CSCI 2011: Predicate Logic

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Logistics

Reading: Rosen

- Now: Ch 1.4 - 1.5
- Next: Ch 1.6 - 1.8

Assignments

- A01 due tonight
- A02 posted tomorrow, due next Tue

Goals

- Finish up Propositional Logic
- Predicate Logic (First-order Logic)

What one can't do in Propositional Logic

- Propositional logic is simple and neat but has major limits
- Example: The following ideas cannot be expressed and manipulated in propositional logic
  1. *All integers that can be written $2\times$ n for some integer n are called Even.*
  2. *14 can be written as $2\times$ 7.*
  3. *14 is therefore Even.*
- Point of trouble: (1) is a "general" statement while (2) is a "specific case" of (1) which allows (3) as a conclusion
- Propositions in their current form have no notion of "general" or "specific"
- So we need a bigger, badder, logic

First Order "Logic"

Source: 800poundproductions

- The old logic could blow up one planet at a time; that failed.
- The First Order blows up whole sets of planets at a time. That's *gotta* work better, right?

Predicate Logic **Adds** the following to Propositional Logic

Predicates / Propositional Functions

Rather than propositions which are true/false, use **Predicates**, a.k.a. Propositional Functions which are true / false

Logic	Notation	Defined to be	Truthiness
Propos.	p	2 is positive.	true but rigid: 2 *always* positive
Predicate	$P(x)$	x is positive.	Don't know yet, need x
Predicate	$P(2)$	2 is positive.	True
Predicate	$P(-7)$	-7 is positive.	False

Quantifiers

New notation that makes a statement about All objects in a set or the Existence of objects in a set. Used to introduce variables

- $\forall x P(x)$: **For All** x, x is positive (**Every** x is positive)
- $\exists x E(x)$: **There Exists** x such that x is positive.

Quantifying is always over some **domain** such as integers

Aside: Functions in First-Order Logic

- Proper 1st order logic includes functions on objects such as
  - $+(x, y) : x + y$ (arithmetic sum)
  - $f(s)$: size of set s
- Allows statements about functional relationships between objects such as

$$\forall x \exists y (x = y + 1)$$

  For all x, there exists a y such x equals y+1.
- **Not covered** in our text or class: we are just dipping our toes in the water of first order logic
- Would be covered in deep dive Mathematical Logic course such as MATH 5165/5166
- Higher-order logic allows quantifiers over functions which gets even more crazy

Predicates

- Asserts true / false about a specific object
- **Define** $E(x)$: *x is even* (don't know x yet)
  - $E(2)$: true, $E(3)$: false, $E(10100101)$: false
  - $E(apple)$: wait, what?
- Predicates usually have an intended **domain**, which should be honored, the kind of object expected
- Used in combination with Logical Connectives

Symbols	English	Truthiness
$E(2) \wedge E(4)$	2 is even AND 4 is even	true
$E(2) \wedge E(7)$	2 is even AND 7 is even	false
$\neg(E(9) \vee E(7))$	NOT the case that 7 OR 9 is even	true
$E(x) \rightarrow E(y)$	IF x is even THEN y is even	unknown

Exercise: Use some Predicates

- **Define** $E(x)$: *x is even* (don't know x yet)
- **Define** $S(x, y)$: the sum of x and y is 5
- Fill in the blanks in the table below
- Truthiness can be: True / False / Unknown

Symbols	English	Truthiness
$E(2) \rightarrow E(4)$		
$S(2, 4)$		
	The sum of 4 and 1 is 5 OR 3 is even	
	x is even	
$\neg E(5)$		
	The sum of x and 1 is 5 OR 7 is NOT even	
	IF the sum of x an y is 5 THEN the sum of y and z is NOT 5	

Answers: Use some Predicates

- **Define** $E(x)$: *x is even* (don't know x yet)
- **Define** $S(x, y)$: the sum of x and y is 5
- Fill in the blanks in the table below
- Truthiness can be: True / False / Unknown

Symbols	English	Truthiness
$E(2) \rightarrow E(4)$	IF 2 is even THEN 4 is even	true
$S(2, 4)$	The sum of 2 and 4 is 5	false
$S(4, 1) \vee E(3)$	The sum of 4 and 1 is 5 OR 3 is even	true
$E(x)$	x is even	unknown
$\neg E(5)$	5 is NOT even	true
$S(x, 1) \vee \neg E(7)$	The sum of x and 1 is 5 OR 7 is NOT even	true
$S(x, y) \rightarrow \neg S(y, z)$	IF the sum of x an y is 5 THEN the sum of y and z is NOT 5	unknown

