

Saha Equation

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What we learn in this supplement

We introduce **Einstein's transition coefficients** and find their ratios by setting the transitions from one state of an atom to be in balance with the reverse transitions between the same two states ("**detailed balance**") in **thermal equilibrium**. For this, we use the Planck function and the Boltzmann formula. We then use these ratios of coefficients, generalized to include the presence of electrons, to obtain the **Saha equation**, which gives the relative proportions of atoms of a certain species that are in two different **states of ionization**, again in thermal equilibrium. To accomplish this, we again apply detailed balance and invoke the Maxwell-Boltzmann distribution for the electrons and the Planck function for the radiation field. As examples, the Saha equation is applied to the solar photosphere and to the decoupling of radiation in the early universe.

The Saha equation may be derived directly from statistical arguments (e.g., L. H. Aller, *Astrophysics, The Atmospheres of the Sun and Stars*, second edit., Ronald Press, 1963). This derivation is not presented here.

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1 Introduction

The Saha equation (4.15 of AP) gives the relative number of atoms in two ionization states as a function of electron density n_e and temperature T .

$$\frac{n_{r+1} n_e}{n_r} = \frac{G_{r+1} g_e}{G_r} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp\left(-\frac{\chi_r}{kT}\right) \quad (1)$$

Here n_r and n_{r+1} are the number densities of atoms in the ionization state r (e.g., with 4

electrons missing) and the ionization state $r + 1$ (5 electrons missing) of a given element. Also, n_e is the electron number density, G_r and G_{r+1} are the partition functions (defined below) of the two states, $g_e = 2$ is the statistical weight of the electron, m_e is the electron mass, and χ_r is the ionization potential from state r to $r+1$. For hydrogen, under most astrophysical conditions, $G_r \approx 2$ and $G_{r+1} \approx 1$.

We approach the Saha equation through the Einstein transition probabilities while making use of the Planck function, the Boltzmann formula, and the Maxwell-Boltzmann distribution. This approach is intended to introduce the reader to the Einstein coefficients and the physical considerations that can lead to the Saha equation. It is not a derivation from first principles of statistical physics. For that, see for example, L. H. Aller, *Astrophysics, The Atmospheres of the Sun and Stars*, second edit., Ronald Press, 1963).

2 Basic concepts

Here, we outline several concepts and expressions that are needed in our derivation of the Saha equation.

Planck function

The Planck blackbody function may be derived with Bose-Einstein statistics. The number of ways to achieve possible distributions of photon energy under these statistics is maximized with the aid of Lagrange multipliers. A thermodynamic argument relates the temperature to one of the multiplier constants. The result for the energy specific intensity $I(\nu, T)$ is (6.6 of AP):

$$\Rightarrow I(\nu, T) = \frac{2 h \nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad \begin{array}{l} \text{(Planck function;} \\ \text{W m}^{-2} \text{ Hz}^{-1} \text{ sr}^{-1}) \end{array} \quad (2)$$

The energy density u ($\text{W m}^{-3} \text{ Hz}^{-1}$) is obtained simply by multiplying (2) by $4\pi/c$,

$$\Rightarrow u(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \quad \begin{array}{l} \text{(Energy density;} \\ \text{W m}^{-3} \text{ Hz}^{-1}) \end{array} \quad (3)$$

In thermal equilibrium, the photon energy distribution is described by these expressions.

Boltzmann formula

An atom with multiple energy states in thermal equilibrium with a radiation field described by (3) will find itself in one or another of these energy states. Frequent transitions to and from other states will occur as photons interact with the atoms. Statistical arguments, similar to those used to derive the Planck Function (as just described), give the relative number densities of atoms in the several excited states in thermal equilibrium (Fig. 1),

$$\Rightarrow \frac{n_i}{n_j} = \frac{g_i}{g_j} \exp\left(-\frac{h\nu_{ij}}{kT}\right) \quad \text{(Boltzmann formula)} \quad (4)$$

where T is the temperature, g_i and g_j are the statistical weights of the two states shown in Fig. 1, and $h\nu_{ij}$ is the energy between them.

