Arithmetic compactifications of PEL-type Shimura varieties

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Abstract

In this thesis, we constructed minimal (Satake-Baily-Borel) compactifications and smooth toroidal compactifications of integral models of *general* PEL-type Shimura varieties (defined as in Kottwitz [79]), with descriptions of stratifications and local structures on them extending the well-known ones in the complex analytic theory. This carries out a program initiated by Chai, Faltings, and some other people more than twenty years ago. The approach we have taken is to redo the Faltings-Chai theory [37] in full generality, with as many details as possible, but without any substantial case-by-case study. The essential new ingredient in our approach is the emphasis on *level structures*, leading to a crucial Weil pairing calculation that enables us to avoid unwanted boundary components in naive constructions.

Contents

7 Algebraic Constructions of Minimal Compactifications 565

A Algebraic Spaces and Algebraic Stacks 609

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謹以此文，

獻給我的父母。

Introduction

Here we give a soft introduction to the background and the status of this work. This is not a summary of the results. To avoid introducing a heavy load of notations and concepts, we shall not attempt to give any precise mathematical statement. Please refer to the main body of the work for more precise information.

Complex Theory

It is classically known, especially after the works of Shimura, that complex abelian varieties with the so-called PEL-structures (polarizations, endomorphisms, and level structures) can be parameterized by unions of arithmetic quotients of (connected) Hermitian symmetric spaces. Simple examples include the modular curves as quotients of the Poincaré upper-half plane, the Hilbert modular spaces as quotients of products of the Poincaré upper-half plane, and the Siegel moduli spaces as quotients of the Siegel upper-half spaces. (In this introduction, we shall not try to include the historical details of the modular or Hilbert modular cases related to only GL_2.)

According to Baily and Borel [16], any such arithmetic quotient can be given an algebro-geometric structure, because it can be embedded as a Zariski-open subvariety of a canonically associated complex normal projective variety called the *Satake-Baily-Borel* or *minimal compactification.* Thus the above-mentioned parameter spaces can be viewed as unions of complex quasi-projective varieties. These parameter spaces are called PEL-type Shimura varieties. These PEL-type Shimura varieties admit canonical models over number fields, as investigated by Shimura and others. (See in particular [31].) Since abelian varieties (with additional structures) make sense over rings of algebraic integers localized at some precise sets of *good primes*, we obtain *integral models* of these PEL-type Shimura varieties by defining suitable moduli problems of abelian varieties. Moreover, the precise sets of good primes can be chosen so that the moduli problems are representable by smooth schemes with non-empty fibers. (See for example [81], [123], and [79].)

Although the minimal compactifications mentioned above are normal and canonical, Igusa [68] and others have discovered that minimal compactifications are in general highly singular. In [15], Mumford and his coworkers constructed a large class of (noncanonical) compactifications in the category of complex algebraic spaces, called *toroidal compactifications*. Within this class, there are plenty of nonsingular compactifications, many among them are projective, hence providing a theory of smooth compactifications for the PEL-type Shimura varieties over the complex numbers. Based on works of many people since Shimura, it is known that both minimal compactifications and toroidal compactifications admit canonical models over the same number fields over which the Shimura varieties are defined. (See Pink's thesis [107], and also [63].)

Integral Theory

In [37], Faltings and Chai studied the theory of degeneration for polarized abelian varieties over complete local rings satisfying certain reasonable normality conditions, and constructed smooth toroidal compactifications for the integral models of Siegel modular varieties (parameterizing principally polarized abelian schemes over base schemes over which the primes dividing the level are invertible). The key point in their construction is the *gluing process* in the étale topology. Such a process is feasible because there exist local charts over which the sheaves of differentials can be explicitly calculated and compared. The above-mentioned theory of degeneration and the theory of toroidal embeddings over arbitrary bases play a major role in the construction of these local charts.

As a byproduct, they obtained the minimal compactifications for the integral models of Siegel modular varieties using the graded algebra generated by automorphic forms of various (parallel) weights, extending the ones over the complex numbers. We would like to remark that, although the local charts for the minimal compactifications can also be written down explicitly, the fact that we do not have a good way to compare the local structures between different local charts makes the gluing process very difficult in practice. This explains the main difference between the complex analytic and the arithmetic geometric stories, and explains why the toroidal compactifications were constructed before the minimal compactifications in the latter case.

These toroidal and minimal compactifications for the integral models of the Siegel modular varieties are the prototypes of our arithmetic compactifications of PEL-type Shimura varieties. It is not surprising that the existence of such integral models are important for arithmetic applications of Shimura varieties.

In Larsen's thesis [83] (see also [84]), he applied the techniques of Faltings and Chai and constructed the arithmetic compactifications of integral models of the Picard modular varieties, namely the Shimura varieties associated to unitary groups defined by Hermitian pairings of real signature $(2, 1)$ over imaginary quadratic fields. This is the so-called $\mathrm{GU}(2, 1)$-case. In this case, there is a unique toroidal compactification for each Shimura variety one considers. (The same phenomenon occurs whenever each of the $\mathbb{Q}$-simple factors of the adjoint quotient of the corresponding algebraic group has $\mathbb{R}$-rank no greater than one.) His compactification theory has been used in the main results of the Montréal volume [80].

Before moving on, let us mention that there is also the revision of the master thesis of Fujiwara [40] on the arithmetic compactifications for PEL-type Shimura varieties not involving any simple component of type D. The main difference between his work and Faltings-Chai's is his ingenious use of rigid-analytic methods in the gluing process. The point is that his gluing method has the potential to be generalized for non-smooth moduli problems. However, as far as we can understand, there are important steps in his boundary construction (*before* the gluing step) that are not fully justified.

What Is New?

In this work, our goal is to carry out the theory of arithmetic compactifications for smooth integral models of PEL-type Shimura varieties, as defined in Kottwitz's paper [79], with no other special restriction on the Hermitian pairings or the groups involved. Our construction is based on a generalization of Faltings and Chai's in [37]. It is a very close imitation from the perspective of algebraic geometry. Thus our work can be viewed as a long student exercise justifying the claims in [37, pp. 95–96 and 137] that their method works for general PEL-type Shimura varieties.

However, there do exist some differences coming from linear algebra and related issues, as long as the readers believe that the theory of modules and pairings over $\mathbb{Z}$ is simpler than the analogous theory over orders in some (not necessarily commutative) semisimple algebras, and that solving equations like $0 = 0$ is easier than solving any other linear equations (which might not have solutions) in a (not necessarily torsion-free) noncommutative algebra. Let us make this more precise.

The main issue is about the level structures. In the work of Faltings and Chai, the moduli problem for Siegel modular varieties is defined for abelian schemes with *principal* polarizations. In particular, they only have to work with self-dual lattices. Moreover, the additional fact that $\mathbb{Q}$ does not have a nontrivial involution makes the isotropic submodules of the lattice mod n always *liftable* to characteristic zero. With some care, the first assumption alone can be harmless, as in the earlier work of Rapoport [109], and the second assumption can continue to be assumed in the Hilbert modular cases. But these convenient assumptions simply do not hold in general. A symplectic isomorphism between modules mod n may not lift to a symplectic isomorphism between the original modules. We need the notion of *symplectic-liftability* to translate the adelic definition of level structures correctly into the language of finite étale group schemes. Accordingly, we need to find the right way to assign degeneration data to level structures, namely symplectic-liftable isomorphisms between finite étale group schemes.

Thus, the main objective of our approach is to formulate certain liftability and pairing conditions on the degeneration data, so that the combination of these two conditions can predict the existence of level structures of a prescribed type on the generic fiber of the corresponding degenerating families. This involves a Weil pairing calculation that we believe has never been mentioned in the past literature. After this important step, we have to construct boundary charts parameterizing the degeneration data we need. We have to incorporate the additional liftability and pairing conditions that are absent in the works of Faltings and Chai. We would like to point out that naive generalizations of their construction, essentially the only one available in the past literature, lead to unwanted additional components along the boundary. We can think of these additional components as belonging to some different Shimura varieties. The question of avoiding these unwanted components, is certainly another complication. Fortunately, our calculation mentioned above suggests that there is a rather elementary and algebro-geometric way to identify and to give meanings to the correct components. Finally, with the correct components, the approximating and gluing steps are exactly the same as in the original work of Faltings and Chai. We shall not pretend that we have any invention in this respect.

We would like to mention that our own approach (with emphasis on conditions on the level structures) emerged from our initial attempts on the case of unitary groups of ranks higher than $\mathrm{GU}(2,1)$. At that time our more naive and rather ad hoc generalization of the method of Faltings and Chai could only handle the cases of unitary groups defined over an imaginary quadratic field *of odd discriminant*. The cases of even discriminants remained problematic for a long while. After some trials in vain, we realized that the essential difficulty is not special for these particular unitary cases. As soon as we have obtained the right approach for the even discriminant cases, it seemed clear to us that it could also work for all other PEL-type cases, without avoiding any particular one. After all, it is important that the strategy we have thus obtained does not require any previous studies on special cases.

Note that there are inevitably some inaccuracies in the main results of [37, Ch. II and III]. As far as we can understand, most of the existing theories of arithmetic compactifications over an integral base scheme depend logically on the theory of degeneration for abelian varieties and on Mumford's construction, both of which have only been sufficiently explained in their full generality in [37, Ch. II and III]. Hence it seems desirable for us to rework through these most fundamental machineries, even if such an effort does not involve any novelty in mathematical ideas. We do not believe that our work can replace or even become partially comparable with the monumental contributions of Mumford, Faltings, and Chai. We do not think there should be any reason to cast any doubt on the importance of their works. But at least we would like to try to make their brilliant ideas more consolidated after they have appeared for so many years. Alongside with other small corrections, modifications, or justifications that we have attempted to offer, we hope that our unoriginal, nonconstructive, and uninspiring effort is not totally redundant even for cases which might be considered *well-known*. (We believe it is sensible to justify the existing works before proceeding to more general cases, anyway.)

At this moment, there are people who are working on also cases of non-smooth integral models of Shimura varieties. Let us explain why we do not consider this further generality in our work. The main reason is not about the theory of degeneration data or the techniques of constructing local boundary charts. It is rather about the definition of the Shimura varieties themselves, and the expectation of the results to have. In some cases, there seem to be more than one reasonable way of stating them. This certainly does not mean that it is impossible to compactify a particular non-smooth integral model. However, it would probably be more sensible if we know why we compactify it, and if we know why it should be compactified in a particular way. It is not easy to provide systematic answers to these questions when the objects we consider are not smooth. (The best we can hope is probably to answer these questions for integral models of Shimura varieties that are flat and regular, as aimed in [108], [105], [106], and sequels to them.) Since we have a much better understanding of the general smooth cases, we shall be content with only the treatment of them in this work.

Finally, we would like to mention that we are not working along the lines of the *canonical compactifications* constructed by Alexeev and Nakamura [1], or by Olsson [103], because it is not plausible that one could define general Hecke actions on their canonical compactifications. Let us explain the reason as follows. The main component in their compactifications can be related to the toroidal compactifications constructed using some particular choices of cone decompositions. However, the collection of such cone decompositions is not invariant under conjugation by rational elements in the group arising naturally from the Hecke actions. It is possible that their definition could be useful for the construction of minimal compactifications (with Hecke actions), but we believe that the argument will be forced to be indirect. Nevertheless, we would like to emphasize that their compactifications are described by moduli problems allowing deformation theoretic considerations along the boundary. Hence their compactifications might be more useful for applications to algebraic geometry.

Structure of The Exposition

In Chapter 1, we lay down the foundations and give the definition of the moduli problems we consider. The moduli problems we define parameterize *isomorphism classes* of abelian schemes over integral bases with additional structures of the above-mentioned types, which are equivalent to the moduli problems defining integral models of PEL-type Shimura varieties using *isogeny classes* as in [79]. Therefore, as already explained in [79], the complex fibers of these moduli problems contain the Shimura varieties associated to the reductive groups mentioned above.

In Chapter 2, we elaborate on the representability of the moduli problems defined in Chapter 1. Our treatment is biased towards the prorepresentability of local moduli and Artin's criterion of algebraic stacks. We do not need the theory of geometric invariant theory or Barsotti-Tate groups. The argument is very elementary, which might be considered outdated by the experts in this area. (Indeed, it may not be enough for the study of bad reductions.) Although the readers might want to skip this chapter as they might be willing to believe the representability of the moduli problems, there are still some reasons to include this chapter. For example, the Kodaira-Spencer maps of abelian schemes with PEL-structures are of fundamental importance in our argument of gluing of boundary charts (in Chapter 6), and they are best understood via the study of deformation theory. Furthermore, the proof of the formal smoothness of local moduli functors illustrates how the linear algebraic assumptions are used. Some of the linear algebraic facts are used again in the construction of the boundary components, and it is an interesting question whether one can propose a satisfactory intuitive explanation to this coincidence.

In Chapter 3, we explain well-known notions important for the study of semi-abelian schemes, such as groups of multiplicative type and torsors under them, biextensions, cubical structures, semi-abelian schemes, Raynaud extensions, and certain *dual objects* for these latter two notions extending the notion of dual abelian varieties. Our main references for these are [33], [59], and in particular [96].

In Chapter 4, we reproduce the theory of degeneration data for polarized abelian varieties, as elaborated in the first three chapters of [37]. In the main theorems (of Faltings and Chai) that we present, we have made some substantial modifications to the statements according to our own understanding of the proof. Notably, we have provided weakened statements in the main definitions and theorems, because we do not need their original stronger versions in [37] for our main result. Examples of this sort include in particular Definitions 4.2.1.1 and 4.5.1.2, Theorems 4.2.1.8 and 4.4.18, and Remarks 4.1.1, 4.2.1.10, and 4.5.1.4.

In Chapter 5, we supply a theory of degeneration data for endomorphism structures, Lie algebra conditions, and level structures, based on the theory of degeneration in Chapter 4. People often claim that the degeneration theory for general PEL-type structures is just a straightforward consequence of the functoriality of the merely polarized case. However, the Weil pairing calculation carried out in this chapter may suggest that it is not necessarily the case. As far as we can see, functoriality does not seem to imply properties about pairings in an explicit way. There are conceptual details to be understood beyond simple implications of functoriality. After all, we are able to present in this chapter a theory of degeneration data for abelian varieties with PEL-structures, together with the notion of cusp labels.

In Chapter 6, we explain the algebraic construction of toroidal compactifications. For this purpose we need one more basic tool, namely the theory of toroidal embeddings for torsors under groups of multiplicative type. Based on this theory, we start the general constructions of local charts on which degeneration data for PEL-structures are tautologically associated. The key ingredient in these constructions is the construction of the tautological PEL-structures, including particularly the level structures. The construction depends heavily on way we classify the degeneration data and cusp labels developed in Chapter 5. As explained above, there are complications that are not seen in special cases such as Faltings and Chai's work. The next important step is the description of good formal models, and good algebraic models approximating them. The correct formulation of necessary properties and actual construction of these good algebraic models are the key to the gluing process in the étale topology. In particular, this includes the comparison of local structures using the Kodaira-Spencer maps alluded above. As a result of gluing, we obtain the arithmetic toroidal compactifications in the category of algebraic stacks. The chapter is concluded by a study of Hecke actions on towers of arithmetic toroidal compactifications.

In Chapter 7, we first study the automorphic forms that are defined as global sections of certain invertible sheaves on the toroidal compactifications. The local structures of toroidal compactifications lead naturally to the theory of Fourier-Jacobi expansions and the Fourier-Jacobi expansion principle. As in the case of Siegel modular schemes, we obtain also the algebraic construction of arithmetic minimal compactifications, which are normal schemes defined over the same integral bases as the moduli problems are. As a byproduct of codimension counting, we obtain Koecher's principle for arithmetic automorphic forms. Furthermore, following Chai's generalization of Tai's result to Siegel moduli schemes over integral base schemes, we show the projectivity of a large class of arithmetic toroidal compactifications by realizing them as normalizations of blowups of the corresponding minimal compactifications. The results in this chapter parallel part of those in [37, Ch. V].

For the convenience of the readers, we have included two appendices containing basic information about algebraic stacks and Artin's criterion for them. There is also an index of notations and terminologies at the end of the document. We hope they will be useful for the readers.

Our overall treatment might seem unreasonably lengthy, and some of the details might have made the arguments more clumsy than they should. Even so, we have tried to provide sufficient information, so that the readers should have no trouble correcting any of the foolish mistakes, or improving any of the unnecessarily inefficient arguments. It is our belief that it is the right of the readers, but not the author, to skip details. At least, we hope that the readers will not have to repeat some of the elementary but tedious tasks we have gone through.

Apology

Due to limitation of time and energy, the proofreading process might not have achieved a satisfactory status at the time that this work is sent to print, and there might be non-mathematical or mathematical typos that are very difficult to correct from the readers' side. We apologize for this inconvenience due to our incompetence. Please contact the author whenever there are unclear or incorrect statements that require justifications or modifications.

Comparison With Submitted Version

The content in this volume is almost identical to the version submitted to Harvard University. However, single-siding and double-spacing are not enforced here, and some changes made (by the date May 21) after the submission has been incorporated.

Notations and Conventions

All rings, commutative or not, will have an identity element. All left or right module structures, or algebra structures, will respect the identity elements. Unless when the violation is clear from the context, or unless otherwise specified, any ring homomorphism will send the identity to the identity element. Unless necessary for the sake of clarity, all modules will be assumed to be left modules by default. An exception is about ideals in noncommutative rings, in which case we shall always describe precisely whether it is a left ideal, a right ideal, or a two-sided ideal. All involutions in this work are anti-automorphisms of order two. The dual of a left module is naturally equipped with a left module structure over the opposite ring, and hence over the same ring if the ring admits an involution (which is an isomorphism from the ring to its opposite ring).

We shall use the notations $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{A}$, $\mathbb{A}^\infty$, $\hat{\mathbb{Z}}$ to denote respectively the ring of rational integers, rational numbers, real numbers, complex numbers, adeles, finite adeles, and the integral adeles without any further explanation.

More generally, for any set $\Box$ of rational primes, which can be either finite or infinite in number, or even empty, we denote by $\mathbb{Z}_{(\Box)}$ its localization at the ideal generated by 0 and $\Box$, and denote by $\hat{\mathbb{Z}}^\Box$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^\Box$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. Note that when $\Box$ is empty, we have simply $\mathbb{Z}_{(\Box)} = \mathbb{Q}$, $\mathbb{A}^{\infty,\Box} = \mathbb{A}^\infty$, and $\mathbb{A}^\Box = \mathbb{A}$.

We say that an integer is prime-to-$\Box$ if it is not divisible by any prime number in $\Box$. We shall also write $\Box | m$ or $\Box \nmid m$, with its obvious meanings to be understood.

These notations and conventions are designed so that the results would be compatible if $\Box$ were literally just a prime number.

The notation $[A : B]$ will mean either the index of B in A as a subgroup when we work in the category of lattices, or the degree of A over B when we work in the category of finite-dimensional algebras over a field. We allow this ambiguity because there is no interesting overlap of these two usages.

The notation δ_{ij}, when i and j are indices, means the Kronecker delta, which is 1 when $i = j$ and 0 when $i \neq j$ as usual.

The terminology *scheme* will almost always mean a separated prescheme in the old fashioned language, unless otherwise specified. When it is convenient we shall also use the term *prescheme* to mean a scheme in the current language that is not necessarily separated. The separateness assumption will be used tacitly whenever we reduce our arguments to affine cases.

All *algebraic stacks* that we will encounter are *Deligne-Mumford stacks* (cf. [28], [37, Ch. I, §4], [85], and Appendix A). The notion of *relative schemes* over a ringed topos can be found in [61], which in particular is necessary when we talk about schemes over formal schemes. We shall generalize this notion tacitly to relative schemes over algebraic stacks or formal algebraic stacks.

Note that we do have to distinguish the terminology of an *algebraic stack* and the terminology of a *separated algebraic stack*. In our definition algebraic stacks will only be generalizations of the weaker notion of *quasi-separated preschemes* (cf. Remark A.2.5). Although we will state many results not just valid for schemes, we will never try to insist the upmost generality in the realm of preschemes. We hope this will not lead to any inconvenience or disappointment.

By a *normal* scheme we mean a scheme whose local rings are all integral and integrally closed in its fraction field. No noetherian hypothesis is imposed in this statement. A ring R is normal if $\mathrm{Spec}(R)$ is normal. We do not need R to be integral and/or noetherian in such a statement.

We shall almost always interpret *points* as *functorial points*, and hence fibers as fibers over functorial points. By a *geometric point* of a point we mean a morphism from a separably closed field to the scheme we consider. We will often use the relative notion of various scheme-theoretic concepts without explicitly stating the convention.

