MATH3283W LECTURE NOTES: WEEK 3

2/1/2010
Proof without words: picture depicts
What is being proved from Fig.3.1?

1 Adding more and more dots gives bigger and bigger squares.
  $\rightarrow$ It is too vague and it is not actually a math statement.
2 Each consecutive line has two more dots than the previous line.
  $\rightarrow$ Nothing to prove.
3 The sum of consecutive odd numbers gives a square number .
  It can be proved by induction:

Observation: $P(n) : 1 + 3 + \cdots + (2n + 1) =?$
$n = 1$, $1 + (1 + 2) = 1 + (1 + 2 \cdot 1) = 4 = 2^2$
$n = 2$, $1 + (1 + 2) + (1 + 4) = 1 + (1 + 2 \cdot 1) + (1 + 2 \cdot 2) = 9 = 3^2$
So we guess that $P(n)$ is $1 + 3 + \cdots + (2n + 1) = (n + 1)^2$ and prove it by induction.
$n = 0$, OK.
Assume $P(n)$ is true, we want to show that $P(n + 1)$ is true.

$$\begin{aligned}
& 1 + 3 + 5 + \cdots + (2n + 1) + (2(n + 1) + 1) \\
\overset{P(n)}{=} \; & (n + 1)^2 + 2n + 3 \\
= \; & n^2 + 2n + 1 + 2n + 3 \\
= \; & n^2 + 4n + 4 \\
= \; & (n + 2)^2 \\
= \; & ((n + 1) + 1)^2
\end{aligned}$$

So $P(n + 1)$ is true.

Upper and lower bounds

Suppose $A(\neq \emptyset) \subset \mathbb{R}$ has an upper bound (bounded above). Let

$$B = \{r \in \mathbb{R} | r : \text{upper bound for } A\} \neq \emptyset$$

Suppose B has a smallest element w. Then w is called the **least upper bound** of A, or the **supremum** of A, write $w = lubA$ or $w = \sup A$. Thus $w = \sup A$ if

(1) w is an upper bound for A and
(2) if r is an upper bound for A, then $r \geq w$.

Another form of (2), using contrapositive:

(2') $\forall r \in \mathbb{R}(r < w \Rightarrow \exists a \in A, r < a)$

Note that any $r < w$ is not an upper bound.

Facts

- If $r > w = \sup A$, then $r \in B$.
  Since w is also an upper bound, B is a ray $[w, \infty)$.
- Let $\varepsilon > 0$, then $w - \varepsilon < w$ and by (2'), $\exists a \in A, w - \varepsilon < a$, so we also have an equivlant condition:
  (2") $\forall \varepsilon > 0, \exists a \in A, w - \varepsilon < a$.

Similar for lower bounds:
Suppose $A(\neq \emptyset) \subset \mathbb{R}$ is bounded below and w is the **greatest lower bound** for A, write $w = glbA$ or $w = \inf A$ (**infemum** of A). Thus $w = \inf A$ if

(1) w is a lower bound for A and
(2) if s is a lower bound for A, then $w \geq s$.

Again, using contrapositive of (2)

(2') $\forall r \in \mathbb{R}(r > w \Rightarrow \exists a \in A, a < r)$

Facts

- If $s < w = \inf A$, then S is a lower bound and

  $s \in C$: set of lower bounds of A (C is the ray $(-\infty, w]$)

- We also have
  (2") $\forall \varepsilon > 0, \exists a \in A, a < w + \varepsilon$.

Note: If $A \neq \emptyset$ has a maximal value w, then $w = \sup A$. If $A \neq \emptyset$ has a minimum value s, then $s = \inf A$. sup's and inf's generalize max/min values.

Examples:

(1) $A = (0, 1)$. Then $\sup(0, 1) = 1$ and $\inf(0, 1) = 0$.
(Observe: 1 is an upper bound. If $r < 1$, we have to show that r is not an upper bound. Or we want to find $s \in (0, 1)$ such that $r < s < 1$. Choose average $\frac{1+r}{2}$, then $r < \frac{1+r}{2} = s < 1$.)
$A' = [0, 1]$ also has $\sup[0, 1] = 1$ and $\inf[0, 1] = 0$. So $\sup A$ and $\inf A$ may or may not be an element of A.