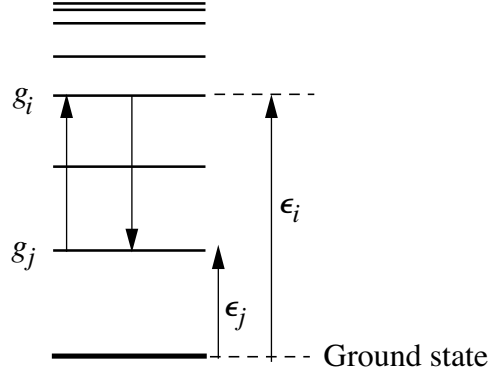


Figure 1. Excitation energy states of an atom. Transitions between the i th and j th states are shown. The energy levels indicate the energies required to excite the state of interest relative to the ground state.

If the two levels have angular momenta quantum numbers J_i and J_j the ratio of statistical weights is

$$\frac{g_i}{g_j} = \frac{2J_i + 1}{2J_j + 1} \quad (5)$$

For example the second energy level of the hydrogen has p states with orbital angular mom $J = 1$ and hence a statistical weight of 3. These states can accommodate three electrons of a given spin and another three of the opposite spin, for a total of six electrons.

A photon emitted during the deexcitation from state i to j has energy

$$h\nu_{ij} = \epsilon_i - \epsilon_j \quad (6)$$

This may occur via *radiative (pontaneous) deexcitation* or via *induced deexcitation* wherein a passing photon of energy $h\nu_{ij}$ induces the downward transition with the emission of another photon of the same energy.

The energy of the photon that can excite the atom to the upper state from the lower is also $h\nu_{ij}$. This is called *photoexcitation*. The atom can be de-excited by a photon (*timulated emission*). In this case, a passing photon of energy $h\nu_{ij}$ can induce a transition with the emission of another photon of the same energy.

What fraction of the atoms are in a given excited state? Restate the relative numbers (4) with aid of (6),

$$\frac{n_i}{n_j} = \frac{g_i e^{-\epsilon_i/kT}}{g_j e^{-\epsilon_j/kT}} \quad (7)$$

To relate the number in the i th state to the total density n of atoms, sum over the n_j in the denominator,

$$\frac{n_i}{n} = \frac{g_i e^{-\epsilon_i/kT}}{\sum_j g_j e^{-\epsilon_j}} = \frac{g_i e^{-\epsilon_i/kT}}{G} \quad (8)$$

where we define the *partition function* for the atom,

$$\begin{aligned} \Rightarrow \quad G &\equiv g_0 + g_1 e^{-\epsilon_1/kT} + g_2 e^{-\epsilon_2/kT} + \dots \quad (\text{Partition function}) \\ &\equiv \sum_j g_j e^{-\epsilon_j/kT} \end{aligned} \quad (9)$$

The sum is over all states of the atom, ground and excited. One can think of the partition function as a kind of statistical weight for the atom, taking into account all its excited states.

Einstein coefficients

Transition probabilities

As stated above, transitions from the upper of the two labeled states of Fig. 1 take place by photo deexcitation and by induced deexcitation. Transition in the upward direction is by photoexcitation. Einstein (1917??) introduced probability coefficients for these transitions that are independent of time and dependent only on the characteristics of the atom. For transitions that require photons from a radiation field (induced deexcitation and photoexcitation), the transition rate is proportional to the energy density of the radiation field.

Let A_{ij} be the probability per unit time of a downward radiative transition and $B_{ij} u(\nu_{ij}, T)$ the probability per unit time of an induced deexcitation from state i to state j . The indices refer to the two states involved in the transition (Fig. 1) and u is the energy density of the radiation field (3) at photon energy $E = h\nu_{ij}$. The total probability for a downward transition per unit time is thus

$$P_{ij} = A_{ij} + B_{ij} u(\nu_{ij}, T) \quad (\text{Downward transitions: radiative and induced deexcitation}) \quad (10)$$

The A and B symbols are the Einstein coefficients. Similarly, let $B_{ji} u(\nu_{ji}, T)$ be the probability for photoexcitation, Note that $\nu_{ij} = \nu_{ji}$.

$$P_{ji} = B_{ji} u(\nu_{ij}, T) \quad (\text{Upward transitions: photoexcitation}) \quad (11)$$

