

**GLOBAL EXISTENCE AND SCATTERING  
FOR ROUGH SOLUTIONS OF A NONLINEAR SCHRÖDINGER EQUATION ON  $\mathbb{R}^3$**

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ABSTRACT. We prove global existence and scattering for the defocusing, cubic nonlinear Schrödinger equation in  $H^s(\mathbb{R}^3)$  for  $s > \frac{4}{5}$ . The main new estimate in the argument is a Morawetz-type inequality for the solution  $\phi$ . This estimate bounds  $\|\phi(x, t)\|_{L_{x,t}^4(\mathbb{R}^3 \times \mathbb{R})}$ , whereas the well-known Morawetz-type estimate of Lin-Strauss controls  $\int_0^\infty \int_{\mathbb{R}^3} \frac{(\phi(x, t))^4}{|x|} dx dt$ .

1. INTRODUCTION AND STATEMENT OF RESULTS

We study the following initial value problem for a cubic defocusing nonlinear Schrödinger equation,

$$(1.1) \quad i\partial_t \phi(x, t) + \Delta \phi(x, t) = |\phi(x, t)|^2 \phi(x, t), \quad x \in \mathbb{R}^3, t \geq 0,$$

$$(1.2) \quad \phi(x, 0) = \phi_0(x) \in H^s(\mathbb{R}^3).$$

Here  $H^s(\mathbb{R}^3)$  denotes the usual inhomogeneous Sobolev space.

It is known [5] that (1.1)-(1.2) is well-posed locally in time in  $H^s(\mathbb{R}^3)$  when<sup>1</sup>  $s > \frac{1}{2}$ . In addition, these local solutions enjoy  $L^2$  conservation,

$$(1.3) \quad \|\phi(\cdot, t)\|_{L^2(\mathbb{R}^3)} = \|\phi_0(\cdot)\|_{L^2(\mathbb{R}^3)},$$

and the  $H^1(\mathbb{R}^3)$  solutions have the following conserved energy,

$$(1.4) \quad E(\phi)(t) \equiv \int_{\mathbb{R}^3} \frac{1}{2} |\nabla_x \phi(x, t)|^2 + \frac{1}{4} |\phi(x, t)|^4 dx = E(\phi)(0).$$

Together, these conservation laws and the local-in-time theory immediately yield global-in-time well-posedness of (1.1)-(1.2) from data in  $H^s(\mathbb{R}^3)$  when  $s \geq 1$ . It is conjectured that (1.1)-(1.2) is in fact globally well-posed in time from all data included in the local theory. Previous work ([16], extending [3]) established this global theory when  $s > \frac{5}{6}$ . Our first goal here is to loosen further the regularity requirements on the initial data which

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<sup>1</sup>In addition, there are local in time solutions from  $H^{\frac{1}{2}}$  data, however, the time interval of existence depends upon the profile of the initial data and not just upon the data's Sobolev norm. Note that the  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$  norm is *critical* in the sense that it is invariant under the natural scaling of solutions to (1.1).

ensure global-in-time solutions. In addition we aim to loosen the symmetry assumptions on the data which were previously used [3] to prove scattering for rough solutions.

Before stating our main result, we recall some terminology (see e.g. [6, 17]). Write  $S^L(t)$  for the flow map  $e^{it\Delta}$  corresponding to the linear Schrödinger equation, and  $S^{NL}(t)$  for the nonlinear flow, that is  $S^{NL}(t)\phi_0 = \phi(x, t)$  with  $\phi, \phi_0$  as in (1.1), (1.2). Given a solution<sup>2</sup>  $\phi \in C((-\infty, \infty), H^s(\mathbb{R}^3))$  of (1.1)-(1.2), define the asymptotic states  $\phi^\pm$  and wave operators  $\Omega^\pm : H^s(\mathbb{R}^3) \rightarrow H^s(\mathbb{R}^3)$  by

$$(1.5) \quad \phi^\pm = \lim_{t \rightarrow \pm\infty} S^L(-t)S^{NL}(t)\phi_0$$

$$(1.6) \quad \Omega^\pm \phi^\pm = \phi_0$$

in so far as these limits exist in  $H^s(\mathbb{R}^3)$ . When the wave operators  $\Omega^\pm$  are surjective we say that (1.1)-(1.2) is *asymptotically complete* in  $H^s(\mathbb{R}^3)$ .

Our main result is the following:

**Theorem 1.1.** *The initial value problem (1.1)-(1.2) is globally-well-posed from data  $\phi_0 \in H^s(\mathbb{R}^3)$  when  $s > \frac{4}{5}$ . In addition, there is scattering for these solutions. More precisely, the wave operators (1.6) exist and there is asymptotic completeness on all of  $H^s(\mathbb{R}^3)$ .*

By *globally-well-posed*, we mean that given data  $\phi_0 \in H^s(\mathbb{R}^n)$  as above, and any time  $T > 0$ , there is a unique solution to (1.1)-(1.2)

$$(1.7) \quad \phi(x, t) \in C([0, T]; H^s(\mathbb{R}^n))$$

which depends continuously in (1.7) upon  $\phi_0 \in H^s(\mathbb{R}^n)$ .

We sketch the relationship of our results here with previous work.

Scattering in the space  $H^1(\mathbb{R}^3)$  was shown in [17]. Theorem 1.1 extends part of the work<sup>3</sup> in [3, 4] where global well-posedness was shown for general  $H^s(\mathbb{R}^3)$  data,  $s > \frac{11}{13}$ . (See [1] for a related result in two space dimensions.) In the case of radially symmetric data, [3, 4] establish global well-posedness and scattering for  $\phi_0 \in H^s(\mathbb{R}^3), s > \frac{5}{7}$ . Theorem 1.1 also extends the result of [16], where we showed global existence for  $s > \frac{5}{6}$ , with no scattering statement.

As in [16], our arguments here preclude growth of  $\|\phi(t)\|_{H^s(\mathbb{R}^3)}$  by showing that the energy of a smoothed version of the solution is almost conserved<sup>4</sup>. We refer to [16] (pages 2-3) for remarks comparing the almost conservation law approach used here with the argument in [3, 4]. See [22, 21, 11, 14] for further applications of almost conservation laws; and [15, 13, 12] for instances where the inclusion of damping correction terms in the almost conserved energy leads to sharp results. Unlike our work in [16], where  $\|\phi(t)\|_{H^s(\mathbb{R}^3)}$  was bounded polynomially in time, we ultimately obtain here a uniform bound. The main new estimate allowing such a uniform bound is the Morawetz-type estimate (2.26) for the solution  $u$  of any relatively general defocusing nonlinear Schrödinger equation, see (2.1) below. Besides yielding the scattering results which come along with such a uniform  $H^s$  bound, this new estimate is also the ingredient which pushes the allowed regularity in Theorem 1.1 below our previously obtained  $s > \frac{5}{6}$ . We do not expect our results here to be sharp. For example, we hope to extend Theorem 1.1 to allow lower values of  $s$ , using the correction terms mentioned above and multilinear estimates (stemming from e.g. [10, 31]) to more tightly bound the increment in the almost-conserved quantity.

<sup>2</sup>We can easily extend the solution in (1.1) to negative times by the equation's time reversibility.

<sup>3</sup>In [3, 4], it is also shown that the difference between the linear and nonlinear evolutions from rough data has finite energy. Our technique neither employs nor implies such smoothing.

<sup>4</sup>The phrase *almost conserved* is made precise in Proposition 3.1 below.

Theorem 1.1 above, like the referenced work on global rough solutions for other dispersive equations, has a number of motivations. We mention here three. First and most obviously, we aim to better understand the global in time evolution properties of known local-in-time solutions. Second, our results for rough solutions yield polynomial in time bounds<sup>5</sup> for the growth of some below-energy Sobolev norms of *smooth* solutions. Such bounds give, for example, a qualitative understanding of how the energy in a smooth solution moves from high frequencies to low frequencies<sup>6</sup>. Third, we hope that the techniques developed for these subcritical, rough initial data problems can be used to address open problems for relatively smooth solutions. For an immediate example, our arguments below give a new proof of the finite energy scattering result of [17]. Also, the bounds we obtain on the global Schrödinger admissible space-time norms of the solution depend polynomially on the energy of the initial data, whereas previous bounds were exponential. (See the remark in [3], page 276, and (2.26), (4.20) below.) There are of course more significant examples where low-regularity techniques have helped to solve open problems for smooth solutions, e.g. [2, 32].

The paper is organized as follows. In Section 2, after recalling the standard Morawetz-type estimates from Lin-Strauss [25], we introduce a Morawetz interaction potential and prove it is bounded and monotone increasing. As a consequence, we obtain the aforementioned spacetime  $L_{x,t}^4$  bound on solutions of (1.1). Section 3 revisits the almost conservation law argument in [16], now in the setting of an a-priori  $L_{x,t}^4$  bound on a spacetime slab. In Section 4, we first show in Proposition 4.1 how the almost conservation law (Proposition 3.1), the interaction Morawetz inequality (2.26), and the assumption  $s > \frac{4}{5}$  combine with a scaling and bootstrap argument to give a uniform bound on  $\|\phi(t)\|_{H^s(\mathbb{R}^3)}$  and the finiteness of  $\|\phi\|_{L^4(\mathbb{R}^3 \times [0, \infty))}$ . The scattering claims in Theorem 1.1 follow from these uniform bounds and by now well-known arguments from earlier scattering results of Brenner, Ginibre, Glassey, Morawetz, Strauss, and Velo (see surveys in [6, 29]).

Note that for finite energy solutions, that is  $s = 1$ , Proposition 4.1 follows immediately from energy conservation and the interaction Morawetz inequality (2.26). Hence in case  $s = 1$ , the arguments in Section 2 and the later part of Section 4 below give a new, relatively direct proof of scattering for (1.1) in the energy class  $H^1(\mathbb{R}^3)$ . This result was first established by Ginibre-Velo [17].

