CSCI 2011: Inference, Proofs, Logic Programming

Chris Kauffman

Last Updated:
Tue Jul 10 16:28:05 CDT 2018

Logistics

Reading: Rosen

- Now: Ch 1.6 - 1.8
- Next: 2.1 - 2.5

Assignments

- A02 due tonight
- A03 posted tomorrow, due next Tue

Quizzes

- Quiz 01 on Thu
- Practice Today

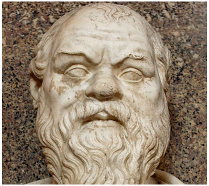
Goals

- Logic and Inference
- Proof Techniques

Formalizing Arguments

- A mathematical **proof** is a formalized argument
- Must start with some given facts
  - Axioms, **premises**, hypotheses, assumptions, knowns
- Makes use of valid logical steps to produce new facts
  - Rules of inference
- Arrives at some logical conclusion

1.	All humans are mortal	Fact
2.	Socrates is a human	Fact
3.	Socrates is mortal.	Inferred



1.	Max is a cat with no hair	Fact
2.	Not all cat's have hair	Inferred

Rules of Inference

- Formal argument uses rules to go from established facts to new facts
- Stated referred to as **rules of inference** as they allow new facts to be *inferred* from existing ones
- Equivalents in both Propositional and Predicate/First Order Logic
- Example in Prop Logic

$$\begin{array}{lll} & p & \text{Fact } (p \text{ is True}) \\ & p \rightarrow q & \text{Fact } (p \rightarrow q \text{ is True}) \\ \hline \therefore & q & \text{Inferred by } \textit{Modus Ponens} \end{array}$$

- "$\therefore$" The "therefore" symbol, used to denote new fact based on application of rule of inference
- *Modus Ponens*: Latin for "method of affirming", based on the following **tautology**

$$(p \land (p \rightarrow q)) \rightarrow q$$

Common Rules of Inference

- There are a variety of rules of inference
- All are based on a tautology
- Not all of them are strictly "needed"
- Variety can make "proofs" shorter

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Example Proof through Inference

	Definitions
p	It's sunny
q	It's cold
r	We're going swimming
s	We're going canoeing
t	We'll be home by sunset

Facts	In English
$\neg p \wedge q$	It's not sunny and it's cold.
$r \rightarrow p$	If we're swimming, it must be sunny.
$\neg r \rightarrow s$	If we're not swimming, we'll be canoeing.
$s \rightarrow t$	If we're canoeing, we'll be home by sunset.

Prove t: *We'll be home by sunset*

	Symbols	Rule
1	$\neg p \wedge q$	Fact
2	$\neg p$	Simplification of (1)
3	$r \rightarrow p$	Fact
4	$\neg r$	Modus tollens of (2) / (3)
5	$\neg r \rightarrow s$	Fact
6	s	Modus ponens of (4) / (5)
7	$s \rightarrow t$	Fact
8	t ■	Modus ponens of (6) / (7)

Exercise: Use Rules of Inference

	Definitions
p	It's sunny
q	It's cold
r	We're going swimming
s	We're going canoeing
t	We'll be home by sunset

Facts	In English
$r \to p$	If we're swimming, it must be sunny.
$\neg s \to r$	If we're not canoeing, we'll be swimming.
$s \to t$	If we're canoeing, we'll be home by sunset.
$\neg t$	We're not home by sunset.

Prove p: *It's sunny*

Note: Facts given above a somewhat different from previous example.

Answers: Use Rules of Inference

	Definitions
p	It's sunny
q	It's cold
r	We're going swimming
s	We're going canoeing
t	We'll be home by sunset

Facts	In English
$r \to p$	If we're swimming, it must be sunny.
$r \vee s$	We're swimming or canoeing.
$s \to t$	If we're canoeing, we'll be home by sunset.
$\neg t$	We're not home by sunset.