The ratios

The ratios of the Einstein coefficients will prove useful to us. In thermal equilibrium, the populations of the several excited states of a given atomic species will be stable or stationary. In that condition, the rate of upward transitions between the two labeled states in Fig. 1 should equal the downward transitions between the two levels. This is called *detailed balance*. The number of transitions per unit time (and volume) is the product of the probability and density of atoms in the initial state. Hence, we equate P_{ij} (10) and P_{ji} (11),

$$n_i(A_{ij} + B_{ij} u) = n_j B_{ji} u \quad (12)$$

Solve for the ratios of the coefficients. First simplify the Planck function (3) (at frequency ν_{ij}) with $a = 8\pi h\nu^3/c^3$ and $x = h\nu_{ij}/kT$.

$$u = a (e^x - 1)^{-1} \quad (13)$$

Divide (12) by n_j and replace the ratio of the Boltzmann formula with (4),

$$\frac{g_i}{g_j} e^{-x} (A_{ij} + B_{ij} u) = B_{ji} u \quad (14)$$

collect the terms in u and replace it with (13) to obtain

$$a \left(e^x B_{ji} - \frac{g_i}{g_j} B_{ij} \right) = (e^x - 1) \frac{g_i}{g_j} A_{ij} \quad (15)$$

The Einstein coefficients are independent of temperature or frequency. They should thus satisfy (15) for any value of e^x . For low frequency, let $x \rightarrow 0$ or $e^x \rightarrow 1$, giving

$$\Rightarrow \frac{B_{ij}}{B_{ji}} = \frac{g_j}{g_i} \quad (16)$$

and, for high frequency, $e^x \rightarrow \infty$,

$$a B_{ji} = \frac{g_i}{g_j} A_{ij} \quad (17)$$

Use (16) and the expression given above (13) for a to obtain

$$\Rightarrow \frac{A_{ij}}{B_{ij}} = \frac{8\pi h\nu^3}{c^3} \quad (18)$$

Two comments are appropriate. First, the ratio (16) of the B coefficients for the transitions involving radiation fields depends only on the ratio of statistical weights. The coefficient, and hence the transition probability, is greater if the final state has greater statistical weight, i.e., there are more substates into which the system can terminate.

Second, the ratio A_{ij}/B_{ij} (18) reflects the fact that the B coefficients are multiplied by an energy density in the probability expressions (10) and (11). The Einstein coefficients are sometimes defined with the Planck specific intensity function (2) rather than the Planck energy density (3). In this case the ratio (18) becomes $A_{ij}/B_{ij} = 2h\nu^3/c^2$. This differs only by the constant $c/4\pi$ and has no effect on the result of our calculation below.

3 Saha equation

We derive the Saha equation (1) with the aid of the above considerations. Here we are dealing with the relative numbers of atoms of a given species that are in different ionization states in thermal equilibrium.

Derivation

Ionization states

Consider an atom in an ionization state r , for example a carbon ion with only 3 electrons (C IV) when normally it would have six in its neutral state. In its ground state, 2 electrons would occupy its $n = 1$ state and one would occupy its $n = 2$ state. This atom can be raised to the next higher ionization state (C V) with the ejection of the latter electron if a collision or photon provides the appropriate energy, the ionization potential, χ_r . Such a transition is indicated as Transition 1 in Fig. 2. It takes the atom from the ground state of the r ionization state to the ground state of the $r+1$ ionization state. The ejected electron will take on any excess energy p the exciting photon may have had beyond χ_r . The photon energy would be

