PRF LET $f_1 : \mathbb{N} \longrightarrow A_1$, $f_2 : \mathbb{N} \longrightarrow A_2$ BE BIJECTIONS. CLAIM: $f : \mathbb{N} \times \mathbb{N} \longrightarrow A_1 \times A_2$, $f((i,j)) = (f_1(i), f_2(j))$ IS A BIJECTION. f ONTO: LET $(a_1, a_2) \in A_1 \times A_2$. $\exists\, i, j \in \mathbb{N}$ ST. $f_1(i) = a_1$, $f_2(j) = a_2$. THEN $f((i,j)) = (f_1(i), f_2(j)) = (a_1, a_2)$

f 1-TO-1: SUPPOSE $f((i,j)) = f((i_1, j_1))$ THEN $(f_1(i), f_2(j)) = (f_1(i_1), f_2(j_1))$, $f_1(i) = f_1(i_1)$ AND $f_2(j) = f_2(j_1)$. SINCE f_1, f_2 ARE 1-TO-1 $i = i_1$ AND $j = j_1$, I.E. $(i,j) = (i_1, j_1)$.

WE ALREADY HAVE $g : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ BIJECTION. SO $f \circ g : \mathbb{N} \longrightarrow A_1 \times A_2$ IS THE DESIRED BIJECTION

EXERCISE IF $f : R \rightarrow S$, $g : S \longrightarrow T$ ARE BIJECTIONS, THEN $g \circ f : R \rightarrow T$ IS A BIJECTION.

COROLLARY LET $A_1, A_2, \ldots A_k$ BE COUNTABLY INF. THEN $A_1 \times A_2 \times \cdots A_k$

__EXAMPLE__ THE SET OF ALL POSITIVE RATIONALS $\mathbb{R} = \{ \frac{p}{q} \mid p, q \in \mathbb{N} \}$ ARE CONNTABLY INFINITE

__PROOF__ EVERY POSITIVE RATIONAL q CAN BE UNIQUELY WRITTEN AS $m + \frac{p_1}{q_1}$, $\frac{p_1}{q_1}$ IN LOWEST TERMS & $1 \le p_1 < q_1$

WE KNOW $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ IS COUNT. INF

HENCE, $R = \{ (n, p, q) \mid n, p, q \in \mathbb{N}$, $\frac{p}{q}$ IN LOWEST TERMS & $p < q \}$ IS ALSO COUNT. INF ($\{ (n, 1, 2) \mid n \in \mathbb{N} \} \subset R$). LET $f : R \to \mathbb{R}$, $f(n, p, q) = n + p/q$. THEN f IS A BIJECTION ONTO $\mathbb{R} - \mathbb{N}$, AND $\mathbb{R} - \mathbb{N}$ IS COUNT INF. SO $\mathbb{R} = (\mathbb{R} - \mathbb{N}) \cup \mathbb{N}$ IS THE UNION OF 2 COUNT. INF SETS AND IS COUNTABLY INFINITE

__EXERCISE__ PROVE THAT IF A, B COUNT INFINITE AND $A \cap B = \phi$, THEN $A \cup B$ IS COUNTABLY INFINITE.

IS COUNTABLY INFINITE

PRF OUTLINE (EXERCISE - ADD DETAILS)

$A_1 \times A_2 = B_2$ CO. INF BY THM. SINCE A_3 CO. INF, $B_2 \times A_3 = (A_1 \times A_2) \times A_3$ $= A_1 \times A_2 \times A_3 = B_3$ CO. INF. SINCE A_4
(SHOW THIS)
CO. INF, $B_3 \times A_4 = B_4 = A_1 \times A_2 \times A_3 \times A_4$ CO. INF. PROCEED IN THIS WAY.