Quantifiers and Variables

- Quantifiers allow statements about all objects in a particular **universe** (mathematical set)
- Introduce a **variable** to represent object instances as in

$$\forall x(\text{some statements about x})$$

- Variables without quantifiers are unbound and considered **syntactically incorrect**
- Quantifiers have very high operator precedence and may require parentheses:

$\forall x A(x) \vee B(x)$	bad syntax, reads:	$(\forall x A(x)) \vee$ B(x)
$\forall x(A(x) \vee B(x))$	kosher	

Example

- Define $C(x)$: x is a comedian, $F(x)$: x is funny
- Assume quantifying over the universe of people
- $\forall x(C(x) \rightarrow F(x))$:
  FOR ALL people, IF person x is a comedian, THEN person x is funny.

Exercise: To and From English in Predicate Logic

- $\forall x$: universal quantifier, "For all..."
- $\exists x$: existential quantifier, "There exists..."
- Define $C(x)$: x is a comedian, $F(x)$: x is funny
- Assume quantifying over the universe of people

Symbols to English

- $\forall x(C(x) \wedge F(x))$
- $\exists x(C(x) \rightarrow F(X))$

English to Symbols

- Among people, there exists a person who is a comedian and is funny.
- For all people, if a person is not funny, that person is not a comedian.

Answers: To and From English in Predicate Logic

- $\forall x$: universal quantifier, "For all..."
- $\exists x$: existential quantifier, "There exists..."
- Define $C(x)$: x is a comedian, $F(x)$: x is funny
- Assume quantifying over the universe of people

Symbols to English

- $\forall x(C(x) \wedge F(x))$
  - All people are comedians and are funny.
- $\exists x(C(x) \rightarrow F(x))$
  - There exists a person that, if that person is a comedian, they are funny.

English to Symbols

- Among people, there exists a person who is a comedian and is funny.
  - $\exists x(C(x) \wedge F(x))$
- For all people, if a person is not funny, that person is not a comedian.
  - $\forall x(\neg F(x) \rightarrow \neg C(x))$

Logical Equivalence in Predicate Logic

- In Propositional Logic, two statements equivalent ($\equiv$) if they had the same truth values for any truth assignment; could construct a table of these
- Predicate Logic is similar: two statements are equivalent if they have the same truth values but must account for
  - *Any* Predicate definition: $P(x)$ might be *x is odd* or *x is > 0*
  - *Any* universe/set over quantifiers including a universe of infinite objects
- Result: can't use truth tables anymore
- Need a formal **proof** of equivalence

Proof that $\forall x(P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$

- Makes intuitive sense but need a more formal description to qualify as a proof
- Recall $A \equiv B$ is identical to $A \leftrightarrow B$ being a tautology
- Showing $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$ gets us there

A: $\forall x(P(x) \wedge Q(x))$ **B:** $(\forall x P(x)) \wedge (\forall x Q(x))$

$A \rightarrow B$

Assume A true: $\forall x(P(x) \wedge Q(x))$
(Implication: don't care if it's false)

- That means for any specific value v, both $P(v)$ and $Q(v)$ are true.
- Since $P(v)$ is true for all elements, have $\forall x P(x)$
- Since $Q(v)$ is true for all elements, have $\forall x Q(x)$
- Then have desired result $B : (\forall x P(x)) \wedge (\forall x Q(x))$

$B \rightarrow A$

Assume B true: $(\forall x P(x)) \wedge (\forall x Q(x))$

- Means that any specific value v, $P(v)$ is true
- AND means that any specific value v, $Q(v)$ is true
- So for any v, $P(v) \wedge Q(v)$ true
- Means that this statement is true for **all** specific values so..
- Have desired result: $A : \forall x(P(x) \wedge Q(x))$

Since $A \leftrightarrow B$ is a tautology, property holds ■

First Example of a "Proper" Proof

- Symbols helped determine the structure of the proof
- Gave some insight into the plan of attack
  - Show A and B are true/false at the same time
  - Used the fact that $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$
  - Allows showing two "smaller" things are true, very common proof structure

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases... mental power
– Alfred North Whitehead, (1911)

- Ultimately part of the proof was **not in symbols** but was based on reasoning outside of the notation

The difficulty that attends mathematical symbolism is the accompanying tendency to take the symbol as exhaustively descriptive of reality.
– Charles Nordmann (1922)

Exercise: What about the or?

