Parallel Sorting

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Logistics

Today

- Parallel Sorting: Quicksort

Reading: Grama Ch 9

- Sorting
- Focus on 9.4: Quicksort

Sorting

- Much loved computation problem
- What is the best complexity of general purpose (comparison-based) sorting algorithms?
- What are some algorithms which have this complexity?
- What are some other sorting algorithms which aren't so hot?
- What issues need to be addressed to parallelize any sorting algorithm?

Parallel Sorting Base algorithm

Prospects of parallelizing standard $O(N \log N)$ sorting algorithms...

Heap Sort

- Manipulates a global array
- Very serial in nature: repeatedly percolate array elements up heap, swap to end of heap, repeat
- Random access to entire array is a must, not good for distributed memory

Merge Sort

- Has a nice recursive decomposition, but...
- Merging two sorted arrays on separate processors to produce a larger array would involve prohibitive communication
- Will look later at Odd-Even sort which has a similar flavor

This leaves the king of sorting for a parallel implementation...

Partition and Quicksort

- Quicksort has $O(N \log N)$ average complexity
- In-place, low overhead sorting, recursive

Partition

- Select a `pivot` value
- Rearrange elements so
  - Left array is $\leq$ `pivot`
  - Right array is $>$ `pivot`
  - `pivot` is in “middle”

```
# A is an array, lo/hi are
# inclusive boundaries
algorithm partition(A, lo, hi):
  pivot := A[hi]
  boundary := lo
  for j = lo to hi do
    if A[j] <= pivot then
      swap A[boundary], A[j]
      boundary++
  swap A[boundary], A[hi]
  return boundary
```

Quicksort

- Partition into two parts
- Recurse on both halves
- Bail out when boundaries `lo/hi` cross

```
algorithm quicksort(A, lo, hi):
 if lo < hi then
   p = partition(A, lo, hi)
   quicksort(A, lo, p - 1)
   quicksort(A, p + 1, hi)
```

Practical Parallel Sorting Setup

- Input array A of size N is already spread across P processors (no need to scatter)

```
P0: A[] = { 84 31 21 28 }
P1: A[] = { 17 20 24 84 }
P2: A[] = { 24 11 31 99 }
P3: A[] = { 13 32 26 75 }
```

- Goal: Numbers sorted across processors. Smallest on P0, next smallest on P1, etc.

```
P0: A[] = { 11 13 17 20 }
P1: A[] = { 21 24 24 26 }
P2: A[] = { 28 31 32 33 }
P3: A[] = { 75 84 84 99 }
```

- Want to use P processors as effectively as possible
- Favor bulk communication over many small messages

Exercise: Parallel Quicksort

- Find a way to parallelize quicksort
- **Hint:** The last step is each processor sorting its own data using a serial algorithm. Try to arrange data so this is possible.

```
START:
P0: A[] = { 84 32 21 28 }
P1: A[] = { 17 20 25 85 }
P2: A[] = { 24 11 31 99 }
P3: A[] = { 13 32 26 75 }

GOAL
P0: A[] = { 11 13 17 20 }
P1: A[] = { 21 24 25 26 }
P2: A[] = { 28 31 32 33 }
P3: A[] = { 75 84 85 99 }
```

```
SERIAL ALGORITHM
algorithm quicksort(A, lo, hi) is
 if lo < hi then
   p := partition(A, lo, hi)
   quicksort(A, lo, p - 1)
   quicksort(A, p + 1, hi)

algorithm partition(A, lo, hi) is
  pivot := A[hi]
  boundary := lo
  for j := lo to hi - 1 do
    if A[j] <= pivot then
      swap A[boundary] with A[j]
      boundary++
  swap A[boundary] with A[hi]
  return boundary
```

Answers: Parallel Quicksort Ideas 1 / 3

- Select a global shared Pivot value and broadcast to all procs
- Select pivot so that half data elements got to lower processors, half got to higher processors
- Redistribute low data to low procs, high data to high procs
- Split procs into low / high group, and recurse
- When each proc is on its own, sort locally