$$h\nu = \chi_r + \frac{p^2}{2m} \quad (19)$$

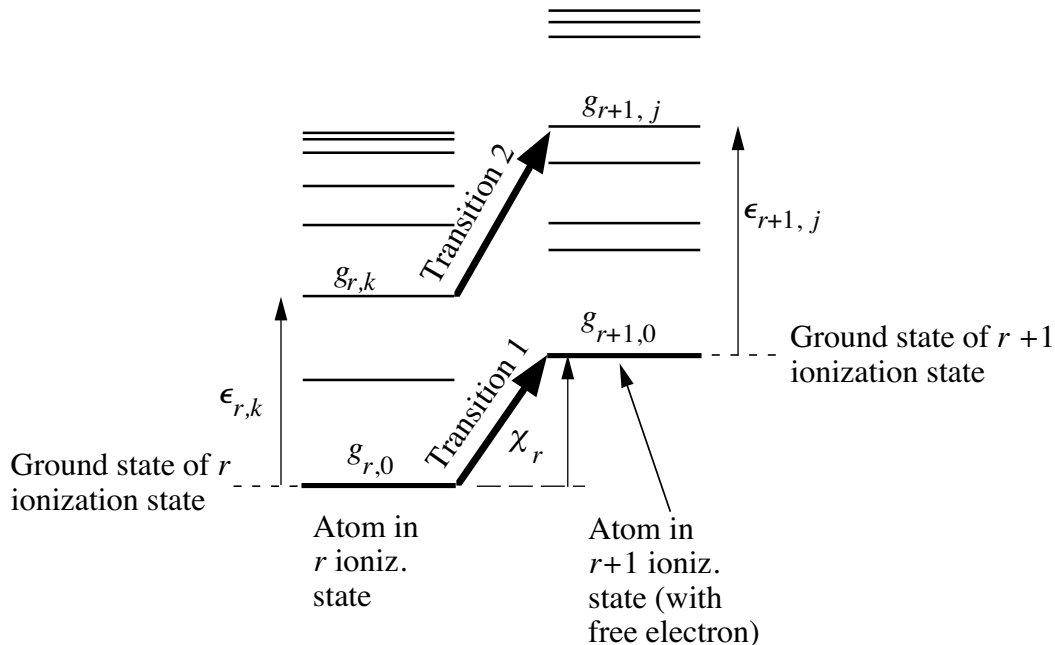


Figure 2. Energy states of an atom in two ionizations states. The $r+1$ state has one less bound electron less than the r state. In the $r+1$ state, that electron is free.

More generally, the transition can initiate in any energy state of the original ion and terminate in any energy state of the final ion as illustrated by Transition 2 of Fig. 2. The energy of the exciting photon would be

$$h\nu = \chi_r + \epsilon_{r+1,j} - \epsilon_{r,k} + \frac{p^2}{2m} \quad (20)$$

if the final momentum of the ejected electron is $p^2/2m$.

We will consider this general case and find, initially, the relative numbers of electrons, r,k ions and $r+1,j$ ions in thermal equilibrium.

Transition rates

The rate $R_{r+1,j \rightarrow r,k}$ of *spontaneous recombinations* – a recombination with an electron of momentum p that results in the emission of a photon of energy $h\nu$ (20) – must be proportional to the electron-ion collision rate which may be expressed as the product of the density of free electrons of momentum p (per unit momentum interval) and the density of $r+1$ ions in state j , namely, $n_{e,p}(p) n_{r+1,j}$. The rate of *induced recombinations* – a recombination induced by a nearby photon of energy $h\nu$ (20) – requires yet another factor, the radiation density $u(\nu)$ at that frequency. The rate of *photoionizations* – ionization by means of a photon of energy $h\nu$ (20) – requires again the radiation energy density at frequency ν and also the density of excited r,k ions. One can thus write the three rates with Einstein coefficients as

$$R_{sr} = n_{r+1,j} n_{e,p}(p) A_{r+1,j \rightarrow r,k} \quad \begin{array}{l} \text{(Spontaneous} \\ \text{recombinations)} \end{array} \quad (21)$$

$$R_{ir} = n_{r+1,j} n_{e,p}(p) u(\nu) B_{r+1,j \rightarrow r,k} \quad \text{(Induced recombination)} \quad (22)$$

$$R_{pi} = n_{r,k} u(\nu) B_{r,k \rightarrow r+1,j} \quad \text{(Photoionization)} \quad (23)$$

In thermal equilibrium, the rate of “downward” transitions must equal the “upward” rate; i.e., detailed balance must exist. Thus $R_{sr} + R_{ir} = R_{pi}$,

$$n_{r+1,j} n_{e,p}(p) A_{r+1,j \rightarrow r,k} + n_{r+1,j} n_{e,p}(p) u(\nu) B_{r+1,j \rightarrow r,k} = n_{r,k} u(\nu) B_{r,k \rightarrow r+1,j} \quad (24)$$

To obtain the desired ratios of densities (1), we take guidance from the ratios of Einstein coefficients obtained above. For the ratio of the B 's (16), the statistical weight of the $r+1,j$ state must include the statistical weight of the free electron which is $g_e 4\pi p^2/h^3$. The g_e factor equals 2 for the two spin states of the electron. The number of cells in phase space (per unit physical volume and unit momentum band) is $4\pi p^2/h^3$ (see Fig. 3.1c and Eq. 6.29 of AP). The A/B ratio for recombinations is the same as for the atomic transitions (18).

