CSCI 2011: Basic Discrete Structures
Sets, Functions, Sequences, Sums, Matrices

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Logistics

Reading: Rosen

- Now: 2.1 - 2.5

Assignments

- A03: post later today
- Due Tuesday

Quizzes

- Quiz 01 today

Goals

- Finish up proof basics
- Sets now, others later

Logistics

Reading: Rosen

- Now: 2.1 - 2.5

Assignments

- A03: post later today
- Due Tuesday

Quizzes

- Quiz 02 thu

Goals

Sets, Functions, Seqs, Sums

Sets

- A collection of **unique** objects (no redundant elements)
- In Mathematics, small sets often written with curlies:

$$A = \{1, 3, 5, 7, 11\}, B = \{apple, orange, banana\}, C = \{1, 2, 3, ..., 99\}$$

- Standard numeric sets denoted with well-established symbols (though context can vary what symbol is used)

Sym	Alt	Name	Values
$\mathbf{N}$	$\mathcal{N}$	Natural Numbers	$\{0, 1, 2, 3, ...\}$
$\mathbf{Z}$	$\mathcal{Z}$	Integers	$\{0, 1, -1, 2, -2, ...\}$
$\mathbf{Q}$	$\mathcal{Q}$	Rational Numbers	$\{0, 1, 2, 1/2, 3, 1/3, 2/3, ...\}$
$\mathbf{R}$	$\mathcal{R}$	Real Numbers	$\{0, 0.1, 0.25, 2.67, \pi, e, \sqrt{2}, ...\}$
$\mathbf{C}$	$\mathcal{C}$	Complex Numbers	Reals plus stuff with $i = \sqrt{-1}$

Set builder notation frequently employed

- $A = \{x \in S | conditions\}$: elements of S with *conditions* true
- $\mathbf{Z}^+ = \{x \in \mathbf{Z} | x > 0\}$: positive integers (standard notation)
- $T = \{x \in \mathbf{R} | 0 \leq x \leq 1\}$: reals between 0 and 1

Sizes of Sets, Nesting

- **Cardinality** describes the size of a set, written $|S|$
- Examples:
  - $E = \{\}, \ |E| = 0$ (the empty set / null set, written $\emptyset$ or $\varnothing$)
  - $A = \{1, 3, 5, 7, 11\}, \ |A| = 5$
  - $B = \{x \in \mathbf{N} | x < 100, x \text{ is even}\}, \ |B| = 50$
- Sets can **nest**, a "set of sets"
  - $D = \{\{1, 2, 3\}, \{3, 4\}\}, \ |D| = 2$ (contains two sets)
  - $F = \{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}, \ |F| = 4$ (contains four sets)
  - $G = \{\{\}, \{\{\}\}, \{\{\{\}\}\}\}, \ |G| = 3$ (contains three sets)
  - The style of G looks really weird but is frequently used when using set theory to establish number theory. Math folks *love* that stuff, get on well with `(lisp '(programmers ()))`
- **Finite** sets have a cardinality in the natural numbers
- **Infinite** sets like $\mathbf{N}$ and $\mathbf{R}$ have infinite cardinality **but** these infinities *come in different sizes*

Membership of Individual Elements

- Statements about membership of an element in a set often use the "in" symbol: $\in$
- This is a Predicate and has a true/false value

$$S = \{1, 3, 5, 7, 11\},\ 7 \in S : \textit{true},\ 12 \in S : \textit{false}$$

- Keep in mind that membership is on individual elements

$$S = \{1, 3, 5, 7, 11\},\ S \in \mathbf{N} : \textit{false}$$

The set of Natural Numbers contains numbers, not sets, so despite all elements of S being in $\mathbf{N}$, $S \notin \mathbf{N}$

Subsets

- A is a **subset** of B if every element in A is also in B
- Notation: $A \subseteq B \equiv \forall x((x \in A) \to (x \in B))$
- Two sets are equal if they are subsets of each other
- Notation: $A = B \equiv \forall x((x \in A) \leftrightarrow (x \in B))$
- A is a **proper subset** of B if it is a subset but not equal
- Notation:
  $A \subset B \equiv \forall x((x \in A) \to (x \in B)) \land \exists x(x \in B \land x \notin A)$