$(-\infty, 0]$	$(0, 1)$	$[1, \infty)$
$\uparrow$	$\uparrow$	$\uparrow$
set of lower bounds	A	set of upper bounds

(2)
$$A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} = \{\frac{1}{n} | n \in \mathbb{N}\}$$
Since $\frac{1}{n}$ is decreasing, 1 is the largest element of A and $\sup(A) = 1$.
calculus: $\frac{1}{x} \to 0$ as $x \to +\infty$
replacing x by n (integer values): $\frac{1}{n} \to 0$
So $\inf(A) = 0$. This means:
$\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

(3)
$$A = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots\}$$
A is bounded above by 1. What is $\sup(A)$? Note that
$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$$
Let $\varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n+1} < \varepsilon$. Then
$$1 - \varepsilon < 1 - \frac{1}{n+1} = \frac{n}{n+1}$$
By (2"), $1 = \sup A$.

(4) Let
$$A = \{x | x^3 < 4\}$$
Now $[x > 0$ and $x^3 < 4]$ iff $0 < x < 4^{\frac{1}{3}}$. If $x < 0$, then $x^3 < 0$ and so $x^3 < 4$. Hence $A = (-\infty, 4^{\frac{1}{3}})$.
$\sup A = 4^{\frac{1}{3}}$ and $\inf A$ **DOES NOT** exist.

(5)
$$A = \{x \cos x | 0 \leq x \leq \pi\}$$

Observe: $f(0) = 0, f(\frac{\pi}{2}) = 0, f(\pi) = -\pi$. The graph may look like Fig.3.2. By calculus, f has a max and min value on $[0, \pi]$.

$f'(x) = x \sin x + \cos x = 0 \;\Rightarrow\; \cos x = x \sin x, \tan x = \dfrac{1}{x}$

Then $x \approx .87$ and $x \cos x \approx .56$. So $\sup A = .56$ (check by graph), $f(\pi) = -\pi$ is the minimum value and $\inf A = -\pi$.

(6)

$$A = \{x | x^2 + x - 6 < 0\}$$

$x^2 + x - 6 = (x+3)(x-2) = 0$ when $x = -3$ or $x = 2$.
For $x = 0$, we can get $x^2 + x - 6 = -6 < 0$. So $A = (-3, 2)$ and

$$\sup A = 2, \;\inf A = -3$$

2/3/2010
Q: Is $2^{2^k} + 1$ prime for any $k \in \mathbb{N}$?
A: No!

$$k = 5, 2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \cdot 6700417$$

Examples:

(1) Let $A \subset \mathbb{R}$ and suppose $\sup A = \inf A$. What can we say about A?
Let $w = \sup A = \inf A$. If $a \in A$, then

$$w = \sup A \Rightarrow w \geq a$$

$$w = \inf A \Rightarrow w \leq a$$

So $w = a$ and $A = \{w\}$.

(2) Let $A \subset \mathbb{R}$ and $B \subset A$. What can we say about $\sup A$ and $\sup B$?
Assuming both A and B are bounded. Let $w = \sup A$, then w is an upper bound for A. Since $B \subset A$, w is also an upper bound for B. Hence $w \geq \sup B$.
Exercise: What can you prove about $\inf A$ and $\inf B$?

(3)

$$A = \{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \cdots\}$$

$a_n = \frac{n}{n+2} = \frac{1}{1+\frac{2}{n}} \to 1$ ($\frac{2}{n} \to 1$ when $n \to 1$)
or $\frac{n}{n+2} = \frac{n+2-2}{n+2} = 1 - \frac{2}{n+2} \to 1$.
So $\sup A = 1$ and $\inf A = \frac{1}{3}$. Note that $\sup A$ is not an element of A but $\inf A \in A$.

(4)
$$B = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \cdots\}$$
$\sup A = 1$ and $\inf A = -\frac{1}{2}$.

(5)
$$A = \{x \in \mathbb{R} | x^2 + x > 0, x > 0\}$$
$x^2 + x = 0 = x(x+1), x = 0, -1$.
$x = -\frac{1}{2}, (-\frac{1}{2})^2 + (-\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{2} < 0$
So $A = (0, +\infty)$. No $\sup A$ and $\inf A = 0$.

(6)
$$B = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots, \frac{1}{3^n}, \cdots\}$$
$\sup A = 1$, $\inf A = 0$ because $\frac{1}{3^n} \to 0$.
In order to prove $\inf A = 0$, we need to show that $\forall \varepsilon > 0, 0+\varepsilon = \varepsilon$ is not a lower bound. So we "need to find" some n such that $\frac{1}{3^n} < \varepsilon$ or $\frac{1}{\varepsilon} < 3^n$.
Take logs: $\ln(\frac{1}{\varepsilon}) < n\ln(3)$, $-\ln(\varepsilon) < n\ln(3)$, $n > -\frac{\ln 3}{\ln \varepsilon}$.
The existence of such integer is given by the next topic.