Answers: Parallel Quicksort Ideas 2 / 3

```
A[] = { 84 32 21 11 | 17 20 25 85 | 24 28 31 99 | 13 33 26 75 }
        P0             P1             P2             P3

Partition(pivot=26) on each processor
A[] = { 21 11 84 32 | 17 20 25 85 | 24 28 31 99 | 13 26 33 75 }
Boundary:    ^                ^        ^                ^
Counts: P0: 2          P1: 3          P2: 1          P3: 2

Calculate which data goes where
A[] = { 21 11 84 32 | 17 20 25 85 | 24 28 31 99 | 13 26 33 75 }
        P0 P0 P2 P2    P0 P0 P1 P2    P1 P2 P3 P3    P1 P1 P3 P3

Re-arrange so values <= 26 on P0 and P1, > 26 on P2 and P3
A[] = { 21 11 17 20 | 25 24 13 26 | 84 32 85 28 | 31 99 33 75 }
        P0             P1             P2             P3

Split the world: 2 groups
A[] = { 21 11 17 20 | 25 24 13 26}|{84 32 85 28 | 31 99 33 75 }
        P0             P1             P2             P3
```

Answers: Parallel Quicksort Ideas 3 / 3

```
Each half partitions on different pivot value
        P0-P1: Partition(pivot=20)  P2-P3: Partition(pivot=33)
A[] = { 11 17 20 21 | 13 25 24 26}|{28 32 84 85 | 31 33 99 75 }
Boundary:      ^       ^                ^              ^
Counts: P0: 3          P1: 1          P2: 2          P3: 2

Calculate which data goes where
A[] = { 11 17 20 21 | 13 25 24 26}|{28 32 84 85 | 31 33 99 75 }
        P0 P0 P0 P1   P0 P1 P1 P1   P2 P2 P3 P3   P2 P2 P3 P3

Re-arrange values to proper processors
A[] = { 11 17 20 13 | 21 25 24 25}|{28 32 31 33 | 84 85 99 75 }
        P0             P1             P2             P3

Split the world: 4 groups
A[] = { 11 17 20 13}|{21 25 24 25}|{28 32 31 33}|{84 85 99 75 }
        P0             P1             P2             P3

4 groups == 4 processors, all processors sort locally
A[] = { 11 13 17 20}|{21 24 25 25}|{28 31 32 33}|{75 84 85 99 }
        P0             P1             P2             P3
```

Quicksort Difficulties

Communication

- Determine which data go to which processors, how many send/receives are required
- Opportunity for **all-to-all communications** in MPI

Recursing

- Recursive step of algorithm requires smaller "worlds"
- Use MPI's **communicator splitting** capability

Pivot Value Selection

- In example, pivot values were cherry-picked to get even distribution of data among processors
- A bad pivot splits data unevenly, is annoying for serial Quicksort, shaves off processors in parallel quicksort destroying efficiency

All-to-All Personalized Communication

All-to-all personalized communication: like every processor scattering to every other processor.

```
                    BEFORE
P0: send[] ={A0, B0, C0, D0}  recv[] = { -,  -,  -,  -,}
P1: send[] ={A1, B1, C1, D1}  recv[] = { -,  -,  -,  -,}
P2: send[] ={A2, B2, C2, D2}  recv[] = { -,  -,  -,  -,}
P3: send[] ={A3, B3, C3, D3}  recv[] = { -,  -,  -,  -,}

                MPI_Alltoall(...);

                    AFTER
P0: send[] ={A0, B0, C0, D0}  recv[] = {A0, A1, A2, A3}
P1: send[] ={A1, B1, C1, D1}  recv[] = {B0, B1, B2, B3}
P2: send[] ={A2, B2, C2, D2}  recv[] = {C0, C1, C2, C3}
P3: send[] ={A3, B3, C3, D3}  recv[] = {D0, D1, D2, D3}
```

MPI_Alltoall

- Standard version: every processor gets a slice of `sendbuf`, same sized data
- Vector version allows different sized slices (appropriate for quicksort)

```
int MPI_Alltoall(
 void *sendbuf, int sendcount, MPI_Datatype sendtype,
 void *recvbuf, int recvcount, MPI_Datatype recvtype,
 MPI_Comm comm);

int MPI_Alltoallv(
 void *sendbuf, int sendcounts[], int sdispls[], MPI_Datatype sendtype,
 void *recvbuf, int recvcounts[], int rdispls[], MPI_Datatype recvtype,
 MPI_Comm comm);
```