Exercise: Set Operations

- $A - B = \{x | x \in A \wedge x \notin B\}$: Difference
- $A \cup B = \{x | x \in A \vee x \in B\}$: Union
- $A \cap B = \{x | x \in A \wedge x \in B\}$: Intersection
- $A \times B = \{(a, b) | a \in A, b \in B\}$: Cartesian Product,
  - Note the result of the Cartesian Product is a set of pairs
- $\mathcal{P}(A) = \{S | S \subseteq A\}$: Power Set of A
  - Note the result is a set of sets, all subsets of A

Do Set Ops

With $A = \{1, 3, 5\}$, $B = \{3, 9\}$, determine the results of the following operations

- $A - B =$
- $A \cup B =$
- $A \cap B =$
- $A \times B =$
- $\mathcal{P}(A) =$

Answers: Set Operations

- $A - B = \{x | x \in A \wedge x \notin B\}$: Difference
- $A \cup B = \{x | x \in A \vee x \in B\}$: Union
- $A \cap B = \{x | x \in A \wedge x \in B\}$: Intersection
- $A \times B = \{(a, b) | a \in A, b \in B\}$: Cartesian Product,
  - Note the result of the Cartesian Product is a set of pairs
- $\mathcal{P}(A) = \{S | S \subseteq A\}$: Power Set of A
  - Note the result is a set of sets, all subsets of A

Do Set Ops

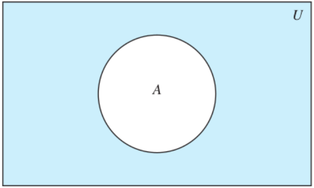
With $A = \{1, 3, 5\}$, $B = \{3, 9\}$, determine the results of the following operations

- $A - B = \{1, 5\}$
- $A \cup B = \{1, 3, 5, 9\}$
- $A \cap B = \{3\}$
- $A \times B = \{(1, 3), (1, 9), (3, 3), (3, 9), (5, 3), (5, 9)\}$
- $\mathcal{P}(A) = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$

Exercise: Complement of a Set

The **complement** of a set is the set of all elements **not** in the set with respect to some larger Universe (also a set)

- Notation involves "overlines"
- $\overline{A} = \{x | x \in U \wedge x \notin A\}$
- $\overline{A \cup B} = \{x | x \in U \wedge x \notin A \cup B\}$



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A.

Exercise

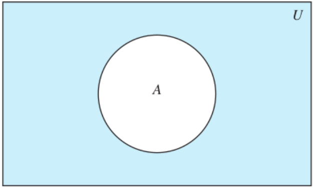
- With $U = \mathbf{N}$
  $E = \{x \in \mathbf{N} | x \text{ is even}\}$, $\overline{E} =??$
- With $U = \mathbf{R}$, $\overline{\mathbf{Q}} =??$
- With $U = \mathbf{C}$, $\overline{\mathbf{R}} =??$

Sym	Name
$\mathbf{N}$	Natural Numbers
$\mathbf{Z}$	Integers
$\mathbf{Q}$	Rational Numbers
$\mathbf{R}$	Real Numbers
$\mathbf{C}$	Complex Numbers

Answers: Complement of a Set

Exercise

- With $U = \mathbf{N}$
  $E = \{x \in \mathbf{N} | x \text{ is even}\}$
  $\overline{E} =$ Odd numbers
- With $U = \mathbf{R}$
  $\overline{\mathbf{Q}} =$ Irrational Numbers
- With $U = \mathbf{C}$
  $\overline{\mathbf{R}} =$ Imaginary numbers, $\mathbf{I}$



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A.

Sym	Name
N	Natural Numbers
Z	Integers
Q	Rational Numbers
R	Real Numbers
C	Complex Numbers

Set Identities

- Like many other mathematical objects (numbers, propositions, etc.), sets can be manipulated through rules
- The following table shows the most common identities for sets
- Examine and then describe anything familiar that appears on this table

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Showing Equivalence

- Common flavor of proof with sets: show two sets are equivalent
- Often done via definitions in set builder notation