SETS OF SUBSETS - POWER SETS

LET $A = \{1, 2\}$. THE POSSIBLE SUBSETS OF A ARE: ϕ, $\{1\}$, $\{2\}$, $\{1,2\} = A$

NOTE $\#A = 2$, $\#$ SUBSETS $= 4 = 2^2 = 2^{\#A}$

$B = \{1, 2, 3\}$. POSSIBLE SUBSETS: ϕ, $\underbrace{\{1\}, \{2\}, \{3\}}_{\text{SINGLETONS}}$, $\underbrace{\{1,2\}, \{1,3\}, \{2,3\}}_{\text{DOUBLE TONS}}$, B

$\#B = 3$, $\#$ SUBSETS $= 8 = 2^3 = 2^{\#B}$

DEFINITION LET A SET. THEN THE POWER SET OF A IS $\mathcal{P}(A) = \{B \mid B \subset A\}$ THE SET OF ALL SUBSETS OF A.

FACTS: 1) ALWAYS ϕ AND A IN $\mathcal{P}(A)$

2) THERE ARE "MANY" MORE SUBSETS OF A THAN ELEMENTS IN A.

3) FOR $\#A = 2$ OR 3, WE HAVE $\#\mathcal{P}(A) = 2^{\#A}$

LET $A = \{1, \ldots, n\}$ HAVE n ELEMENTS. HOW TO DESCRIBE A "TYPICAL" SUBSET B OF A. CONSIDER AN n-TUPLE OF 0'S AND 1'S. LET 0 AT i^{th} COORD MEAN THAT $i \notin B$, 1 MEAN THAT $i \in B$. SO EVERY TUPLE $a_1 \cdots a_n$ GIVES A SET $B = \{i \mid a_i = 1\}$

EXAMPLE $A = \{1, 2, \ldots, 6\}$. LOOK AT 6-TUPLES OF 0'S AND 1'S.

$(0,0,0,1,1,0)$ GIVES $B = \{4,5\}$

$B = \{1, 3, 5, 6\}$ CORRESPONDS TO 6-TUPLE $(1,0,1,0,1,1)$, $(0,0,0,0,0,0)$ CORRES TO ϕ AND $(1,1,1,1,1,1)$ TO A

THUS, WE CAN DESCRIBE $\mathcal{P}(A)$ BY LOOKING AT $\{(a_1, a_2, \ldots a_6) \mid a_i = 0 \text{ OR } 1\}$.

DENOTED BY 2^n

<u>THM</u> LET A FINITE WITH $\#A = n$

THEN $\mathcal{P}(A)$ IS DESCRIBED BY LOOKING AT $2^A = \{(a_1, \ldots, a_n) \mid a_i = 0, 1\}$

MOREOVER, $\# \mathcal{P}(A) = \#(2^A) = 2^{\#A}$

<u>PRF</u> WE LOOK AT $B \subset A = \{w_1, \cdots, w_n\}$ AND $\{i \mid w_i \in B\}$. SET $(a_1, \ldots, a_n)$ WHERE $a_i = 1$ IF $w_i \in B$, $a_i = 0$ IF $w_i \notin B$. CONVERSELY, IF $(a_1, \cdots, a_n) \in 2^A$, SET $B = \{w_i \mid a_i = 1\} \subset A$. THIS GIVES THE CORRESPONDENCE BETWEEN $\mathcal{P}(A)$ AND 2^A.

NOW COUNT $\#(2^A)$. FOR $(a_1, \ldots, a_n)$, THERE ARE 2 POSSIBILITIES (0 OR 1) FOR a_1, 2 FOR $a_2, \ldots$, 2 FOR a_n. THUS,

$$\#(2^A) = \underbrace{2 \cdot 2 \cdots 2}_{n} = 2^n = 2^{\#A}.$$

NOTE THAT $\#A < \#(2^A)$. NOW LOOK AT THESE IDEAS WHEN A IS COUNTABLY INFINITE. WHAT IS $\#(2^A)$?