We conclude this introduction by setting some notation and recalling the Strichartz estimates for the linear Schrödinger operator on  $\mathbb{R}^3$ . Given  $A, B \geq 0$ , we write  $A \lesssim B$  to mean that for some universal constant  $K > 2$ ,  $A \leq K \cdot B$ . We write  $A \sim B$  when both  $A \lesssim B$  and  $B \lesssim A$ . The notation  $A \ll B$  denotes  $B > K \cdot A$ . We write  $\langle A \rangle \equiv (1 + A^2)^{\frac{1}{2}}$ , and  $\langle \nabla \rangle$  for the operator with Fourier multiplier  $(1 + |\xi|^2)^{\frac{1}{2}}$ . The symbol  $\nabla$  will denote the spatial gradient. We will often use the notation  $\frac{1}{2} + \equiv \frac{1}{2} + \epsilon$  for some universal  $0 < \epsilon \ll 1$ . Similarly, we write  $\frac{1}{2} - \equiv \frac{1}{2} - \epsilon$ .

Given Lebesgue space exponents  $q, r$  and a function  $F(x, t)$  on  $\mathbb{R}^{n+1}$ , we write

$$(1.8) \quad \|F\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \equiv \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |F(x, t)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}.$$

This norm will be shortened to  $L_t^q L_x^r$  for readability, or to  $L_{x,t}^r$  when  $q = r$ .

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<sup>5</sup>In this paper, we in fact get a *uniform* bound on the growth.

<sup>6</sup>If one has a smooth solution with large but finite energy, the below-energy Sobolev norms could presumably start relatively small and grow large when the low frequencies of the solution grow in (for example)  $L^2$ , while the high frequencies decrease in  $L^2$ . A polynomial bound on the rough norm's growth puts limits on this movement of energy from high to low frequencies.

The Strichartz estimates involve the following definition: a pair of Lebesgue space exponents are called *Schrödinger admissible* for  $\mathbb{R}^{3+1}$  when  $q, r \geq 2$ , and

$$(1.9) \quad \frac{1}{q} + \frac{3}{2r} = \frac{3}{4}.$$

**Proposition 1.1** (Strichartz estimates in 3 space dimensions (See e.g. [27, 28, 18, 33, 23])). *Suppose that  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  are any two Schrödinger admissible pairs as in (1.9). Suppose too that  $\phi(x, t)$  is a (weak) solution to the problem*

$$\begin{aligned} (i\partial_t + \Delta)\phi(x, t) &= F(x, t), \quad (x, t) \in \mathbb{R}^3 \times [0, T], \\ \phi(x, 0) &= \phi_0(x), \end{aligned}$$

for some data  $u_0$  and  $T > 0$ . Then we have the estimate

$$(1.10) \quad \|\phi\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^3)} \lesssim \|\phi_0\|_{L^2(\mathbb{R}^3)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}([0, T] \times \mathbb{R}^3)},$$

where  $\frac{1}{q} + \frac{1}{\tilde{q}'} = 1, \frac{1}{r} + \frac{1}{\tilde{r}'} = 1$ .

## 2. THE MORAWETZ INTERACTION POTENTIAL AND A SPACETIME $L^4$ ESTIMATE

This section introduces an interaction potential generalization of the classical Morawetz action and associated inequalities. We first recall the standard Morawetz action centered at a point and the proof that this action is monotonically increasing with time when the nonlinearity is defocusing. The interaction generalization is introduced in the second subsection. The key consequence of the analysis in this section for the scattering result is the  $L_{x,t}^4$  estimate (2.26).

The discussion in this section will be carried out in the context of the following generalization of (1.1)-(1.2):

$$(2.1) \quad i\partial_t u + \alpha\Delta u = \mu f(|u|^2)u, \quad u : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{C},$$

$$(2.2) \quad u(0) = u_0.$$

Here  $f$  is a smooth function  $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$  and  $\alpha$  and  $\mu$  are real constants that permit us to easily distinguish in the analysis below those terms arising from the Laplacian or the nonlinearity. We also define  $F(z) = \int_0^z f(s)ds$ .

We will use polar coordinates  $x = r\omega$ ,  $r > 0, \omega \in S^2$ , and write  $\Delta_\omega$  for the Laplace-Beltrami operator on  $S^2$ . For ease of reference below, we record some alternate forms of the equation in (2.1):

$$(2.3) \quad u_t = i\alpha\Delta u - i\mu f(|u|^2)u,$$

$$(2.4) \quad \bar{u}_t = -i\alpha\Delta\bar{u} + i\mu f(|u|^2)\bar{u},$$

$$(2.5) \quad u_t = i\alpha u_{rr} + i\frac{2\alpha}{r}u_r + i\frac{\alpha}{r^2}\Delta_\omega u - i\mu f(|u|^2)u,$$

$$(2.6) \quad (ru_t) = i\alpha(ru)_{rr} + i\frac{\alpha}{r}\Delta_\omega u - i\mu r f(|u|^2)u,$$

$$(2.7) \quad (r\bar{u}_t) = -i\alpha(r\bar{u})_{rr} - i\frac{\alpha}{r}\Delta_\omega\bar{u} + i\mu r f(|u|^2)\bar{u}.$$

**2.1. Standard Morawetz action and inequalities.** We will call the following quantity the *Morawetz action centered at  $\theta$*  for the solution  $u$  of (2.1),

$$(2.8) \quad M_0[u](t) = \int_{\mathbb{R}^3} \operatorname{Im}[\bar{u}(t, x) \nabla u(t, x)] \cdot \frac{x}{|x|} dx.$$

We check using the equation that,

$$(2.9) \quad \partial_t(|u|^2) = -2\alpha \nabla \cdot \operatorname{Im}[\bar{u}(t, x) \nabla u(t, x)],$$

hence we may interpret  $M_0$  as the spatial average of the radial component of the  $L^2$ -mass current. We might expect that  $M_0$  will increase with time if the wave  $u$  scatters since such behavior involves a broadening redistribution of the  $L^2$ -mass. The following proposition of Lin and Strauss indeed gives  $\frac{d}{dt}M_0[u](t) \geq 0$  for defocusing equations.

**Proposition 2.1.** [25] *If  $u$  solves (2.1)-(2.2) then the Morawetz action at  $\theta$  satisfies the identity*

$$(2.10) \quad \partial_t M_0[u](t) = 4\pi\alpha |u(t, \theta)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x|} |\nabla_0 u(t, x)|^2 dx + \mu \int_{\mathbb{R}^3} \frac{2}{|x|} \{|u|^2 f(|u|^2)(t) - F(|u|^2)\} dx.$$

where  $\nabla_0$  is the angular component of the derivative,

$$(2.11) \quad \nabla_0 u = \nabla u - \frac{x}{|x|} \left( \frac{x}{|x|} \cdot \nabla u \right).$$

In particular,  $M_0$  is an increasing function of time if the equation (2.1) satisfies the repulsivity condition,

$$(2.12) \quad \mu \{|u|^2 f(|u|^2)(t) - F(|u|^2)\} \geq 0.$$

Note that for pure power potentials  $F(x) = \frac{2}{p+1} x^{\frac{p+1}{2}}$ , where the nonlinear term in (2.1) is  $|u|^{p-1}u$ , the function  $|u|^2 f(|u|^2) - F(|u|^2) = \frac{p-1}{2} F(|u|^2)$ . Hence condition (2.12) holds.

*Proof.* Clearly, we may write

$$(2.13) \quad M_0(t) = \operatorname{Im} \int_{\mathbb{R}^3} \bar{u}(t, x) \left( \partial_r + \frac{1}{r} \right) u(t, x) dx$$

$$(2.14) \quad = \operatorname{Im} \int_0^\infty \int_{S^2} \overline{ru}(ru)_r d\omega dr,$$

since we are working in three space dimensions. Integrating by parts and using the equation (2.6) gives,

$$\begin{aligned} \frac{d}{dt} M_0 &= \int_0^\infty \int_{S^2} \overline{(ru)}(ru_t)_r + \overline{(ru_t)}(ru)_r d\omega dr \\ &= -2\operatorname{Im} \int_0^\infty \int_{S^2} \overline{(ru)}_r (ru_t) d\omega dr \\ &= -2\operatorname{Im} \int_0^\infty \int_{S^2} \overline{(ru)}_r \left\{ i\alpha (ru)_{rr} + i\frac{\alpha}{r} \Delta_\omega u - i\mu r f(|u|^2) u \right\} d\omega dr \\ &= -2\alpha \operatorname{Re} \int_0^\infty \int_{S^2} \overline{(ru)}_r (ru)_{rr} d\omega dr - 2\alpha \operatorname{Re} \int_0^\infty \int_{S^2} \overline{(ru)}_r \frac{1}{r} \Delta_\omega u d\omega dr \\ &\quad + 2\mu \operatorname{Re} \int_0^\infty \int_{S^2} \overline{(ru)}_r r f(|u|^2) u d\omega dr \\ &= I + II + III. \end{aligned}$$

These three terms are analyzed separately and lead to the three terms on the right side of (2.10).

**Term I:** Since  $\partial_r |(ru)_r|^2 = 2\text{Re}(\overline{(ru)_r} (ru)_{rr})$ , the  $r$  integration in Term I equals  $|(ru)_r|^2|_0^\infty = -|u(t, 0)|^2$  which accounts for the first term in (2.10).

**Term II:** Write  $\Delta_\omega = \nabla_\omega \cdot \nabla_\omega$  and integrate by parts to get,

$$II = \alpha \text{Re} \int_0^\infty \int_{S^2} \left[ \partial_r |\nabla_\omega u|^2 + \frac{2}{r} |\nabla_\omega u|^2 \right] d\omega dr.$$

Since  $|\nabla_\omega u| \sim r|\nabla u|$ , we know that  $|\nabla_\omega u|$  vanishes at the origin. Therefore, the first term integrates to zero. Finally, we can reexpress the remaining term as claimed in (2.10) by inserting  $r^2$  in the numerator and denominator and then absorbing two factors of  $r$  using  $\nabla_\omega u = r\nabla_0 u$ .

