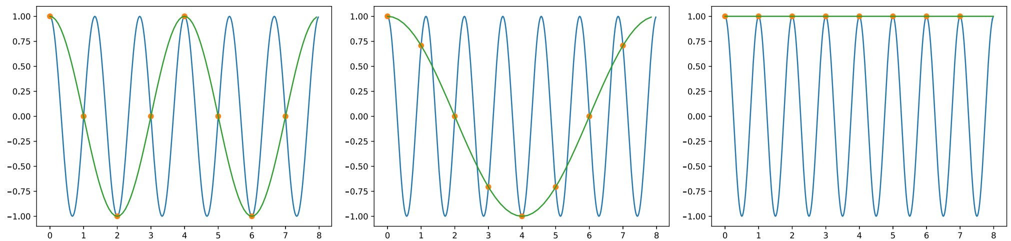
$n=8$




(a) $u_0$ $\ell=0$ (b) $u_1$ $\ell=1$ (c) $u_2$ $\ell=2$




(d) $u_3$ (e) $u_4$ (f) $u_5$




(g) $u_6$ (h) $u_7$ (i) $u_8$

# Aliasing

$$u_{n-\ell}(k) = e^{2\pi i (n-\ell)k/n}$$

$$= w^{(n-\ell)k} \qquad w = e^{2\pi i/n}$$

$$w^n = 1$$

$$= \underbrace{w^{nk}}_{=1} w^{-\ell k}$$

$$= w^{-\ell k} = \overline{w^{\ell k}} = \overline{u_\ell(k)}$$

Since $u_\ell(k) = w^{\ell k} \rightarrow u_{-\ell}(k)$

negative frequencies.

# Orthogonality

**Lemma 1.** *The functions* $u_0, u_1, \dots, u_{n-1}$ *are orthogonal. In particular*

$$\langle u_\ell, u_m \rangle = \begin{cases} n, & \text{if } \ell = m \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Proof: $\langle u_\ell, u_m \rangle = \sum_{k=0}^{n-1} u_\ell(k) \overline{u_m(k)}$

$$= \sum_{k=0}^{n-1} \omega^{\ell k} \overline{\omega^{mk}}$$

$$\omega = e^{2\pi i/n}$$

$$\overline{\omega} = \omega^{-1}$$

$$= \sum_{k=0}^{n-1} \omega^{\ell k} \omega^{-mk}$$

Hence $\langle u_\ell, u_m\rangle = \sum_{k=0}^{n-1} \omega^{(\ell-m)k}$

If $\ell = m$ then $\langle u_\ell, u_\ell\rangle = \sum_{k=0}^{n-1} 1 = n$

If $\ell \neq m$, write $r = \omega^{\ell - m}$.

Then $\langle u_\ell, u_m\rangle = \sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1} = 0$

Note $r^n = (\omega^{\ell-m})^n = (\omega^n)^{\ell-m} = 1$

Since $\omega^n = 1$. So $\langle u_\ell, u_m\rangle = 0$ ▨

Aside: Geometric series

$$S_n = \sum_{k=0}^{n-1} r^k = 1 + r + \cdots + r^{n-1}$$

$$r S_n = \sum_{k=0}^{n-1} r^{k+1} = r + r^2 + \cdots + r^n$$

$$r S_n - S_n = r^n - 1$$

$$\Longrightarrow \quad S_n = \frac{r^n - 1}{r - 1}$$

The Discrete Fourier Transform writes

$$f(k) = \sum_{\ell=0}^{n-1} c_\ell \, u_\ell(k)$$

<u>Note</u>

$$\langle f, u_m \rangle = \sum_{\ell=0}^{n-1} c_\ell \langle u_\ell, u_m \rangle = n c_m$$

So

$$c_m = \frac{1}{n} \langle f, u_m \rangle \quad \text{and}$$

$$\boxed{f(k) = \frac{1}{n} \sum_{\ell=0}^{n-1} \langle f, u_\ell \rangle \, u_\ell(k)}$$

# Definition

**Definition 2.** The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \overbrace{\sum_{k=0}^{n-1} f(k)\omega^{-k\ell}}^{\langle f, u_\ell \rangle} = \sum_{k=0}^{n-1} f(k)e^{-2\pi i k\ell/n},$$

where $\omega = e^{2\pi i/n}$.

