Math 3283W : Week 1

1/20 – Syllabus and course page discussion

– <u>Logic</u>: when are complex statements/arguments true? Start with statements P, Q and build <u>new</u> statements.

$P \wedge Q$ (P and Q) : true when <u>both</u> P and Q are true
$P \vee Q$ (P or Q) : true when <u>either</u> P or Q is true (inclusive "or")
$\neg P$ or $\sim P$ (not P) : true when P is <u>false</u>

(basic building blocks)

These are only formally defined by <u>truth tables</u>. Two statements S, T are equivalent ("$S \equiv T$") if and only if they have the same truth tables.

E.g. Truth table for $\sim P \vee Q$:

(False <u>only</u> if P is true <u>and</u> Q is false)

P	Q	$\sim P$	$\sim P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

<u>Exercise.</u> Are $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ equivalent? Are $\sim(P \vee Q)$ and $\sim P \wedge Q$ equivalent? (2nd one: left for the reader)

Rephrased first question: is NOT(P and Q true) the same as "either P false or Q false"?

Truth table:

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P \vee \sim Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

same truth tables;
therefore, logically equivalent!

Exercise. Find an equivalent statement using "$\vee$" for $\sim(P \wedge Q \wedge R)$. (Guess: $\sim P \vee \sim Q \vee \sim R$; check it on your own)

Exercise. Show that $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (distributive rule).

To define: $P \Rightarrow Q$ ("P implies Q"). (**Want**: If P true, then Q true. If P false, no information for Q.)

We need to build $P \Rightarrow Q$ from statements we already have. **Define:** $\boxed{P \Rightarrow Q \equiv \sim P \vee Q}$ (refer to earlier truth table)

Equivalent forms: (1) $P \Rightarrow Q \equiv \sim(P \wedge \sim Q)$

(2) $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$ (**contrapositive** form)

Proof of (1): $P \Rightarrow Q \underset{\text{def}}{\equiv} \sim P \vee Q \equiv \sim P \vee \sim(\sim Q) \equiv \sim(P \wedge \sim Q)$ (exercise from earlier)

Proof of (2): $P \Rightarrow Q \equiv \sim P \vee Q \equiv \sim(\sim Q) \vee \sim P \equiv \sim Q \Rightarrow \sim P$

Claim: $P \Rightarrow Q$ is NOT equivalent to $Q \Rightarrow P$.

Proof: Suppose P is false and Q is true. Then $P \Rightarrow Q$ is true, but $Q \Rightarrow P$ is false, so they cannot be equivalent.

Claim: $P \equiv Q$ is the same as $[P \Rightarrow Q] \wedge [Q \Rightarrow P]$ (sometimes written "$P \Leftrightarrow Q$")

Proof: Suppose (1) $P \Rightarrow Q$ and (2) $Q \Rightarrow P$. Suppose P is true; then, by (1), Q is true; similarly, if Q is true, by (2) P is true. So P true if and only if Q true. Now suppose P is ~~true~~ false. By the contrapositive form of (2), Q is false. Similarly, if Q is false, by the contrapositive form of (1), P is false. Thus P false if and only if Q false.

Notation: If $P \Rightarrow Q$, P is a sufficient condition for Q, and Q is a necessary condition for P.

($P \Rightarrow Q \wedge R \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R)$: exercise)

Logic and Proofs. Prove: If x is an integer and x^2 is odd, then x is odd. (Take the contrapositive: if x is even, then x^2 is even. $\underbrace{\text{But } 2 \mid x \text{ implies } 2 \mid 4 \mid x^2.}$)

more detail: x even $\Rightarrow x = 2n$ for some n
$\Rightarrow x^2 = 4n^2$, and $4n^2$ is a multiple of 2.

Exercise: If x is an integer, prove that x^2 is even if and only if x is even.
("iff")

[NBS]

1/22 — Quantifiers (universal: for all, $\forall$; existential: there exists, $\exists$) These are used with variable statements.

E.g. Let $P(x)$ be "$x^2 > 7$", $Q(x)$ be "$x^2 - 2x - 3 = 0$".

$(\forall x > 3)(P(x))$, in English, reads "for all real $x > 3$, $x^2 > 7$", which is true.

$(\exists x < 0)(Q(x))$: "there exists $x < 0$ with $x^2 - 2x - 3 = 0$" (true: try $x = -1$).

However, $(\forall x > 2)(P(x))$ and $(\exists x > 4)(Q(x))$ are false.

