(3/1) **Series** [NBS]

Start with a sequence $a_1, a_2, \dots, a_n, \dots$ Goal: Add up all the terms. There's no problem adding finitely many terms: if $m \in \mathbb{N}$, $s_m = a_1 + \dots + a_m = \sum_{i=1}^{m} a_i$ makes sense. Now we "add more and more & take a limit": we define $\sum_{i=1}^{\infty} a_i$ to be $\lim_{m\to\infty} s_m$. Each s_m is the m^{th} partial sum of the terms a_i, and $\sum_{i=1}^{\infty} a_i$ is the infinite series with these terms, which converges (resp. diverges) if $\langle s_m \rangle$ does. If $\lim_{m\to\infty} s_m = L$, then $\sum_{i=1}^{\infty} a_i$ sums to L.

Examples: (1) The geometric series, $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$.

If $|x| < 1$, this series converges to $\frac{1}{1-x}$, because the m^{th} partial sum $s_m = 1 + \dots + x^m$ can be written as $\frac{1-x^{m+1}}{1-x}$, and this last converges to $\frac{1}{1-x}$ as $m \to \infty$.

If $|x| > 1$, this series diverges, since the sequence $\langle x^{m+1} \rangle$ diverges.

Exercise If $x = \pm 1$, this series still diverges.

(2) The harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$.

Remark If $\langle a_n \rangle$ is any sequence with $a_n > 0$ $\forall n$, then $\langle s_m \rangle$ is increasing, so $\sum_{i=1}^{\infty} a_i$ converges $\iff$ $\langle s_m \rangle$ is bounded. (exercise)

Let s_m be the partial sum $1 + \frac{1}{2} + \dots + \frac{1}{m}$ of the harmonic series. By induction, it can be shown that $s_{2^n} \geq 1 + \frac{n}{2}$ if $n \geq 0$. Then $\langle s_m \rangle$ is unbounded, so by the Remark, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Given the partial sums of a sequence $\langle a_n \rangle$, we can recover its terms: $s_{n+1} = \sum_{i=0}^{n+1} a_i = \left(\sum_{i=0}^{n} a_i\right) + a_{n+1}$
($\langle s_m \rangle$)

$$= s_n + a_{n+1},$$

so we can write $a_{n+1} = s_{n+1} - s_n$.

Theorem. If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Proof. Let L be the sum of the series. Then $s_n \to L$ and $s_{n+1} \to L$, so $a_{n+1} = s_{n+1} - s_n \to L - L = 0$.

Corollary. (The contrapositive) If $\langle a_n \rangle$ does not converge to zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

WARNING The converse of this theorem is false; e.g. the harmonic series.

(3/1, cont.) If we remove the first few terms of a series, it may change the SUM (if there is one), but it doesn't affect CONVERGENCE. "The action is in the tail."

<u>Theorem.</u> Let $\langle a_n \rangle$ be a sequence, and consider two series: $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=k}^{\infty} a_n$ (some $k \geq 1$).

(1) $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ $\sum_{n=k}^{\infty} a_n$ converges.

(2) <u>IF</u> they converge, $\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$.

(Idea of <u>proof</u>: if $\langle s_n \rangle$ is the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$, and $\langle r_m \rangle$ is the sequence of partial sums for $\sum_{n=k}^{\infty} a_n$, use the recurrence $r_m + s_{k-1} = s_{k+m}$.)

<u>Example.</u> If $|x| < 1$,
$$\sum_{n=k}^{\infty} x^n = x^k + x^{k+1} + \cdots = x^k(1 + x + x^2 + \cdots) = x^k \cdot \frac{1}{1-x}.$$

(3/3) <u>Repeating decimals</u>

What is $.999\ldots = .\overline{9}$? Use geometric series. $.\overline{9} = \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^n} + \cdots = 9 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$, which is a geometric series (starting at $n=1$, not $n=0$: be careful!), and $\frac{1}{10} < 1$, so the series converges to $9 \cdot \frac{1/10}{1 - 1/10} = 9 \cdot \frac{1}{9} = 1$. Thus $.\overline{9} = 1$.

What is $.171717\ldots = .\overline{17}$? We have $.\overline{17} = \frac{17}{100} + \frac{17}{10000} + \cdots = 17 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{10^2}\right)^n$, another geometric series. It converges to $17 \cdot \frac{1/100}{1 - 1/100} = 17/99$, so $.\overline{17} = 17/99$.

What is $2.1333\ldots = 2.1\overline{3}$?

We have $2.1\overline{3} = 2.1 + 0.0\overline{3} = 2.1 + \frac{1}{10}(0.\overline{3}) = 2 + \frac{1}{10} + \frac{1}{10}\left(\frac{1}{3}\right) = 2 + \frac{2}{15} = 32/15$.

