# SEQUENCES

## PREREQUISITES FROM CALCULUS

**LIMITS** LET $f$ BE DEFINED ON SOME OPEN INTERVAL ABOUT $a \in \mathbb{R}$

THEN $\lim_{x \to a} f(x) = L$ INTUITIVELY MEANS: AS $x$ GETS CLOSER & CLOSER TO $a$, $f(x)$ GETS CLOSER & CLOSER TO $L$. ALTERNATIVELY, WE CAN GET $f(x)$ AS CLOSE AS WE WANT TO $L$ IF WE TAKE $x$ SUFFICIENTLY CLOSE TO $a$.

IN TERMS OF ERROR: GIVEN ANY ERROR/TOLERANCE $\varepsilon > 0$ THAT WE WILL ALLOW $f(x)$ TO DIFFER FROM $L$, WE CAN FIND AN ALLOWABLE DEVIATION $\delta > 0$ (WHICH DEPENDS ON $\varepsilon$) SO THAT IF $x$ DEVIATES FROM $a$ BY LESS THAN $\delta$, THEN $f(x)$ DIFFERS FROM $L$ BY LESS THAN $\varepsilon$.

$|x-a| < \delta$: $x$ DEVIATES FROM $a$ BY LESS THAN $\delta$

$|f(x)-L| < \varepsilon$: $f(x)$ DIFFERS FROM $L$ BY LESS THAN $\varepsilon$

**LIMITS AT INFINITY** LET $f$ BE DEFINED ON SOME RIGHT RAY $[R,+\infty)$

THEN $\lim\limits_{x\to+\infty} f(x) = L$ INTUITIVELY MEANS: AS $X$ GETS CLOSER & CLOSER TO $+\infty$, $f(x)$ GET CLOSER & CLOSER TO $L$. HOW CAN $X$ GET CLOSE TO $+\infty$: $X$ IS VERY LARGE.

ALTERNATIVELY, WE CAN GET $f(x)$ AS CLOSE AS WE WANT TO $L$ IF WE TAKE $X$ SUFFICIENTLY CLOSE TO $+\infty$, I.E., $X$ SUFFICIENTLY LARGE.

GIVEN ANY ERROR/TOLERANCE $\varepsilon>0$ THAT WE ALLOW $f(x)$ TO DIFFER FROM $L$, WE CAN FIND A DEVIATION NUMBER $N>0$ (WHICH DEPENDS ON $\varepsilon$) SO THAT IF $X$ IS $N$-CLOSE TO $+\infty$, I.E., $X > N$, THEN $f(x)$ DIFFERS FROM $L$ BY LESS THAN $\varepsilon$

HERE, A SET WHICH IS "CLOSE TO $+\infty$" IS A RAY $(N,+\infty)$

**CONTINUOUS AT $a$** $\quad \lim\limits_{x\to a} f(x) = f(a)$

FORMAL DEFINITION: <u>CONTINUITY</u>

f IS CONTINUOUS AT $x=a$ IF GIVEN $\varepsilon > 0$, THERE EXISTS $\delta > 0$ SUCH THAT IF $|x-a| < \delta$, THEN $|f(x) - f(a)| < \varepsilon$

f IS CONTINUOUS ON $(a, b)$ OR $[R, +\infty)$ IF f IS CONTINUOUS AT ALL POINTS OF THE INTERVAL OR RAY.

STEWART, E T, 5$^{\text{TH}}$ ED



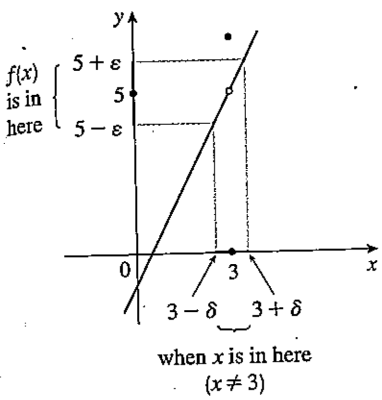


FIGURE 1

must be able to bring it below *any* positive number. And, by the same reasoning, we can! If we write $\varepsilon$ (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

This is a precise way of saying that $f(x)$ is close to 5 when $x$ is close to 3 because (1) says that we can make the values of $f(x)$ within an arbitrary distance $\varepsilon$ from 5 by taking the values of $x$ within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that (1) can be rewritten as

$$5 - \varepsilon < f(x) < 5 + \varepsilon \quad \text{whenever} \quad 3 - \delta < x < 3 + \delta \quad (x \neq 3)$$

and this is illustrated in Figure 1. By taking the values of $x$ ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using (1) as a model, we give a precise definition of a limit.