Exercise: Redistribution during Quicksort

- After `partition`, procs will redistribute data via all-to-all
- Perform All-Gather to get counts in table to the right

Element Count vs Pivot

Proc	<=	>
P0	2	2
P1	3	1
P2	1	3
P3	2	2

Each Proc must calculate its own Count/Displ arrays for all-to-all:

P#		P0	P1	P2	P3		P0	P1	P2	P3	P#
P0	RecvCount	2	2	0	0	SendCount	2	0	2	0	P0
P1		0	1	1	2		2	1	1	0	P1
P2		2	1	1	0		0	1	1	2	P2
P3		0	0	2	2		0	2	0	2	P3
P0	RecvDispl	0	2	4	4	SendDispl	0	2	2	4	P0
P1		0	0	1	2		0	2	3	4	P1
P2		0	2	3	4		0	0	1	2	P2
P3		0	0	0	2		0	0	2	2	P3

- Describe the process of calculating `RecvCount`
- Given `RecvCount`, how can one calculate `RecvDispl`

Answers: Redistribution during Quicksort

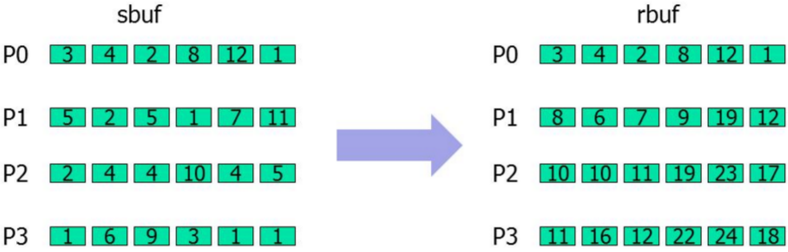
- `RecvCount` can be calculate through an iterative process
- Compute the **prefix sum** of below/above pivot counts

```
El Count vs Pivot        PS[]: PREFIX SUMS
| Proc | <= | > |        | Proc | <= | > |
|------+----+---|        |------+----+---|
| P0   |  2 | 2 |  ===>  | P0   |  2 | 2 |
| P1   |  3 | 1 |        | P1   |  5 | 3 |
| P2   |  1 | 3 |        | P2   |  6 | 6 |
| P3   |  2 | 2 |        | P3   |  8 | 8 |
```

- Know each proc stores `N / P = 4` elements
- Procs receiving `<= pivot`, proc # `i`, scan column 0 for
  - First partner is proc `F` where `PS[F,0] <= 4*i`
  - Last partner is proc `L` where `PS[L,0] >= 4*(i+1)`
- Procs receiving `> pivot`, proc # `i`, scan column 1 for
  - First partner is proc `F` where `PS[F,1] <= 4*(i-2)`
  - Last partner is proc `L` where `PS[L,1] >= 4*(i-2+1)`
- Actual code will need to do additional arithmetic (e.g. P1 receives 1 element from itself)
- `RecvDispl` is the **prefix sum** of `RecvCount`

Prefix Sums / Scan

Prefix Sums or Prefix Scans are supported in parallel via MPI




```c
int MPI_Scan(const void *sendbuf, void *recvbuf, int count,
             MPI_Datatype datatype, MPI_Op op, MPI_Comm comm);
```

- Similar to reduction but only add on values from procs <= `proc_id`
- op can be sum/max/min/etc.
- In simple Quicksort implementations, **don't use parallel prefix scan** as this does not yield enough info to calculate send/receive partners

Overall Flow

1. Pivot selection (open question how to do this right)
2. Broadcast of pivot value
3. Each processor `partition`'s its data
4. All-gather to get element/pivot counts
5. Calculate send/receives
6. Redistribute data via `MPI_Alltoallv()`
7. And then...