- These two statements are NOT logically equivalent

  $$\textbf{A: } \forall x(P(x) \vee Q(x)) \quad \not\equiv \quad \textbf{B: } (\forall x P(x)) \vee (\forall x Q(x))$$

- To see why not, find a counter example as follows
- Pick a universe of discourse (like the integers)
- Define predicates $P()$ and $Q()$ such that one of the above is true while the other is false
- Will need to find two predicates where one or the other or both are true about all integers...

Answer: What about the or?

- These two statements are NOT logically equivalent

$$\textbf{A: } \forall x(P(x) \vee Q(x)) \quad \not\equiv \quad \textbf{B: } (\forall x P(x)) \vee (\forall x Q(x))$$

- Pick a universe of discourse (like the integers)
  - I pick the Integers, as in *For all integers, ...*
- Define predicates $P()$ and $Q()$
  - $P(x)$: x is even
  - $Q(x)$: x is odd
- **A:** $\forall x(P(x) \vee Q(x))$: *For all x where x is an integers, x is even OR x is odd.* **True**
- **B:** $(\forall x P(x)) \vee (\forall x Q(x))$: *All integers are even OR all integers are odd.* **False**

Disproved by counter example. ■

Good Practice

- Showed equivalences for the Universal Quantifier $\forall$ with Conjunction/Disjunction
- Good practice to do the same for the Existential Quantifier $\exists$:

And?	**A:** $\exists x(P(x) \wedge Q(x))$	$? \equiv ?$	**B:** $(\exists x P(x)) \wedge (\exists x Q(x))$
Or?	**C:** $\exists x(P(x) \vee Q(x))$	$? \equiv ?$	**D:** $(\exists x P(x)) \vee (\exists x Q(x))$

This is the kind of thing that might come up on a quiz...

Negating Quantified Expressions

Negation		Equivalent	English
$\neg\exists P(x)$	$\equiv$	$\forall x \neg P(x)$	For all x, P(x) is false.
$\neg\forall x P(x)$	$\equiv$	$\exists x \neg P(x)$	There exists an x for which P(x) is false.

- DeMorgan's Laws for Quantifiers in Predicate Logic
- An example of **logical equivalence** in PredLog
- Can't prove this one with truth tables
- A formal proof would do it

English and Negated Quantifiers

Leads to several ways to phrase the same idea

There is an honest politician

- $H(x)$: x is honest, quantifying over politicians.
- $\exists x H(x)$: *There is an honest politician*
- $\neg \exists x H(x)$: *There is NOT and honest politician.*
- $\forall x \neg H(x)$: *All politicians are DISHONEST.*

All Americans eat cheeseburgers.

- $C(x)$: x eats cheeseburgers honest, quantifying over Americans.
- $\forall x C(x)$: *All Americans eat cheeseburgers.*
- $\neg \forall x C(x)$: *NOT all Americans eat cheeseburgers.*
- $\exists x \neg C(x)$: *There exists an American that does NOT eat cheeseburgers.*

Nested Quantifiers

- Introduce multiple variables using multiple quantifiers
- Creates nesting of variables and **scopes**

Examples

- Domain is integers
- **Define:** $F(x, y, z) : x = y \times z$
- $\forall x \exists y \exists z\, F(x, y, z)$: *For all x, there exists a y and z such a that $x = y \times z$*
- $\exists z \forall y F(0, z, y)$: *There exists an integer z such that for all integers y, $z \times y = 0$.*
- **Define:** $G(x)$: x is even
- $\forall x(G(x) \rightarrow \exists n F(x, n, 2))$: *For all x, IF x is even, THEN there exists an n such that $x = n \times 2$*
- **Note:** also common to use math operators rather than external predicates as in: $\exists z \forall y (z \times y = 0)$

Exercise: Nested quantifier Translation

To Symbols

- The sum of two negative integers is negative.
- There exists an integer i such that for all integer x, $i \times x = x$

To English

- $\exists x \exists y((x^2 > y) \wedge (x < y))$
- $\forall x \exists y((x + y = 0))$

Answers: Nested quantifier Translation

To Symbols

The sum of two negative integers is negative.

- $\forall x \forall y ((x < 0 \land y < 0) \rightarrow (x + y < 0))$

There exists an integer i such that for all integer x, $i \times x = x$

- $\exists i \forall x (i \times x = x)$

To English

$\exists x \exists y ((x^2 > y) \land (x < y))$

- There exist integers x and y such that x^2 is greater than y AND x is less than y.

$\forall x \exists y ((x + y = 0))$

- For every integer x there exists an integer y such that $x + y = 0$.