Example: Show $\overline{A \cap B} = \overline{A} \cup \overline{B}$

De Morgan's Law for Sets: proof below uses De Morgan's Law for logic

1	$\overline{A \cap B}$	$=$	$\{x \mid x \notin A \cap B\}$	Def of Complement
2		$=$	$\{x \mid \neg(x \in A \cap B)\}$	Def of "Not In"
3		$=$	$\{x \mid \neg(x \in A \wedge x \in B)\}$	Def of Intersection
4		$=$	$\{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	**De Morgan's Law**
5		$=$	$\{x \mid (x \notin A) \vee (x \notin B)\}$	Def of $\notin$
6		$=$	$\{x \mid (x \in \overline{A}) \vee (x \in \overline{B})\}$	Def of Complement
7		$=$	$\{x \mid x \in (\overline{A} \cup \overline{B})\}$	Def of Union
8		$=$	$\overline{A} \cup \overline{B}$	Simplify $\blacksquare$

Exercise: Set Equivalences

Show that IF $A \cup B = A$ THEN $B \subseteq A$

- Use set builder notation starting
- Start with known facts
- Derive definition of subset

$$1 \quad A \cup B = A \quad \text{Fact}$$

Exercise: Set Equivalences

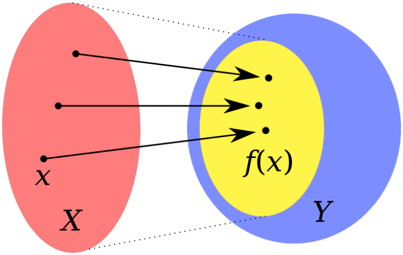
Show that IF $A \cup B = A$ THEN $B \subset A$

- Use set builder notation starting
- Start with known facts
- Derive definition of subset

1	$A \cup B = A$	Fact
2	$A \cup B = \{x \mid x \in A \vee x \in B\}$	Def of Union
3	$A = \{x \mid x \in A\}$	Set Builder Notation
4	$\{x \mid x \in A\} = \{x \mid x \in A \vee x \in B\}$	Equiv of 2/3 by 1
5	$\forall x(x \in B \rightarrow x \in A)$	Meaning of 4
6	$B \subset A$	Def of Subset from 4 ■

Functions

- Most of you are familiar with **functions**
- A simple way to define a function is a **mapping** from one set to another
  - **Domain** is the set of "inputs"
  - **Codomain** is the set of "outputs"
- Programming Examples
  - `int length(String s)`: Domain of Strings, Codomain positive Integers
  - `double halve(int s)`: Domain Integers, Codomain reals (sort of)



$f : X \rightarrow Y$

Functions Map Between Sets

- Function as a set of pairs:

  ```
  strlen = {("hi",2), ("bye",3"), ("yo",2),
            ("hello",5), ("goodbye",7), ...}
  ```
- Important: One input, One Output
  Not a function: `strlen = {("hi",2), ("hi",5),...}`
- Note that the Codomain may be a subset of a larger set:
  - `strlen()` Codomain is Positive Integers which is a subset of Integers
  - Universe of Codomain often referred to as the **Range** of the function

Exercise: Special types of functions

One-to-one
: A function where each element of the Codomain (output) is mapped from a **unique** element of Domain (input)

Onto
: A function which has **all** elements of its Codomain (or Range) mapped from an element of its Domain

Invertible
: A function that is **both** One-to-one and Onto

Are these One-to-One, Onto, Invertible?

- `boolean not(boolean b)`: flip Boolean value
- `boolean is_zero(int i)`: true for zero, false otherwise
- `int strlen(String s)`: length of input string
- `int increment(int i)`: increment integer and return result
- `double halve(int s)`: halve input integer, give real value

Answers: Special types of functions

One-to-one: A function where each element of the Codomain (output) is mapped from a **unique** element of Domain (input)

Onto: A function which has **all** elements of its Codomain (or Range) mapped from an element of its Domain

Invertible: A function that is **both** One-to-one and Onto

Are these One-to-One, Onto, Invertible?

- `boolean not(boolean b)`: 1-to-1, Onto, Invertible
- `boolean is_zero(int i)`: Onto
- `int strlen(String s)`: Onto for pos ints, Not for all ints
- `int increment(int i)`: 1-to-1, Onto, Invertible
- `double halve(int s)`: 1-to-1, Onto*, Invertible

*: If the Codomain is Integers plus Halves, not Onto for Reals

Floor and Ceiling Functions

Several numerical functions come up in computing worth noting

$\lfloor x \rfloor$ or `floor(x)`

- Nearest integer less than or equal to real number `x`
- Sometimes referred to as **truncation**

$\lceil x \rceil$ or `ceil(x)`

- Nearest integer greater than or equal to real number `x`
- Notice for both **behavior at negative values** in graph