Prove p: *It's sunny*

1	$\neg t$	Fact
2	$s \to t$	Fact
3	$\neg s$	Modus tollens of (1) / (2)
4	$r \vee s$	Fact
5	r	Disjunctive Syllogism of (3) / (4)
6	$r \to p$	Fact
7	p ■	Modus ponens of (5) / (6)

A Special Note on Resolution

$$\begin{array}{lll} & p \vee q & \text{Fact} \\ & \neg p \vee r & \text{Fact} \\ \hline \therefore & q \vee r & \text{Resolution} \end{array}$$

- Along with a search algorithm, Resolution leads to a **sound and complete** proving system
    - **Sound**: no false conclusion can be derived
    - **Complete:** all true conclusions can be derived
    - **Search Algorithm:** a way to produce new facts from prior facts using resolution
- Resolution is often used in computer logic systems like **Prolog**
- Will overview Prolog later

Exercise: Another Inference Example

	Definitions	Facts	In English
p	The assignment gets posted.	$p \rightarrow q$	If the assignment gets posted,
q	Dylan finishes the assignment.		Dylan to finishes it.
r	Dylan goes to bed early.	$\neg p \rightarrow r$	If the assignment doesn't get posted,
s	Dylan wakes up feeling great.		Dylan goes to bed early.
		$r \rightarrow s$	If Dylan goes to bed early,
			Dylan wakes up feeling great.

Prove $\neg q \rightarrow s$

If Dylan doesn't finish the assignment, Dylan wakes up feeling great.

1	$p \rightarrow q$	Fact
2	$\neg q \rightarrow \neg p$	Contrapositive of (1)
3	$\neg p \rightarrow r$	Fact
4	$\neg q \rightarrow r$	Hypothetical Syllogism of (2)/(3)
5	$r \rightarrow s$	Fact
6	$\neg q \rightarrow s$ ■	Hypothetical Syllogism of (4)/(5)

- Step 2 converts to the Contrapositive
- How would one show this is valid?
- What property would need to be established about $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$?

Answer: Another Inference Example

What property would need to be established about $p \to q$ and its contrapositive $\neg q \to \neg p$?

- Show that $p \to q$ and $\neg q \to \neg p$ are **logically equivalent**
- By truth table equivalence:

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- Gives another nice proof strategy: **Prove the Contrapositive**
- Rather than directly showing $p \to q$ show $\neg q \to \neg p$
- More on this in a bit

Fallacies

fallacy (noun) a failure in reasoning that renders an argument invalid. Mistakenly made by beginners and often used by expert lawyers and politicians.

Affirming the Conclusion

Fact	IF George does every practice problem, THEN George will learn discrete math.	$p \to q$
Fact	George learned discrete math.	q
Conclusion?	George did every practice problem.	p

Statement $((p \to q) \land q) \to p$ is NOT a tautology so this is NOT a valid Rule of Inference.

Denying the Hypothesis

Fact	IF George does every practice problem, THEN George will learn discrete math.	$p \to q$
Fact	George did NOT do every practice problem.	$\neg p$
Conclusion?	George did NOT learn discrete math.	$\neg q$

Statement $((p \to q) \land \neg p) \to \neg q$ is NOT a tautology so this is NOT a valid Rule of Inference.

Exercise: Identify the Flaw

Identify a fallacy in the reasoning below.

1. If an animal is a cat, it has hair. Rolf has hair. Therefore, Rolf must be a cat.
2. If an animal lives in a burrow and digs holes, it is chipmunk. Alvin does not live in a burrow and does not dig holes. Therefore Alvin is not a chipmunk.

Answers: Identify the Flaw

Identify a fallacy in the reasoning below.

1. If an animal lives in a burrow and digs holes, it is chipmunk. Alvin does not live in a burrow and does not dig holes. Therefore Alvin is not a chipmunk.
   - Denying the Hypothesis: Chipmunks do not necessarily live in burrows and dig holes.
2. If an animal is a cat, it has hair. Rowlf is a hairy animal. Therefore, Rolf must be a cat.
   - Affirming the Conclusion: Rowlf may be some other kind of animal that has hair.