Einstein coefficients

The ratios of coefficients for the ionization problem are thus

$$\frac{B_{r,k \rightarrow r+1,j}}{B_{r+1,j \rightarrow r,k}} = \frac{g_{r+1,j}}{g_{r,k}} \frac{g_e 4\pi p^2}{h^3} \quad (25)$$

$$\frac{A_{r+1,j \rightarrow r,k}}{B_{r+1,j \rightarrow r,k}} = \frac{8\pi h \nu^3}{c^3} \quad (26)$$

The Planck function is given in (3). The electron momentum distribution is taken to be Maxwellian, from (3.11) of AP,

$$n_{e,p}(p) dp = \frac{n_e 4\pi p^2}{(2\pi m k T)^{3/2}} \exp\left(-\frac{p^2}{2m k T}\right) dp \quad (27)$$

where $n_{e,p}(p)dp$ is the number of electrons per unit volume between p and $p + dp$ and $n_e = \int n_{e,p}(p)dp$ is the number of electrons per unit volume integrated over all momenta.

Population of ionized excited states

Now, we proceed to work with the detailed balance equation (24). Rearrange terms,

$$\frac{n_{r+1,j} n_{e,p}(p)}{n_{r,k}} = \frac{B_{r,k \rightarrow r+1,j}}{B_{r+1,j \rightarrow r,k}} \left(1 + \frac{A_{r+1,j \rightarrow r,k}}{u(\nu) B_{r+1,j \rightarrow r,k}}\right)^{-1} \quad (28)$$

Substitute in the expressions for the two coefficient ratios (25) and (26), $n_{e,p}$ (27) and $u(\nu)$ (3),

$$\frac{n_{r+1,j} n_e}{n_{r,k}} = \frac{(2\pi m k T)^{3/2}}{4\pi p^2} e^{p^2/2m k T} \frac{g_{r+1,j}}{g_{r,k}} \frac{g_e 4\pi p^2}{h^3} \left(1 + \frac{8\pi h \nu^3}{c^3} \frac{c^3}{8\pi h \nu^3} (e^{h\nu/kT} - 1)\right)^{-1} \quad (29)$$

Make cancellations and apply the energy relation (20),

$$\frac{n_{r+1,j} n_e}{n_{r,k}} = \frac{(2\pi m k T)^{3/2}}{h^3} \frac{g_{r+1,j} g_e}{g_{r,k}} \exp(-(\chi_r + \epsilon_{r+1,j} - \epsilon_{r,k})/kT) \quad (30)$$

Partition function applied

This gives the relative number densities of particles in the two particular excited states labeled in Fig. 2, namely the $r+1,j$ and r,k states. The Saha equation (1) gives, instead, the relative number densities of atoms in the $r+1$ and r ionization states, that is, n_{r+1} and n_r , regardless of their excited states. The relation (8) for the fractional number

density in a given state in terms of the partition function allows us to make this conversion.

The fraction of atoms in the r,k state relative to the total number in the r ionization state is, from (8),

$$\frac{n_{r,k}}{n_r} = \frac{g_{r,k} e^{-\epsilon_{r,k}/kT}}{G_r} \quad (31)$$

where

$$\begin{aligned} G_r(T) &= g_{r,0} + g_{r,1} \exp\left(\frac{-\epsilon_{r,1}}{kT}\right) + g_{r,2} \exp\left(\frac{-\epsilon_{r,2}}{kT}\right) + \dots \\ &= \sum_k g_{r,k} e^{-\epsilon_{r,k}/kT} \end{aligned} \quad (32)$$