LET $A = \mathbb{N}$. CONSIDER B = EVENS

C = ODDS AND $D = \{n^2 \mid n \in \mathbb{N}\}$

CORRESPONDING TO B IS THE

SEQUENCE OF 0'S AND 1'S $S_B = \langle a_\lambda$

WHERE $a_\lambda = 1$ IF $\lambda \in B$ AND $a_\lambda = 0$ IF $\lambda \notin$

So $\langle a_\lambda \rangle$: $0\underset{2}{1}0\underset{4}{1}0\underset{6}{1}0\underset{8}{1}\ldots$ SIMILARLY S_C

$= \langle b_\lambda \rangle$, b_λ: $\underset{1}{1}0\underset{3}{1}0\underset{5}{1}0\underset{7}{1}\ldots$, $S_D = \langle d_\lambda \rangle$,

d_λ: $\underset{1}{1}00\underset{4}{1}0000\underset{9}{1}\ldots 0\underset{16}{1}0\ldots 0\underset{25}{1}0\ldots$, CONVERSELY,

LET $\langle a_\lambda \rangle_{\lambda=1}^{\infty}$ BE ANY SEQUENCE OF

0'S AND 1'S. CONSIDER THE

SET $A = \{\lambda \mid a_\lambda = 1\} \subset \mathbb{N}$. IF WE

NOW FORM S_A AS ABOVE, THEN

$S_A = \langle a_\lambda \rangle_{\lambda=1}^{\infty}$, THE GIVEN SEQUENCE

WE THUS HAVE SHOW

__THM__ CONSIDER THE SET $\mathcal{P}(\mathbb{N})$

OF ALL SUBSETS OF $\mathbb{N}$. THEN

$\wp(\mathbb{N})$ IS DESCRIBED BY, LOOKING AT $2^{\mathbb{N}} = \{ \langle a_\lambda \rangle_{\lambda=1}^{\infty} \mid a_\lambda = 0 \text{ or } 1 \rangle$, THE SET OF ALL SEQS OF 0'S AND 1'S.

<u>EXAMPLES</u>

① LET $A = \{1, \dots, n\} \subseteq \mathbb{N}$. THEN $S_A = \langle a_\lambda \rangle : \underset{1}{1}, 1, 1, \dots \underset{n}{1}, 0, 0, 0 \dots$. EVERY SUBSET $B \subset A$ HAS 1'S ONLY IN THE FIRST n TERMS OF S_B. THUS, IF $S_B = \langle b_\lambda \rangle$, THEN $b_\lambda = 0$ IF $\lambda > n$. IN THIS WAY, $2^{\{1, \dots, n\}}$ CORRESPONDS TO $\{ \langle a_\lambda \rangle_{\lambda=1}^{\infty} \mid a_\lambda = 0 \text{ IF } \lambda > n \} \subset 2^{\mathbb{N}}$ SO $\#(2^{\mathbb{N}})$ IS "BIGGER" THAN $\# 2^{\{1, \dots, n\}} = 2^n$ FOR ANY $n \in \mathbb{N}$

② SOME SEQUENCES $\langle a_\lambda \rangle_{\lambda=1}^{\infty} \in 2^{\mathbb{N}}$ ARE HARD TO DESCRIBE. LET $A = \{ p \in \mathbb{N} \mid p \text{ IS PRIME} \}$. THEN

$S_A = \langle a_\lambda \rangle : a_\lambda = 1$ IF λ IS PRIME. SINCE WE DON'T KNOW EXACTLY ALL THE PRIME #'S (THERE ARE INFINITE #), WE CAN'T COMPLETELY DESCRIBE ALL TERMS OF S_A.

③ CONSIDER THE SET $M = \{m_n \mid n \geq 1\} \subset \mathbb{N}$ DEFINED RECURSIVELY BY $m_1 = 1$, $m_{n+1} = 2^{m_n}$. SO $m_1 = 1$, $m_2 = 2$, $m_3 = 2^2$, $m_4 = 2^{(2^2)} = 2^4$, $m_5 = 2^{2^4} = 2^{16} = 2^{(2^{2^2})}$

$m_{n+1} = 2^{\left(2^{2^{\cdot^{\cdot^{2}}}}\right)}$: n "LEVELS" OF 2'S

S_M HAS HUGE BLOCKS OF 0'S FOLLOWED BY A 1 AND THEN EVEN A BIGGER BLOCK OF 0'S. $m^5 - m^4 = 2^{16} - 2^4 = 2^4(2^{12} - 1) > 2^4 \cdot 2^{11} = 2^{15}$

WE NOW COME TO THE MAJOR RESULT OF THIS SECTION

<u>THM</u> $\mathcal{P}(\mathbb{N})$ (OR $2^{\mathbb{N}}$) IS INFINITE BUT NOT COUNTABLY INFINITE

PRF THAT $\mathcal{P}(\mathbb{N})$ IS NOT FINITE IS CLEAR. $\forall n \in \mathbb{N}$, CONSIDER $A_n = \{n\}$. THEN $\forall n, m$, $n \neq m \Rightarrow A_n \neq A_m$. So $\{A_n \mid n \in \mathbb{N}\}$ IS AN INFINITE SUBSET OF $\mathcal{P}(\mathbb{N})$.