Additional Inference Rules in Predicate Logic

- Quantifiers and Predicates in 1st Order Logic have additional rules of inference
- Come in pairs that allow a quantifier to be
  - Discarded: **Instantiation**
  - Introduced: **Generalization**

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

1	$\forall x(H(x) \to M(x))$	All humans are mortal.	Fact
2	$H(Socrates) \to M(Socrates)$	If Socrates is a human, he is mortal.	Univ Inst
3	$H(Socrates)$	Socrates is a human.	Fact
4	$M(Socrates)$	Socrates is mortal.	M.P. 2/3

1	$C(Max) \wedge \neg H(Max)$	Max is a Cat and has No Hair	Fact
2	$\exists x(C(x) \wedge \neg H(x))$	There exists a hairless cat.	Exist. Gen.

Exercise: Inference with Quantifiers

	Definitions	Facts	English
$C(x)$	x is in this Class.	$\exists x(C(x) \wedge \neg B(x))$	??
$B(x)$	x read the Book.	$\forall x(C(x) \rightarrow P(x))$	??
$P(x)$	x passed the Quiz.		

Show: $\exists x(P(x) \wedge \neg B(x))$

There someone who passed the quiz did not read the book.

Answer: Inference with Quantifiers

	Definitions	Facts
$C(x)$	x is in this Class.	$\exists x(C(x) \wedge \neg B(x))$
$B(x)$	x read the Book.	*There is someone in the class who*
$P(x)$	x passed the Quiz.	*didn't read the book.*
		$\forall x(C(x) \rightarrow P(x))$
		Everyone in the class passed the quiz.

Show $\exists x(P(x) \wedge \neg B(x))$:

There someone who passed the quiz did not read the book.

1	$\exists x(C(x) \wedge \neg B(x))$	Fact
2	$C(a) \wedge \neg B(a)$	Exist. Inst. (1)
3	$C(a)$	Simplification (2)
4	$\forall x(C(x) \rightarrow P(x))$	Fact
5	$(C(a) \rightarrow P(a))$	Univ. Inst. (4)
6	$P(a)$	Modus Ponens of (3/4)
7	$\neg B(a)$	Simplification (2)
8	$P(a) \wedge \neg B(a)$	Conjunction of (6/7)
9	$\exists x(P(x) \wedge \neg B(x))$ ■	Exist. Gen. (8)

Logic Programming: Prolog

- Prolog is a programming language/system developed in the 1970's
- General Purpose but focused on logic programming specifically
  - Stores **Facts** about the world (true Predicates)
  - Facts may be **Relations** between items (multi-arg Predicate)
  - Allows **Inference** about those facts via a search algorithm
  - Search algorithm is based on **resolution** refutation
- Premise is to write program as a search problem
- Can solve problem by searching for a "proof" that the facts/relations resolve to some conclusion

A Brief Prolog Example

Input file

```
> cat dog.pl
dog(rex) :- true.    % states fact: rex is a dog
dog(fly).            % shorthand for a fact
pig(babe).           % different predicate

%% statement of true two-item relations
barksat(rex,fly).   barksat(rex,babe). barksat(rex,sheep).
barksat(fly,sheep). barksat(fly,babe).
```

Interactive Session

```
> swipl
Welcome to SWI-Prolog
1 ?- consult('dog.pl'). % load facts
 true.
2 ?- dog(rex).        % is rex a dog
 true.
3 ?- dog(fly).        % is fly a dog
 true.
4 ?- dog(babe).       % is babe NOT a dog
 false.
```

```
5 ?- dog(X).  % find a dog
 X = rex.
4 ?- dog(X).  % find all dogs
 X = rex ;
 X = fly.
6 ?- barksat(rex,Z). % rex
 Z = fly ;           % barks
 Z = babe ;          % at...
 Z = sheep.
```