**Term III:** We expand the integrand using the Leibniz rule to find  $(\bar{u} + r\bar{u}_r)rf(|u|^2)u = r|u|^2 f(|u|^2) + r^2 f(|u|^2)u\bar{u}_r$ . The first of these terms is purely real valued. The real part of the second term may be reexpressed using  $2\text{Re}f(|u|^2)u\bar{u}_r = [F(|u|^2)]_r$ . Upon integrating this last term by parts with respect to  $r$ , we obtain the third expression in (2.10).

The remaining claim in the Proposition follows directly from (2.10).  $\square$

We may center the above argument at any other point  $y \in R^3$  with corresponding results. Toward this end, define the *Morawetz action centered at  $y$*  to be,

$$(2.15) \quad M_y[u](t) = \int_{\mathbb{R}^3} \text{Im}[\bar{u}(x)\nabla u(x)] \cdot \frac{x-y}{|x-y|} dx.$$

We shall often drop the  $u$  from this notation, as we did previously in writing  $M_0(t)$ .

**Corollary 2.1.** *If  $u$  solves (2.1) the Morawetz action at  $y$  satisfies the identity*

$$(2.16) \quad \frac{d}{dt} M_y = 4\pi\alpha |u(t, y)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla_y u(t, x)|^2 dx + \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \{ |u|^2 f(|u|^2) - F(|u|^2) \} dx,$$

where  $\nabla_y u \equiv \nabla u - \frac{x-y}{|x-y|} \left( \frac{x-y}{|x-y|} \cdot \nabla u \right)$ . In particular,  $M_y$  is an increasing function of time if the nonlinearity satisfies the repulsivity condition (2.12).

Corollary 2.1 shows that a solution is, on average, repulsed from any fixed point  $y$  in the sense that  $M_y[u](t)$  is increasing with time.

For our scattering results, we'll need the following pointwise bound for  $M_y[u](t)$ .

**Lemma 2.1.** *Assume  $u$  is a solution of (2.1) and  $M_y[u](t)$  as in (2.15). Then,*

$$(2.17) \quad |M_y(t)| \lesssim \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

*Proof.* Without loss of generality we take  $y = 0$ . This is a refinement of the easy bound using Cauchy-Schwarz  $|M_y(t)| \lesssim \|u(t)\|_{L_x^2} \|\nabla u(t)\|_{L_x^2}$ . By duality

$$\left| \text{Im} \int_{\mathbb{R}^3} \overline{u(x, t)} \partial_r u(x, t) dx \right| \leq \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \cdot \|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}.$$

It suffices to show  $\|\partial_r u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \leq \|u\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}$ . By duality and the definition  $\partial_r \equiv \frac{x}{|x|} \cdot \nabla$ , it remains to prove,

$$(2.18) \quad \left\| \frac{x}{|x|} f \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)},$$

for any  $f$  for which the right hand side is finite. Inequality (2.18) follows from interpolating between the following two bounds,

$$\begin{aligned}\left\|\frac{x}{|x|}f\right\|_{L^2(\mathbb{R}^3)} &\leq \|f\|_{L^2(\mathbb{R}^3)} \\ \left\|\frac{x}{|x|}f\right\|_{\dot{H}^1(\mathbb{R}^3)} &\lesssim \|f\|_{H^1(\mathbb{R}^3)}\end{aligned}$$

the first of which is trivial, the second of which follows from Hardy's inequality,

$$\begin{aligned}\left\|\nabla\left(\frac{x}{|x|}f\right)\right\|_{L^2} &\leq \left\|\frac{x}{|x|}\cdot\nabla f\right\|_{L^2} + \left\|\frac{1}{|x|}f\right\|_{L^2} \\ &\lesssim \|\nabla f\|_{L^2}.\end{aligned}$$

□

The well-known Morawetz-type inequalities which have proven useful in proving local decay or scattering for (2.1) arise by integrating the identity (2.10) or (2.16) in time. For nonlinear Schrödinger equations, this argument appears in the work of Lin and Strauss [25], who cite as motivation earlier work on Klein-Gordon equations by Morawetz [26].

**Corollary 2.2** (Morawetz inequalities [25]). *Suppose  $u$  solves (2.1)-(2.2). Then for any  $y \in \mathbb{R}^3$ ,*

$$(2.19) \quad \begin{aligned}2 \sup_{t \in [0, T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 &\gtrsim 4\pi\alpha \int_0^T |u(t, y)|^2 dt + \int_0^T \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla_y u(t, x)|^2 dx dt \\ &+ \int_0^T \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \{|u|^2 f(|u|^2) - F(|u|^2)\} dx dt.\end{aligned}$$

Assuming (2.1) has a repulsive nonlinearity as in (2.12), all terms on the right side of the inequality (2.19) are positive. The inequality therefore gives in particular a bound uniform in  $T$  for the quantity  $\int_0^T \int_{\mathbb{R}^3} \frac{|u(t, x)|^4}{|x-y|} dx dt$ , for solutions  $u$  of (1.1).

In their proof of scattering in the energy space for the cubic defocusing problem (1.1), Ginibre and Velo [17] combine this relatively localized<sup>7</sup> decay estimate with a bound surrogate for finite propagation speed in order to show the solution is in certain global-in time Lebesgue spaces  $L^q([0, \infty), L^r(\mathbb{R}^3))$ . Scattering follows rather quickly.

In the following section, we show how to establish an unweighted, global in time Lebesgue space bound directly. The argument below involves the identity (2.16), but our estimate arises eventually from the linear part of the equation, more specifically from the first term on the right of (2.16), rather than the third (nonlinearity) term.

**2.2. Morawetz interaction potential.** Given a solution  $u$  of (2.1), we define the *Morawetz interaction potential* to be

$$(2.20) \quad M(t) = \int_{\mathbb{R}^3} |u(t, y)|^2 M_y(t) dy.$$

The bound (2.17) immediately implies

$$(2.21) \quad |M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

<sup>7</sup>The bound mentioned here may be considered localized since it implies decay of the solution near the fixed point  $y$ , but doesn't preclude the solution staying large at a point which moves rapidly away from  $y$ , for example.

If  $u$  solves (2.1) then the identity (2.16) gives us the following identity for  $\frac{d}{dt}M(t)$ ,

$$(2.22) \quad \begin{aligned} \frac{d}{dt}M(t) &= 4\pi\alpha \int_y |u(y)|^4 dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |u(y)|^2 |\nabla_y u(x)|^2 dx dy \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} |u(y)|^2 \{ |u(x)|^2 f(|u(x)|^2) - F(|u(x)|^2) \} dx dy \\ &\quad + \int_{\mathbb{R}^3} \partial_t (|u(t, y)|^2) M_y(t) dy. \end{aligned}$$

We write the the right side of (2.22) as  $I + II + III + IV$ , and work now to rewrite this as a sum involving nonnegative terms.

**Proposition 2.2.** *Referring to the terms comprising (2.22), we have*

$$(2.23) \quad IV \geq -II.$$

Consequently, solutions of (2.1) satisfy

$$(2.24) \quad \frac{d}{dt}M(t) \geq 4\pi\alpha \int_{\mathbb{R}^3} |u(t, y)|^4 dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} |u(t, y)|^2 \{ |u|^2 f(|u|^2) - F(|u|^2) \} dx dy.$$

In particular,  $M(t)$  is monotone increasing for equations with repulsive nonlinearities.

Assuming Proposition 2.23 for the moment, we combine (2.21) and (2.24) to obtain the following estimate which plays the major new role in our analysis in Sections 3 and 4 below,

**Corollary 2.3.** *Take  $u$  to be a smooth solution to the initial value problem (2.1)-(2.2) above, under the repulsivity assumption (2.12). Then we have the following interaction Morawetz inequalities,*

$$(2.25) \quad \begin{aligned} 2\|u(t)\|_{L^2}^2 \sup_{t \in [0, t]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 &\gtrsim 4\pi\alpha \int_0^T \int_{\mathbb{R}^3} |u(t, y)|^4 dy dt \\ &\quad + \int_0^T \int_y \int_x \frac{2\mu}{|x-y|} |u(t, y)|^2 \{ |u|^2 f(|u|^2) - F(|u|^2) \} (t, x) dx dy dt. \end{aligned}$$

In particular, we obtain the following spacetime  $L^4([0, \infty) \times \mathbb{R}^3)$  estimate,

$$(2.26) \quad \int_0^T \int_{\mathbb{R}^3} |u(t, y)|^4 dy dt \lesssim \|u_0\|_{L^2(\mathbb{R}^3)}^2 \sup_{t \in [0, t]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2.$$

Of course, for solutions of (1.1) starting from finite energy initial data, the right side of (2.26) is uniformly bounded by energy considerations - leading to a rather direct proof of the result in [17] of scattering in the energy space. This bound (2.26) is also a key part of our rough data scattering argument below.