TO SHOW THAT $\mathcal{P}(\mathbb{N})$ IS NOT COUNTABLE, WE USE $2^{\mathbb{N}}$ AND A PROOF BY CONTRADICTION WE WILL ASSUME THAT $2^{\mathbb{N}}$ IS COUNTABLE AND THEN FIND A SEQUENCE THAT WE DID NOT COUNT. THE PROCESS OF CREATING THIS UNCOUNTED SEQUENCE IS CALLED (CANTOR) DIAGONALIZATION: WE FORM THE SEQUENCE BY CHANGING TERMS ON THE "DIAGONAL"

ASSUME $2^{\mathbb{N}}$ IS COUNTABLE. THEN

WE HAVE $2^{\mathbb{N}} = \{ S_n \mid n \in \mathbb{N} \}$, EACH

$S_n = \langle a_{i,n} \rangle_{i=1}^{\infty}$, A SEQ OF 0's & 1's

WRITE AS AN "INFINITE MATRIX":

$$\begin{array}{lccccc}
S_1: & a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} & \cdots \\
S_2: & a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} & \cdots \\
S_3 & a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} & \cdots \\
S_4 & a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

WE FORM A NEW SEQUENCE $S = \langle \hat{s}_i$

AS FOLLOWS:

CONSIDER $a_{1,1}$. IF $a_{1,1} = 0$, $\hat{s}_1 = 1$

$a_{1,1} = 1$, $\hat{s}_1 = 0$

FOR SIMPLICITY, SAY $\hat{s}_1$ IS THE <u>DUAL</u> OF $a_{1,1}$

NEXT, USE $a_{2,2}$. SET $\hat{s}_2$ TO BE THE DUAL OF $a_{2,2}$

CONTINUE THIS WAY: $\forall n \in \mathbb{N}$,

SET $\hat{s}_n$ TO BE THE DUAL OF $a_{n,n}$

WE NOW HAVE A SEQUENCE S.

CLAIM. $S \neq S_n$ FOR ANY n.

TO SEE THIS, WE HAVE THAT 2 SEQUENCES $\langle C_i \rangle$ AND $\langle D_i \rangle$ ARE EQUAL IF $\forall i \in \mathbb{N}$, $C_i = D_i$. HENCE, $\langle C_i \rangle \neq \langle D_i \rangle$ IF $\exists i \in \mathbb{N}$ $C_i \neq D_i$, I.E., THE SEQUENCES DIFFER AT SOME TERM.

PICK $n \in \mathbb{N}$. THE $n^{\underline{TH}}$ TERM OF S_n IS $a_{n,n}$ AND THE $n^{\underline{TH}}$ TERM OF S IS $\hat{S}_n$, THE DUAL OF $a_{n,n}$. HENCE $\hat{S}_n \neq a_{n,n}$ AND $S_n \neq S$.

BUT S IS A SEQUENCE OF 0'S AND 1'S, AND THUS $S \in 2^{\mathbb{N}}$. THIS IS A CONTRADICTION. SO $2^{\mathbb{N}}$ AND $\mathcal{P}(\mathbb{N})$ ARE NOT COUNTABLY INFINITE

WE NOW LOOK AT OTHER INFINITE SETS WHICH ARE NOT COUNTABLE. CONSIDER THE UNIT INTERVAL $[0,1]$. WE KNOW THAT EVERY INFINITE DECIMAL $.a_1a_2a_3\ldots a_n\ldots\ldots$, WHERE $0 \le a_n \le 9$ AND a_n INTEGER, REPRESENTS A NUMBER IN $[0,1]$. REMEMBER THAT $.a_1a_2a_3\ldots a_n\ldots = \lim_{n\to\infty} .a_1a_2\ldots a_n$ SINCE $\langle .a_1\ldots a_n\rangle$ IS AN INCREASING SEQ AND $\forall n \in \mathbb{N}$, $.a_1\ldots a_n < 1$. $\lim_{n\to\infty} .a_1\ldots a_n$ EXISTS AND ≤ 1.