Similarly, for the $r+1,j$ state,

$$\frac{n_{r+1,j}}{n_{r+1}} = \frac{g_{r+1,j} e^{-\epsilon_{r+1,j}/kT}}{G_{r+1}} \quad (33)$$

where

$$G_{r+1} = \sum_j g_{r+1,j} e^{-\epsilon_{r+1,j}/kT} \quad (34)$$

Substitute (31) and (33) into (30) to find

$$\Rightarrow \frac{n_{r+1} n_e}{n_r} = \frac{G_{r+1} g_e}{G_r} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp\left(-\frac{\chi_r}{kT}\right) \quad (35)$$

where

- n_{r+1} : Density of atoms in *ionization* state $r+1$ (m^{-3})
- n_r : " " " " " " " r
- n_e : Density of electrons (m^{-3})
- G_{r+1} : Partition function of ionization state $r+1$.
- G_r : Partition function of ionization state r :
- $g_e = 2$: Statistical weight of the electron
- m_e : Mass of the electron
- χ_r : Ionization potential of state r (to reach state $r+1$),
ground level to ground level
- T : Temperature
- h : Planck's constant
- k : Boltzmann constant

This is the Saha equation (1), which has been our objective. It yields the value of the product of $n_{r+1} n_e$ divided by n_r , the densities respectively of the atoms in the higher ($r+1$) ionization state, the density of free electrons, and the density of atoms in the lower (r) ionization state (Fig. 3). It is derived from detailed balance and makes use of statistical distributions (Planck function, Boltzmann formula, and the Maxwell-Boltzmann distribution) with which one is likely to be familiar. The reader might wish to search out a derivation that derives the Saha equation from first principles of statistical mechanics, e.g., that in the text by L. H. Aller, *Astrophysics, The Atmospheres of the Sun and Stars*, second edit., Ronald Press, 1963, Ch. 3.

Examples

If one considers a fixed temperature, it is apparent from (35) that the electron density n_e greatly affects the ionization ratio n_{r+1}/n_r . A large electron density results in more collisions and hence more recombinations and thus lowers the ratio n_{r+1}/r . At a fixed n_e , the ionization ratio is enhanced by a larger partition function G_{r+1} and higher temperature T . Both parameters favor the more highly ionized state. The higher temperature also makes available more phase space for the electrons released during ionization. Recall

$$V_{\text{mom}} = \frac{4}{3}\pi p^3 = \frac{4}{3}\pi(mv)^3 \propto (m kT)^{3/2} \quad (36)$$

where we used $mv^2 \propto kT$.

Hydrogen plasma

An example will illustrate how to extract the degree of ionization n_{r+1}/n_r from the Saha equation (35). Consider hydrogen atoms in an optically thick gas in thermal equilibrium at temperature $T = 6400$ K, the temperature of the photosphere of the sun (i.e., at optical depth unity). Consider hydrogen to be the only gas present. First, let's evaluate $G_r(T)$ in (32), where we take the r state to be neutral hydrogen. The first term of G_r (32) is simply the ground-state statistical weight is $g_{r,0} = 2$ because the electron can have 2 spin states (up and down) relative to the proton.

Now consider the second term of G_r for neutral hydrogen, $g_{r,1} \exp(-\epsilon_{r,1}/kT)$. The atom requires $\epsilon_{r,1} = 10.2$ eV to reach the $n = 2$ state from the ground state. For 6400 K, one has $kT = 0.56$ eV. The statistical weight of the $n = 2$ excited state of neutral hydrogen is $g_{r,1} = 8$; (six p states and two s states). The second term thus becomes $8 \exp(-10.2/0.56) = 9.8 \times 10^{-8}$, which is so small compared to $g_{r,0}$ that it can safely be ignored. The terms for the second and higher energy states are similarly negligible.

At high temperatures, it would appear that the partition function would not converge because there are an infinite number of high-energy bound states. In fact it does converge because there is a physical limit to the number of states the atom can have. The highest states have such large atomic sizes that the electron is equally close to adjacent atoms and is no longer associated solely with the atom under consideration. Also, hydrogen becomes ionized at quite low temperatures for a wide range of densities. The higher states of the neutral atom may therefore not be significantly occupied. Thus, for most

astrophysical applications involving hydrogen, the ground-state statistical weight is a satisfactory approximation for G_r .