Logic Programming these Days

- Prolog is not particular popular as a general coding language
- However, has had influence in many areas
  - Database query languages like SQL
  - Natural language processing (English to Computer) including use in Watson, IBM's Jeopardy AI
  - Some branches of Artificial Intelligence with discrete search problems
- Still many usable implementations out there today and for some kinds of problems makes life much easier

Proof Assistants

- Related are **Proof Assistants**: programs that are meant to help humans keep track of mathematical facts as they construct proofs
- Have led to influential inventions
- The Coq Proof Assistant is old and solid example of this
- The ML programming language was developed internally for use in a theorem prover but now has generally usable

Proofs, Theorems, Lemmas

Vocabulary

- A **proof** is a formal argument that some statement is true. It **does not** need to be in symbols and use rules of inference explicitly but it should be readily apparent that this can be done.
- A **theorem** is a fancy name for a true statement. Usually it is somehow *significant* potentially because it requires a lot of work to prove.
- A **lemma** is a fancy name for a little theorem. Often theorems are proved by first proving one or more lemmas then using their validity to show the theorem is true.

Since proving theorems is a search process, get acquainted with the common structures and strategies associated with the proofs.

- The first few times is very difficult
- After gaining experience, re-use structure of related proofs

Proof Strategies and Tactics in the Large

The following tactics show up in proofs often enough to have specific names associated.

Direct Proof
: Show directly that some properties lead to the given conclusion; e.g. $p \to q$ and p leads to q

Proof By Contraposition
: Rather than proving $p \to q$ directly, prove the contrapositive: $\neg q \to \neg p$ and $\neg q$ leads to $\neg p$

Proof By Contradiction
: Prove that if $p \to q$ and p did NOT lead to q, then something false is provable.

Proof by Cases/Exhaustion
: Break the proof into discrete pieces and show properties true/false in each case.

Construction
: Prove something exists by creating it or showing its structure

Counterexamples
: Show some property doesn't hold by constructing something that disobeys the property.

Direct Proofs

- A good first strategy to attempt when starting
- Go straight from properties to conclusion

Example: Show If n is odd then n^2 is odd

$$\forall x(Odd(n) \rightarrow Odd(n^2))$$

- Odd defined as
  - $n = 2a + 1$ for some a and
  - $n^2 = 2b + 1$ for some b
- Do some arithmetic on n^2 to see what comes out

$$\begin{aligned} n^2 &= (2a+1)^2 \\ &= 4a^2 + 4a + 1 \\ &= 2(2a^2 + 2a) + 1 \\ &= 2b + 1 \qquad \text{with } b = 2a^2 + 2a \end{aligned}$$

Since n^2 can be written in the form of an odd number, it is odd. ■

Exercise: n^2 is Even

- Use a direct proof to show that if n is an Even integer, n^2 is also Even.
- Fact: Even integers can be written $n = 2a$ for some integer a.

Answers: n^2 is Even

- Use a direct proof to show that if n is an Even integer, n^2 is also Even.
- Fact: Even integers can be written $n = 2a$ for some integer a.

$$\begin{array}{rcll} n^2 & = & (2a)^2 & \\ & = & 4a^2 & \\ & = & 2(2a^2) & \\ & & 2b & \text{with } b = 2a^2 \end{array}$$

Since n^2 can be written in the form of an even number, it is even.
$\blacksquare$

Exercise: Proof by Contraposition

Prove if $3n + 2$ is odd, then n is odd

p	$3n + 2$ is odd
q	n is odd
$p \to q$	IF $3n + 2$ is odd, THEN n is odd
$\neg q \to \neg p$	IF n is NOT odd, THEN $3n + 2$ is NOT odd
	IF n is EVEN, THEN $3n + 2$ is EVEN
	n even means $n = 2k$ for some k
	$3(2k) + 2 = 6k + 2 = 2(3k + 1)$ which is EVEN
	Assuming $\neg q$, have shown $\neg p$. Completes proof by Contraposition.■