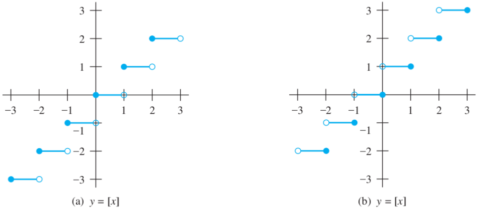


FIGURE 10 **Graphs of the (a) Floor and (b) Ceiling Functions.**

Exercise: Prove or Disprove Ceil Property

Prove or Disprove the following relationship for real numbers x, y

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$$

Answers: Prove or Disprove Ceil Property

Prove or Disprove the following relationship for real numbers x, y

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$$

False: Searching for some counter examples yields the following

- $x = 1/2, \ y = 1/2$
- $\lceil x + y \rceil = \lceil 1/2 + 1/2 \rceil = \lceil 1 \rceil = 1$
- $\lceil x \rceil + \lceil y \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 1 = 2$

Sequences

- Standard Calc I/II topic: sequence of successive numbers
- **Convention:** start index variable n at 0 unless otherwise indicated
- Come in a variety of flavors such as...
- **Arithmetic:** $a, a+d, a+2d, \cdots, a+nd, \cdots$
  - $a_n = 3 + 2n = 3, 5, 7, 9, \cdots$
  - $b_n = 0 + 5n = 0, 5, 10, 15, \cdots$
- **Geometric:** $a, ar^1, ar^2, ar^3, \cdots$
  - $c_n = 1^{-n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$
  - $d_n = 3 \cdot 2^n = 3, 6, 12, 24, 48, \cdots$

Summations and Notation

- In CS, often interested in sequences as they represent **operation counts** in iterations of an algorithm
- Makes sense to sum these as it indicates total operations performed
- Summation notation:

"Sequence"		Notation	Is
$1+2+3+4+5$	$=$	$\sum_{i=1}^{5} i$	15
$2+4+6+8+10$	$=$	$\sum_{i=1}^{n} 2i$	30
$1+2+3+4+\cdots+n$	$=$	$\sum_{i=1}^{n} i$	??

Exercise: Prove Summation of 1 to n

Show that the following equality holds for all positive integers.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Further, show that the right hand side **never** has a remainder in the division that is done.

- First step: Show how to construct sums of n out of pairs of terms in the summation

Exercise: Prove Summation of 1 to n

Show that the following equality holds for all positive integers.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Direct Proof: The summation can be expanded a bit to

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdots + (n-4) + (n-3) + (n-2) + (n-1) + n$$

Pair these terms as follows

0	+	n
1	+	n-1
2	+	n-2
3	+	n-3
..		..

The sum of each term pair is n so we just need to count how many pairs and multiply by n.

Second step: How many pairs are there? Divide into two cases for n even/odd, ensure any extra terms are identified.

Answers: Prove Summation of 1 to n

n is even

There will be $n/2$ such groups and the pairing will end with $n/2$ as below.

$$\begin{array}{ccc} 0 & + & \text{n} \\ 1 & + & \text{n-1} \\ 2 & + & \text{n-2} \\ 3 & + & \text{n-3} \\ .. & & .. \\ & + & \text{n}/2 \end{array}$$

Totaling gives $n(\frac{n}{2}) + n/2$. Rearranging gets the desired result:

$$\frac{n^2 + n)}{2} = \frac{n(n+1)}{2}$$

n is odd

There will be $(n+1)/2$ such groups and the pairing will end

$$\begin{array}{ccc} 0 & + & \text{n} \\ 1 & + & \text{n-1} \\ 2 & + & \text{n-2} \\ 3 & + & \text{n-3} \\ .. & & .. \\ \lfloor n/2 \rfloor & + & \lceil n/2 \rceil \end{array}$$

This gives $n(\frac{n+1}{2})$. Rearranging gets the desired result:

$$\frac{n^2 + n)}{2} = \frac{n(n+1)}{2}$$

Some Useful Summation Closed Forms

Having these in mind can be handy

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Generalized "Big" Ops

- Summing is not the only way to combine elements in a sequence
- Often will see other notations for "big" operations

$$\prod_{i=1}^{10} i = 1 \times 2 \times 3 \cdots \times 10 = 10! \text{ Factorial/Product}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{Union of indexed sets}$$