Splitting the World

```c
int MPI_Comm_split(MPI_Comm comm, int color, int key,
                   MPI_Comm *newcomm);
```

- `comm` is the old communicator (start with `MPI_COMM_WORLD`)
- `color` is which sub-comm to go into
  - Colors `0,1` splits into 2 communicators
  - Colors `0,1,2,3` splits into 4 communicators
  - Etc.
- `key` establishes rank in new sub-comm, usually `proc_id`
- `newcomm` is filled in with a new communicator
- Examine `04-mpi-code/comm_split.c`
- In Quicksort, new comm is different for lower/upper procs

Exercise: Pivot Selection

- So far have assumed a “good” pivot can be found
- Pivot evenly splits N/2 data, half to lower # processors, half to upper

Discuss the following questions with a neighbor

1. What if the pivot is poorly selected? E.g. 1/4 below pivot, 3/4 above? Could the algorithm adapt?
2. How could one avoid a bad pivot? Discuss some strategies
3. Is there a way to avoid recusing entirely?

Answers: Pivot Selection 1/2

Discuss the following questions with a neighbor

1. What if the pivot is poorly selected? E.g. 1/4 below pivot, 3/4 above? Could the algorithm adapt?

   With some additional computation, can split the world unevenly: 1/4 procs assigned to "low" numbers, 3/4 to "high" numbers. Still broken if a tiny fraction of the array is lower/higher than the pivot: should just try another pivot at that point or use a scheme that prevents poor pivot selection.

2. How could one avoid a bad pivot? Discuss some strategies

   Lots of these exist, some mentioned in the textbook such as having a randomly selected processor compute its median and broadcast it as the pivot (main text of Grama) or have processors sample random elements, perform All-Gather, then compute a median as the common pivot (Grama Exercise 9.21).

Answers: Pivot Selection 2/2

3. Is there a way to avoid recusing entirely, e.g. single multiway pivot?

   Grama Exercise 9.20 explores this:

   - *Each proc samples elements, often around* $\log(N)$ *elements, and procs perform an All-Gather*
   - *All procs use common sample to select* $P-1$ *common pivots.*
   - *Elements between pivots are sent directly to final destination procs in an All-to-All communication.*
   - *Local sorting commences.*

   In short: With 4 procs, estimate quartile boundaries based on sampling, give bottom 25% of elements to Proc 0, etc. and sort locally.

Bubble Sort

- Classic CS1 Sorting Algorithm
- Several variants that improve on given pseudocode
  - Limit inner loop bound `i<N-1-r`
  - Terminate when sorted order detected
- Stated version is obviously $O(N^2)$ complexity
- Not a lot of room for parallelism...
- But a variant of this DOES have room for parallelism

```
void bubble_sort(A[]) {
  N = length(A[])
  for(r=0; r < N-1; r++){
    for(i=0; i < N-1; i++){
      compare_exchange(A, i, i+1);
    }
  }
}

void compare_exchange(A[], i, j){
  if(A[i] > A[j]){
    temp = A[i]
    A[i] = A[j]
    A[j] = temp
  }
}
```

Exercise: Odd-Even Sort

- Variant of bubble sort which splits bubbling into odd/even phases
- $O(N^2)$ complexity of serial algorithm
- There is potential for parallelism here: what is it?
  - Consider simple case where each $P = N$: each proc hold a single number
  - What can be parallelized and how?

```
void odd_even_sort(A[]) {
  N = length(A[])
  for(r=0 to N-1){
    if(r is even){
      for(i=0; i<N-1; i+=2){
        compare_exchange(A, i, i+1);
      }
    }
    if(r is odd){
      for(i=1; i<N-1; i+=2){
        compare_exchange(A, i, i+1);
      }
    }
  }
}

void compare_exchange(A[], i, j){
  if(A[i] > A[j]){
    temp = A[i]
    A[i] = A[j]
    A[j] = temp
  }
}
```

Answers: Odd-Even Sort

What can be parallelized and how?