WE CAN HAVE THAT A FINITE DECIMAL CAN EQUAL AN INFINITE DECIMAL: $.5 = .4999\ldots$ SINCE $.499\ldots = .4 + .0999\ldots$ $= .4 + .1$ $\left[.0999\ldots = 9\sum_{i=2}^{\infty}\frac{1}{10^i}\right]$.

HOWEVER, 2 <u>INFINITE</u> DECIMALS

$.a_1a_2\ldots a_n\ldots$ AND $.b_1b_2\ldots b_n\ldots$ ARE EQUAL ONLY IF $a_i=b_i$ $\forall i\in\mathbb{N}$. NOW CONSIDER THE UNCOUNTABLE INFINITE SET $2^{\mathbb{N}}=\{\langle a_i\rangle \mid a_i=0 \text{ OR } 1\}$. WE NOW DEFINING $f: 2^{\mathbb{N}}\to[0,1]$ BY $f(\langle a_i\rangle)=.a_1a_2\ldots a_i\ldots$.

CLAIM: f IS 1-TO-1. SUPPOSE $.a_1a_2\ldots a_i\ldots = .b_1b_2\ldots b_i\ldots$. IF BOTH ARE INFINITE (HAVE INF # OF 1'S), THEN $\forall i.\ a_i=b_i$ AND SO $\langle a_i\rangle=\langle b_i\rangle$. IF BOTH ARE FINITE, THEN $a_i=1$ PRECISELY WHEN $b_i=1$ AND AGAIN $\langle a_i\rangle=\langle b_i\rangle$ SINCE $\frac{1}{10^n}$ <u>CANNOT</u> BE OBTAINED AS $\frac{1}{10^n}=\sum_{k=n+1}^{\infty}\frac{a_k}{10^k}$, $a_k=0$ OR 1, THESE ARE THE ONLY 2 POSSIBILITIES.

THUS, $\{.a_1a_2\ldots a_n\ldots \mid \forall n, a_n=0 \text{ OR } 1\}$

IS AN UNCOUNTABLE INFINITE SUBSET OF $[0,1]$. THIS IMPLIES $[0,1]$ IS ALSO INFINITE AND UNCOUNTABLE, HAVE SHOWN: SUPPOSE $B \subset A$. THEN A COUNTABLY INFINITE AND B INFINITE $\Rightarrow$ B COUNTABLY INFIN.
CONTRAPOSITIVE: B INFINITE, NOT COUNTABLY INFINITE $\Rightarrow$ A NOT COUNTABLY INFINITE.

MORE ON POWER SETS

ALREADY KNOW: IF A FINITE OR $A = \mathbb{N}$, THEN $\# A < \# \mathcal{P}(A) = \# 2^A$. IS THIS ALWAYS TRUE? SUPPOSE $A = [0,1]$, THE UNIT INTERVAL. WE KNOW A IS AN UNCOUNT. INF. SET. WHAT ABOUT $\mathcal{P}(A)$. IS IT TRUE THAT $\# A < \# \mathcal{P}(A)$

THE ANSWER IS ALWAYS YES

THEOREM (CANTOR POWER SET)

$\# A < \# \mathcal{P}(A)$.

THIS LEADS TO SETS OF HUGE CARDINALITY. START WITH $\mathbb{N}$ THEN $\# \mathbb{N} < \# \mathcal{P}(\mathbb{N})$. NOW LET $A = \mathcal{P}(\mathbb{N})$. THEN $\# A < \# \mathcal{P}(A)$ MEANS $\# \mathcal{P}(\mathbb{N}) < \# \mathcal{P}(\mathcal{P}(\mathbb{N}))$

WE CAN CONTINUE AND GET

$\# \mathbb{N} < \# \mathcal{P}(\mathbb{N}) < \# \mathcal{P}(\mathcal{P}(\mathbb{N}) = \# \mathcal{P}^2(\mathbb{N})$
$< \# \mathcal{P}^3(\mathbb{N}) < \dots < \# \mathcal{P}^k(\mathbb{N}) < \dots$

LET'S LOOK AT THE PARTS OF A PROOF.