We will use the notation $\mathbf{G}_{\mathrm{m}}$, $\mathbf{G}_{\mathrm{a}}$, and $\boldsymbol{\mu}_n$ to denote respective the *multiplicative group*, the *additive group*, and the group scheme kernel of $[n] : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ over $\mathrm{Spec}(\mathbb{Z})$. Their base change to other base schemes S will often be denoted by respectively $\mathbf{G}_{\mathrm{m},S}$, $\mathbf{G}_{\mathrm{a},S}$, and $\boldsymbol{\mu}_{n,S}$.

The typesetting of this work will be sensitive to small differences in notations. Although no difficult simultaneous comparison between similar symbols will be required, the differences should not be overlooked when looking for references. More concretely: We have used all the following fonts: A (normal), A (Roman), $\mathbf{A}$ (boldface), $\mathbb{A}$ (blackboard boldface), A (sans serif),

$\mathtt{A}$ (typewriter), $\mathcal{A}$ (calligraphic), $\mathfrak{A}$ (Fraktur), and $\mathscr{A}$ (Ralph Smith's formal script). The tiny difference between A (normal) and A (Italian) in width, which does exist, seems to be extremely difficult to tell. So we shall never bother to use both of them. We distinguish between A and $\underline{A}$, where the latter almost always means the relative version of A (as a sheaf or functor etc). We distinguish between Greek letters in each of the pairs: ϵ and ε, ρ and ϱ, σ and ς, ϕ and φ, and π and ϖ. The musical symbols ${}^\flat$ (flat), ${}^\natural$ (natural), and ${}^\sharp$ (sharp) will be used following Grothendieck (cf. for example [59, IX]) and some other authors. The difference in each of the pairs ${}^\flat$ and ^{b}, and ${}^\sharp$ and ${}^\#$, should not lead to any confusion. The notations ${}^\heartsuit$ and ${}^\diamondsuit$ are used respectively for Mumford families and good formal models, where the convention for the former follows from [37]. We distinguish between the two star signs * and *. The two dagger notations † and ‡ are used as superscripts and are of course different from other notations. The differences among v, ν, υ, and the dual sign ${}^\vee$ should not be confusing because they are never used for similar purposes. The same is true for i, ι, $\imath$, and $\jmath$. Since we will never need calculus in this work, the symbols ∂, $\int$, and $\oint$ are used as variants of d or S.

Finally, unless it comes with "resp.", the content in any parenthesis in text descriptions is *not an option*, but rather a reminder, a remark, or a supplement of information.

Chapter 1

Definition of Moduli Problems

In this chapter, we give the definition of the moduli problems providing integral models of PEL-type Shimura varieties that we will compactify.

Just to make sure that potential logical problems will not arise in our use of categories, we assume that a pertinent choice of a *universe* has been made. (See Section A.1.1 for more details.) This is harmless for our study, and we shall not bother to mention it again in our work.

The main objective in this chapter is to state Definition 1.4.1.4 with justifications. Therefore, we need to include Definition 1.4.2.1, the definition that agrees with the one in [79, §5] when specialized to the same base schemes, and to explain the relation between the two definitions. All sections preceding them are preparatory for them. Technical results worth noting are Propositions 1.1.2.16 and 1.1.5.17 in Section 1.1; Propositions 1.2.2.4, 1.2.3.7, 1.2.3.11, 1.2.5.20, and 1.2.5.22 in Section 1.2; and Proposition 1.4.3.3 in Section 1.4. The Theorem 1.4.1.12 (on the representability of our moduli problems in the category of algebraic stacks) is stated in Section 1.4, but its proof will be carried out only in Chapter 2. The representability of our moduli problem as schemes (when the level is neat) will be deferred until Corollary 7.2.3.10, after we have the construction of the minimal compactifications.

1.1 Preliminaries in Algebra

1.1.1 Lattices and Orders

For the convenience of the readers, let us summarize certain basic definitions and important properties of lattices over an order in a (possibly noncommutative) finite-dimensional algebra over a number field. Our main reference for this purpose will be [112].

Let us begin with the most general setting. Let R be a (commutative) noetherian integral domain with fractional field $\mathrm{Frac}(R)$.

Definition 1.1.1.1. *An* **R-lattice** *M is a finitely generated R-module M with no nonzero R-torsion. Namely, for any nonzero $m \in M$, there is no nonzero element $r \in R$ such that $rm = 0$.*

Note that in this case we have an embedding from M to $M \underset{R}{\otimes} \mathrm{Frac}(R)$.

Definition 1.1.1.2. *Let V be any finite-dimensional* $\mathrm{Frac}(R)$*-vector space. A* **full R-lattice** *is a submodule of V such that* $\mathrm{Frac}(R) \cdot M = V$*. In other words, it contains a* $\mathrm{Frac}(R)$*-basis of V.*

Let A be a (possibly noncommutative) finite-dimensional algebra over $\mathrm{Frac}(R)$.

Definition 1.1.1.3. *An* **R-order** *$\mathcal{O}$ in the* $\mathrm{Frac}(R)$*-algebra A is a subring of A having the same identity element as A, such that $\mathcal{O}$ is also an R-sublattice of A.*

Here are two familiar examples of orders:

1. If $A = \mathrm{M}_n(\mathrm{Frac}(R))$, then $\mathcal{O} = \mathrm{M}_n(R)$ is an R-order in A.

2. If R is a Dedekind domain, and $A = L$ is a separable extension of $\mathrm{Frac}(R)$. Then the integral closure $\mathcal{O}$ of R in L is an R-order in A. In particular, if $R = \mathbb{Z}$, then the rings of algebraic integers $\mathcal{O} = \mathcal{O}_L$ in L is a $\mathbb{Z}$-order in L.

Definition 1.1.1.4. *A* **maximal R-order** *in A is an R-order not properly contained in any other R-order in A.*

Proposition 1.1.1.5 ([112, Thm. 8.7])**.**
1. *If the integral closure of R in A is an R-order, then it is automatically maximal.*

2. *If $\mathcal{O}$ is a maximal R-order in A, then for each integer $n \geq 1$, $\mathrm{M}_n(\mathcal{O})$ is a maximal R-order in $\mathrm{M}_n(A)$. In particular, if R is normal (namely integrally closed in* $\mathrm{Frac}(R)$*), then* $\mathrm{M}_n(R)$ *is a maximal R-order in* $\mathrm{M}_n(\mathrm{Frac}(R))$*.*

Now suppose R is a noetherian *normal* domain, and A is a finite-dimensional separable $\mathrm{Frac}(R)$-algebra. By definition, A is Artinian and semisimple. For simplicity we shall suppress the modifier *reduced* from traces and norms when talking about such algebras. One important invariant of orders in such an algebra A is its discriminant:

Definition 1.1.1.6. *Let $m = [A : \mathrm{Frac}(R)]$. The* **discriminant** $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/R}$ *is the ideal of R generated by the set of elements*

$$\{\mathrm{Det}_{\mathrm{Frac}(R)}(\mathrm{Tr}_{A/\mathrm{Frac}(R)}\, x_i x_j)_{1 \leq i,j \leq m} : x_1, \ldots, x_m \text{ any } m \text{ elements in } \mathcal{O}\}.$$

Remark 1.1.1.7. If $\mathcal{O}$ has a free R-basis $e_1, \ldots, e_m$, then any m elements $x_1, \ldots, x_m$ can be expressed as R-linear combinations of $e_1, \ldots, e_m$. Hence Disc is generated by a single element

$$\mathrm{Det}_{\mathrm{Frac}(R)}(\mathrm{Tr}_{A/\mathrm{Frac}(R)}\, e_i e_j)_{1 \leq i,j \leq m}$$

in this case.

Proposition 1.1.1.8 ([112, Cor. 10.4])**.** *Let R be a noetherian normal domain, and let A be a finite-dimensional separable* $\mathrm{Frac}(R)$*-algebra. Then every R-order in A is contained in a maximal R-order in A. There exists at least one maximal R-order in A.*

Let $\mathfrak{p}$ be any ideal of R. We denote by $R_{\mathfrak{p}}$ the localization of R at $\mathfrak{p}$, and by $\hat{R}_{\mathfrak{p}}$ the completion of $R_{\mathfrak{p}}$ with respect to its maximal ideal $\mathfrak{p}$. (The slight deviation of this convention is that we shall denote by $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at (p), and by $\mathbb{Z}_p$ the completion of $\mathbb{Z}_{(p)}$.) If R is local, then we simply denote by $\hat{R}$ its completion at its maximal ideal.

Proposition 1.1.1.9 ([112, Thm. 11.1 and Cor. 11.2])**.** 1. *Let R be a* **local** *noetherian normal domain. An R-order $\mathcal{O}$ in A is maximal if and only if $\mathcal{O} \underset{R}{\otimes} \hat{R}$ is an $\hat{R}$-maximal order in $A \underset{\mathrm{Frac}(R)}{\otimes} \mathrm{Frac}(\hat{R})$.*

2. *Let R be a noetherian normal domain. An R-order $\mathcal{O}$ in A is maximal if and only if $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-maximal order in A.*

Definition 1.1.1.10. *We say that an R-order $\mathcal{O}$ in A is* **maximal at** $\mathfrak{p}$ *if* $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ *is maximal in A, or if* $\mathcal{O} \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ *is maximal in* $A \underset{\operatorname{Frac}(R)}{\otimes} \operatorname{Frac}(\hat{R})$.

Now suppose R is a *Dedekind domain.* In particular, R is noetherian and normal.

Definition 1.1.1.11. *The* **inverse different** $\operatorname{Diff}^{-1} := \operatorname{Diff}^{-1}_{\mathcal{O}/R}$ *is defined by*

$$\operatorname{Diff}^{-1} := \{x \in A : \operatorname{Tr}_{A/\operatorname{Frac}(R)}(xy) \subset R, \forall y \in \mathcal{O}\}.$$

Note that it is clear from definition that Diff^{-1} is a two-sided ideal in A.

Proposition 1.1.1.12. *Let* $\operatorname{Diff}_{\mathcal{O}/R} := \{z \in A : z \operatorname{Diff}^{-1}_{\mathcal{O}/R} \subset \mathcal{O}\}$ *be the inverse ideal of* $\operatorname{Diff}^{-1}_{\mathcal{O}/R}$. *Then this is a well-defined two-sided ideal of $\mathcal{O}$, and the discriminant* $\operatorname{Disc}_{\mathcal{O}/R}$ *is related to* $\operatorname{Diff}^{-1}_{\mathcal{O}/R}$ *by*

$$\operatorname{Disc}_{\mathcal{O}/R} = \operatorname{Norm}_{A/\operatorname{Frac}(R)}(\operatorname{Diff}_{\mathcal{O}/R}) = [\operatorname{Diff}^{-1}_{\mathcal{O}/R} : \mathcal{O}]_R. \tag{1.1.1.13}$$

If $\mathcal{O}$ is a maximal order, then this is just [112, Thm. 25.2]. The same proof by localizations works in the case where $\mathcal{O}$ is not maximal as well:

Proof of Proposition 1.1.1.12. By replacing R by its localizations, we may assume that every R-lattice is free over R. Let $m = [A : \mathbb{Q}]$. Let us take a free R-basis $x_1, \ldots, x_m$ of $\mathcal{O}$, and take its dual R-basis $y_1, \ldots, y_m$ under the perfect pairing $\operatorname{Tr}_{A/\operatorname{Frac}(R)} : \mathcal{O} \times \operatorname{Diff}^{-1} \to R$, so that $\operatorname{Tr}_{A/\operatorname{Frac}(R)}(x_i y_j) = \operatorname{Tr}_{A/\operatorname{Frac}(R)}(y_j x_i) = \delta_{ij}$ is the Kronecker delta function. Since $\mathcal{O} \subset \operatorname{Diff}^{-1}$, we may express each x_i as $x_i = \sum\limits_{1 \leq j \leq m} a_{ij} y_j$ for some integers a_{ij}. Note that by definition

$$\operatorname{Norm}_{A/\operatorname{Frac}(R)}(\operatorname{Diff}_{\mathcal{O}/R}) = [\operatorname{Diff}^{-1}_{\mathcal{O}/R} : \mathcal{O}]_R = \operatorname{Det}_{\operatorname{Frac}(R)}(a_{ij}).$$

On the other hand,

$$\operatorname{Tr}_{A/\operatorname{Frac}(R)}(x_i x_j) = \sum_{1 \leq k \leq m} a_{ik} \operatorname{Tr}_{A/\operatorname{Frac}(R)}(x_k y_j) = a_{ij}.$$

Hence $\operatorname{Disc}_{\mathcal{O}/R} = \operatorname{Det}_{\operatorname{Frac}(R)}(\operatorname{Tr}_{A/\operatorname{Frac}(R)}(x_i x_j))$, and the equation (1.1.1.13) holds as desired. □

Definition 1.1.1.14. *We say that the prime ideal* $\mathfrak{p}$ *of R is* **unramified** *in* $\mathcal{O}$ *if* $\mathfrak{p} \nmid \operatorname{Disc}_{\mathcal{O}/R}$.

Proposition 1.1.1.15 ([112, Thm. 25.3])**.** *Any two maximal R-orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in A have the same discriminant over R.*

Therefore it makes sense to say:

Definition 1.1.1.16. *An ideal $\mathfrak{p}$ of R is unramified in A if it is* **unramified** *in one (and hence any) maximal R-order of A.*

Proposition 1.1.1.17. *Let R be a Dedekind domain such that* $\mathrm{Frac}(R)$ *is a* **global field***, let A be a finite-dimensional* $\mathrm{Frac}(R)$*-algebra with center E, and let $\mathcal{O}$ be an R-order in A. Suppose $\mathfrak{p}$ is a* **nonzero** *prime ideal of R such that $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$. Then:*

1. *$\mathcal{O}$ is maximal at $\mathfrak{p}$.*
2. *$\mathcal{O} \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ is isomorphic to a product of matrix algebra(s) containing $\mathcal{O}_E \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ as its center.*
3. *$\mathfrak{p}$ is unramified in both A and its center E.*

Proof. If $\mathcal{O} \subset \mathcal{O}'$ are two orders, then necessarily

$$\mathcal{O} \subset \mathcal{O}' \subset \mathrm{Diff}^{-1}_{\mathcal{O}'/R} \subset \mathrm{Diff}^{-1}_{\mathcal{O}/R}.$$

In particular, if $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$, then

$$\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}} = \mathrm{Diff}^{-1}_{\mathcal{O}/R} \underset{R}{\otimes} R_{\mathfrak{p}}$$

forces $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ to be maximal. This proves the first statement.

According to [112, Thm. 10.5], $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order if and only if it is a maximal $\mathcal{O}_E \underset{R}{\otimes} R_{\mathfrak{p}}$-order. Then [112, Thm. 25.7] implies that $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ is a product of matrix algebras containing $\mathcal{O}_E \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ as its center. This is the second statement.

Finally, by taking the matrix with only one element on the diagonal, we see that $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}} = \mathrm{Diff}^{-1}_{\mathcal{O}/R} \underset{R}{\otimes} R_{\mathfrak{p}}$ forces $\mathrm{Diff}^{-1}_{\mathcal{O}_E/R} \underset{R}{\otimes} R_{\mathfrak{p}} = \mathcal{O}_E \underset{R}{\otimes} R_{\mathfrak{p}}$. In other words, $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$ forces $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}_E/R}$. Then the third statement is just a reformulation of what we know in terms of Definition 1.1.1.16. ☐

Definition 1.1.1.18. *A (left) $\mathcal{O}$-module M is called an* **$\mathcal{O}$-lattice** *if it is an R-lattice. Namely, it is finitely generated and torsion-free as an R-module.*

Definition 1.1.1.19. *A commutative noetherian integral domain*

Proposition 1.1.1.20 (see [112, Thm. 21.4 and Cor. 21.5])**.** *Every maximal order over a Dedekind domain is* **hereditary** *in the sense that all lattices over the maximal order are projective.*

Proposition 1.1.1.21 (see [112, Cor. 21.5 and Thm. 2.44])**.** *Every projective module over a maximal order is a direct sum of left ideals.*

Remark 1.1.1.22. Propositions 1.1.1.20 and 1.1.1.21 suggests that, although $\mathcal{O}$-lattices might not be projective, their localizations or completions become projective as soon as $\mathcal{O}$ itself becomes maximal after localization or completion.

1.1.2 Determinantal Conditions

Let C be a finite-dimensional separable algebra over a field k. By definition (such as [112, p. 99]), this means the center E of C is a commutative finite-dimensional *separable* algebra over k. Let K be a (possibly infinite) field extension of k, and K^{sep} be a fixed choice of separable closure of K. Unless otherwise specified, all the homomorphisms below will be k-linear.

Fix any separable closure K^{sep} of K, and consider the possible k-algebra maps τ from E to K^{sep}. Note that $\mathrm{Hom}_k(E, K^{\mathrm{sep}})$ has cardinality $[E : k]$, because E is separable over k. The $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbits $[\tau]$ of such maps $\tau : E \to K^{\mathrm{sep}}$ can be classified in the following way: Consider the equivalence classes of pairs of the form (K_τ, τ), where K_τ is K-isomorphic to the composite of K and the image of τ in K^{sep}, and where τ is the induced map from E to K_τ. Here K_τ is necessarily separable over K with degree at most $[E : k]$. Two such pairs (K_{τ_1}, τ_1) and (K_{τ_2}, τ_2) are considered equivalent if there is a K-isomorphism $\sigma : K_{\tau_1} \xrightarrow{\sim} K_{\tau_2}$ such that $\tau_2 = \sigma \circ \tau_1$. We shall denote such an equivalence class by $[\tau] : E \to K_{[\tau]}$. By abuse of notations, this will also mean an actual representative $\tau : E \to K_\tau$, which can be considered as a map $\tau : E \to K^{\mathrm{sep}}$ as well. Note that $K_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}} \cong \prod_{\tau' \in [\tau]} K^{\mathrm{sep}}_{\tau'}$, where each $K^{\mathrm{sep}}_{\tau'}$ means a copy of K^{sep} with a map $\tau' : E \to K^{\mathrm{sep}}$ in the equivalence class $[\tau]$.

Lemma 1.1.2.1. *We have a decomposition of* $E \underset{k}{\otimes} K$ *into a product of separable extensions* $E_{[\tau]}$ *of* K*,*

$$E \underset{k}{\otimes} K \cong \prod_{[\tau]:E \to K_{[\tau]}} E \underset{E,[\tau]}{\otimes} K_{[\tau]} \cong \prod_{\tau:E \to K_{[\tau]}} K_{[\tau]}.$$

Corollary 1.1.2.2. *We have a decomposition of* $C \underset{k}{\otimes} K$ *into* K*-simple algebras,*

$$C \underset{k}{\otimes} K \cong C \underset{E}{\otimes} (E \underset{k}{\otimes} K) \cong \prod_{[\tau]:E \to K} C \underset{E,[\tau]}{\otimes} K_{[\tau]}.$$

Let us quote the following weaker form of the Noether-Skolem theorem:

Lemma 1.1.2.3 (see for example [67, Lem. 4.3.2])**.** *Let* C' *be any simple Artinian algebra. Then* C' *has only one unique irreducible module.*

If we take the action of the center E' into account, then we can reinterpret Lemma 1.1.2.3 as follows: If M_1 and M_2 are two E'-modules that are irreducible as C'-modules, then there is an E'-isomorphism from M_1 to M_2 that corresponds the two C'-actions. Therefore we can reformulate Lemma 1.1.2.3 as:

Corollary 1.1.2.4. *Irreducible representations of* C' *with coefficients in some field* K' *are determined by its restriction to its center* E'*.*

Applying this to the simple factors of $C \underset{k}{\otimes} K$ as in Corollary 1.1.2.2, we obtain:

Corollary 1.1.2.5.

1. *There is a unique* $C \underset{k}{\otimes} K$*-module* $W_{[\tau]}$ *on which* E *acts via a map* $[\tau] : E \to K_{[\tau]}$*. The* K*-semisimple algebra* $C \underset{k}{\otimes} K$ *acts on* $W_{[\tau]}$ *via its projection to the* K*-simple subalgebra* $C \underset{E,[\tau]}{\otimes} K_{[\tau]}$ *given by Corollary 1.1.2.2.*

2. *Any finite-dimensional* $C \underset{k}{\otimes} K$*-module is of the form*

$$M \cong \underset{[\tau]:E \to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus\, m_{[\tau]}}$$

for some integers $m_{[\tau]}$*.*

When applied to the case $M = C \underset{k}{\otimes} K$*, this decomposition agrees with the one given in Corollary 1.1.2.2.*

3. *The isomorphism class of the* $C \underset{k}{\otimes} K$*-module* M *is determined by the set of integers* $\{m_{[\tau]}\}$*.*

For any particular $C \underset{k}{\otimes} K$-module M_0, we may ask ourselves if we can find its *field of definition* K_0 inside K. To make sure we have sufficiently many automorphisms of K over k, let make use of any choice of K^{sep} and define K_0 to be the fixed field of K^{sep} by the elements σ in $\mathrm{Aut}(K^{\mathrm{sep}}/k)$ such that $M_0 \underset{K}{\otimes} K^{\mathrm{sep}}$ and $(M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}}$ are isomorphic as $C \underset{k}{\otimes} K^{\mathrm{sep}}$-modules. Since M_0 is an object defined over K, this K_0 must be contained in K and independent of the choice of K^{sep}.