Prove IF $n = ab$ THEN $a \leq \sqrt{n}$ OR $b \leq \sqrt{n}$

- Assume a, b are positive integers
- Use a **proof by contraposition**

Answers: Proof by Contraposition

Prove IF $n = ab$ THEN $a \leq \sqrt{n}$ OR $b \leq \sqrt{n}$

- Assume a, b are positive integers
- Use a **proof by contraposition**

p	$n = ab$
$q \vee r$	$a \leq \sqrt{n}$ OR $b \leq \sqrt{n}$
$p \rightarrow (q \vee r)$	IF $n = ab$ THEN $a \leq \sqrt{n}$ OR $b \leq \sqrt{n}$
$\neg(q \vee r) \rightarrow \neg p$	Contrapositive
$(\neg q \wedge \neg r) \rightarrow \neg p$	De Morgan's Law on left side of implication
In English:	IF $a > \sqrt{n}$ AND $b > \sqrt{n}$ THEN $n \neq ab$
	Multiply: $ab > n$ so clearly $ab \neq n$
	Assuming $\neg q$, have shown $\neg p$.
	Completes proof by Contraposition.■

Proof by Contradiction

- To show that p is true, assume $\neg p$ and show that some contradictory result arises result arises.
- Common tactic which indirectly shows that p is true
- Usually involves specific tricks to the domain as Direct Proofs and Proof by Contraposition did

Example: Proof by Contradiction

Definition: *rational number* is one that can be written as the quotient of two integers **without common factors**; i.e. n rational means $n = a/b$. Some important numbers are *irrational* in that they are not the quotient of two integers.

Show $\sqrt{2}$ is Irrational by Contradiction

1. Assume $\sqrt{2}$ is NOT irrational; so $\sqrt{2}$ IS rational
2. Thus $\sqrt{2} = a/b$
3. Squaring (2) gives $2 = a^2/b^2$
4. Re-arrange (3) to get $2b^2 = a^2$
5. Equality in (4) shows a^2 is even
6. By an earlier proof and (5), a must be even
7. a even means $a = 2c$ for some c
8. Substitute $a = 2c$ into (4) to get $b^2 = (2c)^2$
9. Rearrange (8) to get $b^2 = 2(2c^2)$
10. (9) means b^2 is even so b is also even
11. So a and b are even so they have a common factor
12. Contradicts the **no common factors** property of rational numbers
13. Assumption that $\sqrt{2}$ is rational
14. Therefore $\sqrt{2}$ must be irrational. $\blacksquare$

Exercise: Proof by Contradiction

Show that IF $3n + 2$ is Odd THEN n is Odd by **Contradiction**

- Structure your argument to prove $p \rightarrow q$
- Start by assuming the opposite, $(\neg(p \rightarrow q)) \equiv (\neg(\neg p \vee q)) \equiv (p \wedge \neg q)$
- Show this leads to a contradiction

Answers: Proof by Contradiction

Show that IF $3n + 2$ is Odd THEN n is Odd by **Contradiction**

1. Assume is $Odd(3n + 2) \wedge \neg Odd(n)$
2. (1) means $Odd(3n + 2) \wedge Even(n)$
3. By (2), have that $n = 2k$ for some k
4. Substitute (3) into LHS of (1): $Odd(3(2k) + 2)$
5. Rearrange to get $Odd(2(3k + 1))$
6. Contradiction: (5) cannot be Odd with factor of 2
7. Original assumption leads to a contradiction
8. Must be that $Odd(3n + 2) \rightarrow Odd(n)$ ■

A good exercise would be to do this as a **Direct Proof** as well.

Exercise: Debugging Proofs

Consider the following and identify a subtle but significant flaw.

Show that if n^2 is Even, n is Even

Proved Directly.