$$G_r(T) = g_{r,0} = 2 \quad (37)$$

Similarly, because the proton has no bound electrons in the ionized state, we have

$$G_{r+1}(T) = g_{r+1,0} = 1 . \quad (38)$$

The final statistical weight we need is that of the free electron, $g_e = 2$; it has spin up and spin down; thus the combined statistical weight of the free proton and free electron system is $G_{r+1} g_e = 1 \times 2 = 2$. (One could equally well reverse the values of $g_{r+1,0}$ and g_e for this particular case.) Finally, the given values of the ionization potential and temperature in energy units are:

$$\chi_r = 13.6 \text{ eV}; kT = 0.56 \text{ eV} \quad (39)$$

In the (pure) hydrogen plasma of our example, there will be a free electron with each ionized atom (proton) so we can invoke the condition, $n_{r+1} = n_e$. Now evaluate the Saha equation (35),

$$\frac{n_{r+1}^2}{n_r} = 2.41 \times 10^{21} \frac{1 \times 2}{2} (6400)^{3/2} \exp\left(-\frac{13.6}{0.56}\right) = 3.5 \times 10^{16} \text{ m}^{-3} \quad (40)$$

(aha for pure hydrogen; 6400 K)

This does not yet tell us the degree of ionization; another condition on the densities is required, and this could be that the total density n , ionized and neutral, is $n = n_{r+1} + n_r$. Let us reverse the logic and adopt a degree of ionization and find the density. An illustrative condition is that the plasma be exactly 50% ionized, meaning that the number densities of atoms in the two states are equal, $n_{r+1} = n_r$. In this case, we find from (40) that

$$n_{r+1} = n_r = 3.5 \times 10^{16} \text{ m}^{-3} \quad \text{Densities for 50\% ionization)} \quad (41)$$

Thus, at $T = 6400 \text{ K}$, the *total* hydrogen (H I and H II) number density, or equivalently, the total proton number density, must be

$$n = n_{r+1} + n_r = 7 \times 10^{16} \text{ m}^{-3} \quad (42)$$

if our hydrogen plasma is 50% ionized.

At lower densities n , still at the same temperature, one would expect there to be fewer collisions leading to recombination and thus the plasma to be more than 50% ionized, i.e., $n_{r+1}/n_r > 1$. This follows from the constancy of $(n_{r+1})^2/n_r = [(n_{r+1})/n_r]n_{r+1}$ which tells us that an increase in the ionization ratio n_{r+1}/n_r (above unity) implies a reduced n_{r+1} . In

turn, we know that n_r is less than the reduced n_{r+1} because $n_{r+1}/n_r > 1$. The total density $n = n_{r+1} + n_r$ is thus reduced. Conversely, if the density is greater, the hydrogen will be less than 50% ionized. It will be mostly neutral for only moderately higher values.

Ionization in the sun and universe

In the photosphere of the sun, the mass density is $\sim 3 \times 10^{-4} \text{ kg/m}^3$ which corresponds to $2 \times 10^{23} \text{ protons/m}^3$ ($m_p = 1.67 \times 10^{-27} \text{ kg}$). Since this is substantially larger than the 50% density (42) for $T = 6400 \text{ K}$, the plasma must be more than 50% neutral. Indeed, one can impose the criterion $n_{r+1} + n_r = 2 \times 10^{23} \text{ m}^{-3}$ onto (40) to find that $n_{r+1}/n_r = 4 \times 10^{-4}$ at 6400 K; it is mostly neutral. At a somewhat higher temperature, $\sim 23000 \text{ K}$, one finds that the gas would be very much ionized, $n_{r+1}/n_r \sim 50$.

These temperatures for reaching an ionized state are much less than the value one might naively guess, namely that obtained from $kT = 13.6 \text{ eV}$, or $13.6 \times 11,600 \text{ K} = 160,000 \text{ K}$! (Recall that $T = 11,600 \text{ K}$ yields $kT = 1 \text{ eV}$.) In the dilute gases often found in astrophysics, the temperature required for ionization is quite low. Once an atom is ionized, it has few opportunities to become neutral again; an interaction with a free electron is required and these are rare in a dilute gas.