By replacing K by K^{sep} in Corollary 1.1.2.5, we see that there is a unique $C \underset{k}{\otimes} K^{\mathrm{sep}}$-module W_τ for each map $\tau : E \to K^{\mathrm{sep}}$. By checking the action of E, we see that there is a decomposition

$$W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}} \cong \underset{\tau' \in [\tau]}{\oplus} W_{\tau'}. \tag{1.1.2.6}$$

Moreover, each $\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}/k)$ modifies the action of E on W_τ by composition with σ, and hence we have

$$W_\tau \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong W_{\sigma\circ\tau}$$

for any $\tau : E \to K^{\mathrm{sep}}$. Putting together those τ' in an orbit $[\tau]$, we obtain

$$(W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong W_{[\sigma\circ\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}.$$

This is isomorphic to $W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}$ as a $C \underset{k}{\otimes} K^{\mathrm{sep}}$-module only when $[\sigma \circ \tau] = [\tau]$. As a result, if M is any $C \underset{k}{\otimes} K$-module with decomposition $M \cong \underset{[\tau]:E\to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus\, m_{[\tau]}}$ as in Corollary 1.1.2.5, then

$$(M \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong M \underset{K}{\otimes} K^{\mathrm{sep}}$$

if and only if

$$m_{[\sigma^{-1}\circ\tau]} = m_{[\tau]}.$$

for any τ. A useful observation is:

Corollary 1.1.2.7. *The field of definition for* W_τ *(as a* $C \underset{k}{\otimes} K^{\mathrm{sep}}$*-module) is contained in* $\tau(E) \subset K^{\mathrm{sep}}$*.*

Therefore:

Corollary 1.1.2.8. *Let* E^{Gal} *denote the* **Galois closure** E^{Gal} *of* E *in* K^{sep}*, namely the composite field of* $\tau(E)$ *for all possible* $\tau : E \to K^{\mathrm{sep}}$*. Then the field of definition* K_0 *for any* $C \underset{k}{\otimes} K$*-module* M_0 *is contained in the intersection* $E^{\mathrm{Gal}} \cap K$*.*

Proof. Simply note that $\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}/E^{\mathrm{Gal}})$ does not alter any possible maps $E \to K^{\mathrm{sep}}$. ☐

It is desirable to have a way to detect if two $C \underset{k}{\otimes} K$-modules are isomorphic, without having to go through the comparison of the $m_{[\tau]}$'s. In characteristic zero, it is classical to use the *trace* to classify representations:

Lemma 1.1.2.9. *Suppose* $\mathrm{char}(k) = 0$*. Then the maps* $C \to K : x \mapsto \mathrm{Tr}_K(x|W_{[\tau]})$ *are linearly independent over* K*.*

Proof. It suffices to show this by restricting the maps to E. Then, for any $e \in E$, we have

$$\mathrm{Tr}_K(e|W_{[\tau]}) = \mathrm{Tr}_{K^{\mathrm{sep}}}(e|W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}) = \sum_{\tau' \in [\tau]} \mathrm{Tr}_{K^{\mathrm{sep}}}(e|W_{\tau'}) = \sum_{\tau' \in [\tau]} d_{[\tau]} \tau'(e)$$

by (1.1.2.6), where $d_{[\tau]}^2$ is the degree of $C \underset{E,[\tau]}{\otimes} K_{[\tau]}$ over its center $K_{[\tau]}$. This is $d_{[\tau]}$ times a sum of maps from E to K^{sep}, which all factor through a single subfield of E as they are all in the same $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbit of a single map from E to K^{sep}. Now it is a classical lemma of Dedekind that any set of distinct maps from a first field to a second field are linearly independent over the second field. ☐

Corollary 1.1.2.10. *Suppose* $\mathrm{char}(k) = 0$*. Then two* $C \underset{k}{\otimes} K$*-modules* M_1 *and* M_2 *are isomorphic if and only if* $\mathrm{Tr}_K(x|M_1) = \mathrm{Tr}_K(x|M_2)$ *for all* $x \in C$*.*

Proof. If $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules, then certainly the traces are the same. Conversely, suppose the traces are the same for any $x \in C$. In particular, they are the same for any $e \in E$. Let us decompose $M_i = \underset{[\tau]:E\to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus\, m_{[\tau],i}}$ for $i = 1, 2$, as in Corollary 1.1.2.5. Then by Lemma 1.1.2.9,

$$\begin{aligned}\operatorname{Tr}_K(e|M_1) &= \sum_{[\tau]:E\to K_{[\tau]}} m_{[\tau],1} \operatorname{Tr}_K(e|W_{[\tau]}) \\ &= \sum_{[\tau]:E\to K_{[\tau]}} m_{[\tau],2} \operatorname{Tr}_K(e|W_{[\tau]}) = \operatorname{Tr}_K(e|M_2)\end{aligned}$$

for any $e \in E$ if and only if $m_{[\tau],1} = m_{[\tau],2}$ for all $[\tau]$, or equivalently $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules. $\square$

Remark 1.1.2.11. The fact that $\operatorname{char}(k) = 0$ is used in an essential way. We cannot expect the trace comparison to work in any characteristic.

Corollary 1.1.2.12. *Suppose* $\operatorname{char}(k) = 0$. *Then the field of definition* K_0 *of a* $C \underset{k}{\otimes} K$*-module* M_0 *can be given by*

$$K_0 = k(\operatorname{Tr}_K(x|M_0) : x \in C) = k(\operatorname{Tr}_K(x|M_0) : x \in C).$$

Proof. Let σ be any element in $\operatorname{Gal}(K^{\mathrm{sep}}/k)$. We would like to show that

$$(M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong M_0 \underset{K}{\otimes} K^{\mathrm{sep}}$$

as $C \underset{k}{\otimes} K^{\mathrm{sep}}$-modules if and only if σ leaves $\operatorname{Tr}(x|M_0)$ invariant for all $x \in C$. But this follows from Corollary 1.1.2.10 as soon as we observe that

$$\sigma \operatorname{Tr}_K(x|M_0) = \sigma \operatorname{Tr}_{K^{\mathrm{sep}}}(x|M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) = \operatorname{Tr}_{K^{\mathrm{sep}}}(x|(M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}})$$

for any $x \in C$. $\square$

To classify $C \underset{k}{\otimes} K$-modules without the assumption that $\operatorname{char}(k) = 0$, let us introduce the *determinantal conditions* used by Kottwitz in the fundamental paper [79]. To avoid dependence on choices of basis elements, we shall give a definition similar to the one in [108, 3.23(a)].

Definition 1.1.2.13. *Let* L_0 *be any finitely generated locally free module over a commutative ring* R_0*, and let* $L_0^\vee := \operatorname{Hom}_{R_0}(L_0, R_0)$ *be the dual module of* L_0*. Define*

$$R_0[L_0^\vee] := \underset{k\geq 0}{\oplus} \operatorname{Sym}^k(L_0^\vee), ,$$

and define the **associated vector space scheme** $\mathbb{V}_{L_0}$ *to be the scheme*

$$\mathbb{V}_{L_0} := \mathrm{Spec}(R_0[L_0^\vee]).$$

Then, for any R_0-algebra R, we have canonical isomorphisms

$$\begin{aligned}\mathbb{V}_{L_0}(R) &\cong \mathrm{Hom}_{\mathrm{Spec}(R_0)}(\mathrm{Spec}(R), \mathbb{V}_{L_0})\\ &\cong \mathrm{Hom}_{R_0\text{-alg.}}(\bigoplus_{k\geq 0} \mathrm{Sym}^k(L_0^\vee), R)\\ &\cong \mathrm{Hom}_{R_0\text{-mod.}}(L_0^\vee, R) \cong L_0 \underset{R_0}{\otimes} R.\end{aligned}$$

In other words, $\mathbb{V}_{L_0}$ represents the functor that associates to each R_0-algebra R the locally free R-module $L_0 \underset{R_0}{\otimes} R$.

This construction certainly sheafifies and associates to any coherent locally free sheaf $\mathscr{L}$ over any locally noetherian scheme S the dual sheaf $\mathscr{L}^\vee := \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{L}, \mathscr{O}_S)$, the sheaf of graded algebras

$$\mathscr{O}_S[\mathscr{L}^\vee] := \bigoplus_{k\geq 0} \mathrm{Sym}^k(\mathscr{L}^\vee), ,$$

and the vector space scheme

$$\mathbb{V}_{\mathscr{L}} := \underline{\mathrm{Spec}}_{\mathscr{O}_S}\left(\mathscr{O}_S[\mathscr{L}^\vee]\right).$$

Definition 1.1.2.14. *Suppose $\alpha_1, \ldots, \alpha_t$ are any elements of C forming a basis over k. Let $X_1, \ldots, X_t$ be the coordinate functions on $\mathbb{V}_C$ mapping $\sum\limits_j c_j\alpha_j$ to c_i. Then we have a canonical isomorphism $k[C^\vee] \cong k[X_1, \ldots, X_t]$. For any $C \underset{k}{\otimes} K$-module M, we define a polynomial*

$$\mathrm{Det}_{C|M}^{\alpha_1,\ldots,\alpha_t} \in K[X_1, \ldots, X_t]$$

by

$$\mathrm{Det}_{C|M}^{\alpha_1,\ldots,\alpha_t}(X_1, \ldots, X_t) := \mathrm{Det}_K(X_1\alpha_1 + \ldots + X_t\alpha_t|M),$$

which corresponds to an element

$$\mathrm{Det}_{C|M} \in K[C^\vee] := k[C^\vee] \underset{k}{\otimes} K$$

under the canonical isomorphism $K[C^\vee] \cong K[X_1, \ldots, X_t]$. This element $\mathrm{Det}_{C|M}$ is independent of the choice of the basis elements $\alpha_1, \ldots, \alpha_t$.

Lemma 1.1.2.15. *If* $[\tau_1] \neq [\tau_2]$*, then* $\mathrm{Det}_{C|W_{[\tau_1]}} \neq \mathrm{Det}_{C|W_{[\tau_2]}}$ *as elements in* $K[C^\vee]$*. Moreover, they have no common factors over* K^{sep} *when they both decompose into irreducibles.*

Proof. Let $t := [C : k]$, let $t_0 := [E : k]$, and let us take a k-basis $\alpha_1, \ldots, \alpha_t$ of C such that $\{\alpha_1, \ldots, \alpha_{t_0}\}$ is a k-basis of E. (This is always possible up to a k-linear change of coordinates, which does not affect the statement.) To verify the lemma, it suffices to show that, after setting $X_s = 0$ for $s > t_0$, the polynomials $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_1]}}(X_1, \ldots, X_{t_0})$ and $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_2]}}(X_1, \ldots, X_{t_0})$ have no common factors over K^{sep} when they both decompose into irreducibles.

Let $d_{[\tau_i]}$ be the degree of $C \underset{E,[\tau_i]}{\otimes} K_{[\tau_i]}$ over its center $K_{[\tau_i]}$. By (1.1.2.6), we have

$$\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_i]}}(X_1, \ldots, X_{t_0}) = \mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_i]} \underset{K}{\otimes} K^{\mathrm{sep}}}(X_1, \ldots, X_{t_0})$$
$$= \prod_{\tau_i' \in [\tau_i]} \mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_i'}}(X_1, \ldots, X_{t_0}) = \prod_{\tau_i' \in [\tau_i]} (X_1\tau_i'(\alpha_1) + \ldots + X_{t_0}\tau_i'(\alpha_{t_0}))^{d_{[\tau_i]}}$$

for $i = 1, 2$. Since the polynomial algebra over K^{sep} is a unique factorization domain, the factors of the two sides must match after a permutation if $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_1}} = \mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_2}}$. By reason of degree, each linear factor

$$X_1\tau_i'(\alpha_1) + \ldots + X_{t_0}\tau_i'(\alpha_{t_0})$$

is irreducible. Suppose we have $\tau_1' \in [\tau_1]$ and $\tau_2' \in [\tau_2]$ such that

$$X_1\tau_1'(\alpha_1) + \ldots + X_{t_0}\tau_1'(\alpha_{t_0}) = X_1\tau_2'(\alpha_1) + \ldots + X_{t_0}\tau_2'(\alpha_{t_0}).$$

Then in particular

$$c_1\tau_1'(\alpha_1) + \ldots + c_{t_0}\tau_1'(\alpha_{t_0}) = c_1\tau_2'(\alpha_1) + \ldots + c_{t_0}\tau_2'(\alpha_{t_0})$$

for any $c_1, \ldots, c_s \in k$. This simply means $\tau_1' = \tau_2'$ as maps from E to K^{sep}. Since $[\tau_1] \neq [\tau_2]$ are disjoint orbits, this is a contradiction. $\square$

Proposition 1.1.2.16. *Two* $C \underset{k}{\otimes} K$*-modules* M_1 *and* M_2 *are isomorphic if and only if* $\mathrm{Det}_{C|M_1} = \mathrm{Det}_{C|M_2}$*.*

Proof. Let us decompose $M_i = \underset{[\tau]:E\to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus m_{[\tau],i}}$ for $i = 1, 2$, as in Corollary 1.1.2.5. Then we have

$$\mathrm{Det}_{C|M_i} = \prod_{[\tau]:E\to K_{[\tau]}} \mathrm{Det}_{C|W_{[\tau]}}^{m_{[\tau],i}}$$

for $i = 1, 2$. By Lemma 1.1.2.15, different factors $\mathrm{Det}_{C|W_{[\tau_1]}}$ and $\mathrm{Det}_{C|W_{[\tau_2]}}$ have no common factors when they decompose into irreducibles over K^{sep}. Therefore, $\mathrm{Det}_{C|M_1} = \mathrm{Det}_{C|M_2}$ if and only if $m_{[\tau],1} = m_{[\tau],2}$ for all $[\tau]$, or equivalently $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules. ☐

Now suppose R_0 is a commutative noetherian integral domain with fraction field $\mathrm{Frac}(R_0)$. Let $\mathcal{O}$ be an R_0-order in some finite-dimensional $\mathrm{Frac}(R_0)$-algebra A with center F. Suppose the underlying R_0-lattice of $\mathcal{O}$ is free.

Definition 1.1.2.17. *Let S be any locally noetherian scheme over $\mathrm{Spec}(R_0)$, and $\mathscr{M}$ any locally free $\mathscr{O}_S$-module of finite rank on which $\mathcal{O}$ has an action. Suppose $\alpha_1, \ldots, \alpha_t$ are any elements of $\mathcal{O}$ forming a free basis over R_0. Let $X_1, \ldots, X_t$ be the coordinate functions on $\mathbb{V}_{\mathcal{O}}$ mapping $\sum_j c_j\alpha_j$ to c_i. Then we have a canonical isomorphism $R_0[\mathcal{O}^\vee] \cong R_0[X_1, \ldots, X_t]$. Define a polynomial function*

$$\mathrm{Det}_{\mathcal{O}|\mathscr{M}}^{\alpha_1,\ldots,\alpha_t} \in \mathscr{O}_S[X_1, \ldots, X_t]$$

by

$$\mathrm{Det}_{\mathcal{O}|\mathscr{M}}^{\alpha_1,\ldots,\alpha_t}(X_1, \ldots, X_t) := \mathrm{Det}_{\mathscr{O}_S}(X_1\alpha_1 + \ldots + X_t\alpha_t|\mathscr{M}),$$

which corresponds to an element

$$\mathrm{Det}_{\mathcal{O}|M} \in \mathscr{O}_S[\mathcal{O}^\vee] := R_0[\mathcal{O}^\vee] \underset{R_0}{\otimes} \mathscr{O}_S$$

under the canonical isomorphism $\mathscr{O}_S[\mathcal{O}^\vee] \cong \mathscr{O}_S[X_1, \ldots, X_t]$. This element $\mathrm{Det}_{\mathcal{O}|\mathscr{M}}$ is independent of the choice of the basis elements $\alpha_1, \ldots, \alpha_t$.

Remark 1.1.2.18. Definition 1.1.2.17 works in particular when $S = \mathrm{Spec}(R)$ and R is a commutative algebra over R_0. In this case we may consider the same definition for a locally free module M over R, and write $\mathrm{Det}_{\mathcal{O}|M} \in R[\mathcal{O}^\vee] := R_0[\mathcal{O}^\vee] \underset{R_0}{\otimes} R$ instead of $\mathrm{Det}_{\mathcal{O}|\mathscr{M}} \in \mathscr{O}_S[\mathcal{O}^\vee]$. If $S = \mathrm{Spec}(k)$, k is a field, $C := \mathcal{O} \underset{R_0}{\otimes} k$ is semisimple over k, and $E := F \underset{R_0}{\otimes} k$ is a separable k-algebra, then Definition 1.1.2.17 agrees with Definition 1.1.2.14 if we consider C, E, k as before with $K = k$.

1.1.3 Projective Modules

Lemma 1.1.3.1. *Let R_0 be a commutative noetherian integral domain with fraction field* $\mathrm{Frac}(R_0)$*, and let $\mathcal{O}$ be an R_0-order in some finite-dimensional* $\mathrm{Frac}(R_0)$*-algebra A with center F, so that underlying R_0-lattice of $\mathcal{O}$ is projective. Let R be a noetherian local R_0-algebra with residue field k. Let M_1 and M_2 be two $\mathcal{O} \underset{R_0}{\otimes} R$-modules such that M_1 is projective as an $\mathcal{O} \underset{R_0}{\otimes} R$-module and M_2 is projective as an R-module. Then M_1 is isomorphic to M_2 if and only if $M_1 \underset{R_0}{\otimes} k$ and $M_2 \underset{R_0}{\otimes} k$ are isomorphic as $\mathcal{O} \underset{R_0}{\otimes} k$-modules.*

Proof. The direction from R to k is obvious. Conversely, suppose there is an isomorphism $\bar{f} : M_1 \underset{R}{\otimes} k \cong M_2 \underset{R}{\otimes} k$. By assumption that M_1 is projective as an $\mathcal{O} \underset{R_0}{\otimes} R$-module, we have a map $f : M_1 \to M_2$ of $\mathcal{O} \underset{R_0}{\otimes} R$-modules such that $f \underset{R}{\otimes} k = \bar{f}$. Note that this is in particular a map of R-modules. Since the underlying R_0-lattice of $\mathcal{O}$ is projective over R_0, it is a direct summand of a free R_0-module. Therefore, being projective as an $\mathcal{O} \underset{R_0}{\otimes} R$-module, or equivalent being a direct summand of a free $\mathcal{O} \underset{R_0}{\otimes} R$-module, implies being a direct summand of a free R-module, or equivalently being projective as a R-module. Now the projectivity of the two R-modules implies that f is an isomorphism by the usual NAK (Nakayama's lemma) for R-modules. (There is nevertheless a noncommutative NAK (Nakayama's lemma). See [112, Thm. 6.11]. The proof is the same well-known one.) ☐

Remark 1.1.3.2. Being projective, namely being a direct summand of a free module, is not equivalent to being locally free, as the semisimple algebra $\mathcal{O} \underset{R_0}{\otimes} k$ might have more than one simple components.

There is a general theory of left ideals of maximal orders over Dedekind domains. Combining with Proposition 1.1.1.21, this gives a complete classification of the projective modules that we will encounter when the order is maximal. However, for our purpose we shall only need the following simple special case:

Suppose R_0 is the ring of integers in a number field. In particular, R_0 is a Dedekind domain, and therefore the underlying R_0-lattice of $\mathcal{O}$ is projective over R_0. Let k be either a field of characteristic $p = 0$ or a finite field of characteristic $p > 0$, such that there is a nonzero ring homomorphism from

R_0 to k. Suppose $p \nmid \mathrm{Disc}$. Let $\Lambda = k$ when $p = 0$, and let $\Lambda = W(k)$ when $p > 0$. Let R be any noetherian local Λ-algebra with residue field k.