$\boxed{2}$ **Definition** Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then we say that the **limit of $f(x)$ as $x$ approaches $a$ is $L$**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Another way of writing the last line of this definition is

$$\boxed{\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon}$$

Since $|x - a|$ is the distance from $x$ to $a$ and $|f(x) - L|$ is the distance from $f(x)$ to $L$, and since $\varepsilon$ can be arbitrarily small, the definition of a limit can be expressed in words as follows:

> $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and $L$ can be made arbitrarily small by taking the distance from $x$ to $a$ sufficiently small (but not 0).

Alternatively,

> $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to $L$ by taking $x$ close enough to $a$ (but not equal to $a$).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

> $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small $\varepsilon$ is) we can find $\delta > 0$ such that if $x$ lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where $f$ maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

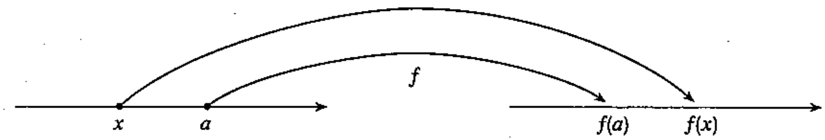


**FIGURE 2**

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around $L$, then we can find an interval $(a - \delta, a + \delta)$ around $a$ such that $f$ maps all the points in $(a - \delta, a + \delta)$ (except possibly $a$) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

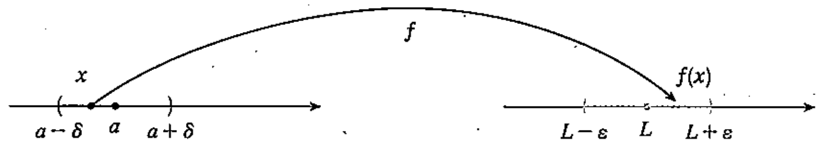


**FIGURE 3**

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of $f$ (see Figure 4). If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict $x$ to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$. (See Figure 5.) You can see that if such a $\delta$ has been found, then any smaller $\delta$ will also work.

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number $\varepsilon$ no matter how small it is chosen. Figure 6 shows that if a smaller $\varepsilon$ is chosen, then a smaller $\delta$ may be required.

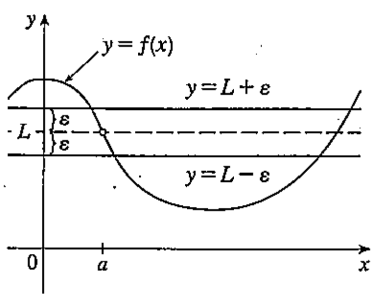


**FIGURE 4**

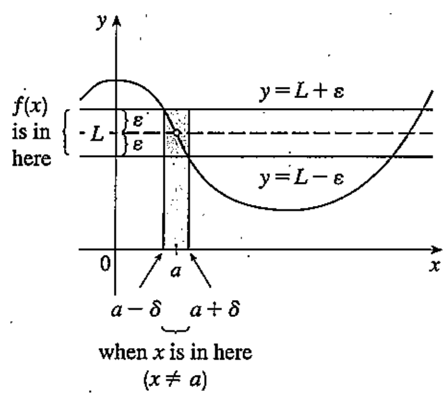


**FIGURE 5**

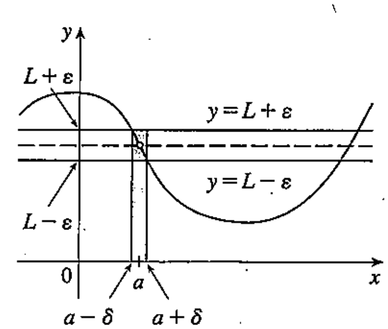


**FIGURE 6**

**EXAMPLE 1** Use a graph to find a number $\delta$ such that

$$|(x^3 - 5x + 6) - 2| < 0.2 \quad \text{whenever} \quad |x - 1| < \delta$$

In other words, find a number $\delta$ that corresponds to $\varepsilon = 0.2$ in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

STEWART, ET, 5TH ED
SECTION 2.6

**1 Definition** Let $f$ be a function defined on some interval $(a, \infty)$. Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.