1) $\# A \leq \# \mathcal{P}(A)$. TO SHOW THIS, WE NEED TO FIND A SUBSET $\hat{A} \subset \mathcal{P}(A)$ [I.E., $\hat{A}$ IS A SET WHOSE MEMBERS ARE THEMSELVES SUBSETS OF A]

AND A BIJECTION $f: A \to \hat{A}$

THEN $\# A = \# \hat{A} \le \# \mathcal{P}(A)$, SINCE

$\hat{A} \subset \mathcal{P}(A)$. THIS IN NOT HARD.

$\forall x \in A$, SET $A_x = \{x\} \subset A$. NOTE:

THIS CHANGES AN ELEMENT OF A

TO A SINGLETON SUBSET OF A.

DEFINE $\hat{A} = \{A_x \mid x \in A\} = \{\{x\} \mid x \in A\}$

$\subset \mathcal{P}(A)$, SINCE $\{x\} \in \mathcal{P}(A)$, AND

$f: A \to \hat{A}$, $f(x) = \{x\}$. THEN

f IS 1-TO-1: IF $x \ne x_1$, THEN $\{x\}$

$\ne \{x_1\}$, AND $f(A) = \hat{A}$. f IS

THE DESIRED BIJECTION.

) TO SHOW THAT $\# \mathcal{P}(A)$ IS

STRICTLY GREATER THAN A,

IT IS SUFFICIENT TO SHOW:

LET $f: A \to \mathcal{P}(A)$ BE ANY

FUNCTION. THEN f IS NEVER

ONTO: I.E, $\exists B \in \mathcal{P}(A). \forall a \in A.\ B \ne f(a)$.

THIS PROVES THE RESULT, SINCE IF $\#A = \#\mathcal{P}(A)$, WE COULD FIND A BIJECTION $g: A \longrightarrow \mathcal{P}(A)$, WHICH IS THEN ONTO.

THE PROOF IS AMAZINGLY SHORT.

LET f BE GIVEN. DEFINE $B = \{x \in A \mid x \notin f(x)\}$. REMEMBER $f(x) \in \mathcal{P}(A) \Rightarrow f(x)$ IS A SUBSET OF A

SUPPOSE $\exists\, a \in A$ ST $B = f(a)$.

IF $a \in B = f(a)$, THEN BY DEFN OF B, $a \notin f(a)$, WHICH IS A CONTRADICTION. IF $a \notin B = f(a)$ THEN $a \in A - B = \{x \in A \mid x \in f(x)\}$ AND SO $a \in f(a)$, AGAIN A CONTRADICTION. THIS MEANS THAT FOR ALL $a \in A$, $f(a) \neq B$ AND f IS NOT ONTO $\blacksquare$