*Proof.* We now turn to the proof of Proposition 2.2. Use (2.9) to write

$$\begin{aligned} IV &= - \int_{\mathbb{R}_y^3} \nabla \cdot \text{Im}[2\alpha \bar{u}(y) \nabla u(y)] M_y(t) dy \\ &= - \int_y \int_x \partial_{y_i} \text{Im}[2\alpha \bar{u}(y) \partial_{y_i} u(y)] \text{Im}[\bar{u}(x) \frac{x_m - y_m}{|x-y|} \partial_{x_m} u(x)] dx dy, \end{aligned}$$



where repeated indices are implicitly summed. We integrate by parts in  $y$ , moving the the leading  $\partial_{y_l}$  to the unit vector  $\frac{x-y}{|x-y|}$ . Note that,

$$(2.27) \quad \partial_{y_l} \left( \frac{x_m - y_m}{|x-y|} \right) = \frac{-\delta_{lm}}{|x-y|} + \frac{(x_l - y_l)(x_m - y_m)}{|x-y|^3}.$$

Write  $\mathbf{p}(x) = \text{Im}[\bar{u}(x)\nabla u(x)]$  for the mass current at  $x$  and use (2.27) to obtain

$$(2.28) \quad IV = -2\alpha \int_y \int_x \left[ \mathbf{p}(y) \cdot \mathbf{p}(x) - (\mathbf{p}(y) \cdot \frac{x-y}{|x-y|})(\mathbf{p}(x) \cdot \frac{x-y}{|x-y|}) \right] \frac{dxdy}{|x-y|}.$$

The preceding integrand has a natural geometric interpretation. We are removing the inner product of the components of  $\mathbf{p}(y)$  and  $\mathbf{p}(x)$  parallel to the vector  $\frac{x-y}{|x-y|}$  from the full inner product of  $\mathbf{p}(y)$  and  $\mathbf{p}(x)$ . This amounts to taking the inner product of  $\pi_{(x-y)^\perp} \mathbf{p}(y) \cdot \pi_{(x-y)^\perp} \mathbf{p}(x)$  where we have introduced the projections onto the subspace of  $\mathbb{R}^3$  perpendicular to the vector  $\frac{x-y}{|x-y|}$ . But

$$(2.29) \quad |\pi_{(x-y)^\perp} \mathbf{p}(y)| = \left| \mathbf{p}(y) - \frac{x-y}{|x-y|} \left( \frac{x-y}{|x-y|} \cdot \mathbf{p}(y) \right) \right| = |\text{Im}[\bar{u}(y)\nabla_x u(y)]| \leq |u(y)| \cdot |\nabla_x u(y)|.$$

A similar identity and inequality holds upon switching the roles of  $x$  and  $y$  in (2.29). We have thus shown that

$$(2.30) \quad IV \geq -2\alpha \int_y \int_x |u(x)| \cdot |\nabla_y u(x)| \cdot |u(y)| \cdot |\nabla_x u(y)| \frac{dxdy}{|x-y|}.$$

The conclusion (2.23) follows by applying the elementary bound  $|ab| \leq \frac{1}{2}(a^2 + b^2)$  with  $a = |u(y)| \cdot |\nabla_y u(x)|$  and  $b = |u(x)| \cdot |\nabla_x u(y)|$ .  $\square$

### 3. ALMOST CONSERVATION LAW.

Keeping in mind that the energy (1.4) of our solutions might be infinite, our aim will be to control the growth in time of  $E(I\phi)(t)$ , where  $I\phi$  is a smoothed version of  $\phi$ . The operator  $I$  depends on a parameter  $N \gg 1$  to be chosen later, and the level of regularity  $s < 1$  at which we are working<sup>8</sup>. We write,

$$(3.1) \quad \widehat{I}f(\xi) \equiv m_N(\xi)\hat{f}(\xi),$$

where the multiplier  $m_N(\xi)$  is smooth, radially symmetric, nonincreasing in  $|\xi|$  and

$$(3.2) \quad m_N(\xi) = \begin{cases} 1 & |\xi| \leq N \\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi| \geq 2N. \end{cases}$$

The following two inequalities follow quickly from the definition of  $I$ , the  $L^2$  conservation (1.3), and by considering separately those frequencies  $|\xi| \leq N$  and  $|\xi| \geq N$ .

$$(3.3) \quad E(I\phi)(t) \lesssim \left( N^{1-s} \|\phi(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^3)} \right)^2 + \|\phi(t, \cdot)\|_{L^4(\mathbb{R}^3)}^4,$$

$$(3.4) \quad \|\phi(\cdot, t)\|_{H^s(\mathbb{R}^3)}^2 \lesssim E(I\phi)(t) + \|\phi_0\|_{L^2(\mathbb{R}^3)}^2.$$

In studying the possible growth of our solution in time, we will estimate  $E(I\phi)(t)$  rather than bounding  $\|\phi(t)\|_{H^s(\mathbb{R}^3)}$  directly. Of course, since (1.1) is a nonlinear equation,  $I\phi(x, t)$  is not a solution. In particular, one doesn't expect  $E(I\phi)(t)$  to be constant. One of the main ingredients of Theorem 1.1 is proving that this quantity is uniformly bounded in time. The local in time result which contributes to the proof of such a bound is what we mean by an *almost conservation law*. Global well-posedness follows from (3.4), a uniform bound on  $E(I\phi)(t)$  in terms of  $\|\phi_0\|_{H^s(\mathbb{R}^3)}$ , the fact that (1.1)-(1.2) is locally well posed when  $s > \frac{1}{2}$ , and a density argument.

<sup>8</sup>We abuse notation and suppress this dependence, writing simply  $I$  instead of  $I_{s,N}$ .

**Proposition 3.1** (Almost Conservation Law). *Assume we have  $s > \frac{1}{2}$ ,  $N \gg 1$ ,  $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ , and a solution of (1.1)-(1.2) on a time interval  $[0, T]$  for which*

$$(3.5) \quad \|\phi\|_{L_{x,t}^4([0,T] \times \mathbb{R}^3)} \lesssim \epsilon.$$

*Assume in addition that  $E(I\phi_0) \lesssim 1$ .*

*We conclude that for all  $t \in [0, T]$ ,*

$$(3.6) \quad E(I\phi)(t) = E(I\phi_0) + O(N^{-1+}).$$

Equation (3.6) asserts that  $I\phi$ , though not a solution of the nonlinear problem (1.1), enjoys something akin to energy conservation. If one could replace the increment  $N^{-1+}$  in  $E(I\phi)$  on the right side of (3.6) with  $N^{-\alpha}$  for some  $\alpha > 0$ , one could repeat the argument we give below to prove global well-posedness of (1.1)-(1.2) for all  $s > \frac{3+\alpha}{3+2\alpha}$ . In particular, if  $E(I\phi)(t)$  were conserved (i.e.  $\alpha = \infty$ ), one could show that (1.1)-(1.2) is globally well-posed when  $s > \frac{1}{2}$ . Recall that the scale-invariant Sobolev space is  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ .

Proposition 3.1 is a modification of a similar statement (also labelled Proposition 3.1) in [16]. The statement in [16] establishes a uniform time step, determined by the size of the modified energy of the data  $E(I\phi)$ , on which there is almost conservation of  $E(I\phi)(t)$ . Here we obtain an almost conservation property in time intervals  $[0, T]$  on which  $\phi$  is assumed small in  $L_{x,t}^4$ . Note that these intervals may have various lengths, and that the constant implicit in (3.6) is independent of these lengths.

The proof of Proposition 3.1 proceeds by pretending that  $I\phi$  is a solution of (1.1) and using the usual proof of energy conservation. We look at the resulting space-time integral in Fourier space, where we estimate various frequency interactions separately. In doing so, we'll need control of a local-in-time norm  $Z_I(t)$  involving the indices in (1.9),

$$(3.7) \quad Z_I(t) \equiv \sup_{q,r \text{ admissible}} \|\nabla I\phi\|_{L_t^q L_x^r([0,t] \times \mathbb{R}^3)}$$

similar to those norms that are usually bounded by the local in time existence theorem for (1.1). (See e.g. [5]). Since the norm here includes the operator  $I$ , and as mentioned above, we will control  $Z_I(t)$  on time intervals of varying lengths, we think of the following lemma as a modified local existence theory.

**Lemma 3.1.** *Consider  $\phi(x, t)$  as in (1.1)-(1.2) defined on  $[0, T^*] \times \mathbb{R}^3$  where*

$$(3.8) \quad \|\phi\|_{L_{x,t}^4([0,T^*] \times \mathbb{R}^3)} \leq \epsilon,$$

*for some universal constant  $\epsilon$ . Assume too  $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ . Then for  $s > \frac{1}{2}$  and sufficiently large<sup>9</sup>  $N$ ,*

$$(3.9) \quad Z_I(T^*) \lesssim C(\|\phi_0\|_{H^s(\mathbb{R}^3)}).$$

**Proof of Lemma 3.1:** Apply  $I\nabla$  to both sides of (1.1). Choosing  $\tilde{q}', \tilde{r}' = \frac{10}{7}$ , (1.10) and a fractional Leibniz rule<sup>10</sup> give us that for all  $0 \leq t \leq T$ ,

$$Z_I(t) \lesssim \|\nabla I\phi_0\|_{L^2(\mathbb{R}^3)} + \|\nabla I\phi\|_{L_{x,t}^{\frac{10}{3}}([0,t] \times \mathbb{R}^3)} \cdot \|\phi\|_{L_{x,t}^5([0,t] \times \mathbb{R}^3)}.$$

<sup>9</sup>Recall that  $I \equiv I_{N,s}$  was defined in (3.1)-(3.2).

<sup>10</sup>Since  $s > \frac{1}{2}$ , the multiplier for  $\nabla^\alpha I$  is increasing in  $|\xi|$  when  $\frac{1}{2} \leq \alpha \leq 1$ . Using this fact, one can easily modify the usual proof of the fractional Leibniz rule so this rule holds for the operators  $\nabla^\alpha I$ . (See e.g. page 105 of the exposition in [30], or the articles [7], [20].)

The  $L^{\frac{10}{3}}$  factor here is bounded by  $Z_I(t)$ . We claim that the remaining  $L^5_{x,t}$  factors are bounded by,

$$(3.10) \quad \|\phi\|_{L^5_{x,t}([0,T^*] \times \mathbb{R}^3)} \lesssim \epsilon^{\delta_1} \cdot (Z_I(T^*))^{\delta_2}$$

for some  $\delta_1, \delta_2 > 0$ , and  $Z_I$  as in (3.7). Assuming (3.10) for the moment, we conclude that for  $N$  sufficiently large,

$$(3.11) \quad Z_I(t) \lesssim 1 + \epsilon^{\delta_3} (Z_I(t))^{1+\delta_4},$$

for some constants  $\delta_3, \delta_4 > 0$ . For sufficiently small choice of  $\epsilon$ , the bound (3.11) yields (3.9) for all  $0 \leq t \leq T$ , as desired.