L HORIZONTAL ASYMPTOTE FOR f

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol $\infty$ does not represent a number. Nonetheless, the expression $\lim\limits_{x \to \infty} f(x) = L$ is often read as

"the limit of $f(x)$, as $x$ approaches infinity, is $L$"

or "the limit of $f(x)$, as $x$ becomes infinite, is $L$"

or "the limit of $f(x)$, as $x$ increases without bound, is $L$"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the $\varepsilon$, $\delta$ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of $f$ to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

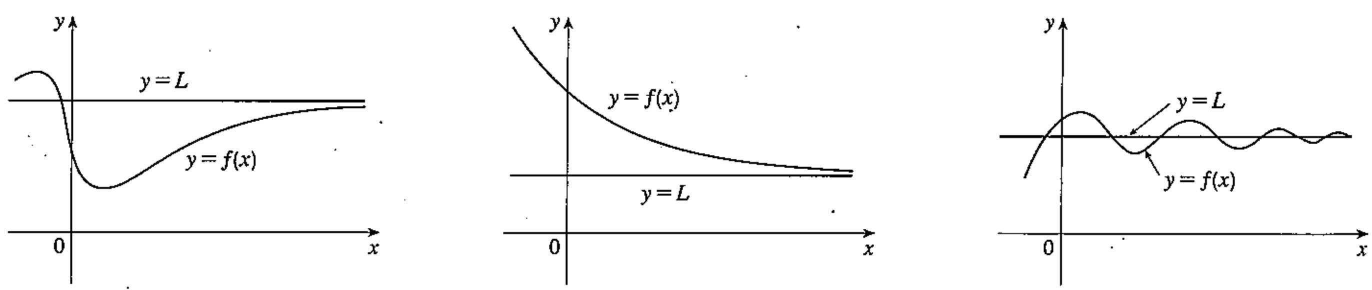


**FIGURE 2**
Examples illustrating $\lim\limits_{x \to \infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of $x$, the values of $f(x)$ are close to 1. By letting $x$ decrease through negative values without bound, we can make $f(x)$ as close as we like to 1. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

**2 Definition** Let $f$ be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large negative.

## Precise Definitions

Definition 1 can be stated precisely as follows.

> **7 Definition** Let $f$ be a function defined on some interval $(a, \infty)$. Then
>
> $$\lim_{x \to \infty} f(x) = L$$
>
> means that for every $\varepsilon > 0$ there is a corresponding number $N$ such that
>
> $$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > N$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to $L$ (within a distance $\varepsilon$, where $\varepsilon$ is any positive number) by taking $x$ sufficiently large (larger than $N$, where $N$ depends on $\varepsilon$). Graphically it says that by choosing $x$ large enough (larger than



some number $N$) we can make the graph of $f$ lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14. This must be true no matter how small we choose $\varepsilon$. Figure 15 shows that if a smaller value of $\varepsilon$ is chosen, then a larger value of $N$ may be required.

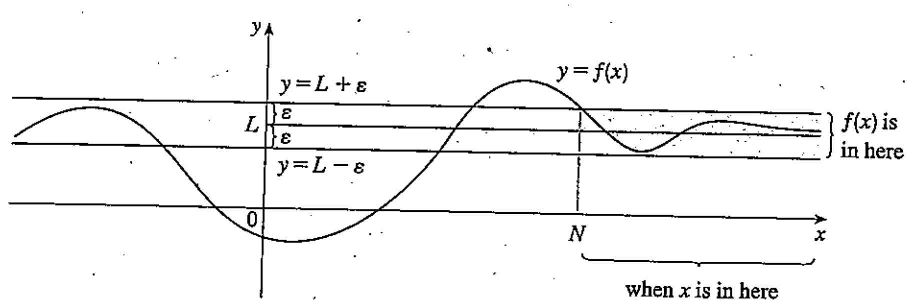


**FIGURE 14**
$\lim_{x \to \infty} f(x) = L$

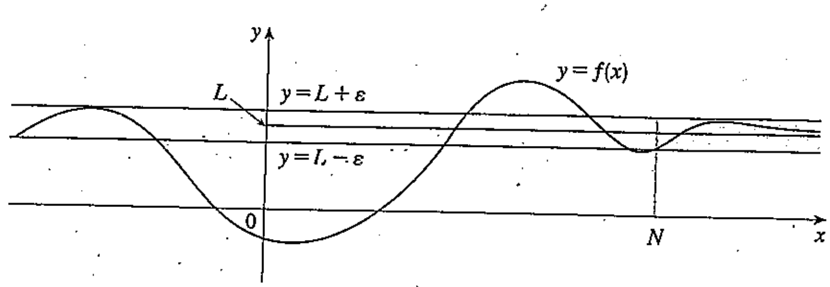


**FIGURE 15**
$\lim_{x \to \infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 8, which is illustrated in Figure 16.