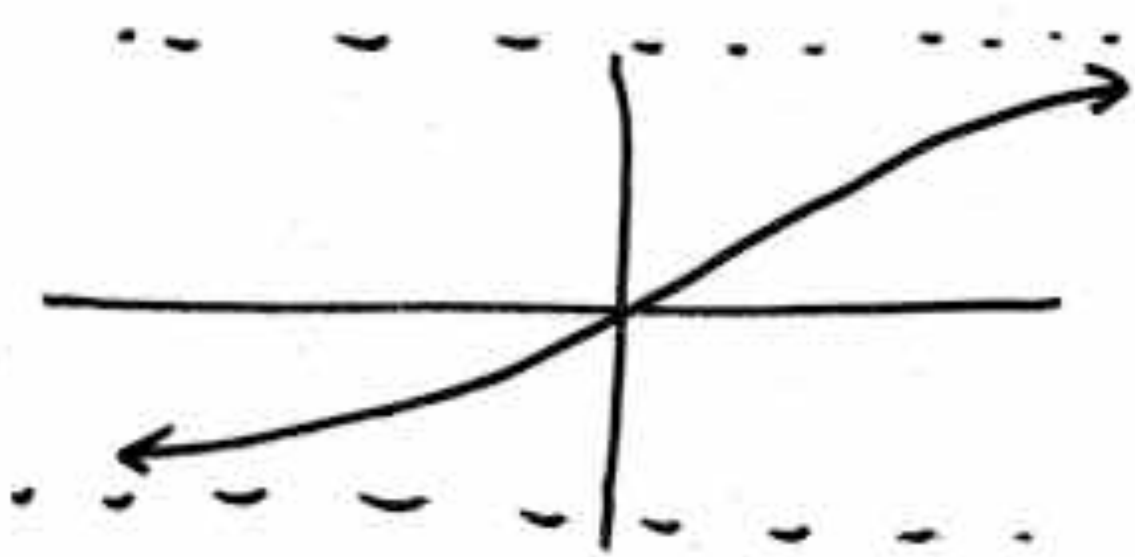
SAVIOR THIS ELEGANT PROOF!!

SO WE NOW KNOW : $\#[0,1) <$
$\#\mathcal{P}([0,1))$ OR $\#\mathbb{R} < \#\mathcal{P}(\mathbb{R})$
THIS SAYS THE SAME THING TWICE
SINCE $\#(0,1) = \#\mathbb{R}$. TO SHOW
THIS, WANT $f: \mathbb{R} \to (0,1)$ A BIJECTN
WE FIRST FIND $g: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$
BY $g(x) = \text{ARCTAN } x$. THEN g IS
1-TO-1, ONTO AND A BIJECTIO
WE ALSO HAVE A BIJECTION
$h: (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0,1)$, $h(x) = \frac{1}{\pi}x + \frac{1}{2}$
EXERCISE. THEN $f = h \circ g : \mathbb{R} \to (0,1)$
IS THE DESIRED BIJECTON.
<u>NOTE</u> $f(x) = h(g(x)) = h(\text{ARCTAN } x)$
$= \frac{1}{\pi} \text{ARCTAN } x + \frac{1}{2}$.

$$h\left(-\frac{\pi}{2}\right) = \frac{1}{\pi}\left(-\frac{\pi}{2}\right) + \frac{1}{2} = 0$$

$$h\left(\frac{\pi}{2}\right) = \frac{1}{\pi}\cdot\frac{\pi}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$f = h \circ g(x) = \frac{1}{\pi} \text{ ARCTAN } x + \frac{1}{2}$$



A CANTOR-TYPE PROOF THAT $[0,1]$ IS UNCOUNTABLE

1. PROVE THAT $\frac{1}{10^n} = \sum_{j=n+1}^{\infty} \frac{9}{10^j}$

2. SHOW THAT EVERY FINITE DECIMAL $.a_1 \cdots a_n$ CAN BE WRITTEN AS AN INFINITE DECIMAL $.b_1 \cdots b_n b_{n+1} \cdots$ WHERE $a_j = b_j$ $1 \le j \le n-1$, $b_n = a_n - 1$, $b_j = 9$, $j \ge n+1$

3. SHOW THAT $\forall x \in [0,1]$, x CAN BE WRITTEN $x = .a_1 a_2 \ldots a_n \ldots$ UNIQUELY. SO $[0,1] = \{ .a_1 a_2 \ldots a_n \ldots \mid a_i \neq 0$ INF MANY $i \}$

4. SHOW $\{ .a_1 a_2 \ldots a_n \ldots \mid a_i \neq 0$ INF MANY $i \}$ IS UNCOUNTABLY INFINITE

<u>HINT</u> IF COUNTABLE, SAY $\{ \hat{a}_j \mid j \in \mathbb{N} \}$ WITH $\hat{a}_j = .a_{j,1} a_{j,2} a_{j,3} \cdots a_{j,n} \cdots$ SHOW THAT $\exists$ INF DECIMAL $\hat{b} = .b_1 b_2 \ldots b_n \ldots$ WITH $\hat{b} \neq \hat{a}_j$, $\forall j \in \mathbb{N}$