Recurrence Relations

- So far have seen terms in sequences have been expressed as functions of index: $a_n = 3n^2$
- Can also create **Recurrent Sequences** where terms depend on previous elements
- Example: $f_n = 2 \cdot f_{n-1}2$ with $f_0 = 1$
- Resulting sequence can be computed by building up from previous elements
- What's the example of this that CS folks love?
- **Recurrence Relations** define these sequences and will be studied later in the course as they pertain to recursive algorithms

Exercise: Sizes of Infinity

- Now acquainted with the idea of an infinite set
- Worthwhile to consider that there are different sizes of infinity
- Want to know about the **cardinality** of various infinite sets
- If $|A| = \infty$ and $|B| = \infty$, is $|A| = |B|$?
- Consider:
    1. Are there more Even Natural #'s or Odd Natural #'s?
    2. Are there more Natural Numbers or Integers?
    3. Are there more Integers or Rational Numbers?
    4. Are there more Real Numbers than Integers?
- How would one argue about the relative sizes of these?

Answers: Sizes of Infinity

- Argue about infinite cardinality sets by deriving **mapping** between the two sets
- Mapping is an **one-to-one function** that maps elements of one set to another

Example Mappings

- Evens and Odds have are the same size: $\mathit{Increment}(e) : e + 1$ maps all even numbers to an odd numbers; it is a one-to-one function, same size
- Natural Numbers and Integers have the same size by using the following mapping / inverse mapping

```
nat_to_int(nat):
  if(nat is even)
    return nat / 2      // Pos
  else
    return -((nat+1)/2) // Neg
```

```
int_to_nat(int):
  if(int > 0)
    return 2*int         // Even
  else
    return 2*abs(int)+1 // Odds
```

Exercise: Countable Sets

- A set is called **countable** if one can derive a mapping (one-to-one function) from the Natural Numbers **N** to its elements
- The set has the same **cardinality** as the Natural numbers then
- Formally, set X is countable if $\exists f(f\colon \mathbf{N} \to X \wedge f \text{ is one-to-one})$

Map It

Construct a mapping function from the Naturals to the following sets to show that they are countable

- Mapping between **N** and **Even Natural #s**
- Mapping between **N** and **Character Strings**
- Mapping between **N** and **Positive Rational Numbers Q^+**

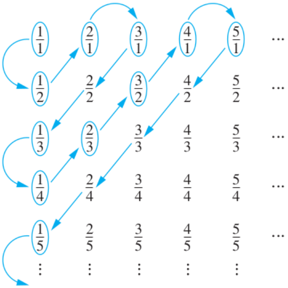
Answers: Countable Sets

- Mapping between **N** and **Even Natural #s** `double(nat): return 2*nat`
- Mapping between **N** and **Character Strings**
  - Informally, let each letter in the alphabet be numbered 1 to `size`
  - Let the empty string be string 0
  - All 1-character strings are numbered 1 to `size`
  - All 2-character strings are numbered `size+1` to `size*size`
  - Continue the pattern for all 3-char, 4-char, etc. strings

- Mapping between **N** and **Positive Rational Numbers** $\mathbf{Q}^+$
- Rational $impliesp/q$
- List in order of

$$p+q=1 \quad p+q=2$$
$$p+q=3 \quad p+q=4 \quad \ldots$$

skipping repeats: $\frac{2}{4} = \frac{1}{2}$



Uncountable Sets

- Important idea: a countable set has some algorithm to **enumerate** all of its elements (print them), alternately called **enumerable sets**
- May take a long time, but will eventually print any element
-
- An **uncountable set** is one for which no algorithm exists to enumerate its elements
  - Fundamentally **larger than Natural numbers**
  - Uncountable is a **bigger infinity** than the one associated with Naturals, Integers, Rationals, Strings
- The **Real Numbers** are uncountable and here's why...

