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CS 2011: Practice Quiz 4 SOLUTION

Summer 2018

University of Minnesota

Quiz period: 15 minutes

Points available: 30

Problem 1 (20 pts): Recall that a Full Binary Tree is one in which each node has 0 or 2 children. Use a proof by strong/structural induction to show that the size $n(T)$ (number of nodes) and height $h(T)$ of a full binary tree are related as follows.

$$n(T) \geq 2h(T) + 1$$

Define the height of a single root node as 0: $h(r) = 0$.

Clearly illustrate the Base and Induction Cases and identify the Inductive Hypothesis. *Hint: Recall that full binary trees have a recursive definition in terms of smaller full binary trees.*

SOLUTION:

BASE CASE: The smallest full binary tree is a single root for which $n(T) = 1$ *and* $h(T) = 0$. *This gives* $1 \geq 2(0) + 1 = 1$ *which is true.*

INDUCTION CASE:

1. *T is formed by joining root r to left/right full trees T_L, T_R. Show that the property holds for T.*
2. *The Inductive hypothesis states that the property holds for all smaller full binary trees, in particular* $n(T_L) \geq 2(h(T_L) - 1) + 1$ *and* $n(T_R) \geq 2(h(T_R) - 1) + 1$
3. *We can then derive the following*

$$\begin{aligned}
n(T) &= 1 + n(T_L) + n(T_R) && \textit{Def. of } n(T) \\
&\geq 1 + 2h(T_L) + 1 + 2h(T_R) + 1 && \textit{IH for } n(T_L), n(T_R) \\
&= 2(h(T_L) + h(T_R) + 1) + 1 && \textit{Rearrange} \\
&\geq 2(\max(h(T_L), h(T_R)) + 1) + 1 && \textit{Def. of Max vs Sum} \\
&= 2h(T) + 1 && \textit{Def of height}
\end{aligned}$$

4. *Since valid transformations of the left side led to the right side the inequality holds proving the induction case.*

By the combination of base and induction cases, the property holds. ■

Problem 2 (5 pts): Consider the characters XYZQRT. Determine how many strings can be constructed according to the following criteria. Give your answers in a symbolic form such as through use of permutations/combinations. Justify your answers with a sentence or two.
(A) Any ordering of 4 of the characters with repetitions allowed.
SOLUTION: Permutations with replacement, 4 choices with 6 possibilities each:

$$6^4$$

(B) Any ordering of 5 of the characters so long as Q and R are among the characters, no repetition allowed.
SOLUTION: QR is required leaving 4 chars. Choose 3 from 4 the permutations of length 5.

$$C(4,3)P(5,5)$$

Problem 3 (5 pts): Consider length 10 bit strings. How many bit strings have the following properties. Describe and justify your answers as in the last problem.
(A) Exactly 4 1's in any position.
SOLUTION: Choose 4 indices from 10 at which to put 1's; order of index selection doesn't matter:

$$C(10,4)$$

(B) At least 7 1's in any position except last bit which must be a 0.
SOLUTION: Last bit is 0 leaving 9 slots. There can be 7, 8, or 9 1's in the bitstring then and the indices at which the 1's appear are chosen as before without accounting for order of index selection:

$$C(9,7) + C(9,8) + C(9,9)$$

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