By Proposition 1.1.1.17, we know that $\mathcal{O} \underset{R_0}{\otimes} k$ is a separable algebra over k, and $\mathcal{O}_\Lambda$ is a maximal order over Λ. By [112, Thm. 10.5] (see also Section 1.1.2), we have decompositions

$$\mathcal{O}_{F,\Lambda} \cong \coprod_\tau \mathcal{O}_{F_\tau}$$

and

$$\mathcal{O}_\Lambda \cong \coprod_\tau \mathcal{O}_\tau$$

into simple factors, where τ is parameterized by orbits of embeddings of F into the separable closure of $\mathrm{Frac}(\Lambda)$. In the first decomposition, each simple factor $\mathcal{O}_{F_\tau}$ is the maximal Λ-order in some separable field extension F_τ of $\mathrm{Frac}(\Lambda)$. By Proposition 1.1.1.17 again, we know that each $\mathcal{O}_\tau$ is isomorphic to $\mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau})$ for some integer $d_\tau \geq 1$. We may identify $\mathcal{O}_\tau$ with the endomorphism algebra $\mathrm{End}_\Lambda(M_\tau)$ of the free $\mathcal{O}_{F,\tau}$-module $M_\tau := \mathcal{O}_{F_\tau}^{\oplus d_\tau}$, so that M_τ is considered as an $\mathcal{O}_\Lambda$-module via the projection from $\mathcal{O}_\Lambda$ to $\mathcal{O}_\tau$.

For simplicity we shall allow two different interpretations of subscripts R:

Convention 1.1.3.3. *The notations such as $\mathcal{O}_R$, $\mathcal{O}_{F,R}$, $(\mathrm{Diff}^{-1})_R$, ... will stand for tensor product from R_0 to R as in Section 1.1.4, while the notations such as $\mathcal{O}_{\tau,R}$, $\mathcal{O}_{F_\tau,R}$, $M_{\tau,R}$, ... will stand for tensor product from Λ to R.*

We hope that such an abuse of notations is acceptable.

Lemma 1.1.3.4. *Assumptions as above, every projective $\mathcal{O}_R$-module M is isomorphic to*

$$\oplus\, M_{\tau,R}^{\oplus\, m_\tau}$$

for some uniquely determined integers $m_\tau \geq 0$.

Proof. By Lemma 1.1.3.1, we may replace M by $M \underset{R}{\otimes} k$ and reduce the problem to the classification of finite-dimensional modules over a finite-dimensional semisimple algebra with separable center over a field. This is already addressed in Corollary 1.1.2.5, with the W_τ there replaced by $M_{\tau,R} \underset{R}{\otimes} k$. □

Motivated by Lemma 1.1.3.4:

Definition 1.1.3.5. *Assumptions as above, the* **multi-rank** *of a projective* $\mathcal{O}_R$*-module* M *is defined to be the integers* (m_τ) *appearing in the decomposition* $M \cong \oplus M_{\tau,R}^{\oplus m_\tau}$ *in Lemma 1.1.3.4.*

It is useful to have the following generalized form of Noether-Skolem theorem in our context:

Lemma 1.1.3.6. *Suppose* $p \nmid \mathrm{Disc}$, *and suppose* M *is any projective* $\mathcal{O}_R$*-module. Let* C *be any* $\mathcal{O}_{F,R}$*-subalgebra of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *containing the image of* $\mathcal{O}_{F,R}$*. Then any* C*-automorphism of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *(namely an automorphism inducing the identity on* C*) is an inner automorphism* $\mathrm{Int}(a)$ *for some invertible element* a *in* $\mathrm{End}_C(M)$*.*

The proof we give here is an imitation of the proof in [112, Thm. 7.21]. (We cannot simply refer to [112, Thm. 7.21] because $\mathcal{O}_{F,R}$ is a field there.)

Proof of Lemma 1.1.3.6. For simplicity, let us denote $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ by E, and denote image of $\mathcal{O}_{F,R}$ in $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ by Z. Let $\varphi : E \xrightarrow{\sim} E$ be any C-automorphism of E.

Note that M is an E-module by definition. Let M' be another E-module, with same elements as M, but with the E-action twisted by φ. Namely, for $b \in E$ and $m \in M$, we replace the action $m \mapsto bm$ by $m \mapsto \varphi(b)m$. Note that since φ is C-linear, we know that M and M' are isomorphic as C-modules. By assumption, and by Lemma 1.1.3.4, we know that E is a product of matrix algebras, which is the base extension $\tilde{\mathcal{O}}_R$ of a product $\tilde{\mathcal{O}}$ of matrix algebras of the same form over $\mathcal{O}_{F,\Lambda}$. By Proposition 1.1.1.5, we know $\tilde{\mathcal{O}}$ is a maximal order over Λ. By Lemma 1.1.3.1, we know that M and M' are isomorphic as E-modules if $M \underset{R}{\otimes} k$ and $M' \underset{R}{\otimes} k$ are isomorphic as $E \underset{R}{\otimes} k$-modules. Since $p \nmid \mathrm{Disc}$, which in particular implies that p is unramified in $\mathcal{O}_F$, we know that the center $Z \underset{R}{\otimes} k$ of the matrix algebra $E \underset{R}{\otimes} k$ is separable over k. In particular, the classification in Section 1.1.2 (based on Lemma 1.1.2.3) shows that $M \underset{R}{\otimes} k$ and $M' \underset{R}{\otimes} k$ are isomorphic as $E \underset{R}{\otimes} k$-modules if they are isomorphic as $Z \underset{R}{\otimes} k$-modules. This is true simply because M and M' are isomorphic as C-modules, and because C contains Z. As a result, we see that there is an isomorphism $\theta : M \xrightarrow{\sim} M'$ of E-modules, which by definition satisfies $\theta(bm) = \varphi(b)\theta(m)$ for any $b \in E$ and any $m \in M$. Since M and M' are identical as $\mathcal{O}_{F,R}$-modules, we may interpret θ as an element a in $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. Since $\varphi(b) = b$ for any $b \in C$, we see that a lies in $\mathrm{End}_C(M)$. Since θ is an isomorphism, we see that a is invertible. Finally, $\theta(bm) = \varphi(b)\theta(m)$ for any $m \in M$ means simply $ab = \varphi(b)a$, or $\varphi(b) = aba^{-1} = \mathrm{Int}(a)(b)$, for any $b \in E$. This shows that $\varphi = \mathrm{Int}(a)$, as desired. $\square$

1.1.4 Generalities of Pairings

In this section, let R_0 be a commutative noetherian integral domain with fraction field $\mathrm{Frac}(R_0)$, and let $\mathcal{O}$ be an R_0-order in some finite-dimensional $\mathrm{Frac}(R_0)$-algebra A with center F. Suppose moreover that A is equipped with an *involution* * that sends $\mathcal{O}$ to itself. Then it is automatic that * sends the center F of A to itself. For simplicity, let us denote by F^+ the fixed subalgebra of F by *.

Let R be any commutative R_0-algebra, and let M be any R-module. For simplicity, we shall adopt Convention 1.1.3.3 in this section, so that for example $\mathcal{O}_R$ stands for $\mathcal{O} \underset{R_0}{\otimes} R$.

Let Diff^{-1} be the inverse different of $\mathcal{O}$ over R_0 (defined as in Definition 1.1.1.11). By definition, the restriction of $\mathrm{Tr}_{A/\mathrm{Frac}(R_0)} : A \to \mathrm{Frac}(R_0)$ to Diff^{-1} defines an R_0-linear map $\mathrm{Tr}_{\mathcal{O}/R_0} : \mathrm{Diff}^{-1} \to R_0$. We denote by the same notation $\mathrm{Tr}_{\mathcal{O}/R_0} : L_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1} \to L_0$ its natural base extension to any R_0-module L_0.

Lemma 1.1.4.1. *Suppose $\mathcal{O}$ is locally free as an R_0-module. Then* Diff^{-1} *is also locally free, and* $\mathrm{Tr}_{\mathcal{O}/R_0}$ *induces a perfect pairing* $\mathrm{Tr}_{\mathcal{O}/R_0} : \mathcal{O} \times \mathrm{Diff}^{-1} \to R_0$*. More precisely, if $\mathcal{O}_R$ has a free basis $\{e_i\}_{1\leq i\leq t}$ over some localization R of R_0, where t is the rank of $\mathcal{O}$ over R_0, then* $(\mathrm{Diff}^{-1})_R$ *has a free basis $\{f_j\}_{1\leq i\leq t}$ over R dual to $\{e_i\}_{1\leq i\leq t}$ in the sense that* $\mathrm{Tr}_{\mathcal{O}/R_0}(e_i f_j) = \delta_{ij}$ *for any $1 \leq i, j \leq t$.*

Proof. We may localize R_0 and assume that $\mathcal{O}$ is free. Let $\{e_i^0\}_{1\leq i\leq t}$ be any basis of $\mathcal{O}$. Let us proceed by induction on k. Suppose we have found pairs of elements $\{(e_i, f_i)\}_{1\leq i<k}$ such that the R_0-span of $\{e_i\}_{1\leq i<k}$ is the same as the R_0-span of $\{e_i^0\}_{1\leq i<k}$, and such that $\mathrm{Tr}_{\mathcal{O}/R_0}(e_i f_j) = \delta_{ij}$ for any $1 \leq i, j < k$. Set $e_k := e_k^0 - \sum\limits_{1\leq i\leq k-1} \mathrm{Tr}_{\mathcal{O}/R_0}(e_k^0 f_i) e_i$. Then $\mathrm{Tr}_{\mathcal{O}/R_0}(e_k f_j) = 0$ for any $1 \leq j < k$. Let f_k^0 be any element in Diff^{-1} such that $\mathrm{Tr}_{\mathcal{O}/R_0}(e_k f_k^0) = 1$. Set $f_k := f_k^0 - \sum\limits_{1\leq j\leq k-1} \mathrm{Tr}_{\mathcal{O}/R_0}(e_j f_k^0) f_j$. Then $\mathrm{Tr}_{\mathcal{O}/R_0}(e_i f_k) = 0$, and $\mathrm{Tr}_{\mathcal{O}/R_0}(e_k f_k) = \mathrm{Tr}_{\mathcal{O}/R_0}(e_k f_k^0) = 1$ for any $1 \leq i < k$. Proceeding as above, we obtain a new basis $\{e_i\}$ of $\mathcal{O}$, and a set of elements $\{f_j\}_{1\leq j\leq t}$ in Diff^{-1} dual to $\{e_i\}_{1\leq i\leq t}$

in the sense that $\mathrm{Tr}_{\mathcal{O}/R_0}(e_i f_j) = \delta_{ij}$ for any $1 \leq i, j \leq t$. Note that this forces $\{f_j\}_{1 \leq j \leq t}$ to be a basis of A over $\mathrm{Frac}(R_0)$. If $y = \sum_j c_j f_j$ is an element in A such that $\mathrm{Tr}_{\mathcal{O}/R_0}(xy) \in R_0$ for any $x \in \mathcal{O}$, then in particular $c_j = \mathrm{Tr}_{\mathcal{O}/R_0}(e_j y) \in R_0$ for any $1 \leq j \leq t$. This shows that $\{f_j\}_{1 \leq j \leq t}$ is also a basis of Diff^{-1}, as desired. ☐

Corollary 1.1.4.2. *Suppose $\mathcal{O}$ is locally free as an R_0-module. Let L_0 be any R_0-module, and let z be any element of $L_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$. If $\mathrm{Tr}_{\mathcal{O}/R_0}(bz) = 0$ for any $b \in \mathcal{O}$, then $z = 0$.*

Proof. We may localize and assume that both $\mathcal{O}$ and Diff^{-1} are free as R_0-modules. Let $\{e_i\}_{1 \leq i \leq t}$ be a basis of $\mathcal{O}$ over R_0, and let $\{f_j\}_{1 \leq j \leq t}$ be a basis of Diff^{-1} over R_0 dual to $\{e_i\}_{1 \leq i \leq t}$ as in Lemma 1.1.4.1. Then the element z in $L_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ can be written uniquely in the form $z = \sum_{1 \leq j \leq t} z_j \otimes f_j$, where $x_j \in L_0$. By assumption, $z_i = \mathrm{Tr}_{\mathcal{O}/R_0}(e_i z) = 0$ for any $1 \leq i \leq t$. This shows that z is zero, as desired. ☐

Definition 1.1.4.3. *Let R be any commutative R_0-algebra, and let M and N be any two R-modules.*

1. *An R-bilinear pairing $\langle \cdot, \cdot \rangle : M \times M \to N$ is called* **symmetric** *if $\langle x, y \rangle = \langle y, x \rangle$ for any $x, y \in M$.*

2. *An R-bilinear pairing $\langle \cdot, \cdot \rangle : M \times M \to N$ is called* **skew-symmetric** *if $\langle x, y \rangle = -\langle y, x \rangle$ for any $x, y \in M$.*

3. *An R-bilinear pairing $\langle \cdot, \cdot \rangle : M \times M \to N$ is called* **alternating** *if $\langle x, x \rangle = 0$ for any $x \in M$.*

4. *An R-bilinear pairing $(\!| \cdot, \cdot |\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is called* **Hermitian** *if $(\!| x, y |\!) = (\!| y, x |\!)^\star$ and $(\!| x, by |\!) = b(\!| x, y |\!)$ for any $x, y \in M$ and any $b \in \mathcal{O}$. Here * and $\mathcal{O}$ acts only on the second factor* Diff^{-1} *of $N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$.*

5. *An R-bilinear pairing $(\!| \cdot, \cdot |\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is called* **skew-Hermitian** *if $(\!| x, y |\!) = -(\!| y, x |\!)^\star$ and $(\!| x, by |\!) = b(\!| x, y |\!)$ for any $x, y \in M$ and any $b \in \mathcal{O}$. Here * and $\mathcal{O}$ acts only on the second factor* Diff^{-1} *of $N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$.*

Remark 1.1.4.4. An alternating form is always skew-symmetric, but the converse might not be true when 2 is a zero-divisor.

Definition 1.1.4.5. *Let* ϵ *be either* $+1$ *or* -1. *An* R*-bilinear pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ *is called* ϵ**-symmetric** *if* $\langle x, y\rangle = \varepsilon\langle y, x\rangle$ *for all* $x, y \in M$. *An* R*-bilinear pairing* $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ *is called* ϵ**-Hermitian** *if* $(\!|x, y|\!) = \epsilon(\!|y, x|\!)^\star$ *and* $(\!|x, by|\!) = b(\!|x, y|\!)$ *for all* $x, y \in M$ *and* $b \in \mathcal{O}$.

Lemma 1.1.4.6. *Let* ϵ *be either* $+1$ *or* -1, *and let* M *and* N *be finitely generated modules over* R. *Suppose* $\mathcal{O}$ *is locally free over* R_0, *and suppose* $\mathcal{O}$ *acts on* M. *Then there is a one-one correspondence between the set of* ϵ*-Hermitian pairings*

$$(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$$

(*that is* $\mathcal{O}_R$*-linear in the second variable according to our definition*) *and the set of* ϵ*-symmetric pairings*

$$\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N,$$

such that $\langle bx, y\rangle = \langle x, b^\star y\rangle$ *for any* $x, y \in M$ *and any* $b \in \mathcal{O}_R$. *The map in one direction can be given explicitly as follows: Let* $\mathrm{Tr}_{\mathcal{O}/R_0} : N \underset{R_0}{\otimes} \mathrm{Diff}^{-1} \to N$ *be. the natural base extension of* $\mathrm{Tr}_{\mathcal{O}/R_0} : \mathrm{Diff}^{-1} \to R_0$. *We associate to each* ϵ*-Hermitian pairing* $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ *the* ϵ*-symmetric pairings* $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ *defined by* $\langle\,\cdot\,,\,\cdot\,\rangle := \mathrm{Tr}_{\mathcal{O}/R_0}(\!|\,\cdot\,,\,\cdot\,|\!)$.

Proof. It is obvious that if $(\!|\,\cdot\,,\,\cdot\,|\!)$ is ϵ-Hermitian, then the associated $\langle\,\cdot\,,\,\cdot\,\rangle := \mathrm{Tr}_{\mathcal{O}/R_0}(\!|\,\cdot\,,\,\cdot\,|\!)$ is ϵ-symmetric and satisfies $\langle bx, y\rangle = \langle x, b^\star y\rangle = \epsilon\langle b^\star y, x\rangle$ for any $x, y \in L$ and any $b \in \mathcal{O}$.

This association is injective, because if $(\!|\,\cdot\,,\,\cdot\,|\!)' : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is another ϵ-Hermitian pairing such that $\mathrm{Tr}_{\mathcal{O}/R_0}(\!|x, y|\!) = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x, y|\!)'$ for any $x, y \in M$, then we have $\mathrm{Tr}_{\mathcal{O}/R_0}[b(\!|x, y|\!)] = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x, by|\!) = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x, by|\!)' = \mathrm{Tr}_{\mathcal{O}/R_0}[b(\!|x, y|\!)']$ for any $x, y \in M$ and any $b \in \mathcal{O}$. By Corollary 1.1.4.2, this implies that $(\!|x, y|\!) = (\!|x, y|\!)'$ for any $x, y \in L$, which shows the two pairings are identical. Note that this uses only the $\mathcal{O}_R$-linearity in the second variable for the injectivity.

For the remaining proof let us assume that $\mathcal{O}_R$ is free over R by localization. If the result is true after all localizations, then it is also true before localization, because the modules M and N we consider are finitely generated over R.

Let $\{e_i\}$ be a basis of $\mathcal{O}_R$ over R, with $\{f_i\}$ a dual basis of $(\mathrm{Diff}^{-1})_R$ over R as in Lemma 1.1.4.1. Then we can write any $\llparenthesis \cdot , \cdot \rrparenthesis : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1} = N \underset{R}{\otimes} (\mathrm{Diff}^{-1})_R$ as a sum $\llparenthesis \cdot , \cdot \rrparenthesis = \sum\limits_i [\langle \cdot , \cdot \rangle_i \ f_i]$, where $\langle \cdot , \cdot \rangle_i : M \times M \to N$ is determined by taking $\langle x, y \rangle_i = \mathrm{Tr}_{\mathcal{O}/R_0}[e_i \llparenthesis x, y \rrparenthesis]$ for any $x, y \in M$. By $\mathcal{O}_R$-linearity of $\llparenthesis \cdot , \cdot \rrparenthesis$ in the second variable, this means $\langle x, y \rangle_i = \mathrm{Tr}_{\mathcal{O}/R_0} \llparenthesis x, e_i y \rrparenthesis = \langle x, e_i y \rangle$, and so we must have $\llparenthesis x, y \rrparenthesis = \sum\limits_i [\langle x, e_i y \rangle \ f_i]$ for any $x, y \in M$.

Now we are ready to show the surjectivity: If $\langle \cdot , \cdot \rangle := \mathrm{Tr}_{\mathcal{O}/R_0} \llparenthesis \cdot , \cdot \rrparenthesis$ is any ϵ-symmetric pairing such that $\langle bx, y \rangle = \langle x, b^\star y \rangle$ for any $x, y \in M$ and any $b \in \mathcal{O}_R$. Consider $\llparenthesis \cdot , \cdot \rrparenthesis : M \times M \to N \underset{R}{\otimes} (\mathrm{Diff}^{-1})_R$ defined by $\llparenthesis x, y \rrparenthesis = \sum\limits_i [\langle x, e_i y \rangle \ f_i]$ for any $x, y \in M$. Suppose $1 = \sum\limits_i u_i e_i$, where $u_i \in R$. Then $\mathrm{Tr}_{\mathcal{O}/R_0} f_i = \mathrm{Tr}_{\mathcal{O}/R_0}(\sum\limits_k u_k e_k f_i) = u_i$, and we have $\mathrm{Tr}_{\mathcal{O}/R_0} \llparenthesis x, y \rrparenthesis = \sum\limits_i [\langle x, e_i y \rangle \, \mathrm{Tr}_{\mathcal{O}/R_0} f_i] = \langle x, [\sum\limits_i u_i e_i] y \rangle = \langle x, y \rangle$.

Moreover, $\llparenthesis \cdot , \cdot \rrparenthesis$ is $\mathcal{O}_R$-linear in the second variable: For any $b \in \mathcal{O}_R$, assume that $e_i b = \sum\limits_j a_{ij} e_j$ for all i, where $a_{ij} \in R$. Then we know from $a_{ij} = \mathrm{Tr}_{\mathcal{O}/R_0}(e_i b f_j)$ that $b f_j = \sum\limits_i a_{ij} f_i$ for all j. As a result, we have $\llparenthesis x, by \rrparenthesis = \sum\limits_i \langle x, e_i b y \rangle f_i = \sum\limits_{i,j} \langle x, e_j y \rangle a_{ij} f_i = \sum\limits_j \langle x, y e_j \rangle b f_j = b \llparenthesis x, y \rrparenthesis$.