<u>Exercises</u> What are (i) $.123123123\ldots = .\overline{123}$, (ii) $.23042304\ldots = .\overline{2304}$, (iii) $.\overline{59} + .\overline{73}$, (iv) $3.14\overline{27}$?

<u>Telescoping series</u>

<u>Problem</u> Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converge? If so, find its value.

Look at the m^{th} partial sum, $s_m = \frac{1}{1\cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{m(m+1)}$

$$\underset{\text{partial fractions}}{=} \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{2}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right)$$

$$\underset{\text{cancel!}}{=} 1 - \frac{1}{m+1}$$

$$\to 1, \text{ as } m \to \infty. \text{ So } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m \to \infty} s_m = 1.$$

 A harder telescoping series: $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$.

We can write $\frac{1}{n^2-1}$ as $\frac{1}{(n+1)(n-1)}$, or $\frac{1}{j(j+2)}$ if we set $j = n-1$. Thus, we can reindex the series to be $\sum_{j=1}^{\infty} \frac{1}{j(j+2)}$: now j starts at 1, not 2.

Look at the m^{th} partial sum: $s_m = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \cdots + \frac{1}{m(m+2)} = \frac{1}{2}\left(1-\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) + \cdots + \frac{1}{2}\left(\frac{1}{m}-\frac{1}{m+2}\right)$

$$= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{m+1}-\frac{1}{m+2}\right) \xrightarrow[m\to\infty]{} 3/4,$$

so $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{j=1}^{\infty} \frac{1}{j(j+2)} = \lim_{m\to\infty} s_m = \frac{3}{4}$.

NB It is true, but much harder to prove, that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(Nick's note: see the book "Proofs from THE BOOK" by Aigner & Ziegler for a proof using multivariable calculus.)

Annuities (Promised regular payments after a fixed deposit)

"Immediate" How much money (A) is needed, invested at $r\%$ interest, to insure yearly payments (P) for the rest of your life, assuming you'll live forever?

You need to invest $A_n = \frac{P}{(1+r)^n}$ for the n^{th} year (since $r\%$ interest, compounded n times, on $P/(1+r)^n$ dollars returns P dollars).

Total required investment: $A = \sum_{n=1}^{\infty} \frac{P}{(1+r)^n} \underset{\text{geom. series}}{=} P\left(\frac{\frac{1}{1+r}}{1-\frac{1}{1+r}}\right) = \underline{\frac{P}{r}}$.

"Deferred" What if you're willing to wait M years before collecting any payments? If the initial investment is A_0, then you'll have $A = A_0(1+r)^M$ when the payments begin. So $A = \frac{P}{r} \Rightarrow A_0 = \frac{P}{r(1+r)^M}$.

"Fixed term" What if you only want guaranteed payments for n_0 years, not forever? In this case, the sum is finite.

$$A = \sum_{n=1}^{n_0} A_n = \frac{P}{1+r}\left(\sum_{j=0}^{n_0-1} \frac{1}{(1+r)^j}\right) = \frac{P}{1+r}\,\frac{\left(1-\frac{1}{(1+r)^{n_0}}\right)}{1-\frac{1}{1+r}} = \frac{P}{r}\left(1-\frac{1}{(1+r)^{n_0}}\right).$$

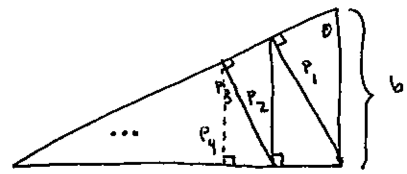
Example Suppose you want $P = \$6000$ per year, and the policy has a 4% interest rate.

You need to invest: \$150,000 for an "immediate" plan,

\$101,335 for a 10-year deferred plan,

\$93,725 for a 25-year fixed term.

(3/5) A <u>geometric example</u>: consider a right triangle with angle θ and leg length b, and drop successive perpendiculars: [NBS]



$P_1 = b\sin\theta$

$P_2 = P_1 \sin\theta = b\sin^2\theta$

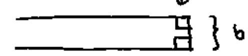
$\vdots$

$P_n = b\sin^n\theta$

Total length of all P_i's: $P = \displaystyle\sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} b\cdot\sin^n\theta = b\cdot\sum_{n=1}^{\infty}(\sin\theta)^n = \frac{b\cdot\sin\theta}{1-\sin\theta}$

E.g. if $\theta = \frac{\pi}{4}$, $P \simeq (2.4)b$; if $\theta = \frac{\pi}{3}$, $P \simeq (6.5)b$. As $\theta \to \frac{\pi}{2}$, $P \to \infty$.

$\theta = 90°$



<u>Question</u> Does $\displaystyle\sum_{n=1}^{100} 2^n + \sum_{n=101}^{\infty} 2^{-n}$ converge? Yes, to $2535301200456458802993406410750 + \frac{1}{2^{100}}$.

(where'd this come from?)