It remains to prove (3.10). All space-time norms in this proof will be taken on the slab  $[0, T^*] \times \mathbb{R}^3$ , even when, for legibility, this isn't explicitly written. Write

$$\phi = \psi_0 + \sum_{i=1}^{\infty} \psi_j$$

where  $\psi_0$  has spatial frequency support on  $\langle \xi \rangle \lesssim N_1 \equiv N$  and the remaining  $\psi_j$  each have dyadic spatial frequency support  $\langle \xi_j \rangle \sim N_j \equiv 2^{k_j}$ , where  $k_j \gtrsim \log(N)$  are integers and  $j = 1, 2, \dots$ . The argument given below estimates the low frequency constituent  $\psi_0$  with the available  $L^4$  and  $L^{10}$  bounds; and the high frequency pieces  $\psi_j, j \geq 1$  with the  $L^{\frac{10}{3}}$  and  $L^{10}$  bounds.

Specifically, the definition of  $I$  in (3.2) gives,

$$\|I\psi_j\|_{L^{10}_{x,t}} \sim \begin{cases} \|\psi_j\|_{L^{10}_{x,t}} & j = 0 \\ N^{1-s} (N_j)^{s-1} \|\psi_j\|_{L^{10}_{x,t}} & j = 1, 2, \dots \end{cases}$$

Using Sobolev's inequality, the left hand side here is bounded by  $Z_I(T^*)$ . Rewriting gives,

$$(3.12) \quad \|\psi_j\|_{L^{10}_{x,t}([0,T^*] \times \mathbb{R}^3)} \lesssim \begin{cases} Z_I(T^*) & j = 0 \\ N_j^{1-s} N^{s-1} Z_I(T^*) & j = 1, 2, \dots \end{cases}$$

Similarly,

$$\|\nabla I\psi_j\|_{L^{\frac{10}{3}}_{x,t}} \sim N_j^s N^{1-s} \|\psi_j\|_{L^{\frac{10}{3}}_{x,t}} \quad j = 1, 2, \dots$$

Hence we get the following  $L^{\frac{10}{3}}$  bounds,

$$(3.13) \quad \|\psi_j\|_{L^{\frac{10}{3}}_{x,t}} \lesssim N^{s-1} (N_j)^{-s} Z_I(T^*), \quad j \geq 1.$$

We now have the ingredients for our desired  $L^5_{x,t}$  bound of  $\phi$ . By the triangle inequality,

$$(3.14) \quad \|\phi\|_{L^5_{x,t}} \leq \sum_{j=0}^{\infty} \|\psi_j\|_{L^5_{x,t}}.$$

Interpolating between the  $L^{10}$  and  $L^4$  bounds of (3.12), (3.8) gives,

$$(3.15) \quad \|\psi_0\|_{L^5_{x,t}} \lesssim \|\psi_0\|_{L^4_{x,t}}^{\frac{2}{3}} \cdot \|\psi_0\|_{L^{10}_{x,t}}^{\frac{1}{3}}$$

$$(3.16) \quad \lesssim \epsilon^{\frac{2}{3}} (Z_I(T^*))^{\frac{1}{3}}.$$

For  $N_j \gtrsim N$  interpolation between (3.12) and (3.13) yields,

$$\begin{aligned} \sum_{j=1}^{\infty} \|\psi_j\|_{L_{x,t}^5} &\lesssim \sum_{j=1}^{\infty} \|\psi_j\|_{L_{x,t}^{\frac{10}{3}}}^{\frac{1}{2}} \cdot \|\psi_j\|_{L_{x,t}^{10}}^{\frac{1}{2}} \\ &\lesssim \sum_{j=1}^{\infty} (N^{s-1} (N_j)^{-s} Z_I(T^*))^{\frac{1}{2}} \cdot ((N_j)^{1-s} \cdot N^{s-1} Z_I(T^*))^{\frac{1}{2}} \\ &\lesssim N^{s-1} Z_I(T^*), \end{aligned}$$

since  $s > \frac{1}{2}$ . Choosing  $N$  sufficiently large, depending on  $\epsilon$ , yields (3.10) for these high frequency contributions as well.  $\square$

### Proof of Proposition (3.1):

For sufficiently smooth solutions, the usual energy (1.4) is shown to be conserved by differentiating in time, integrating by parts, and using the equation (1.1),

$$\begin{aligned} \partial_t E(\phi) &= \operatorname{Re} \int_{\mathbb{R}^3} \overline{\phi}_t (|\phi|^2 \phi - \Delta \phi) dx \\ &= \operatorname{Re} \int_{\mathbb{R}^3} \overline{\phi}_t (|\phi|^2 \phi - \Delta \phi - i\phi_t) dx \\ &= 0. \end{aligned}$$

We begin to estimate  $E(I\phi)(t)$  in the same way. We need to pay attention when we use the equation (1.1) since of course  $I\phi$  is not a solution. Repeating our steps above gives,

$$\begin{aligned} \partial_t E(I\phi)(t) &= \operatorname{Re} \int_{\mathbb{R}^3} \overline{I(\phi)}_t (|I\phi|^2 I\phi - \Delta I\phi - iI\phi_t) dx \\ &= \operatorname{Re} \int_{\mathbb{R}^3} \overline{I(\phi)}_t (|I\phi|^2 I\phi - I(|\phi|^2 \phi)) dx. \end{aligned}$$

When we integrate in time and apply the Parseval formula it remains for us to bound

$$(3.17) \quad E(I\phi(t)) - E(I\phi(0)) = \int_0^t \int_{\sum_{j=1}^4 \xi_j=0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) \cdot m(\xi_3) \cdot m(\xi_4)} \right) \widehat{I\partial_t \phi}(\xi_1) \widehat{I\phi}(\xi_2) \widehat{I\phi}(\xi_3) \widehat{I\phi}(\xi_4).$$

We use the equation (1.1) to substitute for  $\partial_t I(\phi)$  in (3.17). Our aim is to show that

$$(3.18) \quad \text{Term}_1 + \text{Term}_2 \lesssim N^{-1+} (Z_I(T))^P,$$

for some  $P > 0$ , where the two terms on the left are

$$(3.19) \quad \text{Term}_1 \equiv \left| \int_0^T \int_{\sum_{i=1}^4 \xi_i=0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) (\widehat{\Delta I\phi})(\xi_1) \cdot \widehat{I\phi}(\xi_2) \cdot \widehat{I\phi}(\xi_3) \cdot \widehat{I\phi}(\xi_4) \right|$$

$$(3.20) \quad \text{Term}_2 \equiv \left| \int_0^T \int_{\sum_{i=1}^4 \xi_i=0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) (\widehat{I(|\phi|^2 \phi)})(\xi_1) \cdot \widehat{I\phi}(\xi_2) \cdot \widehat{I\phi}(\xi_3) \cdot \widehat{I\phi}(\xi_4) \right|.$$

In both cases we break  $\phi$  into a sum of dyadic constituents  $\phi_j$ , each with spatial frequency support  $\langle \xi \rangle \sim 2^{k_j} \equiv N_j$ ,  $k_j \in \{0, \dots\}$ , and employ the following estimate of Coifman-Meyer for a class of multilinear operators.

Consider an infinitely differentiable symbol  $\sigma : \mathbb{R}^{nk} \rightarrow \mathbb{C}$  so that for all  $\alpha \in N^{nk}$  and all  $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{nk}$ , there is a constant  $c(\alpha)$  with,

$$(3.21) \quad |\partial_\xi^\alpha \sigma(\xi)| \leq c(\alpha)(1 + |\xi|)^{-|\alpha|}.$$

Define the multilinear operator  $T$  by,

$$(3.22) \quad [T(f_1, \dots, f_k)](x) = \int_{\mathbb{R}^{nk}} e^{ix(\xi_1 + \dots + \xi_k)} \sigma(\xi_1, \dots, \xi_k) \hat{f}_1(\xi_1) \cdots \hat{f}_k(\xi_k) d\xi_1 \cdots d\xi_k.$$

**Theorem 3.1** ([8], Page 179). *Suppose  $p_j \in (1, \infty)$ ,  $j = 1, \dots, k$ , are such that  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} \leq 1$ . Assume  $\sigma(\xi_1, \dots, \xi_k)$  a smooth symbol as in (3.21). Then there is a constant  $C = C(p_i, n, k, c(\alpha))$  so that for all Schwarz class functions  $f_1, \dots, f_k$ ,*

$$(3.23) \quad \|T(f_1, \dots, f_k)\|_{L^p(\mathbb{R}^n)} \leq C \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|f_k\|_{L^{p_k}(\mathbb{R}^n)}$$

Remark: The estimate (3.23) is also available for operators whose symbols obey much weaker bounds than (3.21), see e.g. [9], page 55.

When we estimate below the terms which constitute both Term<sub>1</sub> (3.19) and Term<sub>2</sub> (3.20), we will first seek a pointwise bound on the symbol,

$$(3.24) \quad \left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \leq B(N_2, N_3, N_4).$$

We factor  $B(N_2, N_3, N_4)$  out of the left side of (3.24), leaving a symbol  $\sigma$  that satisfies the estimate (3.21)<sup>11</sup>. We are left to estimate a quantity of the form

$$\left| B(N_2, N_3, N_4) \int_0^T \int_{\mathbb{R}^3} [T(f_1, f_2, f_3)]^\wedge(\xi_4) \overline{\widehat{f_4}(\xi_4)} d\xi_4 dt \right|,$$

for some multilinear operator  $T$  of the form (3.22), (3.21). We estimate this using the Plancherel formula, Hölder's inequality, Theorem 3.1, and the Strichartz estimates. We can sum over the all  $\phi_i$  since our bounds will be seen to decay sufficiently fast in the frequencies  $N_i$ . We suggest that the reader at first ignore this summation issue, and so ignore on first reading the appearance below of all factors such as  $N_i^{0-}$  which we include only to show explicitly why our frequency interaction estimates sum. The main goal of the analysis is to establish the decay of  $N^{-1+}$  in each class of frequency interactions below. In what follows we drop the complex conjugates as they don't affect the analysis used here<sup>12</sup>.