Finally, since $\{e_i^\star\}$ is also a basis for $\mathcal{O}_R$ over R with $\{f_i^\star\}$ its dual basis for $(\mathrm{Diff}^{-1})_R$ over R respect to $\mathrm{Tr}_{\mathcal{O}/R_0}$, we can also consider $\llparenthesis \cdot , \cdot \rrparenthesis' : M \times M \to N \underset{R}{\otimes} (\mathrm{Diff}^{-1})_R$ defined by $\llparenthesis x, y \rrparenthesis' = \sum\limits_i [\langle x, e_i{}^\star y \rangle \ f_i{}^\star]$ for any $x, y \in M$, which then also satisfies $\mathrm{Tr}_{\mathcal{O}/R_0} \llparenthesis \cdot , \cdot \rrparenthesis' = \langle \cdot , \cdot \rangle$ and $\mathcal{O}_R$-linearity in the second variable. By the injectivity above, we have $\llparenthesis \cdot , \cdot \rrparenthesis = \llparenthesis \cdot , \cdot \rrparenthesis'$. As a result, we have $\llparenthesis y, x \rrparenthesis = \llparenthesis y, x \rrparenthesis' = \sum\limits_i [\langle y, e_i^\star x \rangle f_i^\star] = \sum\limits_i [\langle e_i y, x \rangle f_i^\star] = \epsilon \sum\limits_i \langle x, e_i y \rangle f_i^\star = \epsilon \llparenthesis x, y \rrparenthesis^\star$. $\square$

Remark 1.1.4.7. Under the assumption that $\mathcal{O}$ is locally free over R_0, this sets up the relation between certain symmetric pairings and Hermitian pairings when $\epsilon = 1$, and the relation between certain skew-symmetric pairings and skew-Hermitian pairings when $\epsilon = -1$.

Remark 1.1.4.8. In the works of Kottwitz (such as [79]) and some other authors, Hermitian pairings in their definitions are symmetric or skew-symmetric pairings that correspond to Hermitian or skew-Hermitian pairings in our definition. We shall not follow their conventions in our work.

Definition 1.1.4.9. *Suppose $\mathcal{O}$ is locally free over R_0. Let R be a commutative R_0-algebra, let M be a finitely generated $\mathcal{O}_R$-module, and let N be a finitely generated R-module. An R-bilinear pairing*

$$\langle \cdot , \cdot \rangle : M \times M \to N$$

is called an $(\mathcal{O}_R, {}^\star)$**-pairing***, or simply an $\mathcal{O}_R$-pairing, if it satisfies $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for any $x, y \in M$ and any $b \in \mathcal{O}_R$.*

Definition 1.1.4.10. *Suppose $\mathcal{O}$ is locally free over R_0. Let R be a commutative R_0-algebra. A* **symplectic $\mathcal{O}_R$-module** *$(M, \langle \cdot , \cdot \rangle, N)$ is a $\mathcal{O}_R$-module M together with an* **alternating** *$\mathcal{O}_R$-pairing*

$$\langle \cdot , \cdot \rangle : M \times M \to N$$

(*as in Definitions 1.1.4.3 and 1.1.4.9*)*, where M and N are both finitely generated R-modules.*

Suppose N is locally free of rank one over R. We say $(M, \langle \cdot , \cdot \rangle, N)$ is **nondegenerate** (*resp.* **self-dual**) *if the pairing $\langle \cdot , \cdot \rangle$ is* **non-degenerate** (*resp.* **perfect**)*, in the sense that the R-module morphism*

$$M \to \mathrm{Hom}_R(M, N) : x \mapsto (y \mapsto \langle x, y\rangle)$$

induced by the pairing is an injection (*resp. an isomorphism*)*.*

If $N = R$, or if it is clear from the context, then we often omit N from the notations, and denote simply by $(M, \langle \cdot , \cdot \rangle)$.

Definition 1.1.4.11. *A* **symplectic morphism** *$f : (M_1, \langle \cdot , \cdot \rangle_1, N_1) \to (M_2, \langle \cdot , \cdot \rangle_2, N_2)$ is a pair of morphisms $f : M_1 \to M_2$ and $\nu(f) : N_1 \to N_2$ such that $\langle f(x), f(y)\rangle_2 = \nu(f)\langle x, y\rangle_1$ for any $x, y \in M_1$. A symplectic morphism $(f, \nu(f))$ is a* **symplectic isomorphism** *if both f and $\nu(f)$ are isomorphisms.*

Remark 1.1.4.12. The datum of a symplectic isomorphism consists of not only the morphism f between the underlying modules, but also the morphism $\nu(f)$ between the values. Certainly, if f is a symplectic isomorphism between nonzero modules, and if N_1 and N_2 are isomorphic locally free modules of rank one over R, then the datum $\nu(f)$ is a unit and is uniquely determined by f, and hence redundant. However, if for example $M_1 = M_2 = 0$ and $N_1 = N_2 = R$, then any two different units in R can be attached to the unique trivial underlying isomorphism from M_1 to M_2. The morphism f does *not* determine $\nu(f)$ in general. We are enforcing an abuse of notations here.

Definition 1.1.4.13. *Conventions as above, assume moreover that R is a noetherian integral domain. A* **symplectic $\mathcal{O}_R$-lattice** *$(M, \langle \cdot, \cdot \rangle, N)$ is a symplectic $\mathcal{O}_R$-module whose underlying $\mathcal{O}_R$-module M is an R-lattice.*

Definition 1.1.4.14. *Let $(M, \langle \cdot, \cdot \rangle, N)$ be a nondegenerate symplectic $\mathcal{O}_R$-lattice with value in a locally free sheaf N of rank one over R. The* **dual lattice** *$M^\#$ (with respect to $\langle \cdot, \cdot \rangle$ and N) is defined by*

$$M^\# := \{x \in M \underset{R}{\otimes} \mathrm{Frac}(R) : \langle x, y \rangle \in N, \forall y \in M\}.$$

By definition, the dual lattice contains M as a sublattice.

Definition 1.1.4.15. *Let M_1 and M_2 be two finitely generated $\mathcal{O}_R$-modules with respectively two $\mathcal{O}_R$-pairings $\langle \cdot, \cdot \rangle_i : M_i \times M_i \to N$, $i = 1, 2$, with the images in the same finitely generated R-module N. For simplicity, let us use the notations $(M_i, \langle \cdot, \cdot \rangle_i, N)$ as in the case of symplectic $\mathcal{O}_R$-modules. The* **orthogonal direct sum** *of $(M_i, \langle \cdot, \cdot \rangle_i, N)$, $i = 1, 2$, denoted by $(M_1, \langle \cdot, \cdot \rangle_1, N) \overset{\perp}{\oplus} (M_2, \langle \cdot, \cdot \rangle_2, N)$, is a pair $(M, \langle \cdot, \cdot \rangle, N)$ whose underlying $\mathcal{O}_R$-module M is $M_1 \oplus M_2$ and whose pairing $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \overset{\perp}{\oplus} \langle \cdot, \cdot \rangle_2$ is defined so that $\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2$ for any $(x_1, x_2), (y_1, y_2) \in M_1 \overset{\perp}{\oplus} M_2$.*

If $N = R$, or if it is clear from the context, then we often omit N from the notations.

Lemma 1.1.4.16. *Let M be any $\mathcal{O}_R$-module. Let $M^\vee := \mathrm{Hom}_R(M, R)$ denote the dual module of M with $\mathcal{O}_R$-action given by $(bf)(y) = f(b^\star y)$ for any $b \in \mathcal{O}_R$ and any $f \in \mathrm{Hom}_R(M, R)$. Then the canonical pairing*

$$M \times M^\vee \to R : (x, f) \mapsto f(x)$$

defines a canonical pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathrm{can.}} : (M \oplus M^\vee) \times (M \oplus M^\vee) &\to R \\ ((x_1, f_1), (x_2, f_2)) &\mapsto (f_2(x_1) - f_1(x_2)) \end{aligned}$$

which gives $M \oplus M^\vee$ a canonical structure of a symplectic $\mathcal{O}_R$-module. If the canonical morphism $M \to (M^\vee)^\vee$ is an isomorphism, then $(M \oplus M^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\text{can.}})$ is self-dual.

It is useful to have an interpretation of the pairings we shall consider in terms of anti-automorphisms of the endomorphism algebra of $\mathcal{O}_R$-modules. Assuming no longer that R is a domain, let M be a finitely generated $\mathcal{O}_R$-module, and let N be locally free of rank one over R. Having an R-bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$$

(which for the moment we allow to be symmetric, skew-symmetric, or neither) on M is equivalent to having an R-linear morphism

$$\langle\,\cdot\,,\,\cdot\,\rangle^* : M \to \mathrm{Hom}_R(M, N) : x \mapsto (y \mapsto \langle x, y\rangle).$$

To say that we have a *perfect* pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is equivalent to require that $\langle\,\cdot\,,\,\cdot\,\rangle^*$ is an *isomorphism.* Once we know that $\langle\,\cdot\,,\,\cdot\,\rangle^*$ is an isomorphism, we can define an anti-automorphism ${}^{\maltese}$ of $\mathrm{End}_R(M)$ by sending an endomorphism $b : M \to M$ to $b^{\maltese}$ defined by the composition

$$\begin{tikzcd}
M \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle^*"] \arrow[d, dotted, "b^{\maltese}"'] & \mathrm{Hom}_R(M, N) \arrow[d, "b^\vee"] \\
M & \mathrm{Hom}_R(M, N) \arrow[l, "(\langle\,\cdot\,,\,\cdot\,\rangle^*)^{-1}"]
\end{tikzcd} .$$

In other words, we have $\langle x, b(y)\rangle = \langle b^{\maltese}(x), y\rangle$ for any $x, y \in M$ and any $b \in \mathrm{End}_R(M)$.

If we equip $\mathrm{Hom}_R(M, N)$ with an action of $\mathcal{O}_R$ given by $(bf)(y) = f(b^\star y)$ for any $b \in \mathcal{O}_R$ and any $f \in \mathrm{Hom}_R(M, N)$, then the condition that $\langle bx, y\rangle = \langle x, b^\star y\rangle$ means exactly that $\langle\,\cdot\,,\,\cdot\,\rangle^*$ is $\mathcal{O}_R$-linear. In this case, the diagram above commutes with $b \in \mathcal{O}_R$ and $b^{\maltese} = b^\star \in \mathcal{O}_R$. Hence ${}^{\maltese}$ maps the image of $\mathcal{O}_R$ in $\mathrm{End}_R(M)$ to itself and restricts to the involution induced by *. Since $\mathcal{O}_{F,R}$ is the center of $\mathcal{O}_R$, and since $\mathrm{End}_{\mathcal{O}_R}(M)$ and $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ are respectively the centralizers of the images of $\mathcal{O}_R$ and $\mathcal{O}_{F,R}$, we see that any of them is mapped to itself under ${}^{\maltese}$. For simplicity, we shall denote the restrictions of the anti-automorphism ${}^{\maltese}$ to $\mathrm{End}_{\mathcal{O}_R}(M)$ and $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ by the same notation.

The general structure of $\mathrm{End}_{\mathcal{O}_R}(M)$ and $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ for arbitrary $\mathcal{O}_R$-modules could be rather complicated. However, when M is projective, and when $\mathcal{O}_R$ satisfies certain reasonably strong condition, there is a nice classification of pairings (with values in R) in terms of involutions on $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. We shall explore this classification in the next section.

1.1.5 Classification of Pairings By Involutions

With the setting as in Section 1.1.4, suppose moreover that R_0 is the ring of integers in a number field. In particular, R_0 is a Dedekind domain, and therefore the underlying R_0-lattice of $\mathcal{O}$ is projective over R_0 (and hence locally free). Let k be either a field of characteristic $p = 0$ or a finite field of characteristic $p > 0$, such that there is a nonzero ring homomorphism from R_0 to k. Suppose $p \nmid \mathrm{Disc}$. Let $\Lambda = k$ when $p = 0$, and let $\Lambda = W(k)$ when $p > 0$.

Let R be any noetherian local Λ-algebra with residue field k, with Convention 1.1.3.3 as in Sections 1.1.3 and 1.1.4. Let M be a *projective* $\mathcal{O}_R$-module. By Lemma 1.1.3.4, $M \cong \bigoplus\limits_{\tau} M_{\tau,R}^{\oplus m_\tau}$ for some integers m_τ, and hence $\mathrm{End}_{\mathcal{O}_R}(M) \cong \prod\limits_{\tau} \mathrm{M}_{m_\tau}(\mathcal{O}_{F_\tau,R})$ and $\mathrm{End}_{\mathcal{O}_{F_\tau,R}}(M) \cong \prod\limits_{\tau} \mathrm{M}_{m_\tau}(\mathcal{O}_{\tau,R}) \cong \prod\limits_{\tau} \mathrm{M}_{m_\tau d_\tau}(\mathcal{O}_{F_\tau,R})$. For convenience, we shall denote by $\overline{\mathcal{O}}_R$ (resp. $\overline{\mathcal{O}}_{F,R}$) the image of $\mathcal{O}_R$ (resp. $\mathcal{O}_{F,R}$) in $\mathrm{End}_R(M)$. Then $\overline{\mathcal{O}}_R$ (resp. $\overline{\mathcal{O}}_{F,R}$) is the product of those $\mathcal{O}_{\tau,R}$ (resp. $\mathcal{O}_{F_\tau,R}$) acting faithfully on M.

Note that any locally free module N of rank one over R is isomorphic to R, because R is local. Therefore, for the purpose of classifying perfect pairings $\langle \cdot , \cdot \rangle : M \times M \to N$ up to isomorphism, it suffices to assume that $N = R$.

Definition 1.1.5.1. *Assumptions as above, we say that two perfect $\mathcal{O}_R$-pairings $\langle \cdot , \cdot \rangle_i : M \times M \to R$, $i = 1, 2$, are* **weakly isomorphic** *if $\langle \cdot , \cdot \rangle_1^*$ and $\langle \cdot , \cdot \rangle_2^*$ differ only up to multiplication by an element in $\overline{\mathcal{O}}_{F,R}^{\times}$. We say they are* **weakly symplectic isomorphic** *if the pairings are alternating pairings.*

Lemma 1.1.5.2. *Assumptions as above, two perfect $\mathcal{O}_R$-pairings $\langle \cdot , \cdot \rangle_i : M \times M \to R$, $i = 1, 2$, are weakly isomorphic if and only if the anti-automorphisms $\maltese_i$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ determined by the pairings are identical: $\maltese_1 = \maltese_2$.*

Proof. Since one of the implication is clear, it suffices to show the other implication that if $\maltese_1 = \maltese_2$ as anti-automorphisms of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$, then $\langle\,\cdot\,,\,\cdot\,\rangle_1^*$ and $\langle\,\cdot\,,\,\cdot\,\rangle_2^*$ differ only up to multiplication by an element in $\mathcal{O}_{F,R}^\times$. Consider $a := (\langle\,\cdot\,,\,\cdot\,\rangle_1^*)^{-1} \circ \langle\,\cdot\,,\,\cdot\,\rangle_2^*$ as an element in $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. The inner automorphism $\mathrm{Int}(a)$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ defined by $b \mapsto a \circ b \circ a^{-1}$ satisfies $\mathrm{Int}(a) = \maltese_1 \circ (\maltese_2)^{-1}$. If $\maltese_1$ and $\maltese_2$ are equal, then $\mathrm{Int}(a)$ is trivial. This means a lies in the center of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$, which is simply $\overline{\mathcal{O}}_{F,R}$. Finally, $a \in \overline{\mathcal{O}}_{F,R}^\times$ simply because it is invertible. $\square$

Note that we have made a choice of the two variables when defining $\langle\,\cdot\,,\,\cdot\,\rangle^*$: If we take alternatively $x \mapsto (y \mapsto \langle y, x\rangle)$, then we obtain an anti-automorphism $\maltese'$ such that $\langle b(y), x\rangle = \langle y, b^{\maltese'}(x)\rangle$ for any $x, y \in M$ and any $b \in \mathrm{End}_R(M)$. Then we have $\langle x, b(y)\rangle = \langle b^{\maltese}(x), y\rangle = \langle x, (b^{\maltese})^{\maltese'}(y)\rangle$ for any $x, y \in M$ and any $b \in \mathrm{End}_R(M)$. In other words, we have $\maltese' \circ \maltese = \mathrm{Id}_{\mathrm{End}_R(M)}$.

If $(\maltese)^2 = \mathrm{Id}_{\mathrm{End}_R(M)}$, namely if $\maltese$ is an involution of $\mathrm{End}_R(M)$, then $\maltese' = \maltese$, and hence, if we repeat the proof of Lemma 1.1.5.2 with $\mathcal{O}_R$ replaced by R, there is some $\gamma \in R^\times$ such that $\langle x, y\rangle = \langle y, \gamma x\rangle$ for any $x, y \in M$. Then $\langle x, y\rangle = \langle y, \gamma x\rangle = \langle \gamma x, \gamma y\rangle = \langle x, \gamma^2 y\rangle$ for any $x, y \in M$ implies $\gamma^2 = 1$. As a result, we see that the anti-automorphism $\maltese$ it induces on $\mathrm{End}_\Lambda(M)$ is an involution if and only if there is an element $\gamma \in R^\times$ such that $\gamma^2 = 1$ and $\langle x, y\rangle = \langle y, \gamma x\rangle$ for any $x, y \in M$. (The converse is clear.)

If we only consider the restriction of the anti-automorphism $\maltese$ to $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$, then we may only conclude in the argument above that $\maltese$ is an involution if and only if there is some $\gamma \in \mathcal{O}_{F,R}^\times$ such that $\gamma^\star\gamma = 1$ and $\langle x, y\rangle = \langle y, \gamma x\rangle$ for any $x, y \in M$.

If $F = F^+$, namely if * acts trivially on F, this implies as above that $\langle x, y\rangle = \langle y, \gamma x\rangle$ for some $\gamma \in \overline{\mathcal{O}}_{F,R}^\times$ with $\gamma^2 = 1$. If we write $\mathcal{O}_{F,R} = \prod_\tau \mathcal{O}_{F_\tau}$ and write γ accordingly as $\gamma = (\gamma_\tau)$, then we see that $\gamma_\tau^2 = 1$ for any τ. The case where $\gamma_\tau = 1$ (resp. $\gamma_\tau = -1$) for all τ implies in particular that $\langle\,\cdot\,,\,\cdot\,\rangle$ is symmetric (resp. alternating). More generally:

Definition 1.1.5.3. *Let $\epsilon \in \overline{\mathcal{O}}_{F,R}^\times$ be an element such that $\epsilon^2 = 1$. Then we say that an $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ is ϵ-symmetric if it satisfies $\langle x, y\rangle = \langle y, \epsilon x\rangle$ for any $x, y \in M$.*

Then we see that a perfect $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ induces an involution of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ if and only if $\langle\,\cdot\,,\,\cdot\,\rangle$ is ϵ-symmetric for some $\epsilon \in \overline{\mathcal{O}}_{F,R}^\times$ such that $\epsilon^2 = 1$. Moreover, if $\langle\,\cdot\,,\,\cdot\,\rangle$ is ϵ-symmetric, then $\langle x, ry\rangle = \langle ry, \epsilon x\rangle =$

$\langle y, r\epsilon x\rangle = \langle y, \epsilon r x\rangle$ shows that any perfect pairing that is weakly isomorphic to $\langle\,\cdot\,,\,\cdot\,\rangle$ is also ϵ-symmetric. Therefore it makes sense to consider the following:

Definition 1.1.5.4 (cf. the classification in [75] in the case of algebras)**.** *Assumptions as above, suppose moreover that $F = F^+$. Then we say that an involution* $^{\maltese}$ *of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *is* **of ϵ-symmetric type** (*resp.* **of symplectic type**, *resp.* **of orthogonal type**) *if there exists a perfect ϵ-symmetric* (*resp. alternating, resp. symmetric*) *$\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ inducing* $^{\maltese}$*.*

Lemma 1.1.5.5. *Suppose $F = F^+$. Then an alternating $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to \mathcal{O}_R$ satisfies $\langle x, rx\rangle = 0$ for any $x \in M$ and any $r \in \mathcal{O}_{F,R}$ such that $r = r^\star$.*

Proof. By replacing Λ by an sufficiently large étale extension, we may assume that, in the product $\mathcal{O}_{F,\Lambda} = \prod\limits_{\tau} \mathcal{O}_{F_\tau}$, we have $\mathcal{O}_{F_\tau} = \Lambda$ for any τ. Then it suffices to prove the lemma by replacing $\mathcal{O}_R$ by $\mathcal{O}_{\tau,R}$ and by replacing $\mathcal{O}_{F,R}$ by $\mathcal{O}_{F_\tau,R} = R$, and the result is automatic. ☐

Remark 1.1.5.6. By Lemmas 1.1.5.2 and 1.1.5.5, the definitions of being of ϵ-symmetric, symplectic, and orthogonal types do not depend on the particular perfect pairing we choose that induces the involution.