- *Suppose each of N elements is stored on P processors in a line/ring with $N = P$*
- *The inner loops of `compare_exchange()` can be executed in parallel as it involves communication between 2 procs to potentially exchange elements but only with a single partner.*
- *Even iterations, lower evens exchange with higher odds*
- *Odd iterations lower odds exchange with higher evens*
- *Exchange can be done via a Send/Receive of elements and then "keeping" the appropriate element, min on lower proc, max on higher proc*

Answers: Odd-Even Sort

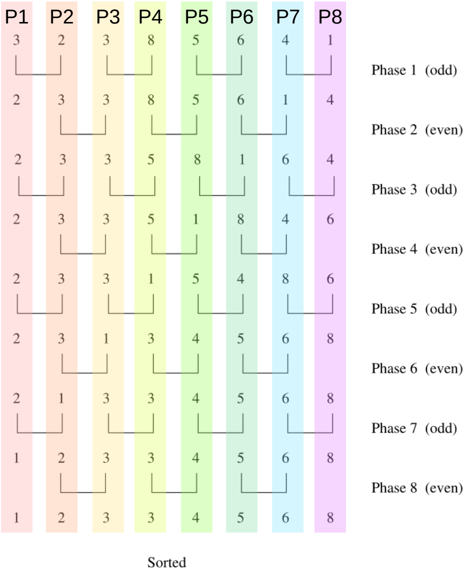


Figure 9.13 Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.

Odd-Even Sort with $N > P$

- As before, unrealistic to have $P = N$, rather each proc holds N/P elements of the array `A[]`
- `COMPARE_EXCHANGE()` becomes `COMPARE_SPLIT()`

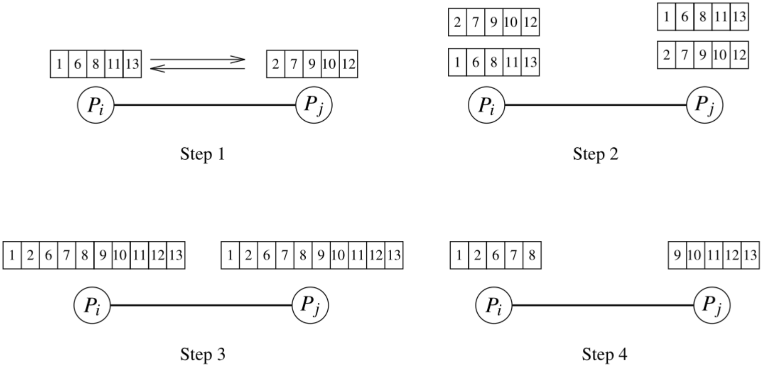


Figure 9.2 A compare-split operation. Each process sends its block of size n/p to the other process. Each process merges the received block with its own block and retains only the appropriate half of the merged block. In this example, process P_i retains the smaller elements and process P_j retains the larger elements.

Analysis of Odd-Even Sort

- Initially all procs sort their local array: $O(\frac{N}{P}\log\frac{N}{P})$
- Conduct P Outer iterations of `ODD_EVEN_SORT()`
- Each odd/even inner loop is done in parallel by all procs communicating with a neighbor
- Neighbor procs exchange ararys: $O(t_s + t_w\frac{N}{P})$
- Each proc then performs a compare/split: $O(\frac{N}{P})$
- Overall complexity of parallel algorithm:

$$
\begin{aligned}
T_{par} &= O\left(\frac{N}{P}\log\frac{N}{P}\right) + P \times \left(O\left(t_s + t_w\frac{N}{P}\right) + O\left(\frac{N}{P}\right)\right) \\
&= O\left(\frac{N}{P}\log\frac{N}{P}\right) + O(N) + O(N)
\end{aligned}
$$

Isoefficiency? : Reported in textbook as $O(P2^P)$, linear increase in P requires exponential increase in problem size to maintain efficiency. Verifying this is a good exercise.

Sorting Extras

Odd-Even Sort to Shell Sort

- Allowing bigger "moves" in odd-even sort can improve practical efficiency of algorithm
- Shell Sort provides a mechanism for this: neighbors selected according to a "gap" scheme, less known sort with yet mysterious complexity analysis

Sorting Hardware

- Grama Ch 9.1 discusses Sorting networks, specialized hardware which can implement sorting
- With N processors, can implement Bitonic Sort in a sorting network and achieve $T_{par} = O(\log^2 N)$
- Hardware that implements sorting networks is not common but...
- GPUs provide interesting hardware, large numbers of procs, will revisit sorting on studying CUDA