The Least upper bound axiom

Math statement that the reals $\mathbb{R}$ have no "holes". Equivalently, if we approach a number as a l.u.b, then that number exists.

Least upper bound/complete axiom

Every non-empty set of real numbers that is bounded above has a least upper bound.

From this, we get a version of the well-ordering theorem for the reals.

Theorem 0.1. *Let $A \neq \emptyset$, $A \subset \mathbb{R}$ and A bounded below. Then $glbA$ exists.*

Proof. Consider $B = \{-a | a \in A\}$. Since A is bounded below, $\exists x \in \mathbb{R}$, $\forall a \in A, a \geq x$. Then $\forall a \in A, -a \leq -x$ and $-x$ is an upper bound for B. By LUB axiom, B has a l.u.b., say $y = lub(B)$.
Claim: $-y = glb(A)$.
First, we want to show that $-y$ is a lower bound.
$$\forall a \in A, -a \leq y \Rightarrow \forall a \in A, a \geq -y$$
and $-y$ is a lower bound.
Second, we have to show that $-y$ is the greatest one. Suppose $-y < r$, then $y > -r$. Since $y = lub(B), \exists a \in A, y > -a > -r$. Then $a < r$ and r is not a lower bound for A. So $-y = glb(A)$. □

An important consequence is:
The natural numbers $\mathbb{N}$ and in fact the set $A_r = \{nr|n \in \mathbb{N}\}$ for any positive real r are unbounded above.

Theorem 0.2 (Archimedean Property of Reals)**.** *Let a, b be positive real numbers, then $\exists n \in \mathbb{N}, na > b$.*

Proof. By contradiction. Suppose $\forall n \in \mathbb{N}, na \leq b$. Then $A = \{na|n \in \mathbb{N}\}$ is bounded above. Let $b^* = lub(A)$. Since $a > 0$, $b^* - a$ is not an upper bound. So $\exists m \in \mathbb{N}, b^* - a < ma$. This implies

$$b = (b^* - a) + a < ma + a = (m + 1)a$$

contradicts that b^* is an upper bound for A. So $\exists n \in \mathbb{N}, na > b$. □

Corollary 0.3. *(1) $\mathbb{N}$ is unbounded above.*
(2) $glb\{\frac{1}{n}|n \in \mathbb{N}\} = 0$

Proof. (1) $\mathbb{N} = \{n \cdot 1|n \in \mathbb{N}\}$ is unbounded by A.P. ($a = 1$).
(2) Foe any $r > 0$, we want to show $\exists n, \frac{1}{n} < r$. Since

$$\frac{1}{n} < r \Leftrightarrow 1 < nr$$

This follows from A.P. ($b = 1, a = r$). So 0 is the greatest lower bound. □

Exercises

(1) Let $a > 0$. Then

$$glb\{\frac{a}{n}|n \in \mathbb{N}\} = 0$$

(2) Prove the following variant of A.P.:

$$\text{Let } a, b > 0, \text{ then } \exists n \in \mathbb{N}, -na < -b$$

This means $\{-na|n \in \mathbb{N}\}$ and $\{-n|n \in \mathbb{N}\}$ are unbounded in **NEG SENSE** (goes to $-\infty$).

(3) Prove: If $a > 0$, then $lub\{-\frac{a}{n}|n \in \mathbb{N}\} = 0$

2/5/2010

Theorem 0.4. *There is a real number x such that $x^2 = 2$.*

Proof. Let $S = \{s \in \mathbb{R} | s > 0 \text{ and } s^2 < 2\}$.
Since $1 \in S$, S is not empty. Moreover, 2 is an upper bound. This can be proved by contrapositive:
If $r \geq 2$, then $r^2 \geq 2^2 = 4 > 2 \Rightarrow r$ is not in S.
By LUB axiom, $\sup S$ exists. Let $x = \sup S > 1$.
Claim: $x^2 \geq 2$ and $x^2 \leq 2$, which says $x^2 = 2$.
Suppose $x^2 < 2$, then $b = 2 - x^2 > 0$. Set $a = 2x + 1$. By exercise (1), $\exists n \in \mathbb{N}, \frac{a}{n} < b$ i.e. $\frac{1}{n}(2x+1) < 2 - x^2$
$\Rightarrow \frac{1}{n}(2x + \frac{1}{n}) \leq \frac{1}{n}(2x+1) < 2 - x^2$
$\Rightarrow x^2 + \frac{2}{n} + (\frac{1}{n})^2 < 2, (x + \frac{1}{n})^2 < 2$ and $x + \frac{1}{n} \in S$.
This contradicts to the fact that $x = \sup S$, so $x^2 \geq 2$.
A similar argument shows that if $x^2 > 2$, we can find $n \in \mathbb{N}$ with $(x - \frac{1}{n})^2 > 2$, contradicting that x is the smallest upper bound. So we also have $2 \leq x^2$ and hence $x^2 = 2$. □

Now we want to show that there are rational numbers everywhere.