If $[F : F^+] = 2$, namely if * is nontrivial on $\mathcal{O}_F$, then we need a different approach.

Lemma 1.1.5.7. *Assumptions as above, suppose moreover that $[F : F^+] = 2$. Then the there is an element $e \in \mathcal{O}_{F,R}$ such that $e + e^\star = 1$.*

Proof. By Proposition 1.1.1.17, the assumption that $p \nmid \mathrm{Disc}$ implies that $\mathrm{Diff}^{-1}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}} = \mathcal{O}_{F,\Lambda}$. In particular, there exists some $e \in \mathcal{O}_{F,\Lambda}$ such that $\mathrm{Tr}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}}(e) = e + e^\star = 1$. ☐

Corollary 1.1.5.8. *Assumptions as in Lemma 1.1.5.7, if $\gamma \in \mathcal{O}_{F,R}$ satisfies $\gamma^\star + \gamma = 0$, then there is an element $\delta \in \mathcal{O}_{F,R}$ such that $\gamma = \delta - \delta^\star$.*

Proof. If we take e as in Lemma 1.1.5.7 and take $\delta = e\gamma$, then $\delta - \delta^\star = e\gamma - e^\star\gamma^\star = (e + e^\star)\gamma = \gamma$, as desired. ☐

Lemma 1.1.5.9. *Assumptions as in Lemma 1.1.5.7, suppose moreover that R is a* **complete** *noetherian local R_0-algebra, and that the extension F/F^+ is split over k when* $\mathrm{char}(k) = p = 0$*. Let $\epsilon = \pm 1$, so that $x^\epsilon = x$ or x^{-1} depending on whether $\epsilon = 1$ or -1. If $\gamma \in \mathcal{O}^\times_{F,R}$ is an element such that $\gamma^\star = \gamma^\epsilon$, then $\gamma = \delta(\delta^\star)^\epsilon$ for some $\delta \in \mathcal{O}^\times_{F,R}$.*

The proof we give here is essentially the same as the one for $R = \Lambda$.

Proof. Let us investigate this situation for each τ in the decomposition $\mathcal{O}_{F,R} = \prod\limits_{\tau} \mathcal{O}_{F_\tau}$.

If the involution interchanges $\mathcal{O}_{F_\tau}$ and $\mathcal{O}_{F_{\tau'}}$, and the two factors of $\gamma \in \mathcal{O}_{F,R}^\times$ are of the form $(\gamma_\tau, \gamma_{\tau'}) \in \mathcal{O}_{F_\tau,R} \times \mathcal{O}_{F_{\tau'},R}$, then the condition that $\gamma^\star = \gamma^\epsilon$ shows that $\gamma_{\tau'} = \gamma_\tau^\epsilon$, and hence we may take $\delta \in \mathcal{O}_{F,R}^\times$ with the two factors $(\gamma_\tau, 1)$.

If the involution is nontrivial on F_τ, then it is a nontrivial degree two unramified extension of some local field F_τ^+. By our assumption that the extension F/F^+ is split over $\Lambda = k$ when $p = 0$, this can happen only when $p > 0$, in which case k is a finite field by assumption. Let $N := \mathrm{Norm}_{\mathcal{O}_{F_\tau,R}/\mathcal{O}_{F_\tau^+,R}} : \mathcal{O}_{F_\tau,R}^\times \to \mathcal{O}_{F_\tau^+,R}^\times : x \mapsto xx^\star$ be the *norm map*, and let $D : \mathcal{O}_{F_\tau,R}^\times \to \mathcal{O}_{F_\tau,R}^\times : x \mapsto x(x^\star)^{-1}$. Hence our goal is to show that $\mathrm{image}(N) = \ker(D)$ and $\mathrm{image}(D) = \ker(N)$.

Let $\mathfrak{m}$ be the maximal ideal of R. Since F_τ is unramified over Λ, we see that $\mathfrak{m}$ generates the maximal ideals in $\mathcal{O}_{F_\tau,R}$ and in $\mathcal{O}_{F_\tau^+,R}$. Let k_τ and k_τ^+ be respectively the residue fields of $\mathcal{O}_{F_\tau,R}$ and of $\mathcal{O}_{F_\tau^+,R}$. Let $U_i := \mathcal{O}_{F_\tau,R}^\times \cap [1 + (\mathfrak{m} \cdot \mathcal{O}_{F_\tau,R})^i]$, $U_i^D := \ker(D) \cap U_i = \mathcal{O}_{F_\tau^+,R}^\times \cap [1 + (\mathfrak{m} \cdot \mathcal{O}_{F_\tau^+,R})^i]$, and $U_i^N := \ker(N) \cap U_i$. Let $\mathrm{Gr}_U^i := U_i/U_{i+1}$, $\mathrm{Gr}_{U^D}^i := U_i^D/U_{i+1}^D$, and $\mathrm{Gr}_{U^N}^i := U_i^N/U_{i+1}^N$. Then $\mathrm{Gr}_U^0 = k_\tau^\times$ and $\mathrm{Gr}_{U^D}^0 = (k_\tau^+)^\times$. If we identify the multiplication $(1+x)(1+y) \equiv 1+x+y \pmod{\mathfrak{m}^{i+1}}$ with the addition $x+y$, then Gr_U^i, $\mathrm{Gr}_{U^D}^i$, and $\mathrm{Gr}_{U^N}^i$ are all vector spaces over k. The map N (resp. D) sends U_i to U_i^D (resp. to U_i^N) and induces a map $N_i : \mathrm{Gr}_U^i \to \mathrm{Gr}_{U^D}^i$ (resp. $D_i : \mathrm{Gr}_U^i \to \mathrm{Gr}_{U^N}^i$) for all $i \geq 0$.

Let us claim that $N_i : \mathrm{Gr}_U^i \to \mathrm{Gr}_{U^D}^i$ and $D_i : \mathrm{Gr}_U^i \to \mathrm{Gr}_{U^N}^i$ are surjective for all $i \geq 0$. If $i = 0$, then k_τ and k_τ^+ are finite fields. Suppose q is the cardinality of k_τ^+. Then $k_\tau^\times$ is the *cyclic group* of solutions to $x^{q^2-1} = 1$, and the involution $x \mapsto x^\star$ can be identified with $x \mapsto x^q$. From these we see that $N_0(x) = x^{1+q}$ and $D_0(x) = x^{1-q}$, and the assertion follows simply by counting: $(1+q)(1-q) = 1-q^2$. Now suppose $i \geq 1$. Let us first treat the case of N_i. By flatness of $\mathcal{O}_{F_\tau}$ and $\mathcal{O}_{F_\tau^+}$ over Λ, we have $\mathrm{Gr}_U^i \cong (\mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} \mathfrak{m}^i)/(\mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} \mathfrak{m}^{i+1}) \cong \mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$, and similarly $\mathrm{Gr}_{U^D}^i = \mathcal{O}_{F_\tau^+} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$. Hence we may reinterpret $N_i = \mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F_\tau^+}} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \mathrm{Tr}_{k_\tau/k_\tau^+} \underset{k}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$ as the *base change* of

$\mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F_\tau^+}}$ from Λ to k to $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Then the surjectivity of N_i onto $\mathrm{Gr}^i_{U^D}$ follows from the surjectivity of $\mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F_\tau^+}}$ (by assumption that $p \nmid \mathrm{Disc}$). On the other hand, consider any element $e \in \mathcal{O}_{F,R}$ given by Lemma 1.1.5.7 such that $e + e^\star = 1$. Let x be any element in $\mathrm{Gr}^i_{U^N}$, which by definition is an element in Gr^i_U such that $N_i(x) = x + x^\star = 0$. Then $D_i(ex) = ex - (ex)^\star = (e + e^\star)x = x$. This shows D_i is surjective onto $\mathrm{Gr}^i_{U^N}$. Hence the claim follows.

Since R is complete, we see that $U_0 = \mathcal{O}_{F_\tau,R}$, $U_0^D = \mathcal{O}_{F_\tau^+,R}$, and U_0^N are all complete with respect to their topologies defined by $\mathfrak{m}$. By successive approximation (as in for example [114, Ch. V, Lem. 2]), the surjectivity of N (resp. D) follows from the surjectivity of N_i (resp. D_i) for all $i \geq 0$, as desired. ☐

Corollary 1.1.5.10. *Assumptions as in Lemma 1.1.5.9, the anti-automorphism* $\maltese$ *of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *induced by* $\langle \cdot , \cdot \rangle$ *is an involution if and only if there is an element* $\delta \in \mathcal{O}_{F,R}^\times$ (*resp.* $\delta' \in \mathcal{O}_{F,R}^\times$) *such that the pairing* $\langle \cdot , \cdot \rangle'$ *defined by* $\langle x, y\rangle' := \langle x, \delta y\rangle$ (*resp. by* $\langle x, y\rangle' := \langle x, \delta' y\rangle$) *is symmetric* (*resp. skew-symmetric*).

Proof. If we take δ as in Lemma 1.1.5.9 (with $\epsilon = -1$) so that $\delta(\delta^\star)^{-1} = \gamma$, then $\langle x, \delta y\rangle = \langle \delta y, \gamma x\rangle = \langle y, \delta^\star \gamma x\rangle = \langle y, \delta x\rangle$ for any $x, y \in M$. If we take δ' so that $\delta'((\delta')^\star)^{-1} = -\gamma$, then $\langle x, \delta' y\rangle = -\langle \delta' y, \gamma x\rangle = -\langle y, (\delta')^\star \gamma x\rangle = -\langle y, \delta' x\rangle$ for any $x, y \in M$. ☐

Definition 1.1.5.11. *Assumptions as in Lemma 1.1.5.9, we say that an involution* $\maltese$ *of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *is* **of unitary type** *if there exists a perfect symmetric* $\mathcal{O}_R$*-pairing* $\langle \cdot , \cdot \rangle : M \times M \to R$ *inducing* $\maltese$.

Remark 1.1.5.12. By Corollary 1.1.5.10, we may replace symmetric pairings by skew-symmetric ones in Definition 1.1.5.11 without changing the class of involutions we consider.

To proceed further, let us record a consequence of Lemma 1.1.3.6 as follows:

Corollary 1.1.5.13. *Suppose* $p \nmid \mathrm{Disc}$, *and suppose* M *is any projective* $\mathcal{O}_R$*-module. Then any two involutions of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *inducing the same involution* * *on* $\mathcal{O}_R$ *is conjugate to each other by an element* $a \in \mathrm{End}_{\mathcal{O}_R}(M)$.

Proof. Let $\maltese_1$ and $\maltese_2$ be two such involutions. Then $\maltese_2 \circ (\maltese_1)^{-1}$ is an $\mathcal{O}_R$-automorphism of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. By Lemma 1.1.3.6, with $C = \mathcal{O}_R$, there is an invertible element $a \in \operatorname{End}_{\mathcal{O}_R}(M)$ such that $\operatorname{Int}(a) = {}^{\maltese_2} \circ ({}^{\maltese_1})^{-1}$, which means $\operatorname{Int}(a) \circ ({}^{\maltese_1}) = {}^{\maltese_2}$, as desired. □

Corollary 1.1.5.14. *Suppose* $p \nmid \mathrm{Disc}$*, and suppose* M *is any projective* $\mathcal{O}_R$*-module. Let* ${}^{\maltese_i}$*,* $i = 1, 2$*, be two involutions of* $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ *inducing the same involution* $\star$ *on* $\overline{\mathcal{O}}_R$*. Suppose* ${}^{\maltese_1}$ *is induced by some perfect* $\mathcal{O}_R$*-pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_1 : M \times M \to R$*. Then there is an invertible element* $a \in \operatorname{End}_{\mathcal{O}_R}(M)$ *such that* ${}^{\maltese_2}$ *is induced by the perfect* $\mathcal{O}_R$*-pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_2 := \langle\,\cdot\,,\,\cdot\,\rangle_1 \circ a$*.*

Proof. By Corollary 1.1.5.13, there is an invertible $a \in \operatorname{End}_{\mathcal{O}_R}(M)$ such that ${}^{\maltese_1} = \operatorname{Int}(a) \circ ({}^{\maltese_2})$. In this case, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_3$ defined by $\langle x, y\rangle_3 = \langle x, a(y)\rangle_1$ satisfies $\langle\,\cdot\,,\,\cdot\,\rangle_3^* = \langle\,\cdot\,,\,\cdot\,\rangle_1^* \circ a$, and hence the involution ${}^{\maltese_3}$ it induces on $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ satisfies ${}^{\maltese_1} = \operatorname{Int}(a) \circ ({}^{\maltese_3})$. That is, ${}^{\maltese_2} = {}^{\maltese_3}$. Then ${}^{\maltese_2}$ is induced by $\langle\,\cdot\,,\,\cdot\,\rangle_3$, as desired. □

Remark 1.1.5.15. Suppose $p \nmid \mathrm{Disc}$, and let $M_{\tau,R}$ be as in Lemma 1.1.3.4. If we denote the restriction of $\star$ to $\mathcal{O}_F$ by c, then $\operatorname{Hom}_R(M_{\tau,R}, R) \cong M_{\tau \circ c, R}$, because its $\mathcal{O}_{F,R}$-action is twisted by $\star$. This shows that for our purpose of study pairings we need to consider τ and τ' at the same time only when $\tau' = \tau \circ c$.

Lemma 1.1.5.16. *Suppose* $p \nmid \mathrm{Disc}$*, and suppose* M *is any projective* $\mathcal{O}_R$*-module that decomposes as* $M \cong M_{\tau,R}^{\oplus m_\tau}$ *as in Lemma 1.1.3.4. Suppose* $m_\tau = m_{\tau \circ c}$ *for all* τ*, then there exists a perfect* $\mathcal{O}_R$*-pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ *that induces an involution on* $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$*. (Conversely, the condition* $m_\tau = m_{\tau \circ c}$ *is automatic if there exists any perfect bilinear* $\mathcal{O}_R$*-pairing on* M*.)*

Proof. By base change from Λ to R, it suffices to construct the pairing when $R = \Lambda$. By Remark 1.1.5.15, by forming orthogonal direct sums as in Definition 1.1.4.15, we may construct the pairing by constructing a perfect pairing between M_τ and $M_{\tau \circ c}$ over Λ for each τ. Recall that $M_\tau = \mathcal{O}_{F_\tau}^{\oplus d_\tau}$, and that $\mathcal{O}_\tau = \operatorname{End}_{\mathcal{O}_{F_\tau}}(M_\tau) \cong \mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau})$. When $\tau = \tau \circ c$, we set $M_0 = M_\tau$, $\mathcal{O}_{F,0} = \mathcal{O}_{F_\tau}$, and $\mathcal{O}_0 \cong \operatorname{End}_{\mathcal{O}_{F_0}}(M_0) = \mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau})$. When $\tau \neq \tau \circ c$, we set $M_0 := M_\tau \oplus M_{\tau \circ c}$, $\mathcal{O}_{F,0} := \mathcal{O}_{F_\tau} \times \mathcal{O}_{F_{\tau \circ c}}$, and $\mathcal{O}_0 := \operatorname{End}_{\mathcal{O}_{F_0}}(M_0) \cong \mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau}) \times \mathrm{M}_{d_{\tau \circ c}}(\mathcal{O}_{F_{\tau \circ c}})$.

If $\tau = \tau \circ c$, then there is a natural involution $b \mapsto b'$ of $\mathcal{O}_0 = \mathcal{O}_\tau$ given simply by transposing and complex conjugating the matrix entries. If $\tau \neq \tau \circ c$, then the fact that $\star$ sends $\mathcal{O}_\tau$ to $\mathcal{O}_{\tau \circ c}$ forces $d_\tau = d_{\tau \circ c}$. Then there is a natural involution $b \mapsto b'$ of $\mathcal{O}_0 = \mathcal{O}_\tau \times \mathcal{O}_{\tau \circ c}$ given by switching the two factors. By Lemma 1.1.3.6 with $C = \mathcal{O}_{F,0}$, we see that $b' = ab^\star a^{-1}$ for some invertible $a \in \mathrm{End}_{\mathcal{O}_{F,0}}(M_0) = \mathcal{O}_0$.

Let us write elements x of M_0 in column forms as elements in $\mathcal{O}_{F,0}^{\oplus d_\tau}$, and consider its transposed conjugation ${}^t x^\star$. Then we define a pairing $\langle \,\cdot\,, \,\cdot\,\rangle_0 : M_0 \times M_0 \to \Lambda$ by $\langle x, y\rangle_0 := \mathrm{Tr}_{\mathcal{O}_{F,0}/\Lambda}({}^t x^\star a y)$, which then satisfies

$$\begin{aligned} \langle bx, y\rangle_0 &= \mathrm{Tr}_{\mathcal{O}_{F,0}/\Lambda}({}^t (bx)^\star a y) = \mathrm{Tr}_{\mathcal{O}_{F,0}/\Lambda}({}^t x^\star b' a y) \\ &= \mathrm{Tr}_{\mathcal{O}_{F,0}/\Lambda}({}^t x^\star a (a^{-1} b' a) y) = \mathrm{Tr}_{\mathcal{O}_{F,0}/\Lambda}({}^t x^\star a b^\star y) = \langle x, b^\star y\rangle_0. \end{aligned}$$

This pairing is perfect because a is invertible in $\mathcal{O}_\Lambda$. It induces an involution on $\mathcal{O}_0 \cong \mathrm{End}_{\mathcal{O}_{F,0}}(M_0)$ because it induces in particular the restriction of * to $\mathcal{O}_0$. $\square$

Combining Lemma 1.1.5.2, Corollary 1.1.5.14, and Lemma 1.1.5.16, we obtain:

Proposition 1.1.5.17. *Suppose* $p \nmid \mathrm{Disc}$, R *is a complete noetherian* R_0*-algebra, and suppose* M *is any projective* $\mathcal{O}_R$*-module. Consider the association of anti-automorphisms* $\maltese$ *of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *to weak isomorphism classes of perfect* $\mathcal{O}_R$*-pairings* $\langle \,\cdot\,, \,\cdot\,\rangle : M \times M \to R$ *on* M*.*

1. *Suppose that* $F = F^+$*. Then, for each* $\epsilon \in \mathcal{O}_{F,R}^\times$ *such that* $\epsilon^2 = 1$*, there is a well-defined bijection from weak equivalence classes containing at least one* ϵ*-symmetric (resp. alternating, resp. symmetric)* $\mathcal{O}_R$*-pairing to involutions of* ϵ*-symmetric type (resp. of symplectic type, resp. of orthogonal type).*

2. *Suppose that* $[F : F^+] = 2$*, and that the extension* F/F^+ *is split over* k *when* $\mathrm{char}(k) = p = 0$*. Then there is a well-defined bijection from weak equivalence classes containing at least one symmetric* $\mathcal{O}_R$*-pairing to involutions of unitary type. The same statement is true if we consider classes containing at least one skew-symmetric pairing instead, or classes containing at least one alternating pairing instead.*

In both cases, the images of the bijections exhaust all possible involutions of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *that induce* * *on* $\overline{\mathcal{O}}_R$*.*

1.2 Linear Algebraic Data

1.2.1 PEL-Type $\mathcal{O}$-Lattices

Let B be a finite-dimensional semisimple algebra over $\mathbb{Q}$ with positive involution * and center F. Here *positivity* of * means $\mathrm{Tr}_{B/\mathbb{Q}}(xx^\star) > 0$ for any $x \neq 0$ in B.

Let $\mathcal{O}$ be an order in B invariant under *. Then $\mathcal{O}$ has an involution given by the restriction of *. Let Disc be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$.

Let

$$\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times),$$

which is a free $\mathbb{Z}$-module of rank one. Any choice $\sqrt{-1}$ of a square-root of -1 in $\mathbb{C}$ determines an isomorphism

$$\frac{1}{\sqrt{-1}} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}, \tag{1.2.1.1}$$

but there is no canonical isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$. For any commutative $\mathbb{Z}$-algebra R, we denote by $R(1)$ the module $R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}(1)$.

Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle, \mathbb{Z}(1))$ be a symplectic $\mathcal{O}$-pairing (defined as in Definition 1.1.4.9) valued in $\mathbb{Z}(1)$. For simplicity, we shall suppress $\mathbb{Z}(1)$ from the notations and write simply $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$. Similarly, for any (commutative) $\mathbb{Z}$-algebra R, we write simply $(L \underset{\mathbb{Z}}{\otimes} R, \langle\,\cdot\,,\,\cdot\,\rangle)$, with the understanding that the pairing takes value in $R(1)$.