Proof by Contradiction: Reals are Uncountable

Referred to as *Cantor's Diagonalization Argument*

1. Suppose that Reals are countable.
2. Then any subset of them is also countable such as all $0 < x < 1$
3. If $0 < x < 1$ are countable, then we can enumerate them in a list like the following

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\ &\vdots \end{aligned}$$

   with each d_{ij} being a digit of the number like 1,2,3,...,9
4. Let the real number p have digits p_i defined this enumeration:

$$p_i = \left\{ \begin{array}{ll} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{array} \right\}$$

   Note p's dependency on the "diagonal" elements
5. Suppose that p appeared in the listing of (3) as r_k. This **leads to a contradiction**: the digits p_k and d_{kk} do not match by the definition of p.
6. Therefore p is not in the listing in (4) so there is no enumeration of all real numbers: the **Reals Numbers are uncountable**

An Interesting Application: Uncomputable Functions

- The set of valid *Programs (P)* in some language is a subset of the strings in a some appropriate character set
  - Is P countable or uncountable?
- The set of mathematical *Functions (F)* (mappings) from Integers to Integers has the same cardinality as the reals
  - Is F countable or uncountable?
- Can there be Program in P for every Function in F?
  - Is $|P| = |F|$?

Some mathematical functions are NOT computable

- Referred to as **Uncomputable**
- Disturbing but true
- We'll look at one later: the halting problem

Matrices and their Notation

- A "grid of numbers" with some associated operations and rules
- Dimension is in #rows and #cols
- Notated in various ways (use {} or [] or ()) but usually obvious it's a matrix

Fat Matrix: rows < cols

$$A = \begin{Bmatrix} 1 & 0 & 2 \\ 9 & 4 & 3 \end{Bmatrix}$$

Skinny Matrix: rows > cols

$$C = \begin{pmatrix} 2 & 3 \\ 1 & 7 \\ 8 & 4 \\ 9 & 6 \end{pmatrix}$$

Square Matrix: rows = cols

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Basic Arithmetic

- Element-wise operations on matrices are obvious:
- Addition, Subtraction, Multiplication, Division

Addition (Element-Wise)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{vmatrix} = \begin{vmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{vmatrix}$$

Multiplication (Element-Wise)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \otimes \begin{vmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{vmatrix} = \begin{vmatrix} 7 & 16 & 27 \\ 40 & 55 & 72 \end{vmatrix}$$

Transposition

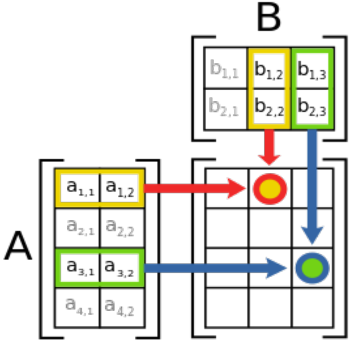
- Mirror a matrix across it's **main diagonal**
  - Main diagonal are elements $a_{11}, a_{22}, a_{33}, \ldots$
- Flips row and column counts: fat to skinny, skinny to fat, square stays square
- Notated as A^T for "transpose"

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 9 & 4 & 3 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 9 \\ 0 & 4 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$

Matrix Multiplication

- The important and "different" operation
- Multiply corresponding elements of rows of first matrix by columns of second matrix, sum to form a new element
- **Dimensions must match**: in $A \times B$, columns of A must equal rows of B



$$\overset{4\times 2\text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix}} \overset{2\times 3\text{ matrix}}{\begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix}} = \overset{4\times 3\text{ matrix}}{\begin{bmatrix} \cdot & x_{12} & x_{13} \\ \cdot & \cdot & \cdot \\ \cdot & x_{32} & x_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}}$$

Some results elements:

$$x_{12} = a_{11} b_{12} + a_{12} b_{22}$$
$$x_{33} = a_{31} b_{13} + a_{32} b_{23}$$

Exponentiation are Repeated Multiplication

- $x^2 = x \cdot x \cdot x$, $x^4 = x \cdot x \cdot x \cdot x \cdot x$
- Same goes for matrices
- $A^3 = A \cdot A \cdot A$
- Since dimensions must match, exponentiation only works for **square matrices**

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$A^2 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1+8+21 & 2+10+24 & 3+12+27 \\ 4+20+42 & 8+25+48 & 12+30+54 \\ 7+32+42 & 14+40+72 & 21+48+81 \end{vmatrix}$$

Menagerie of Matrices

Zero-One and Boolean Matrices Interpret element as True/False and use logical and/or operations as in $A \wedge B$ or generalize multiplication $C = A \odot B$ so that $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots$

Symmetric Matrices Element $a_{ij} = a_{ji}$, has some special properties such as real eigenvalues (not studied here but prominent in linear algebra)

Column and Row Vectors A matrix with 1 column or 1 row, multiplying by an appropriate matrix results in another vector

Many other aspects of matrices that we'll touch on at later times including **algorithm complexity** of matrix operations.