Consider first Term<sub>1</sub>. We will conclude that Term<sub>1</sub>  $\leq N^{-1+}$  once we prove

$$(3.25) \quad \left| \int_0^T \int_{\sum_{i=1}^4 \xi_i = 0} \left( 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) \cdot m(\xi_3) \cdot m(\xi_4)} \right) \widehat{\phi}_1(\xi_1) \widehat{\phi}_2(\xi_2) \widehat{\phi}_3(\xi_3) \widehat{\phi}_4(\xi_4) \right| \lesssim N^{-1+} C(N_1, N_2, N_3, N_4) (Z_I(T))^4$$

where  $C(N_1, N_2, N_3, N_4)$  is sufficiently small. By symmetry, we may assume  $N_2 \geq N_3 \geq N_4$ . The precise extent to which  $C(N_1, N_2, N_3, N_4)$  decays in its arguments, and the fact that this decay allows us to sum over all dyadic shells, will be described below.

**Term<sub>1</sub>, Case 1:**  $N \gg N_2$ . According to (3.2), the symbol  $1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) \cdot m(\xi_3) \cdot m(\xi_4)}$  on the right of (3.17) is in this case identically zero and the bound (3.25) holds trivially.

<sup>11</sup>The required  $L^\infty$  bound is clear, and we leave the reader to check that the derivatives are bounded as in (3.21).

<sup>12</sup>A more detailed argument exploiting the complex conjugates as in [24, 10, 31] might obtain a better exponent in (3.6)

**Term<sub>1</sub>, Case 2:**  $N_2 \gtrsim N \gg N_3 \geq N_4$ . Since  $\sum_i \xi_i = 0$ , we have  $N_1 \sim N_2$ . We aim for (3.25) with

$$(3.26) \quad C(N_1, N_2, N_3, N_4) = N_2^{0-}.$$

With this decay factor, and the fact that we are considering here terms where  $N_1 \sim N_2$ , we may immediately sum over all the  $N_i$ .

By the mean value theorem,

$$(3.27) \quad \left| \frac{m(\xi_2) - m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)} \right| \lesssim \frac{|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}.$$

After estimating the symbol with (3.27), we view the  $N_3$  in the numerator as resulting from a derivative falling on the  $I\phi_3$  factor in the integrand. Hence these interactions can be estimated using Hölder's inequality, Theorem 3.1, and the definition (3.7) of  $Z_I(t)$ ,

$$\begin{aligned} |\text{Left Side of (3.25)}| &\lesssim \frac{N_3}{N_2} \left| \int_0^T \int_{\mathbb{R}^3} T[\Delta I\phi_1, I\phi_2, I\phi_3] \cdot I\phi_4 dx dt \right| \\ &\leq \frac{1}{N_2} \|\Delta I\phi_1\|_{L_{x,t}^{\frac{10}{3}}} \cdot \|I\phi_2\|_{L_{x,t}^{\frac{10}{3}}} \cdot \|\nabla I\phi_3\|_{L_{x,t}^{\frac{10}{3}}} \cdot \|I\phi_4\|_{L_{x,t}^{10}} \\ &\leq \frac{N_1}{N_2 \cdot N_2} \cdot (Z_I(t))^4 \\ &\leq \frac{1}{N_1} (Z_I(t))^4 \\ &\leq N^{-1+} \cdot N_2^{0-} (Z_I(t))^4 \end{aligned}$$

by our assumptions on the  $N_i$ . This establishes (3.25), (3.26).

**Term<sub>1</sub>, Case 3:**  $N_2 \geq N_3 \gtrsim N$ . In this case the only pointwise bound available for the symbol is the straightforward one: when  $|\xi_1|, |\xi_2|$  are not comparable, no cancellation can occur in the numerator of (3.24). When  $|\xi_1| \sim |\xi_2|$ , we then also need  $|\xi_3|, |\xi_4| \leq N$  in order to get cancellation. If any of these conditions fail, our pointwise estimate will be simply,

$$(3.28) \quad \left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}.$$

The frequency interactions here fall into two subcategories, depending on which frequency is comparable to  $N_2$ . **Case 3(a):**  $N_1 \sim N_2 \geq N_3 \gtrsim N$ . By assumption,  $s > \frac{1}{2} + \delta$  for some small  $\delta$ . In this case we prove the decay factor

$$(3.29) \quad C(N_1, N_2, N_3, N_4) = N^{-1+2\delta} N_3^{0-2\delta}$$

in (3.25). This allows us to directly sum in  $N_3, N_4$ , and sum in  $N_1, N_2$  after applying Cauchy-Schwarz to those factors. Estimate the symbol using (3.28). Use Hölder's inequality and Theorem 3.1 to take the factors involving  $\phi_i, i = 1, 2, 3$  in  $L_{x,t}^{\frac{10}{3}}$ , and the  $\phi_4$  factor in  $L_{x,t}^{10}$ . It remains to show

$$(3.30) \quad \frac{m(N_1)N_1N^{1-2\delta}N_3^{2\delta}}{m(N_2)m(N_3)m(N_4)N_2N_3} \lesssim 1.$$

When proving such estimates here and in the sequel, we shall frequently use the following two elementary facts without further mention: for any  $p > \frac{1}{2} - \delta$ , the function  $m(x)|x|^p$  is increasing, and  $m(x)\langle x \rangle$  is bounded below.

The bound (3.30) is now straightforward,

$$\begin{aligned}
\text{Left Side of (3.30)} &\lesssim \frac{N^{1-2\delta} N_3^{2\delta}}{m(N_3)m(N_4)N_3} \\
&\lesssim \frac{N^{1-2\delta} N_3^{2\delta}}{(m(N_3))^2 N_3} \\
&\lesssim \frac{N^{1-2\delta} N_3^{2\delta}}{(m(N_3))N_3^{\frac{1}{2}-\delta} m(N_3)N_3^{\frac{1}{2}-\delta} N_3^{2\delta}} \\
&\lesssim \frac{N^{1-2\delta} N_3^{2\delta}}{N^{1-2\delta} N_3^{2\delta}}
\end{aligned}$$

which gives (3.25), (3.29).

**Case 3(b):**  $N_2 \sim N_3 \gtrsim N$ . We aim in this case for the decay factor

$$(3.31) \quad C(N_1, N_2, N_3, N_4) = N^{-1+2\delta} N_2^{-2\delta}$$

where  $\delta$  is as in Case 3(a) above. This will allow us to sum directly in all the  $N_i$ . Once again we use (3.28) and apply Hölder's inequality and (3.23) exactly as in the preceding discussion.

$$\begin{aligned}
\frac{m(N_1)N_1 N^{1-2\delta} N_2^{2\delta}}{m(N_2)m(N_3)m(N_4)N_2 N_3} &\lesssim \frac{m(N_1)N_1 N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^3 N_2 N_2} \\
&\lesssim \frac{m(N_2)N_2 N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^3 N_2 N_2} \\
&= \frac{N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^2 \cdot N_2} \\
&\leq \frac{N^{1-2\delta} N_2^{2\delta}}{N_2^{2\delta} \cdot N^{1-2\delta}} \\
&\leq 1,
\end{aligned}$$

as desired. It remains to prove bounds of the form (3.18) for  $\text{Term}_2$ (3.20).

When decomposing the integrand of  $\text{Term}_2$  in frequency space, write  $N_{123}$  for the dyadic frequency into which we project the nonlinear factor  $I(\phi^3)$ . Note that in the treatment of  $\text{Term}_1$  above, we always took the  $\Delta\phi_1$  factor in  $L^{\frac{10}{3}}$ , estimating this by  $N_1 Z_I(T)$ . The analysis above for  $\text{Term}_1$  therefore applies unmodified to  $\text{Term}_2$  once we prove the following,

**Lemma 3.2.** *Assume  $\phi, T, Z_I(T), N_{123}$  as defined above, and  $P_{N_{123}}$  the Littlewood-Paley projection onto the  $N_{123}$  frequency shell. Then*

$$(3.32) \quad \|P_{N_{123}}(I(\phi^3))\|_{L_{x,t}^{\frac{10}{3}}([0,T] \times \mathbb{R}^3)} \lesssim N_{123} (Z_I(T))^3.$$

**Proof:** We write  $\phi = \phi_L + \phi_H$  where

$$\begin{aligned}
\text{supp } \hat{\phi}_L(\xi, t) &\subseteq \{|\xi| < 2\} \\
\text{supp } \hat{\phi}_H(\xi, t) &\subseteq \{|\xi| > 1\}.
\end{aligned}$$

Consider first the bound (3.32) when all three factors on the left are  $\phi_L$ ,

$$\begin{aligned} \|P_{N_{123}}(I(\phi_L^3))\|_{L_{x,t}^{\frac{10}{3}}} &\lesssim \|\phi_L\|_{L_{x,t}^{10}}^3 \\ &= \|I\phi_L\|_{L_{x,t}^{10}}^3 \\ &\leq (Z_I(T))^3 \\ &\lesssim N_{123}(Z_I(T))^3, \end{aligned}$$

since  $N_{123} \geq 1$ . When instead all three components on the left of (3.32) are  $\phi_H$ , we have by Littlewood-Paley theory, Sobolev embedding, and the Leibniz rule mentioned in the proof of Proposition 3.1,

$$\begin{aligned} \left\| \frac{1}{N_{123}} P_{N_{123}} I(\phi_H^3) \right\|_{L_{x,t}^{\frac{10}{3}}} &\lesssim \|\nabla^{-1} P_{N_{123}} I(\phi_H^3)\|_{L_{x,t}^{\frac{10}{3}}} \\ &\lesssim \|\nabla^{\frac{1}{2}} I(\phi_H^3)\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{8}}} \lesssim \|\nabla^{\frac{1}{2}} I\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}}^3 \\ &\lesssim \|\nabla I\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}}^3 \lesssim (Z_I(T))^3 \end{aligned}$$

as desired.