**Note:** $\langle f, u_\ell \rangle = \sum_{k=0}^{n-1} f(k)\overline{u_\ell(k)} = \sum_{k=0}^{n-1} f(k)\,\omega^{-k\ell}$

**Proposition 3.** *If $f \in L^2(\mathbb{Z}_n)$ is real-valued (i.e., $f(k) \in \mathbb{R}$ for all $k$), then*

$$\mathcal{D}f(\ell) = \overline{\mathcal{D}f(n-\ell)}.$$

**Proof:** Recall $u_\ell(k) = \overline{u_{n-\ell}(k)}$

$$Df(\ell) = \langle f, u_\ell \rangle = \langle f, \overline{u_{n-\ell}} \rangle$$

since $f \in \mathbb{R}$

$$= \overline{\langle f, u_{n-\ell} \rangle}$$

$$= \overline{Df(n-\ell)} \qquad \blacksquare$$

# Inverse Fourier Transform

**Theorem 4** (Fourier Inversion Theorem)**.** *For any* $f \in L^2(\mathbb{Z}_n)$ *we have*

$$f(k) = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)\omega^{k\ell} = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)e^{2\pi ik\ell/n}. \tag{5}$$

**Definition 5** (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n} \sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n} \sum_{k=0}^{n-1} f(k)e^{2\pi ik\ell/n}.$$

## Proof of Thm 4:

$$\frac{1}{n}\sum_{\ell=0}^{n-1} Df(\ell)\,\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} \underbrace{\left(\sum_{j=0}^{n-1} f(j)\,\omega^{-j\ell}\right)}_{Df(\ell)}\omega^{k\ell}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} f(j) \sum_{\ell=0}^{n-1} \underset{\substack{\uparrow \\ u_k(\ell)}}{\omega^{k\ell}}\ \underset{\substack{\uparrow \\ \overline{u_j(\ell)}}}{\omega^{-j\ell}}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} f(j)\,\langle u_k, u_j\rangle = f(k)$$

∎

# Matrix version

$L^2(\mathbb{Z}_n) \approx \mathbb{C}^n$

**Remark 6.** Define the $n \times n$ complex-valued matrix with entries $W(k, \ell) = \omega^{k\ell}$, that is

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \tag{6}$$

$\omega = e^{2\pi i/n}$

Then the DFT can be expressed via matrix multiplication as $\mathcal{D}f = \overline{W}f$. The inverse DFT can be expressed as $\mathcal{D}^{-1}f = \frac{1}{n}Wf$. In both cases we treat $f$ as a vector $f \in \mathbb{C}^n$. Theorem 4 (Fourier Inversion) can be restated as saying that $W\overline{W} = nI$.

# Basic properties

**Exercise 7.** Show that the DFT enjoys the following basic shift properties.

1. Recall that $u_\ell(k) := \cancel{e^{-2\pi i k\ell/n}} = \cancel{\omega^{-k\ell}}$. Show that
   $\omega^{k\ell}$
   $e^{2\pi i k\ell/n}$
$$\mathcal{D}(f \cdot u_\ell)(k) = \mathcal{D}f(k+\ell).$$

2. Let $T_\ell : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ be the translation operator $T_\ell f(k) = f(k-\ell)$. Show that
$$\mathcal{D}(T_\ell f)(k) = e^{-2\pi i k\ell/n}\mathcal{D}f(k).$$
   [Hint: You can equivalently show that $\mathcal{D}^{-1}\left(f \cdot u_\ell\right)(k) = \mathcal{D}^{-1}f(k-\ell)$, using an argument similar to part 1. ]

$\triangle$

# Intro to DFT (.ipynb)