1. Assume n^2 is Even
2. By (1), $n^2 = 2a$ for some a
3. Let $n = 2b$ for some integer b
4. (3) shows n can be written in the form of an Even number
5. This shows $Even(n^2) \rightarrow Even(n)$. $\square$

Exercise: Debugging Proofs

Flawed: Show that if n^2 is Even, n is Even

1. Assume n^2 is Even
2. By (1), $n^2 = 2a$ for some a
3. Let $n = 2b$ for some integer b : *Why can I do that again?*
4. (3) shows n can be written in the form of an Even number
5. This shows $Even(n^2) \rightarrow Even(n)$. $\square$

Begging the Question / Circular Arguments

- Step (3) assumes that n can be written as an Even number
- Evenness of n is *not* a Fact, it's the target of the proof
- Assuming the conclusion is true makes proofs pointless
- **Begging the Question:** Any form of argument where the conclusion is assumed in one of the premises (given facts).[1]

[1] Logically Falacious (dot com): Begging the Question

Proof by Cases

- Break the space of possibilities into distinct categories
- Apply separate arguments to each category as needed

Example: Prove the Triangle Inequality

Show for all real numbers x, y the inequality $|x| + |y| \geq |x + y|$ holds where $|a|$ denotes the absolute value of a.
Case 1: $x \geq 0, y \geq 0$. This leads to the inequality $x + y \geq x + y$ which is equal and therefore holds.
Case 2: $x < 0, y < 0$. This again leads to the inequality $x + y \geq x + y$ which is equal and therefore holds.
Case 3: $x \geq 0, y < 0$. The inequality becomes $x + |y| \geq x - |y|$ which holds.
Case 4: $x < 0, y \geq 0$. The inequality becomes $|x| + y \geq y - |x|$ which holds.

Exercise: Prove using Cases

Show that the following equality holds for all real numbers

$$\max(x, y) + \min(x, y) = x + y$$

where max() and min() become the maximum or minimum of their arguments.
Use a Proof by Cases.

Answers: Prove using Cases

Show that the following equality holds for all real numbers x, y

$$\max(x, y) + \min(x, y) = x + y$$

where max() and min() become the maximum or minimum of their arguments.
Cases are based on the relative magnitudes of x, y
Case 1: $x = y$. In this case the equality becomes
$x + y = x + x = y + y = x + y$ which holds.
Case 2: $x > y$. In this case the equality becomes $x + y = x + y$ which holds.
Case 3: $x < y$. In this case the equality becomes $y + x = x + y$ which holds.

"Without Loss of Generality"

- A technical phrase for "here's a small additional assumption that could be discarded but it would make the proof longer"
- Often employed when there would be symmetric cases
  - Case 1: $x > y$, Case 2: $x < y$

  where the **same reasoning** would apply
- Introduces a new fact like *Assume* $x > y$
- Use with care as it is easy to actually lose generality if the same reasoning does not apply in both cases

Exercise: Combining Several Proof Techniques 1

- Show that if sum and product of integers x, y are Even, then both x and y Even

$$\forall x \forall y ((Even(x \cdot y) \wedge Even(x + y)) \rightarrow (Even(x) \wedge Even(y)))$$

- Proof by contraposition: if at least one of x, y is Odd, then sum or product is Odd
- Give a symbolic version of the contrapositive
- Do not need to change the quantifiers, just the implication from $p \rightarrow q$ to $\neg q \rightarrow \neg p$

Answers: Combining Several Proof Techniques 1

- Show that if sum and product of integers x, y are Even, then both x and y Even

$$\forall x \forall y((Even(x \cdot y) \wedge Even(x + y)) \rightarrow (Even(x) \wedge Even(y)))$$

- Proof by contraposition: if at least one of x, y is Odd, then sum or product is Odd
- Give a symbolic version of the contrapositive

$$\forall x \forall y((Odd(x) \vee Odd(y)) \rightarrow (Odd(x \cdot y) \vee Odd(x + y)))$$

- **Without loss of Generality assume that x is Odd**
  - Ignore y is Odd as reasoning would be symmetric
- Show $Odd(x \cdot y) \vee Odd(x + y)$ to complete contraposition

Exercise: Combining Several Proof Techniques 2

- **Without loss of Generality assume that x is Odd**
- Show $Odd(x \cdot y) \vee Odd(x + y)$ to complete contraposition
- Divide into 2 cases based on properties of y
- What are these two cases?
- What facts would one start with in the cases?