<u>Example</u> Suppose $a_n \neq 0$ $\forall n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. What about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$? Because $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, we know $a_n \to 0$, so $\frac{1}{a_n}$ must diverge (not necessarily to ∞ or $-\infty$; for example, $a_n = \frac{(-1)^n}{2^n}$), and thus $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ diverges by the contrapositive of the result "$\sum a_n$ converges $\Rightarrow a_n \to 0$". It can also be shown (it's harder!) that $\displaystyle\sum_{n=1}^{\infty} a_n^{1/n}$ diverges.

<u>Sierpinski carpet</u> (see diagram below): at the n^{th} stage, the total area removed is $\frac{8^{n-1}}{9^n}$: 8^{n-1} new squares, each of area $\frac{1}{9^n}$. So the total area removed at all stages is given by a geometric series:

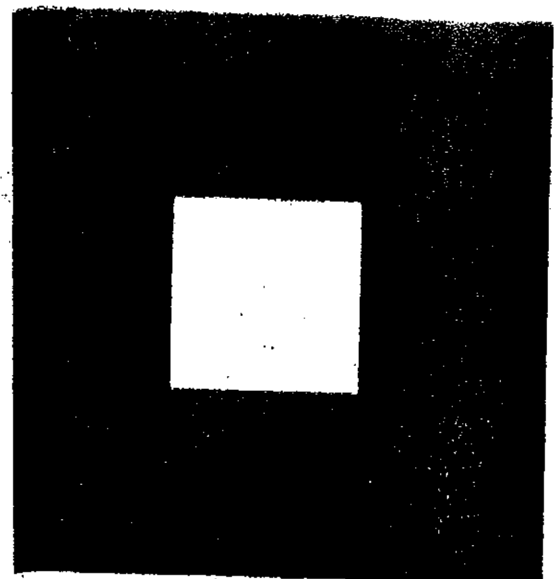
$$\sum_{n=1}^{\infty}\frac{8^{n-1}}{9^n} = \frac{1/9}{1-8/9} = 1.$$ (So what remains is a "web" of zero area.)

<u>The comparison test</u>

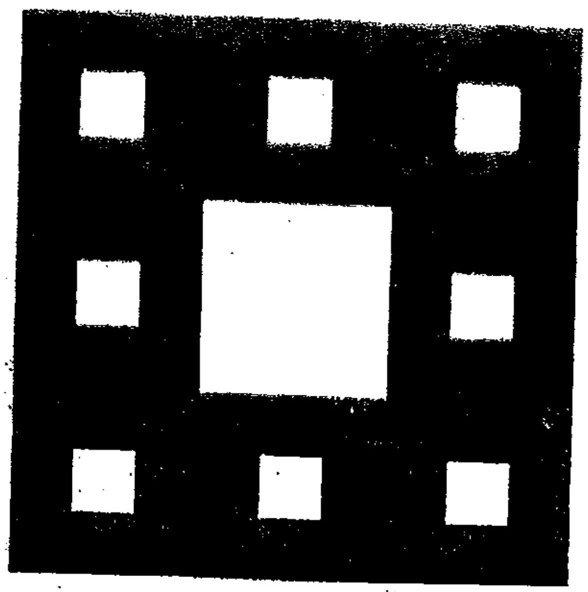
Sometimes it's easier to say "$\sum a_n$ converges" than to find its sum.

<u>Theorem.</u> Suppose $0 \leq a_n \leq b_n$ $\forall n \in \mathbb{N}$. Then if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$. (Corollary/contrapositive: if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} b_n$.)

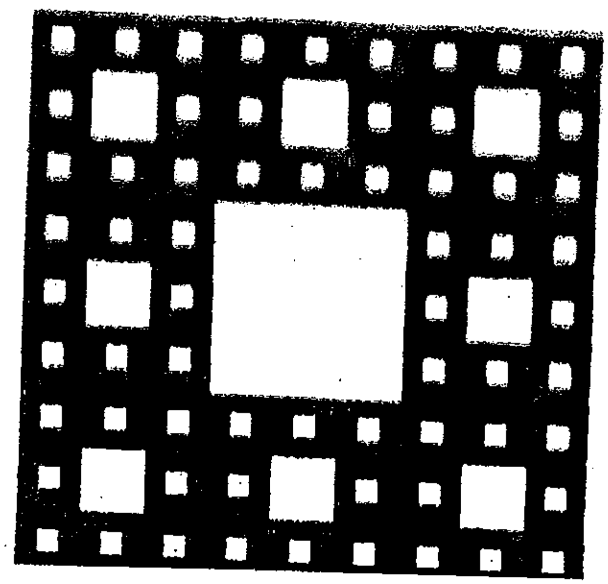
The Sierpinski Carpet is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) SUPPOSE WE COUTINUE THE PROCESS INDEFINITELY. WHAT IS THE AREA OF THE REMAINING FIGURE?



STAGE 1



STAGE 2



STAGE 3