Theorem 0.5. *Let a, b be real numbers with $0 < a < b < 1$, then $\exists r \in \mathbb{Q}$ with $a < r < b$.*

Proof. Since $b > a, b - a > 0$. Since $glb\{\frac{1}{n} | n \in \mathbb{N}\} = 0$, we have $n_1, n_2 \in \mathbb{N}$ with $\frac{1}{n_1} < b - a$ and $\frac{1}{n_2} < a$. Let $n = n_1 n_2$, then $\frac{1}{n} < b - a$ and $\frac{1}{n} < a$ (see Fig.3.3). Let $B = \{\frac{j}{n} | 1 \leq j \leq n \text{ and } \frac{j}{n} \leq a\}$
$B \neq \emptyset$ since $\frac{1}{n} \in B$ and bound above by 1.. By LUB axiom, B has a max element $\frac{j_0}{n}$. (since B is finite, $lub(B) \in B$). Then $\frac{j_0+1}{n} > a$. Also $\frac{j_0+1}{n} = \frac{j_0}{n} + \frac{1}{n} < a + (b - a) = b$. So we can choose $r = \frac{j_0+1}{n}$. □

Theorem 0.6 (n^{th} roots of positive numbers)**.** *Let $n \in \mathbb{N}$ and $y > 0$. Then $\exists x > 0$ such that $x^n = y$ i.e. $x = y^{\frac{1}{n}} = \sqrt[n]{y}$.*

Examples: find lub and glb if they exist:

(1)
$$A = \{x | x^2 < 4\} = \{x | |x| < 2\} = (-2, 2)$$
$lub(A) = 2, glb(A) = -2$.

(2)
$$B = \{x | x^5 > 9\}$$
(x negative $\Rightarrow x^5$ negative) $\Rightarrow x > 0$
If $9 \leq x^5$, then $9^{\frac{1}{5}} \leq (x^5)^{\frac{1}{5}} = x$ (Why? Check it). So
$B = \{x | x > 9^{\frac{1}{5}}\} = [9^{\frac{1}{5}}, +\infty)$
No lub, $glb(B) = 9^{\frac{1}{5}} \in B$.

(3) $C = \{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \cdots\} = \{2 + \frac{1}{n} | n \geq 2\}$
$lub(C) = 2\frac{1}{2} \in C$.
$glb(C) = 2 + glb\{\frac{1}{n} | n \geq 2\} = 2 + 0 = 2$ not in C.

(4) $D = \{x|x > 0 \text{ and } \ln x < 1\}$.
$\ln x = 1 \Rightarrow x = e$.
So $D = (0, e), glb(D) = 0, lub(D) = e$.

(5)
$$A = \{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{1}{3}, \frac{3}{4}, -\frac{1}{4}, \cdots\}$$
$\sup(A) = 1, \inf(A) = -\frac{1}{2}$

(6)
$$A = \{0, \frac{1}{2}, -\frac{1}{2}, \frac{3}{4}, -\frac{1}{4}, \frac{7}{8}, -\frac{1}{8}, \frac{15}{16}, -\frac{1}{16}, \cdots, \frac{2^n - 1}{2^n}, -\frac{1}{2^n}, \cdots\}$$
$\sup(A) = 1, \inf(A) = -\frac{1}{2}$.
Find $a \in A$ with $a > .99$:
$n = 7, \frac{2^7-1}{2^7} = 1 - \frac{1}{128} > 1 - \frac{1}{100} = .99$
Find $a \in A$ with $a > .999$:
$n = 10, \frac{2^{10}-1}{2^{10}} = 1 - \frac{1}{1024} > 1 - \frac{1}{1000} = .999$

(7)
$$A = \{x|x^3 + x > 0\} = \{x|x > 0\}$$
$x^3 + x = x(x^2 + 1) = 0 \Rightarrow x = 0$
No $\sup B, \inf B = 0$ (see Fig.3.4).

(8)
$$B = \{x|x^3 - x > 0\}$$
$x^3 - x = 0 = x(x - 1)(x + 1)$
$$B = \{x|x > 1 \text{ or } -1 < x < 0\} = (-1, 0) \cup (1, +\infty)$$
No $\sup B, \inf B = -1$ (see Fig.3.5).