Note. $(\forall x > 3)(P(x))$ is shorthand for $(\forall x)(x > 3 \Rightarrow P(x))$.

De Morgan's laws: negating $\forall$ and $\exists$:
$$\begin{cases} (1)\ \sim[\forall x\ P(x)] \equiv \exists x\ \sim(P(x)) \\ (2)\ \sim[\exists x\ P(x)] \equiv \forall x\ \sim(P(x)) \end{cases}$$

More statements: (3) $\sim[(\forall x \in A) P(x)] \equiv \exists x\,(x \in A \wedge \sim P(x))$

(4) $\sim[(\exists x \in A) P(x)] \equiv (\forall x \in A)(\sim P(x))$

Proof of 3: $\sim[(\forall x \in A) P(x)] \Leftrightarrow \sim[\forall x\,(x \in A \Rightarrow P(x))] \Leftrightarrow \exists x \sim (x \in A \Rightarrow P(x))$

$\Leftrightarrow$ ~~$\exists x (\sim(x \in A)$~~ $\exists x \sim(\sim(x \in A) \vee P(x)) \Leftrightarrow \exists x\,(x \in A \wedge \sim P(x))$.

(4 is an exercise)

E.g. Quantify: (1) Every positive number less than 1 is less than its square root.

(2) Every positive number is greater than the square of some number.

Also (3): if $P(x,y)$ is "$y^2 < x$", is $\sim[\forall x > 0\ \exists y > 0\ P(x,y)]$ true?

(1): $\forall x > 0\ (x < 1 \Rightarrow x^2 < x)$

(2): $\forall x > 0\ \exists y > 0\ y^2 < x$ (true: just take $0 < y < \sqrt{x}$)

(3): this is the negation of what we just proved in (2), so false.

Exercise. Every positive number is greater than the cube of some (positive?) number. (Quantify and prove)

Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ = non-negative integers. Are the following statements true?

(1) $\forall n \in \mathbb{Z}^+$, $n^2 + n + 37$ is prime FALSE: $n = 1 \Rightarrow n^2 + n + 37 = 39 = 13 \cdot 3$ (true for $n = 0$)

(2) $\forall n \in \mathbb{Z}^+$, $n^2 + n + 31$ is prime FALSE: $n = 1 \Rightarrow n^2 + n + 31 = 33 = 11 \cdot 3$ (true for $n = 0$)

(3) $\forall n \in \mathbb{Z}^+$, $n^2 + n + 41$ is prime $\left.\begin{matrix} n=0 \Rightarrow 41 & n=2 \Rightarrow 47 \\ n=1 \Rightarrow 43 & n=3 \Rightarrow 53 \end{matrix}\right\}$ all primes!

We need something better. Try $n = 41$: $41^2 + 41 + 41 = 41 \cdot 43$ is not prime. So (3) is FALSE.

Exercise. Let p be a prime number. Then it's not true that n^2+n+p is prime for all $n \in \mathbb{Z}^+$. If m is any integer, it's not true that n^2+mn+p is prime for all $n \in \mathbb{Z}^+$. (stated ambiguously in lecture)

- Functions and quantifiers Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Let $P(x)$ be "f is continuous at x". Quantify the following: (1) f has a maximum value

(2) f does not have a minimum value

(3) if f is continuous on $[a,b]$, then f has both a maximum and minimum value on $[a,b]$.

Answers: (1) $\exists y \in \mathbb{R} \ \forall x \in \mathbb{R} \ (f(y) \geq f(x))$

(2) $\forall y \in \mathbb{R} \ \exists x \in \mathbb{R} \ (f(y) > f(x))$

(3) To say "f is continuous on $[a,b]$" is to say $\forall x \in [a,b] \ P(x)$.

To say "f has both a maximum and minimum value on $[a,b]$" is to say $\exists y, w \in [a,b]$ such that $\forall x \in [a,b]$, $f(w) \leq f(x) \leq f(y)$

Putting them together,

$(\forall x \in [a,b] \ P(x)) \Rightarrow (\exists y \in [a,b] \ \exists w \in [a,b] \ \forall x \in [a,b] \ f(w) \leq f(x) \leq f(y))$

(Is this the same as "factoring out"

$\forall x \in [a,b] \ \exists y \in [a,b] \ \exists w \in [a,b] \ (P(x) \Rightarrow f(w) \leq f(x) \leq f(y))$? NO!)