(2/8) **Sequences** (Look at exercises 1.17–1.26 in section 3.1) [NBS]

LIMITS

Some prerequisites from calculus:

Limits at a point $a \in \mathbb{R}$	**Limits at infinity**
Let f be defined in an open interval about $a \in \mathbb{R}$.	Let f be defined on some ray $[R, +\infty)$.

Intuitively,

"$\lim_{x \to a} f(x) = L$" means "as x gets closer and closer to a, $f(x)$ gets closer and closer to L"	"$\lim_{x \to +\infty} f(x) = L$" means "as x gets larger and larger, i.e. 'closer to $+\infty$', $f(x)$ gets closer and closer to L"

$\varepsilon > 0$ is an "error" or "tolerance" by which we allow $f(x)$ to differ from L.

$\delta > 0$, or N, is the "deviation number"; it depends on ε.

In terms of error,

"$\lim_{x \to a} f(x) = L$" means that given $\varepsilon > 0$, we can find $\delta > 0$ such that if $\lvert x-a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$	"$\lim_{x \to +\infty} f(x) = L$" means that given $\varepsilon > 0$, we can find N such that if $x > N$, then $\lvert f(x) - L \rvert < \varepsilon$ (i.e. L is a horizontal asymptote)

Formal definition of continuity: ("$\lim_{x \to a} f(x) = f(a)$")

f is continuous at $x = a$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. f is continuous on (a,b) or $[R, +\infty)$ if f is continuous at all points of the interval or ray.

Definition of a sequence

A sequence is a function $a : \{ n \in \mathbb{Z}^+ \mid n \geq K \} \to \mathbb{R}$ for some $K \in \mathbb{Z}^+$. We typically use $K = 0$ (domain: $\mathbb{Z}^+$) or $K = 1$ (domain: $\mathbb{N}$). We usually write $a_n = a(n)$, and we write the sequence as $\langle a_n \rangle$ or $\langle a_n \rangle_{n=K}^{+\infty}$. Note that the sequence $\langle a_n \rangle$ and the set $\{ a_n \mid n \geq K \}$ of its values are not the same thing. (E.g. if $a_n = \begin{cases} 1, & n \text{ odd}; \\ -1, & n \text{ even} \end{cases}$ is defined for $n \geq 1$, then $\{ a_n \mid n \geq 1 \}$ is just the two-element set $\{1, -1\}$.)

Limits of sequences This concept is similar to the "limit at infinity" of a function. We say that "$\lim_{n\to\infty} a_n = L$", or "$a_n \to L$ as $n \to \infty$", or "$\langle a_n \rangle$ converges to L", if a_n gets closer and closer to L as n gets larger and larger.

$\boxed{\text{Formal Definition}}$ The sequence $\langle a_n \rangle$ converges to the limit L (written $\lim_{n\to\infty} a_n = L$) if $\underbrace{\text{for every } \varepsilon > 0}_{(\text{"error"})}$, $\underbrace{\text{there exists } N \in \mathbb{N}}_{(\text{"cutoff point"})}$ such that <u>if</u> $n \geq N$, <u>then</u> $|a_n - L| < \varepsilon$.

Connection between functions and sequences <u>If</u> $f : [0, +\infty) \to \mathbb{R}$, $\lim_{x \to +\infty} f(x) = L$, and we define a sequence $a_n = f(n)$ (the values of f on $\mathbb{N}$), <u>then</u> $\lim_{n\to\infty} a_n = L$. (Just compare the definitions, and "round up" the N to the nearest integer if necessary.)

This will allow us to use techniques of calculus (e.g. L'Hôpital's Rule) to investigate sequences.

Convergence/ Divergence We say that a sequence <u>converges</u> if $\lim_{n\to\infty} a_n$ exists, and <u>diverges</u> otherwise. There are multiple ways for a sequence to diverge:
($\langle a_n \rangle$)

(1) $\lim_{n\to\infty} a_n$ "=" $+\infty$ or $-\infty$; (examples: $a_n = n^2$ ($a_n \to +\infty$); $a_n = \ln(\frac{1}{n})$ ($a_n \to -\infty$))

(2) More than one possible limiting value; (example: $\langle a_n \rangle = 1, -1, 1, -1, 1, -1, \dots$)

(3) No possible limiting value; (example: $a_n = n^{\text{th}}$ digit in the decimal expansion of $\pi = 3.14159\dots$)

Boundedness The sequence $\langle a_n \rangle$ is <u>bounded</u> if the set $\{a_n \mid n \in \mathbb{N}\}$ (i.e. the range of the function a) is bounded (so $\exists R \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $|a_n| < R$).

<u>Theorem.</u> If $\langle a_n \rangle$ is convergent, then it is bounded.