The remaining terms are bounded using similar arguments,

$$\begin{aligned} \left\| \frac{1}{N_{123}} P_{N_{123}} I(\phi_H \cdot \phi_H \cdot \phi_L) \right\|_{L_{x,t}^{\frac{10}{3}}} &\lesssim \|\nabla^{\frac{1}{2}} I(\phi_H \cdot \phi_H \cdot \phi_L)\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{8}}} \\ &\lesssim \|\nabla^{\frac{1}{2}} I\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}} \cdot \|\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} + \|\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}} \cdot \|\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}} \cdot \|\nabla^{\frac{1}{2}} I\phi_L\|_{L_t^{10} L_x^{\frac{30}{8}}} \\ &\lesssim \|\nabla I\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}} \cdot \|\nabla I\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}} \cdot \|I\phi_L\|_{L_t^{10} L_x^{10}} + \|\nabla^{\frac{1}{2}} I\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}} \cdot \|\nabla^{\frac{1}{2}} I\phi_H\|_{L_t^{10} L_x^{\frac{30}{8}}} \cdot \|\nabla I\phi_L\|_{L_t^{10} L_x^{\frac{30}{13}}} \\ &\lesssim (Z_I(T))^3. \end{aligned}$$

$$\begin{aligned} \left\| \frac{1}{N_{123}} P_{N_{123}} I(\phi_H \cdot \phi_L \cdot \phi_L) \right\|_{L_{x,t}^{\frac{10}{3}}} &\lesssim \|\phi_H \cdot \phi_L \cdot \phi_L\|_{L_t^{\frac{10}{3}} L_x^{\frac{30}{19}}} \\ &\lesssim \|\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \lesssim \|\nabla I\phi_H\|_{L_t^{10} L_x^{\frac{30}{13}}} \cdot \|\nabla I\phi_L\|_{L_t^{10} L_x^{\frac{30}{13}}}^2 \\ &\lesssim (Z_I(T))^3. \end{aligned}$$

This completes the proof of Lemma 3.2, and hence Proposition 3.1.  $\square$

#### 4. PROOF OF MAIN THEOREM

We combine the interaction Morawetz estimate (2.26) and Proposition 3.1 with a scaling argument to prove the following statement giving uniform bounds in terms of the rough norm of the initial data.

**Proposition 4.1.** *Suppose  $\phi(x, t)$  is a global in time solution to (1.1)-(1.2) from data  $\phi_0 \in C_0^\infty(\mathbb{R}^3)$ . Then so long as  $s > \frac{4}{5}$ , we have*

$$(4.1) \quad \|\phi\|_{L^4([0, \infty] \times \mathbb{R}^3)} \lesssim C(\|\phi_0\|_{H^s(\mathbb{R}^3)})$$

$$(4.2) \quad \sup_{0 \leq t < \infty} \|\phi(t)\|_{H^s(\mathbb{R}^3)} \leq C(\|\phi_0\|_{H^s(\mathbb{R}^3)}).$$



**Remark:** As mentioned at the outset of the paper, energy conservation (1.4) and the local in time well-posedness of (1.1)-(1.2) from data in  $H^s(\mathbb{R}^3)$ ,  $s > \frac{1}{2}$  imply that the solution  $\phi$  considered here is smooth and exists globally in time. Since the estimate (4.2) involves only the rough norm  $\|\phi_0\|_{H^s(\mathbb{R}^3)}$  on the right hand side, the global well-posedness portion of Theorem 1.1 follows from (4.2), the local existence theory (see [5] for a proof and further references), and a standard density argument.

*Proof.* The first step is to scale the solution: if  $\phi$  is a solution to (1.1), then so is

$$(4.3) \quad \phi^{(\lambda)}(x, t) \equiv \frac{1}{\lambda} \phi\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

We choose  $\lambda$  so that  $E(I\phi_0^{(\lambda)}) \leq \frac{1}{2}$ . This is possible since we are working with subcritical  $s$ , so long as we choose  $\lambda$  in terms of the parameter<sup>13</sup>  $N$ ,

$$\begin{aligned} E(I\phi_0^{(\lambda)}) &= \frac{1}{2} \|\nabla I\phi_0^{(\lambda)}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4} \|I\phi^{(\lambda)}\|_{L_x^4}^4 \\ &\lesssim N^{2-2s} \lambda^{1-2s} (1 + \|\phi_0\|_{H^s(\mathbb{R}^3)})^4. \end{aligned}$$

Hence we choose

$$(4.4) \quad \lambda \approx N^{\frac{1-s}{s-\frac{1}{2}}}.$$

We claim that the set  $W$  of times for which (4.1) holds is all of  $[0, \infty)$ . In the process of proving this, we will also show (4.2) holds on  $W$ .

For some universal constant  $C_1$  to be chosen shortly, define

$$(4.5) \quad W \equiv \left\{ T : \|\phi^{(\lambda)}\|_{L^4([0, T] \times \mathbb{R}^3)} \leq C_1 \lambda^{\frac{3}{8}} \right\}.$$

The set  $W$  is clearly closed and nonempty. It suffices then to show it is open. For example, suppose that for some  $T_0$  we have

$$(4.6) \quad \|\phi^{(\lambda)}\|_{L_{x,t}^4([0, T_0] \times \mathbb{R}^3)} \leq 2C_1 \lambda^{\frac{3}{8}}.$$

We claim  $T_0 \in W$ : by (2.26),

$$(4.7) \quad \|\phi^{(\lambda)}\|_{L_{x,t}^4([0, T_0] \times \mathbb{R}^3)} \lesssim \|\phi_0^{(\lambda)}\|_{L_x^2}^{\frac{1}{2}} \cdot \sup_{0 \leq t \leq T_0} \|\phi^{(\lambda)}(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{\frac{1}{2}}.$$

$$(4.8) \quad \leq C(\|\phi_0\|_{L_x^2}) \lambda^{\frac{1}{4}} \cdot \sup_{0 \leq t \leq T_0} \|\phi^{(\lambda)}(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}^{\frac{1}{2}}$$

where we've taken into account the  $L^2$  conservation law (1.3). To bound the second factor in (4.8), decompose  $\phi^{(\lambda)}(t)$  as,

$$(4.9) \quad \phi^{(\lambda)}(t) = P_{\leq N} \phi^{(\lambda)}(t) + P_{\geq N} \phi^{(\lambda)}(t).$$

That is, a sum of functions supported on frequencies  $|\xi| \leq N$  and  $|\xi| \geq N$ , respectively. Interpolation and the fact that  $I$  is the identity on low frequencies gives us the bound,

$$\begin{aligned} \|P_{\leq N} \phi^{(\lambda)}(t)\|_{\dot{H}_x^{\frac{1}{2}}} &\lesssim \|P_{\leq N} \phi^{(\lambda)}(t)\|_{L_x^2}^{\frac{1}{2}} \cdot \|P_{\leq N} \phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{2}} \\ &\lesssim \|\phi_0^{(\lambda)}\|_{L_x^2}^{\frac{1}{2}} \cdot \|IP_{\leq N} \phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{2}} \\ (4.10) \quad &\lesssim C(\|\phi_0\|_{L_x^2}) \lambda^{\frac{1}{4}} \|I\phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{2}}. \end{aligned}$$

<sup>13</sup>The parameter  $N$  will be chosen at the very end of the argument, where it is shown to depend only on  $\|\phi_0\|_{H^s(\mathbb{R}^3)}$ .

We interpolate the high frequency constituent between  $\dot{H}_x^s$  and  $L_x^2$ , and use the definition (3.2) of  $I$  to get,

$$\begin{aligned}
\|P_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}_x^{\frac{1}{2}}} &\lesssim \|P_{\geq N}\phi^{(\lambda)}(t)\|_{L_x^2}^{1-\frac{1}{2s}} \cdot \|P_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}_x^s}^{\frac{1}{2s}} \\
&= \|P_{\geq N}\phi^{(\lambda)}(t)\|_{L_x^2}^{1-\frac{1}{2s}} \cdot N^{\frac{s-1}{2s}} \|IP_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{2s}} \\
(4.11) \qquad &\lesssim C(\|\phi_0\|_{L_x^2}) \cdot \|I\phi^{(\lambda)}\|_{\dot{H}_x^1}^{\frac{1}{2s}},
\end{aligned}$$

where we've used both the  $L^2$  conservation (1.3) and our choice of  $\lambda$ , (4.4). Putting together (4.11), (4.10), (4.9), and (4.8) gives us

$$(4.12) \qquad \|\phi^{(\lambda)}\|_{L_{x,t}^4([0,T]\times\mathbb{R}^3)} \lesssim C(\|\phi_0\|_{L_x^2}) \left( \lambda^{\frac{3}{8}} \sup_{0\leq t\leq T_0} \|I\phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{4}} + \sup_{0\leq t\leq T_0} \|I\phi^{(\lambda)}(t)\|_{\dot{H}_x^1}^{\frac{1}{4s}} \right).$$

We conclude  $T_0 \in W$  if we establish

$$(4.13) \qquad \sup_{0\leq t\leq T_0} \|I\phi^{(\lambda)}(t)\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim 1$$

where, as always, the implicit constant is allowed to depend on  $\|\phi_0\|_{H^s(\mathbb{R}^3)}$ .

By (4.6) we may divide the time interval  $[0, T_0]$  into subintervals  $I_j, j = 1, 2, \dots, L$  so that for each  $j$ ,

$$(4.14) \qquad \|\phi^{(\lambda)}\|_{L_{x,t}^4(I_j\times\mathbb{R}^3)} \leq \epsilon.$$

Apply the almost conservation law in Proposition 3.1 on each of the subintervals  $I_j$  to get

$$(4.15) \qquad \sup_{0\leq t\leq T_0} \|\nabla I\phi^{(\lambda)}(t)\|_{L^2(\mathbb{R}^3)} \lesssim E(I\phi_0) + L \cdot N^{-1+}.$$

We get (4.13) from (4.15) if we can show

$$(4.16) \qquad L \cdot N^{-1+} \leq \frac{1}{4}.$$

Recall  $L$  was defined essentially by (4.14). Since

$$\|\phi^{(\lambda)}\|_{L_{x,t}^4([0,T_0]\times\mathbb{R}^3)}^4 \lesssim \lambda^{\frac{3}{2}},$$

we can be certain that  $L \approx \lambda^{\frac{3}{2}}$ . If we put this together with (4.16) and (4.4), we see that we need to be able to choose  $N$  so that

$$(N^{\frac{1-s}{2}})^{\frac{3}{2}} \cdot N^{-1+} \lesssim \frac{1}{4}.$$

This is possible since for  $s > \frac{4}{5}$  the exponent on the left is negative. Notice that (4.2) holds on the set  $W$  using (4.13), the definition of  $I$ , and  $L^2$  conservation.