A graphical presentation of the behavior of the Saha equation is given in Fig. 3. These curves were obtained from an evaluation of (35) at a number of temperatures, for the

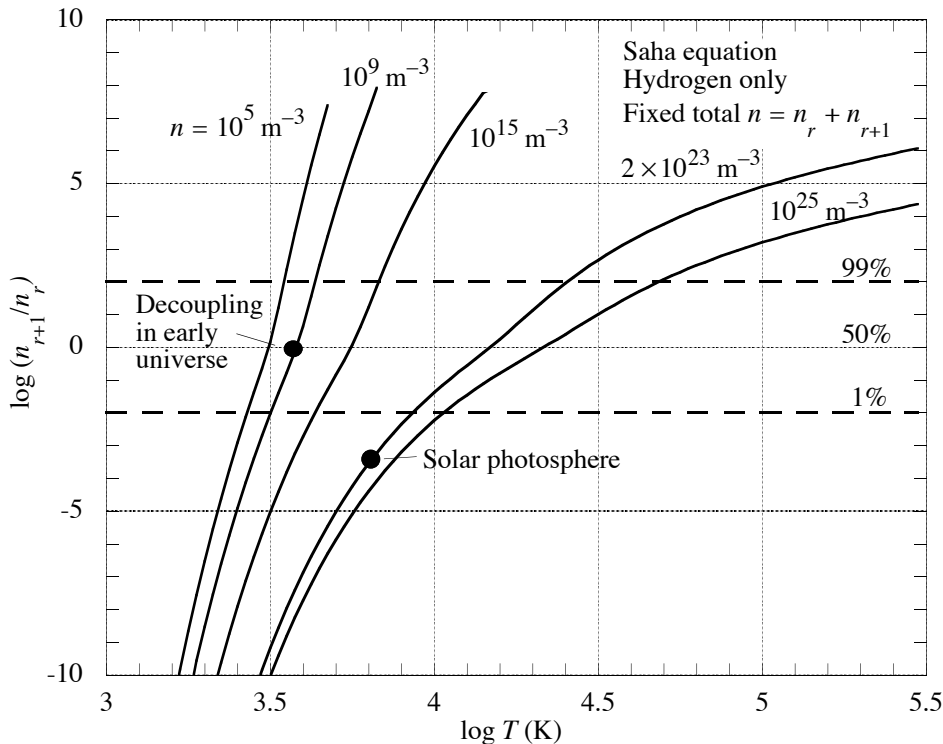


Figure 3. Plot of log ionization fraction n_{r+1}/n_r for various fixed total densities, $n = n_r + n_{r+1}$ (i.e., $n = n_H + n_p$) as a function of temperature. partition functions (37) and (38). This produced a value of $n_{r+1}n_e/n_r = n_{r+1}^2/n_r$ for each temperature. Then, for a given value of the total density $n = n_r + n_{r+1}$, one can solve for

both n_r and n_{r+1} . This allows one to plot $\log(n_{r+1}/n_r)$ in terms of n and T . The ordinate of Fig. 3 is the log of the ratio of ionized to neutral densities. A value of zero indicates 50% of the atoms are ionized. The curves illustrate that the gas in the photosphere of the sun is quite neutral; $n_{r+1}/n_r \approx 4 \times 10^{-4}$ as stated above.

The plot (Fig. 3) covers a wide range of densities from a very low density of $n = 10^5$ nucleons m^{-3} , about the average of visible matter in the universe, to 10^{25}m^{-3} , the number density at which the physical spacing of the atoms is beginning to approach (within a factor of 30 or so) the size of the atoms. At higher densities, the ideal gas law no longer applies. At the density of the center of the sun, $\sim 10^{32} \text{m}^{-3}$, the spacing is less than the size of the ground-state orbital of the electron. The electrons can no longer be associated with a given atom and the Saha equation becomes unreliable.

The Saha equation can also give insight into the state of ionization of hydrogen in the early universe. Standard models indicate that, as the expanding universe cools, the ionized plasma recombines to neutral hydrogen. The universe then becomes transparent to optical photons. At this time ($\sim 10^5$ y), the density is $n \approx 10^9 \text{m}^{-3}$. The curve for this total density of e^- and p crosses the 50% axis at temperature ~ 4000 K. Thus the hydrogen becomes neutral at about this temperature. The relatively cool photons can no longer ionize the hydrogen so they are free to travel freely through the universe. They are seen today as the microwave background radiation. They have cooled from 4000 K to 3 K due to the expansion of the universe in the meantime.

Proper calculations of the states of ionization require knowledge of the ionization states of other elements since all elements provide the free electrons that tend to force the ionized atoms toward neutrality. Furthermore the statistical weights of the individual states and the values of the partition functions are not intuitively obvious; they are derived from the quantum structure of the particular atom under consideration.

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