<u>Sketch of proof.</u> Take $\varepsilon = 1$ in the definition of $\lim_{n\to\infty} a_n = L$. Then $L - 1 < a_n < L + 1$ for all but a finite number of the a_n; consider the maximum absolute value of these, and compare with $|L-1|$ and $|L+1|$. ▨

(2/10) Last time, we proved the following: if $\langle a_n \rangle$ is convergent, then $\langle a_n \rangle$ is bounded. (The converse is not true: why?) The contrapositive – if $\langle a_n \rangle$ is unbounded, then $\langle a_n \rangle$ is divergent – gives us a test for divergence.

E.g. Let $r > 1$ be given. We'll show that $\langle r^n \rangle$ diverges by showing it's unbounded: let $M > 0$ be given, and find $n \in \mathbb{N}$ such that $r^n > M$. But $r^n > M \iff n \ln r > \ln M \iff n > \frac{\ln r}{\ln M}$, and such an n certainly exists since $\mathbb{N}$ is $\overset{\text{not}}{\wedge}$ bounded above.

Exercise. What happens if $0 < r < 1$? If $r = 1$?

The algebra of limits

Limit Laws Suppose $\langle a_n \rangle, \langle b_n \rangle$ are sequences, with $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

(1) $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

(2) If $c \in \mathbb{R}$, $\lim_{n\to\infty} (ca_n) = c \cdot L$

(3) $\lim_{n\to\infty} (a_n b_n) = LM$

(4) if $M > 0$ and $b_n > 0$ $\forall n \in \mathbb{N}$,
$$\lim_{n\to\infty} \left(\frac{1}{b_n}\right) = \frac{1}{M}$$

(5) if $M > 0$ and $b_n > 0$ $\forall n \in \mathbb{N}$,
$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$$

Proofs (1), (2) and (3): see course notes.

(5) follows from (3) and (4) by writing
$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}.$$

We prove (4): let $\varepsilon > 0$ be given. We need to show $\exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\left| \frac{1}{b_n} - \frac{1}{M} \right| < \varepsilon$.

Now $\frac{1}{b_n} - \frac{1}{M} = \frac{M - b_n}{M b_n} = \frac{1}{M b_n}(M - b_n)$. Since $b_n \to M$, $\exists n_1 \in \mathbb{N}$ such that if $n \geq n_1$, then $b_n > \frac{M}{2}$. (Why?) So, if $n \geq n_1$, $\frac{1}{M b_n} < \frac{1}{M} \cdot \frac{1}{M/2} = \frac{2}{M^2}$. On the other hand, since $b_n \to M$ and $\frac{M^2}{2} \cdot \varepsilon$ is positive, $\exists n_2 \in \mathbb{N}$ such that if $n \geq n_2$, then $|b_n - M| = |M - b_n| < \frac{M^2}{2} \cdot \varepsilon$.

Finally, put $n_0 = n_1 n_2$, so $n_0 \geq n_1$ and $n_0 \geq n_2$.

Whenever $n \geq n_0$,
$$\left| \frac{1}{b_n} - \frac{1}{M} \right| = \left| \frac{1}{M b_n} \right| \cdot |M - b_n| < \frac{2}{M^2} \cdot \frac{M^2}{2} \varepsilon = \varepsilon,$$
and so $\frac{1}{b_n} \to \frac{1}{M}$ as claimed. ▨

Corollary. If $\lim_{n\to\infty} a_n = M$, then $\lim_{n\to\infty} (a_n - M) = 0$ and $\lim_{n\to\infty} |a_n - M| = 0$.

Pinching/Squeeze Theorem Let $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ be sequences such that, for some $K \in \mathbb{N}$, $a_n \leq c_n \leq b_n$ for all $n \geq K$. Then if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, we must $\overset{\text{have}}{\wedge}$ $\lim_{n\to\infty} c_n = L$.

E.g. By using the limit laws, and the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$, we can find limits of rational functions.

For example, we can show (easily) that $\lim_{n\to\infty} \frac{5n^5+7n^3+2n+8}{4n^5-9n^4-3n^2} = 5/4$ and $\lim_{n\to\infty} \frac{n^3+2n^2+3n}{n^4-n^3+2n-1} = 0$.

(first step: pull out the largest power from numerator and denominator)

True or False? (1) $\langle a_n\rangle, \langle b_n\rangle$ divergent $\Rightarrow$ $\langle a_nb_n\rangle$ divergent? (FALSE: take $a_n = b_n = \begin{cases} 1, & n \text{ odd}; \\ -1, & n \text{ even} \end{cases}$)

(2) $\langle a_nb_n\rangle$ divergent $\Rightarrow$ at least one of $\langle a_n\rangle, \langle b_n\rangle$ divergent? (TRUE: contrapositive of Limit Law (3))

E.g. Show $\frac{2^n}{n!} \to 0$. (Idea: Pinch $\frac{2^n}{n!}$ between 0 and a_n, with $a_n \to 0$.

But $\frac{2^n}{n!} = 2\cdot \underbrace{\frac{2}{2}\cdot\frac{2}{3}\cdots\frac{2}{n-1}}_{\text{all} \leq 1}\cdot\frac{2}{n} \leq 2\cdot\frac{2}{n}$, so $a_n = \frac{4}{n} \to 0$ works.)