□

We have already explained why the global well-posedness statement in Theorem 1.1 follows from (4.2). It remains only to prove scattering using the following well-known arguments. (See e.g. [25, 17, 3, 6].) Asymptotic completeness will follow quickly once we establish a uniform bound of the form,

$$(4.17) \qquad Z(t) \equiv \sup_{q,r \text{ admissible}} \| \langle \nabla \rangle^s \phi \|_{L_t^q L_x^r([0,t]\times\mathbb{R}^3)}$$

$$(4.18) \qquad \lesssim C(\|\phi_0\|_{H^s(\mathbb{R}^3)}).$$

This is established much as in the proof of Lemma 3.1. By (4.1), we can decompose the time interval  $[0, \infty)$  into a finite number of disjoint intervals  $J_1, J_2, \dots, J_K$  where for  $i = 1, \dots, K$  we have

$$(4.19) \quad \|\phi\|_{L^4_{x,t}(J_i \times \mathbb{R}^3)} \leq \epsilon$$

for a constant  $\epsilon(\|\phi_0\|_{H^s(\mathbb{R}^3)})$  to be chosen momentarily.

Apply  $\langle \nabla \rangle^s$  to both sides of (1.1). Choosing  $\tilde{q}', \tilde{r}' = \frac{10}{7}$ , the Strichartz estimates (1.10) give us that for all  $t \in J_1$ ,

$$Z(t) \lesssim \|\langle \nabla \rangle^s \phi_0\|_{L^2(\mathbb{R}^3)} + \|\langle \nabla \rangle^s (\phi \phi \phi)\|_{L^{\frac{10}{7}}_{t,x}([0,t] \times \mathbb{R}^3)}.$$

Apply the fractional Leibniz rule to the last term on the right, taking the factor with  $\langle \nabla \rangle^s$  in  $L^{\frac{10}{3}}$ , and the other two in  $L^5$ . The factor ending up in  $L^{\frac{10}{3}}$  is bounded by  $Z(t)$ . The remaining  $L^5_{x,t}$  factors are bounded by interpolating between  $\|\phi\|_{L^4_{x,t}}$  and  $\|\phi\|_{L^6_{x,t}}$ . The latter norm is bounded by  $Z(t)$  using Sobolev embedding:

$$\|\phi\|_{L^6_{x,t}} \lesssim \|\langle \nabla \rangle^{\frac{2}{3}} \phi\|_{L^6_t L^{\frac{18}{5}}_x} \leq Z(t).$$

We conclude

$$(4.20) \quad Z(t) \lesssim \|\phi_0\|_{H^s(\mathbb{R}^3)} + \epsilon^{\delta_1} Z(t)^{(1+\delta_2)},$$

for some constants  $\delta_1, \delta_2 > 0$ . For sufficiently small choice of  $\epsilon$ , the bound (4.20) yields (4.18) for all  $t \in J_1$ , as desired. Since we are assuming the bound (4.2), we may repeat this argument to handle the remaining intervals  $J_i$ .

The asymptotic completeness claim in Theorem 1.1 follows quickly from (4.18). Given  $\phi_0 \in H^s(\mathbb{R}^3)$ , we look for a  $\phi^+$  satisfying (1.5). Set,

$$(4.21) \quad \phi^+ \equiv \phi_0 - i \int_0^\infty S^L(-\tau) (|\phi|^2 \phi) d\tau$$

which will make sense once we show the integral on the right hand side converges in  $H^s(\mathbb{R}^3)$ . Equivalently, we want

$$(4.22) \quad \lim_{t \rightarrow \infty} \left\| \int_t^\infty \langle \nabla \rangle^s S^L(-\tau) (|\phi|^2 \phi) d\tau \right\|_{L^2(\mathbb{R}^3)} = 0.$$

With this,

$$\begin{aligned} \lim_{t \rightarrow \infty} \|S^L(t)\phi^+ - \phi(t)\|_{H^s(\mathbb{R}^3)} &= \lim_{t \rightarrow \infty} \|\langle \nabla \rangle^s S^L(t) \int_t^\infty S^L(-\tau) (|\phi|^2 \phi) d\tau\|_{L^2(\mathbb{R}^3)} \\ &= 0 \end{aligned}$$

since we are assuming (4.22). To prove (4.22), test the time integral on the left against an arbitrary  $L^2(\mathbb{R}^3)$  function  $F(x)$ . Using the fractional Leibniz rule,

$$\begin{aligned} \left\langle F(x), \int_t^\infty \langle \nabla \rangle^s S^L(-\tau) (|\phi|^2 \phi) d\tau \right\rangle_{L^2(\mathbb{R}^3)} &\approx \langle S^L(\tau)F(x), \langle \nabla \rangle^s \phi \phi \phi \rangle_{L^2_{x,t}([t,\infty) \times \mathbb{R}^3)} \\ &\leq \|S^L(\tau)F(x)\|_{L^{\frac{10}{3}}_{x,\tau}} \|\langle \nabla \rangle^s \phi\|_{L^{\frac{10}{3}}_{x,t}} \|\phi\|_{L^5_{x,t}([t,\infty) \times \mathbb{R}^3)}^2 \\ &\rightarrow 0, \end{aligned}$$

where in the last step we've used (4.18) and the  $L^5_{x,t}$  argument before (4.20).

For completeness we include an argument proving the existence of wave operators on  $H^s(\mathbb{R}^3)$ , following closely the exposition of [17] in [6], §7.6. Given  $\phi^+ \in H^s(\mathbb{R}^3)$ , we are looking for a solution  $\phi(x, t)$  of (1.1) and data  $\phi_0$  which, heuristically at least, satisfy,

$$\begin{aligned}
(4.23) \quad \phi(x, t) &= S^L(t)\phi_0 - i \int_0^t S^L(t-\tau)|\phi|^2\phi d\tau \\
&= S^L(t) (S^{NL}(-\infty)S^L(\infty)\phi^+) - i \int_0^t S^L(t-\tau)|\phi|^2\phi d\tau \\
&= S^L(t) \left( \phi^+ - i \int_{-\infty}^0 S^L(0-\tau)|\phi|^2\phi d\tau \right) - i \int_0^t S^L(t-\tau)|\phi|^2\phi d\tau \\
(4.24) \quad &= S^L(t)\phi^+ + i \int_t^{\infty} S^L(t-\tau)|\phi|^2\phi d\tau.
\end{aligned}$$

Heuristics aside, we now sketch how this last integral equation is solved for  $\phi(x, t)$  using a fixed point argument, and prove that  $\phi(x, t)$  does in fact approach  $S^L(t)\phi^+$  as  $t \rightarrow \infty$ .

By Strichartz estimates, we have  $S^L(t)\phi^+ \in L_t^{\frac{8}{3}}W_x^{s,4} \cap L_t^8W_x^{s,\frac{12}{5}}([0, \infty) \times \mathbb{R}^3)$ . Set,

$$(4.25) \quad K_{t_0} = \|S^L(t)\phi^+\|_{L_t^{\frac{8}{3}}W_x^{s,4}([t_0, \infty) \times \mathbb{R}^3)} + \|S^L(t)\phi^+\|_{L_t^8W_x^{s,\frac{12}{5}}([t_0, \infty) \times \mathbb{R}^3)}.$$

Clearly  $K_{t_0} \rightarrow 0$  as  $t_0 \rightarrow \infty$ . Define,

$$(4.26) \quad X = \left\{ u \in L_t^{\frac{8}{3}}W_x^{s,4} \cap L_t^8W_x^{s,\frac{12}{5}}((t_0, \infty) \times \mathbb{R}^3) \mid \|u\|_{L_t^{\frac{8}{3}}W_x^{s,4}((t_0, \infty) \times \mathbb{R}^3)} + \|u\|_{L_t^8W_x^{s,\frac{12}{5}}((t_0, \infty) \times \mathbb{R}^3)} \leq 2K_{t_0} \right\}$$

with norm  $\|\cdot\|_{L_t^{\frac{8}{3}}W_x^{s,4}} + \|\cdot\|_{L_t^8W_x^{s,\frac{12}{5}}}$ . For functions  $u \in X$  we have

$$(4.27) \quad \| |u|^2 u \|_{L_t^{\frac{8}{3}}W_x^{s,\frac{4}{3}}} \leq \| \nabla^s u \|_{L_t^{\frac{8}{3}}L_x^4} \cdot \| u \|_{L_t^8L_x^4}^2$$

$$(4.28) \quad \leq C(2K_{t_0})^3,$$

where we've bounded the second two factors on the right of (4.27) using Sobolev embedding. It is straightforward<sup>14</sup> to conclude from (4.28) that the function

$$(4.29) \quad \Phi_u(t) \equiv i \int_t^{\infty} S^L(t-\tau)|u|^2 u d\tau$$

is well defined for all  $u \in X$ , and that

$$(4.30) \quad \Phi_u(t) \in C((t_0, \infty); H^s(\mathbb{R}^3)) \cap X,$$

with,

$$(4.31) \quad \|\Phi_u\|_X \leq C(2K_{t_0})^3 \leq K_{t_0},$$

when  $K_{t_0}$  is small enough - that is, for  $t_0$  large enough. Hence the map,

$$(4.32) \quad A : u(t) \rightarrow S^L(t)\phi^+ + \Phi_u(t),$$

takes  $X$  into itself. It can be similarly argued that  $A$  is a contraction. We conclude there is a unique solution  $\phi \in X$  of (4.24). By our global existence result and time reversibility, we may extend this solution  $\phi$ , starting from data at time  $t_0$ , to all of  $[0, \infty)$ . It is now straightforward to verify that

$$\lim_{t \rightarrow \infty} \|\phi(t) - S^L(t)\phi^+\|_{H^s(\mathbb{R}^3)} = 0,$$

as desired. □

<sup>14</sup>The proof of Corollary 3.2.7 in [6] can be followed without modification.

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