For reasons that will become clear in Section 1.3.4, let us introduce the following condition on the associated symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-lattices $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$:

Condition 1.2.1.2. *There exists an $\mathbb{R}$-algebra homomorphism*

$$h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$$

such that the following two conditions are satisfied:

1. *For any $z \in \mathbb{C}$ and $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we have*

$$\langle h(z)x, y\rangle = \langle x, h(z^c)y\rangle,$$

where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.

2. *For any choice of* $\sqrt{-1}$ *in* $\mathbb{C}$ *defining an isomorphism* $\mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$ *as in (1.2.1.1), the* $\mathbb{R}$*-bilinear pairing*

   $$\frac{1}{\sqrt{-1}} \circ \langle \cdot\,, h(\sqrt{-1})\cdot \rangle : (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$$

   is symmetric and positive-definite. (This last condition forces $\langle \cdot\,, \cdot \rangle$ *to be nondegenerate.)*

Definition 1.2.1.3. *A* **PEL-type $\mathcal{O}$-lattice** $(L, \langle \cdot\,, \cdot \rangle)$ *is a symplectic* $\mathcal{O}$*-lattice* $(L, \langle \cdot\,, \cdot \rangle)$ *such that* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)$ *satisfies Condition 1.2.1.2.*

Remark 1.2.1.4. What we have in mind is that the datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h)$ is an integral version of $(B, \star, V, \langle \cdot\,, \cdot \rangle, h)$ in [79] and related works.

Let us now define the reductive group that will be associated to the Shimura variety we will define:

Definition 1.2.1.5. *Let a PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot\,, \cdot \rangle)$ *be given as in Definition 1.2.1.3. For any* $\mathbb{Z}$*-algebra* R*, set*

$$\mathrm{G}(R) := \left\{ (g, r) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \langle gx, gy \rangle = r \langle x, y \rangle, \forall x, y \in L \right\}.$$

In other words, $\mathrm{G}(R)$ *is the group of symplectic automorphisms of* $L \underset{\mathbb{Z}}{\otimes} R$ *(defined as in Definition 1.1.4.11). For any* $\mathbb{Z}$*-algebra map* $R \to R'$*, we have by definition a natural map* $\mathrm{G}(R) \to \mathrm{G}(R')$*. This defines in particular a group functor over* $\mathbb{Q}$*, which is a closed subgroup of the product of the general linear group over the* $\mathbb{Q}$*-vector space* $V := L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *with* $\mathbf{G}_{\mathrm{m}}$ *over* $\mathbb{Q}$*, and hence a linear algebraic group over* $\mathbb{Q}$*.*

The projection to the second fact $(g, r) \mapsto r$ *defines a morphism* $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$*, which we call the* **similitude character***. For simplicity, we shall often denote elements* (g, r) *in* G *by simply* g*, and denote by* $\nu(g)$ *the value of* r *when we need it. (If* $L \neq \{0\}$*, then the value of* r *is uniquely determined by* g*. Hence there is little that we lose when suppressing* r *from the notations. However, this is indeed an abuse of notation when* $L = \{0\}$*, in which case we have* $\mathrm{G} = \mathbf{G}_{\mathrm{m}}$*.)*

Remark 1.2.1.6. For a general non-flat $\mathbb{Z}$-algebra R, the pairing induced by $\langle \cdot\,, \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} R$ is not necessarily nondegenerate. This suggests that G is not necessarily a smooth functor over the whole base $\mathbb{Z}$.

Remark 1.2.1.7. This gives the definitions for $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^{\infty,\Box})$, $\mathrm{G}(\mathbb{A}^{\infty})$, $\mathrm{G}(\mathbb{R})$, $\mathrm{G}(\mathbb{A}^{\Box})$, $\mathrm{G}(\mathbb{A})$, $\mathrm{G}(\mathbb{Z})$, $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$, $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, $\mathrm{G}(\hat{\mathbb{Z}})$,

$$\Gamma(n) := \ker(\mathrm{G}(\mathbb{Z}) \to \mathrm{G}(\mathbb{Z}/n\mathbb{Z})),$$

$$\mathcal{U}^{\Box}(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}^{\Box}) \to \mathrm{G}(\hat{\mathbb{Z}}^{\Box}/n\hat{\mathbb{Z}}^{\Box}) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$$

for any n prime-to-$\Box$, and

$$\mathcal{U}(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z})).$$

Remark 1.2.1.8. As explained in [79, §5], all the simple factors of $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$ are either of type A, type C, or type D. Moreover, if B is simple, then all factors of $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$ are of the same type. It is not necessary to know this classification over $\mathbb{C}$ in our work. Nevertheless, we will obtain it later as a byproduct of our investigations. (See Proposition 1.2.3.11 below.)

Remark 1.2.1.9. Only $\mathrm{G}(\mathbb{A}^{\Box})$ will be relevant to the moduli problems that we will define. In particular, we have to face the so-called *failure of Hasse's principle*, which means we cannot always identify exactly what the group G is from the points of $\mathrm{G}(\mathbb{A}^{\Box})$, or rather $\mathrm{G}(\mathbb{A})$ (as the difference between $\mathrm{G}(\mathbb{A}^{\Box})$ and $\mathrm{G}(\mathbb{A})$ is immaterial by Proposition 1.2.3.11 below). See Remark 1.4.3.11 below for its consequence on the definition of moduli problems.

Now let us take a closer look at the pairs $(B,^{\star})$ that we consider. As in Section 1.1.2, the $\mathbb{Q}$-algebra F decomposes into a product

$$F = \prod_{[\tau]:F\hookrightarrow\mathbb{Q}_{[\tau]}} F_{[\tau]}$$

of fields finite-dimensional over $\mathbb{Q}$ giving the Galois orbits of embeddings $F \hookrightarrow \mathbb{Q}^{\mathrm{sep}}$, and we obtain accordingly a decomposition

$$B = \prod_{[\tau]:F\hookrightarrow\mathbb{Q}_{[\tau]}} B_{[\tau]}, \tag{1.2.1.10}$$

where B_{τ} is the simple factor consisting of elements in B commuting with F_{τ}.

Lemma 1.2.1.11. *Any simple factor of B is mapped by * to itself.*

Proof. If any of the simple factor of B is mapped to a different simple factor, then any element x in the former simple factor must satisfy $xx^{\star} = 0$, which contradicts the positivity condition that $\mathrm{Tr}_{B/\mathbb{Q}}(xx^{\star}) > 0$. □

Remark 1.2.1.12. It is clear that modules over semisimple algebras can be decomposed as a direct sum of modules over its simple factors. By Lemma 1.2.1.11, we see that we can decompose positive involutions, and hence symplectic modules (in a way compatible with the involutions) into products over simple factors. Note however that the group of similitudes defined by a general semisimple datum (as in Definition 1.2.1.5) is only a subgroup of a product of groups of similitudes defined by simple data, because the similitude factors have to be identical in all factors.

By Wedderburn's Structure Theorem (see for example [112, Thm. 7.4]), any finite-dimensional simple algebra B over $\mathbb{Q}$ is of the form $\mathrm{M}_k(D)$ for some integer k and some division algebra D over $\mathbb{Q}$. Let us record the fundamental classification of division algebras with positive involutions exhibited in [99, §21], originally due to Albert:

Proposition 1.2.1.13 (Albert)**.** *Suppose D is a* **division algebra** *with a positive involution $\diamond$. Then the elements in the center F invariant under $^\diamond$ form a totally real extension F^+ of $\mathbb{Q}$, and there are only four possibilities:*

1. *$D = F = F^+$ is totally real.*
2. *$F = F^+$ is totally real, and $D \underset{F,\tau}{\otimes} \mathbb{R}$ is isomorphic to $\mathrm{M}_2(\mathbb{R})$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$, with the involution $^\diamond$ given by conjugating the natural involution $x \mapsto x' := \mathrm{Tr}_{D/F}(x) - x$ by some element $a \in D$ such that $a^\diamond = -a$. In this case $a^2 = -aa^\diamond$ is totally negative in F.*
3. *$F = F^+$ is totally real, and $D \underset{F,\tau}{\otimes} \mathbb{R}$ is isomorphic to the real Hamilton quaternion algebra $\mathbb{H}$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$, with the natural involution $^\diamond$ given by $x \mapsto x^\diamond := \mathrm{Tr}_{D/F}(x) - x$.*
4. *F is totally imaginary over the totally real F^+, with complex conjugation c, and D satisfies the condition that if $\upsilon = \upsilon \circ c$ then $\mathrm{inv}_\upsilon(D) = 0$, and if $\upsilon \neq \upsilon \circ c$ then $\mathrm{inv}_\upsilon(D) + \mathrm{inv}_{\upsilon\circ c}(D) = 0$.*

Some rough analogue of Proposition 1.2.1.13 for simple algebras (which nevertheless suffices for our purpose) can be given as follows:

Proposition 1.2.1.14. *Suppose B is a* **simple algebra** *with a positive involution *. Then the restrictions of * and $^\diamond$ to F are the same, and the elements in F invariant under * form a totally real extension F^+ of $\mathbb{Q}$. Moreover, there are only four possibilities of B:*

1. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(F)$ for some integer $k \geq 1$, with the involution * given by conjugating the natural involution $x \mapsto {}^t x$ by some element $a \in B$ such that ${}^t a = a$ and such that a is totally positive in the sense that $a = {}^t kk$ for some element $k \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$.*

2. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(D)$ for some integer $k \geq 1$ and some quaternion division algebra D over $\mathbb{Q}$, with $D \underset{F,\tau}{\otimes} \mathbb{R}$ isomorphic to $\mathrm{M}_2(\mathbb{R})$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$. In this case, $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is a product of copies of $\mathrm{M}_{2k}(\mathbb{R})$ indexed by embeddings $\tau : F \hookrightarrow \mathbb{R}$, and the involution * is given by conjugating $x \mapsto {}^t x$ by some element $b \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ that is totally positive in the sense that $b = {}^t kk$ for some element $k \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$.*

3. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(D)$ for some integer $k \geq 1$ and some quaternion division algebra D over $\mathbb{Q}$, with $D \underset{F,\tau}{\otimes} \mathbb{R}$ isomorphic to the real Hamilton quaternion algebra $\mathbb{H}$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$. Let us denote by ${}^\diamond x$ the standard involution $x \mapsto \mathrm{Tr}_{D/F}\, x - x$ on D. Then $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is a product of copies of $\mathrm{M}_k(\mathbb{H})$ indexed by embeddings $\tau : F \hookrightarrow \mathbb{R}$, and the involution * is given by conjugating $x \mapsto {}^t x^\diamond$ by some element $a \in B$ such that ${}^t a^\diamond = a$ and such that a is totally positive in the sense that $a = {}^t k^\diamond k$ for some element $k \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{H})$.*

4. *F is totally imaginary over the totally real F^+.*

Note that Proposition 1.2.1.14 is not as comprehensive as Proposition 1.2.1.13 when we specialize to the case that B is a division algebra.

Proof of Proposition 1.2.1.14. Following the classification for division algebras with positive involutions in [99, §21], the case $F = F^+$ implies (for general simple algebra) that B is isomorphic to its opposite algebra in the Brauer group over F. This shows that $B = \mathrm{M}_k(D)$ for some division algebra over F that is either just F or quaternion over F. For the justification of the statements about the involutions over $\mathbb{R}$, combine Lemma 1.1.3.6 with the classification of real positive involutions in [79, §2], especially [79, Lem. 2.11]. $\square$

Combining Proposition 1.2.1.14 with Lemma 1.2.1.11, we see all the possibilities of semisimple algebras with positive involutions.

Although we claimed in Remark 1.2.1.8 that it is not necessary to know the classification of simple factors of $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$, it is nevertheless convenient to introduce the following terminologies based on the classification:

Definition 1.2.1.15. *Let B be a finite-dimensional semisimple over $\mathbb{Q}$ with a positive involution $\star$. Let $B \cong \prod\limits_{[\tau]:F\to\mathbb{Q}_{[\tau]}} B_{[\tau]}$ be a decomposition of B into simple factors as in (1.2.1.10).*

1. *We say that B involves simple factors of* **type** C *if, for some morphism $\tau : F \to \mathbb{R}$, we have an isomorphism $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{R})$ for some integer $k \geq 1$ respecting their positive involutions. In this case, we say that the factor $B_{[\tau]}$ with $[\tau] : F \to \mathbb{Q}_{[\tau]}$ determined by $\tau : F \to \mathbb{R}$ is of type* C.
2. *We say that B involves simple factors of* **type** D *if, for some morphism $\tau : F \to \mathbb{R}$, we have an isomorphism $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{H})$ for some integer $k \geq 1$ respecting their positive involutions. In this case, we say that the factor $B_{[\tau]}$ with $[\tau] : F \to \mathbb{Q}_{[\tau]}$ determined by $\tau : F \to \mathbb{R}$ is of type* D.
3. *We say that B involves simple factors of* **type** A *if, for some morphism $\tau : F \to \mathbb{C}$ such that $c \circ \tau \neq \tau$, (where $c : \mathbb{C} \to \mathbb{C}$ is the complex conjugation,) we have an isomorphism $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{C})$ for some integer $k \geq 1$ respecting their positive involutions. In this case, we say that the factor $B_{[\tau]}$ with $[\tau] : F \to \mathbb{Q}_{[\tau]}$ determined by $\tau : F \to \mathbb{C}$ is of type* A.

Remark 1.2.1.16. We shall see in Proposition 1.2.3.11 below that the condition that B involves simple factors of type C (resp. type D, resp. type A) is equivalent to the condition that the reductive group G defined in Definition 1.2.1.5 by $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ satisfy the condition that $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$ has a simple factor of type C (resp. type D, resp. type A).

For later references we shall define an invariant $\mathrm{I}_{\mathrm{bad}}$ that describes the so-called *bad prime* phenomenon related to a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ at the prime number 2.

Definition 1.2.1.17. *If B involves simple factors of type* D *(defined as in Definition 1.2.1.15), then we set $\mathrm{I}_{\mathrm{bad}} := 2$. Otherwise we set simply $\mathrm{I}_{\mathrm{bad}} := 1$.*

This definition will be justified by the computations in Sections 1.2.2, 1.2.3, and 1.2.5, which will be essential to the proofs of Theorem 1.4.1.12 and our main result Theorem 6.4.1.1.

Since B is semisimple, it is possible to define the notion of *multi-rank* for $\mathcal{O}$-lattices, even if $\mathcal{O}$-lattices are not necessarily projective: By Lemma 1.1.2.3, we know that each simple factor $B_{[\tau]}$ of B in (1.2.1.10) has only one unique irreducible module $W_{[\tau]}$. As a result, it makes sense to classify finite-dimensional B-modules W over $\mathbb{Q}$ by its multi-rank, namely the integers $(m_{[\tau]})$ such that

$$W \cong \underset{[\tau]:F\to\mathbb{Q}_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus m_{[\tau]}}. \tag{1.2.1.18}$$

Definition 1.2.1.19. *Let R be a noetherian $\mathbb{Z}$-algebra. An $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$-module is called* **integrable** *if it is isomorphic to $M\underset{\mathbb{Z}}{\otimes} R$ for some $\mathcal{O}$-***lattice** *M.*

Definition 1.2.1.20. *The* **multi-rank** *$(m_{[\tau]})$ of an $\mathcal{O}$-lattice M is the multi-rank of its induced B-module $W := M\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ (as explained above). If R is a commutative noetherian $\mathbb{Z}$-algebra, then the multi-rank of an integrable $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$-module, which is by definition isomorphic to $M\underset{\mathbb{Z}}{\otimes} R$ for some $\mathcal{O}$-lattice M, is defined as the multi-rank of M. (This definition is independent of the choice of the $\mathcal{O}$-lattice M, as the integers are essentially determined by $M\underset{\mathbb{Z}}{\otimes} k$)*

Remark 1.2.1.21. If B is simple, then the multi-rank of integrable $\mathcal{O}$-lattices M are given by a single integer, which we call the *rank* of M.

Suppose now that R is a noetherian complete local ring with residue field k, and let $p := \mathrm{char}(k)$. For simplicity, let us temporarily adopt Convention 1.1.3.3, with $R_0 = \mathbb{Z}$.

Suppose $p \nmid \mathrm{Disc}$. As in Lemma 1.1.3.4, if we let M_τ be defined as in Lemma 1.1.3.4, then any projective $\mathcal{O}_R$-module M admit a decomposition

$$M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau} \tag{1.2.1.22}$$

similar to (1.2.1.18) above. The two decompositions (1.2.1.18) and (1.2.1.22) are indexed respectively by morphisms $[\tau] : F \to \mathbb{Q}_{[\tau]}$ and $\tau : F \to \mathrm{Frac}(\Lambda)_\tau$, which should be interpreted respectively as Galois orbits of morphisms $F \to \mathbb{Q}^{\mathrm{sep}}$ and $F \to \mathrm{Frac}(\Lambda)^{\mathrm{sep}}$. Any orbit τ in the latter case determines a unique well-defined orbit $[\tau]$ in the former case. Let us write this determination symbolically as $\tau \in [\tau]$.

Lemma 1.2.1.23. *Suppose* $p \nmid \mathrm{Disc}$. *Let* M_τ *be defined as in Lemma 1.1.3.4, let*

$$M_{[\tau],R} := \underset{\tau\in[\tau]}{\oplus} M_{\tau,R},$$

and let

$$\mathcal{O}_{[\tau],R} := \mathrm{End}_{\mathcal{O}_{F,R}}(M_{[\tau],R}).$$

Then

$$\mathcal{O}_R \cong \prod_{[\tau]} \mathcal{O}_{[\tau],R}, \tag{1.2.1.24}$$

and there exists an element $x_{[\tau]}$ *in* $M_{[\tau],R}$ *such that* $M_{[\tau],R} = (\mathcal{O}_{[\tau],R})x_{[\tau]}$.

Proof. It suffices to treat the universal case $R = \Lambda$, where $\Lambda = k$ when $p = 0$, and where $\Lambda = W(k)$ when $p > 0$. Then the lemma is clear from the explicit realization of $\mathcal{O}_{F,\Lambda}$ as the product of $\mathcal{O}_{F_\tau}$, M_τ as $\mathcal{O}_{F_\tau}^{\oplus d_\tau}$, $\mathcal{O}_\tau$ as $\mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau})$, and $\mathcal{O}_\Lambda$ as the product of $\mathcal{O}_\tau$. For each fixed $[\tau]$, we may take an explicit choice of $x_{[\tau]} = (x_{\tau'})$ to be with $x_{\tau'} = (1,0,0,\ldots,0)$ for any $\tau' \in [\tau]$, and $x_{\tau'} = 0$ for any $\tau' \notin [\tau]$. ☐

Lemma 1.2.1.25. *Suppose* $p \nmid \mathrm{Disc}$. *Then an* $\mathcal{O}_R$*-module* M *is integrable of multi-rank* $(m_{[\tau]})$ *if and only if* M *is projective of multi-rank* (m_τ) *where* $m_\tau = m_{\tau'}$ *for any* τ *and* τ' *that determine the same orbit* $[\tau] = [\tau']$*, if and only if* M *is the direct sum of copies of modules of the form of* $M_{[\tau],R}$. *Moreover, this condition can be checked modulo the maximal ideal of* R.

Proof. The first statement follows simply from Lemma 1.1.3.4 and the definition of integrability. Since R is local, the statement that this condition can be checked modulo the maximal ideal of R follows from Lemma 1.1.3.1. ☐

1.2.2 Torsion of Universal Domains

Let us continue with the setting in Section 1.2.1, in which we have associated an invariant $\mathrm{I}_{\mathrm{bad}}$ to the PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ in Definition 1.2.1.17, which is either 1 or 2 depending on whether B involves simple factors of type D (defined as in Definition 1.2.1.15).

Proposition 1.2.2.1. *Let* k *be either a field of characteristic* $p = 0$ *or a finite field of characteristic* $p > 0$*, such that* $p \nmid \mathrm{GCD}(2, \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc})$. *Let* $\Lambda = k$ *when* $p = 0$*, and let* $\Lambda = W(k)$ *when* $p > 0$. *Let* R *be a noetherian local*

Λ-algebra with residue field k. Let $x \in \mathcal{O}_R := \mathcal{O} \underset{\mathbb{Z}}{\otimes} R$ be any element such that $x = -x^\star$. When B involves simple factors of type C *(defined as in Definition 1.2.1.15), we assume moreover that* 2 *is not a zero-divisor in R. Then x is equal to $z - z^\star$ for some $z \in \mathcal{O}_R$.*

Proof. Throughout the proof, the subscript Λ will mean tensor product with Λ, and the subscript R will mean tensor product with R.