Exercise: Combining Several Proof Techniques 3

- **Without loss of Generality assume that x is Odd**
- Show $Odd(x \cdot y) \vee Odd(x + y)$ to complete contraposition
- Divide into 2 cases based on properties of y

Case 1: y is Even

1	$Odd(x)$ so $x = 2a + 1$ for some a	Fact
2	$Even(y)$ so $y = 2b$ for some b	Fact

Fill in the proof that $Odd(x \cdot y) \vee Odd(x + y)$

Case 2: y is Odd

1	$Odd(x)$ so $x = 2a + 1$ for some a	Fact
2	$Odd(y)$ so $y = 2b + 1$ for some b	Fact

Fill in the proof that $Odd(x \cdot y) \vee Odd(x + y)$

Answers: Combining Several Proof Techniques 3

Case 1: y is Even

1	$Odd(x)$ so $x = 2a + 1$ for some a	Fact
2	$Even(y)$ so $y = 2b$ for some b	Fact
3	$x + y = 2a + 1 + 2b$	Substitute 1/2 into sum
4	$x + y = 2(a + b) + 1$	Rearrange 3
5	$x + y = 2c + 1, c = a + b$ so $Odd(x + y)$	Rearrange 4
6	$Odd(x + y) \vee Odd(x \cdot y)$	Addition (Rule of Inference)

Case 2: y is Odd

1	$Odd(x)$ so $x = 2a + 1$ for some a	Fact
2	$Odd(y)$ so $y = 2b + 1$ for some b	Fact
3	$x \cdot y = (2a + 1) \cdot (2b + 1)$	Substitute 1/2 into product
4	$x \cdot y = 4ab + 2a + 2b + 1$	Rearrange 3
5	$x \cdot y = 2c + 1, c = 2ab + a + b$ so $Odd(x \cdot y)$	Rearrange 4
6	$Odd(x + y) \vee Odd(x \cdot y)$	Addition (Rule of Inference)

- In both cases, $Odd(x) \rightarrow Odd(x + y) \vee Odd(x \cdot y)$
- Completes the proof of contrapositive, original property holds:

$$\forall x \forall y((Even(x \cdot y) \wedge Even(x + y)) \rightarrow (Even(x) \wedge Even(y)))$$

More to Come

Existence Proofs

- Show that something exists out there
- Can be done **Constructively** (here it is!)
- Or **Nonconstructively**, often by contradiction (if it didn't exist, the something we know is true would appear false)
- **Uniqueness** proofs also come up: this thing exists and there is only one of it

Proof by Induction

- Important enough to get its own chapter
- Often comes up in CS due to our use of recursive algorithms and recursive data structures like trees

Conjectures and Open Problems

- A **Conjecture** is a proposed truth often without a proof
- Sometimes may be asked to "form a conjecture" then prove it
- Some **open problems** are a conjecture that hasn't yet been proved

Example: Collatz Conjecture

	Algorithm: integer input $x > 0$	Pseudo Code
1	If x is even, halve it	$x \leftarrow x/2$
2	If x is odd, triple and add 1	$x \leftarrow 3x + 1$
3	Repeat steps 1/2 until $x = 1$	$while(x > 1)$

Conjecture: The sequence of numbers produced by this algorithm **always converges to 1.**

> *Mathematics is not yet ripe enough for such questions.*
> *– Richard K. Guy (1983a) Don't try to solve these problems!, Amer. Math. Monthly 90 (1983), 35–41.*