(2/12) Given a sequence $\langle a_n\rangle$, we can form the subsequences $\langle e_n\rangle$, $e_n = a_{2n}$, of even-index terms and $\langle o_n\rangle$, $o_n = a_{2n+1}$, of odd-index terms.

Exercises. Prove: (1) if $a_n \to L$, then $e_n \to L$ and $o_n \to L$.

(2) if $o_n \to L$ and $e_n \to M$, with $L \neq M$, then $\langle a_n\rangle$ diverges.

Claim. If $e_n \to L$ and $o_n \to L$, then $a_n \to L$.

Proof. Let $\varepsilon > 0$ be given. $\exists\, n_1 \in \mathbb{N}$ such that $n \geq n_1 \Rightarrow |a_{2n+1} - L| < \varepsilon$, and $\exists\, n_2 \in \mathbb{N}$ such that $n \geq n_2 \Rightarrow |a_{2n} - L| < \varepsilon$. Put $n_0 = 2n_1n_2$; then if $n \geq n_0$, $|a_n - L| < \varepsilon$ whether n is even or odd, so $a_n \to L$. ▨

Sequences and functions

Theorem. If $\langle a_n\rangle$ is a sequence such that $a_n \to L$, and $f:\mathbb{R}\to\mathbb{R}$ is a function which is continuous at L, then $f(a_n) \to f(L)$.

Proof. Let $\varepsilon > 0$. Since f is continuous at L, $\exists\, \delta > 0$ such that $|x - L| < \delta \Rightarrow |f(x) - f(L)| < \varepsilon$. Since $a_n \to L$, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |a_n - L| < \delta \Rightarrow |f(a_n) - f(L)| < \varepsilon$, so in fact $f(a_n) \to f(L)$. ▨

This result, together with the fact that if $\lim_{x\to\infty} f(x) = M$ and $a_n = f(n)$ then $a_n \to M$, will allow us to compute many limits.

Examples (1) $\lim_{n\to\infty} \sin^2\left(\frac{\pi}{n^3}\right) = 0$. Why? $\frac{1}{x^3} \to 0$ as $x \to \infty$, so $\frac{1}{n^3} \to 0$ as $n \to \infty$, and thus [NBS]

$\frac{\pi}{n^3} \to 0$. But $\sin^2(x)$ is continuous at 0, so $\sin^2\left(\frac{\pi}{n^3}\right) \to \sin^2(0) = 0$.

(2) $\lim_{n\to\infty} \cos(\ln(1-\frac{1}{n})) = 1$, because: $1-\frac{1}{n} \to 1$. $\ln(1) = 0$, $\cos(0) = 1$,
ln is continuous at 1,
and cos is continuous at 0.

(3) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

If we can show $\lim_{x\to\infty} f(x) = e$, where $f(x) = \left(1+\frac{1}{x}\right)^x$, we're done. Consider $\ln f(x) = x \cdot \ln(1+\frac{1}{x})$.

Rewrite this as $\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$, which is a "$\frac{0}{0}$ form" as $x \to \infty$; thus L'Hôpital's Rule applies.

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \underset{\boxed{\text{LHR}}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\frac{-1}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1,$$

so $\lim_{x\to\infty} f(x) = e^1 = e$, as claimed.

(Similar reasoning shows that $\lim_{n\to\infty} n^{1/n} = 1$, and for any $a \neq 0$, $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.)

Remark. Suppose $\lim_{x\to 0} f(x) = L$. If $b_n = f\left(\frac{1}{n}\right)$, then $b_n = L$ (exercise). For example, since $\frac{\sin x}{x} \to 1$ as $x \to 0$, we have $n \cdot \sin\frac{1}{n} = \dfrac{\sin(\frac{1}{n})}{\frac{1}{n}} \to 1$ as $n \to \infty$.

Limits at infinity

Formally, a sequence $\langle a_n \rangle$ diverges to ∞ if $\forall M \in \mathbb{N}\ \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n \geq M$, and diverges to $-\infty$ if $\forall M \in \mathbb{N}\ \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n \leq -M$. (Intuitively, $a_n \to \infty$ if a_n gets arbitrarily larger and larger as $n \to \infty$, and $a_n \to -\infty$ if a_n gets "larger in the negative sense" as $n \to \infty$.)

Examples Find $\langle a_n \rangle$, $\langle b_n \rangle$ such that $a_n \to \infty$, $b_n \to \infty$, and (i) $\frac{a_n}{b_n} \to \infty$ (ii) $\frac{a_n}{b_n} \to 0$ (iii) $\frac{a_n}{b_n} \to -7$ (iv) $\frac{a_n}{b_n} \to -\infty$.

(i): take $a_n = n^2$, $b_n = n$.

(ii): take $a_n = n$, $b_n = n^2$.

(iii, iv): impossible!

Exercises to look at: 2.11 (a, b), 2.14, 2.15
in section 3.2;
3.7, 4.4, 4.5 in sections 3.3 and 3.4.