If $p \neq 2$, then there is an element e in Λ such that $2e = 1$. Then by taking $z = ex$, we have $x = 2z$ and $z = -z^\star$, and hence $x = 2z = z - z^\star$, as desired.

If $p = 2$, then the assumption is that $p \nmid \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}$. By Lemmas 1.2.1.11 and 1.2.1.23, we have a decomposition

$$\mathcal{O}_\Lambda \cong \prod_{[\tau]} \mathcal{O}_{[\tau],\Lambda}, \tag{1.2.2.2}$$

and the involution * maps each factor $\mathcal{O}_{[\tau],\Lambda}$ in the decomposition (1.2.2.2) into itself, which induces the corresponding situation over R (as in (1.2.1.24)). Therefore it suffices to prove the proposition for each factor $\mathcal{O}_{[\tau],\Lambda}$. For simplicity of notations, let us assume that B is simple. According to Proposition 1.2.1.14, we have four cases of the simple of B with its positive involution.

Suppose $F = F^+$ is a totally real field over $\mathbb{Q}$. Then B is either of type C or of type D (defined as in Definition 1.2.1.15). Since $p = 2 \nmid \mathrm{I}_{\mathrm{bad}}$, we see that B is of type C, which implies that 2 is not a zero divisor in R by assumption.

Since $p \nmid \mathrm{Disc}$, we may assume that $\mathcal{O}_\Lambda \cong \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$. There is another involution of $\mathcal{O}_\Lambda$ given by $x \mapsto {}^t x$. By Lemma 1.1.3.6, there exists an invertible element $c \in \mathcal{O}_\Lambda$ such that $x^\star = c\,{}^t x c^{-1}$ for any $x \in \mathcal{O}_\Lambda$. Then, as in [99, §21, p. 195], since $x = (x^\star)^\star = c\,{}^t(c\,{}^t x c^{-1})c^{-1} = c\,{}^t c^{-1} x\,{}^t c c^{-1}$ for any $x \in \mathcal{O}_\Lambda$, we must be have $c\,{}^t c^{-1} = e \in \mathcal{O}_{F,\Lambda}$ for some e. Then $c = e\,{}^t c = e^2 c$ implies $e = \pm 1$.

If we are in the case that $B \cong \mathrm{M}_k(F)$ for some integer k, with involution given by $x^\star = a\,{}^t x a^{-1}$ for some $a \in B$ such that ${}^t a = a$. Comparing with $x^\star = c\,{}^t x c^{-1}$ in $B \underset{\mathbb{Q}}{\otimes} \Lambda = \mathcal{O}_\Lambda \underset{\Lambda}{\otimes} \mathrm{Frac}(\Lambda)$, we obtain $bc^{-1} \in F \underset{\mathbb{Q}}{\otimes} \Lambda = \mathcal{O}_{F,\Lambda} \underset{\Lambda}{\otimes} \mathrm{Frac}(\Lambda)$, and hence ${}^t a = a$ implies ${}^t c = c$.

If we are in the case that $B \cong \mathrm{M}_k(D)$ for some quaternion division algebra D over F, then $D \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$, because B is of type C. Over $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{R})$, the involution * is given by conjugating

$x \mapsto {}^t x$ by some element $b \in B \underset{F,\tau}{\otimes} \mathbb{R}$ such that ${}^t b = b$. Let us work in any field containing both $\mathbb{R}$ and Λ such as $\mathbb{C}$, and let us assume that there is an isomorphism $B \underset{F,\tau}{\otimes} \mathbb{C} \cong \mathrm{M}_k(\mathbb{C})$ compatible with the existing ones over Λ and over $\mathbb{R}$. (This is always possible by taking some $\tau : F \hookrightarrow \mathbb{C}$ extending both $\tau : F \hookrightarrow \mathbb{R}$ and $\tau : F \hookrightarrow \mathrm{Frac}(\Lambda)_\tau$. Beware of the abuse of notations here.) Then the comparison between the relations $x^\star = b\,{}^t x b^{-1}$ and $x^\star = c\,{}^t x c^{-1}$ shows that ${}^t c = c$.

Now that we know $x^\star = c\,{}^t x c^{-1}$ for some $c = {}^t c$, the relation $x = -x^\star = -c\,{}^t x c^{-1}$ implies that $xc = -c\,{}^t x = -{}^t c\,{}^t x = -{}^t(xc)$. In this case, all diagonal entries of xc is zero, because we assume that 2 is not a zero divisor in R. Set y to be the element in $\mathcal{O}_R$ with only the upper-triangle entries of xc, so that $xc = y - {}^t y$. Then $x = (y - {}^t y)c^{-1} = yc^{-1} - c(c^{-1}\,{}^t y)c^{-1} = yc^{-1} - c\,{}^t(yc^{-1})c^{-1} = yc^{-1} - (yc^{-1})^\star = z - z^\star$ for $z = yc^{-1}$, as desired.

Finally, suppose $[F : F^+] = 2$. By Lemma 1.1.5.7, there exists some element $e \in \mathcal{O}_{F,\Lambda}$ such that $\mathrm{Tr}_{F_\Lambda/F_\Lambda^+}(e) = e + e^\star = 1$. Note that since $p \nmid \mathrm{Disc}$, $\mathcal{O}_\Lambda$ contains $\mathcal{O}_{F,\Lambda}$ in its center. Therefore $\mathcal{O}_\Lambda$ contains e. If there is any $x \in \mathcal{O}_R$ such that $x = -x^\star$. Then $x = (e+e^\star)x = ex - x^\star e^\star = ex - (ex)^\star$. Hence we may simply take $z = ex$. ☐

Remark 1.2.2.3. Let us explain why the case $p = 2 | \mathrm{I}_{\mathrm{bad}}$ has to be excluded when B involves simple factors of type D, in the simplest case when B is a division algebra. In this case, B is quaternion over its center F, a totally field over $\mathbb{Q}$, and $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathbb{H}$ for any embedding $\tau : F \hookrightarrow \mathbb{R}$. Even when $p \nmid \mathrm{Disc}$, in which case $\mathcal{O}_\Lambda$ is isomorphic to $\mathrm{M}_2(\mathcal{O}_{F,\Lambda})$, the image of $z \mapsto z - z^\star$ sends any matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \alpha-\delta & 2\beta \\ 2\gamma & \delta-\alpha \end{smallmatrix}\right)$. On the other hand, elements x satisfying $x = -x^\star$ are of the form $\left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & -\alpha' \end{smallmatrix}\right)$. The same is true on $\mathcal{O}_R \cong \mathrm{M}_2(\mathcal{O}_{F,R})$ by base change from Λ to R. Since $2\mathcal{O}_{F,R}$ is always strictly smaller than $\mathcal{O}_{F,R}$ when $p = 2$, we see that there must be some element $x \in \mathcal{O}_R$ such that $x = -x^\star$ but $x \neq z - z^\star$ for any $z \in \mathcal{O}_R$.

Now suppose that Λ is either $\mathbb{Z}$, or a field of characteristic zero, or $W(k)$ for some finite field k of characteristic $p > 0$. Suppose we have two $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$-lattices L_1 and L_2 and an embedding $\varrho : L_1 \hookrightarrow L_2$ with a cokernel of finite cardinality. Let us denote by $[L_2 : \varrho(L_1)]$ the cardinality of this cokernel. Let $\epsilon = 1$ or 0. Let us define a finitely generated Λ-module $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2)$

by

$$\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2) := (L_1 \underset{\Lambda}{\otimes} L_2) / \begin{pmatrix} x \otimes \varrho(y) - y \otimes \varrho(x) \\ (bx) \otimes z - x \otimes (b^{\star} z) \end{pmatrix}_{\substack{x,y \in L_1, \\ z \in L_2, b \in \mathcal{O}}}$$

when $\epsilon = 1$ and

$$\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2) := (L_1 \underset{\Lambda}{\otimes} L_2) / \begin{pmatrix} x \otimes \varrho(x) \\ (bx) \otimes z - x \otimes (b^{\star} z) \end{pmatrix}_{\substack{x,y \in L_1, \\ z \in L_2, b \in \mathcal{O}}}$$

when $\epsilon = 0$.

Proposition 1.2.2.4. *Let assumptions on Λ be as above. When $\epsilon = 1$ (resp. $\epsilon = 0$), the maps from the finitely generated Λ-module $\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2)$ to Λ-modules L_3 parameterizes all symmetric (resp. alternating) Λ-bilinear pairings*

$$\langle \cdot , \cdot \rangle : L_1 \times L_2 \to L_3$$

such that

$$\langle bx, y \rangle = \langle x, b^{\star} y \rangle$$

for any $x \in L_1$, $y \in L_2$, and $b \in \mathcal{O}$. This $\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2)$ is the so-called **universal domain** *of such pairings. Then the Λ-module $\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2)$ can have p-torsion only when Λ is not a field, and when $p | \mathrm{I}^{\epsilon}_{\mathrm{bad}} \operatorname{Disc}[L_2 : \varrho(L_1)]$.*

Here $\mathrm{I}^{\epsilon}_{\mathrm{bad}}$ is given its literal meaning: $\mathrm{I}^{\epsilon}_{\mathrm{bad}} = \mathrm{I}_{\mathrm{bad}}$ when $\epsilon = 1$ and $\mathrm{I}^{\epsilon}_{\mathrm{bad}} = 1$ when $\epsilon = 0$.

Proof. The universality of $\mathbf{Sym}^{\epsilon}_{\varrho}(L_1, L_2)$ is clear from its definition. The statement is also clear if Λ is a field. If $\Lambda = \mathbb{Z}$, then it suffices to check that if $p \nmid \mathrm{I}^{\epsilon}_{\mathrm{bad}} \operatorname{Disc}[L_2 : \varrho(L_1)]$, then $\mathbf{Sym}_{\varrho}(L_1, L_2) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is $\mathbb{Z}_p$-torsion free. This can be checked by taking the p-adic completions of various lattices, and therefore the problem is reduced to the case $\Lambda = W(k)$ with $k = \mathbb{F}_p$. Hence it suffices to treat the remaining case $\Lambda = W(k)$ for some finite field k of characteristic $p > 0$, such that $p \nmid \mathrm{I}^{\epsilon}_{\mathrm{bad}} \operatorname{Disc}[L_2 : \varrho(L_1)]$. We shall denote by a subscript Λ whenever we form a tensor product with Λ.

Since $p \nmid [L_2 : \varrho(L_1)]$, we have an isomorphism

$$\varrho_{\Lambda} : L_{1,\Lambda} \xrightarrow{\sim} L_{2,\Lambda}.$$

Therefore we may set $L_\Lambda := L_{1,\Lambda} \overset{\varrho_\Lambda}{\tilde{\to}} L_{2,\Lambda}$ and consider the Λ-module $\mathbf{Sym}^\epsilon(L_\Lambda)$ defined by

$$\mathbf{Sym}^\epsilon(L_\Lambda) := (L_\Lambda \underset{\Lambda}{\otimes} L_\Lambda) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_\Lambda, b \in \mathcal{O}_\Lambda}$$

when $\epsilon = 1$, and

$$\mathbf{Sym}^\epsilon(L_\Lambda) := (L_\Lambda \underset{\Lambda}{\otimes} L_\Lambda) / \begin{pmatrix} x \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_\Lambda, b \in \mathcal{O}_\Lambda}$$

when $\epsilon = 0$.

The $\mathcal{O}_\Lambda$-lattice L_Λ, being projective by Proposition 1.1.1.20, factors as the sum of copies M_τ by Lemma 1.1.3.4. Let us write this as

$$L_\Lambda = \underset{i}{\oplus} M_i,$$

where each M_i is isomorphic to some M_τ. Then we see that the Λ-span of the images of $(b_1 x_i) \otimes (b_2 x_j)$ is the whole module $\mathbf{Sym}(L_p)$, for all possible $x_i \in M_i$, $x_j \in M_j$, and $b_1, b_2 \in \mathcal{O}_\Lambda$. Note that the first relation shows that we only need those $i \leq j$, and the second relation shows that we can transform any $(b_1 x_i) \otimes (b_2 x_j)$ to $(b_2^\star b_1 x_i) \otimes x_j$, and hence we only need the Λ-span of elements of the form $(bx_i) \otimes x_j$, for all possible $b \in \mathcal{O}_p$. Let us denote by c the restriction of * to $\mathcal{O}_{F,\Lambda}$. If $M_i \cong M_\tau$ and $M_j \cong M_{\tau'}$ but $\tau' \neq \tau \circ c$, then $(b_1 x_i) \otimes (b_2 x_j) = (b_2^\star b_1 x_i) \otimes x_j = 0$ for $b_1 \in \mathcal{O}_\tau$ and $b_2 \in \mathcal{O}_{\tau'}$ shows that $M_i \otimes M_j = 0$. On the other hand, if $\tau' = \tau \circ c$, then $M_i \otimes M_j$ can be identified with the Λ-span of $(b_1 x_i) \otimes (b_2 x_j)$, for various $b_1, b_2 \in \mathcal{O}_\tau$. Writing either $\mathcal{O}_{ij} = 0$ or $\mathcal{O}_{ij} := \mathcal{O}_\tau$ in these two cases, we have arrived at a direct sum of Λ-modules

$$\mathbf{Sym}^\epsilon(L_\Lambda) \cong \left[\underset{i<j}{\oplus} \mathcal{O}_{ij} \right] \oplus \left[\underset{i=j}{\oplus} \mathbf{Sym}^\epsilon(\mathcal{O}_{ij}) \right]$$

labeled by $i \leq j$, in which

$$\mathbf{Sym}^\epsilon(\mathcal{O}_{ij}) := \mathcal{O}_{ij} / (b - b^\star)_{b \in \mathcal{O}_{ij}}$$

when $\epsilon = 1$, and

$$\mathbf{Sym}^\epsilon(\mathcal{O}_{ij}) := \mathcal{O}_{ij} / (b)_{b \in \mathcal{O}_{ij}} = 0$$

when $\epsilon = 0$. These $\mathbf{Sym}^\epsilon(\mathcal{O}_{ij})$, defined when $i = j$, is the only possible source of torsion of $\mathbf{Sym}^\epsilon(L_\Lambda)$. This already completes the proof when $\epsilon = 0$.

Assume now that $\epsilon = 1$. Since each $\mathbf{Sym}^\epsilon(\mathcal{O}_{ij})$ is either 0 or a direct factor of

$$\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda) := \mathcal{O}_\Lambda/(b - b^\star)_{b\in\mathcal{O}_\Lambda},$$

it suffices to show that $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$ is torsion-free. If $x \in \mathcal{O}_\Lambda$ is mapped to any torsion element in $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$, then $rx = y - y^\star$ for some $y \in \mathcal{O}_\Lambda$ and some nonzero r in Λ. This implies $rx = -rx^\star$, and hence $x = -x^\star$ in $\mathcal{O}_\Lambda$. By Proposition 1.2.2.1, there is some element z in $\mathcal{O}_\Lambda$ such that $x = z - z^\star$. This means x is also mapped to 0 in $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$. This shows that $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$ is torsion-free and completes the proof. □

1.2.3 Self-Dual Symplectic Modules

Let k and Λ be either of the following two types:

1. k is a field of characteristic $p = 0$, and $\Lambda = k$.
2. k is a finite field of characteristic $p > 0$, and $\Lambda = W(k)$.

Suppose $p \nmid \mathrm{Disc}$. Let R be a complete noetherian local Λ-algebra. Throughout this section, the subscript Λ will mean tensor product with Λ, and the subscripts of R will have the two possible meaning as in Convention 1.1.3.3.

Let M be a projective $\mathcal{O}_R$-module of multi-rank (m_τ) (defined as in Definition 1.1.3.5), and let M_0 be the projective $\mathcal{O}_R$-module with multi-rank (1). Namely, $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ and $M_0 \cong \underset{\tau}{\oplus} M_{\tau,R}$ as in Lemma 1.1.3.4. If we replace $\mathcal{O}_R$ by $\mathcal{O}_{F,R}$ in Lemma 1.1.3.4, then projective $\mathcal{O}_{F,R}$-modules N also admits decompositions as $N \cong \mathcal{O}_{F,R}^{\oplus n_\tau}$, and it is straightforward that $M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N \cong M_\tau^{\oplus n_\tau}$ in this case. Conversely, it is straightforward to have the following:

Lemma 1.2.3.1. *Assumptions on R, M, M_0 as above, there is a unique projective $\mathcal{O}_{F,R}$-module N such that $M \cong M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N$, with its $\mathcal{O}_{F,R}$-action given by the first tensor factor alone. Explicitly, $N = \mathcal{O}_{F_\tau,R}^{\oplus m_\tau}$ if (m_τ) is the multi-rank of M. Using the definition of M_0 and the explicit description of N, we have the following canonical isomorphisms:*

$$\mathrm{End}_{\mathcal{O}_{F,R}}(M) \cong \mathrm{End}_{\mathcal{O}_{F,R}}(M_0) \underset{\mathcal{O}_{F,R}}{\otimes} \mathrm{End}_{\mathcal{O}_{F,R}}(N) \cong \mathcal{O}_R \underset{\mathcal{O}_{F,R}}{\otimes} \mathrm{End}_{\mathcal{O}_{F,R}}(N).$$

Suppose $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ is any perfect $\mathcal{O}_R$-pairing that induces an involution ${}^{\maltese}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$, so that the involution ${}^{\maltese}$ sends $\mathcal{O}_R$ to itself and induces * on $\mathcal{O}_R$. (Note that for our purpose it suffices to consider perfect pairings with values in R because locally free rank one modules over R are automatically free.) Then the composition

$$({}^{\maltese}) \circ [({}^{\star}) \otimes (\mathrm{Id}_{\mathrm{End}_{\mathcal{O}_{F,R}}(N)})]$$

is an involution of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ that restricts to the identity on $\overline{\mathcal{O}}_R$. Hence, by Lemma 1.2.3.1, it defines an involution ${}^{\maltese_N}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(N)$, from which we obtain the decomposition

$${}^{\maltese} = ({}^{\star}) \otimes ({}^{\maltese_N}).$$

As a result, by Proposition 1.1.5.17, the classification of those perfect $\mathcal{O}_R$-pairings $\langle\,\cdot\,,\,\cdot\,\rangle$ on M that do induce involutions on $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ can be reduced to the analogous problem of $\mathcal{O}_{F,R}$-pairings $\langle\,\cdot\,,\,\cdot\,\rangle_N$ on N.

Lemma 1.2.3.2. *With the setting of M and N as above, assume that $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$. Suppose that ${}^{\maltese}$ is associated to some alternating $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$, which decomposes as ${}^{\maltese} = ({}^{\star}) \otimes ({}^{\maltese_N})$ as above. Suppose that B is* **simple**. *Then we have the following cases corresponding to classification in Proposition 1.2.1.14 and Definition 1.2.1.15:*

1. *Suppose that B is of type* C. *Then the classification of involutions ${}^{\maltese}$ of symplectic type is the same as the classification of involutions ${}^{\maltese_N}$ of symplectic type.*

2. *Suppose that B is of type* D. *Then the classification of involutions ${}^{\maltese}$ of symplectic type is the same as the classification of involutions ${}^{\maltese_N}$ of orthogonal type.*

3. *Suppose that B is of type* A*, in which case $[F : F^+] = 2$. Then the classification of involutions ${}^{\maltese}$ of unitary type is the same as the classification of involutions ${}^{\maltese_N}$ of unitary type.*

Proof. Suppose B is of type C. Then we have seen in the proof of Proposition 1.2.2.1 that we may assume that $\mathcal{O}_R = \mathrm{M}_k(\mathcal{O}_{F,R})$ and that the involution * is given by $x \mapsto x^{\star} = c\,{}^t x c$ for some $c \in \mathrm{M}_k(\mathcal{O}_{F,R})$ such that ${}^t c = c$. Since ${}^t(bx)c^{-1}y = {}^t x\,{}^t b c^{-1} y = {}^t x c^{-1}(c\,{}^t b c^{-1})y = {}^t x c^{-1}(b^{\star} y)$ for any $x, y, b \in \mathcal{O}_R$, this involution * is associated to the perfect symmetric bilinear pairing on the column vectors $\mathcal{O}_{F,R}^{\oplus k}$ given by $(x, y) \mapsto \mathrm{Tr}_{\mathcal{O}_F/\mathbb{Z}}({}^t x c^{-1} y)$ for $x, y \in \mathcal{O}_{F,R}^{\oplus k}$. Hence we see that ${}^{\maltese_N}$ has to be associated to an alternating pairing in this case, which by definition is of symplectic type.

Suppose B is of type D. Since we assume that $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$, and $\mathrm{I}_{\mathrm{bad}} = 2$ exactly in this case, we see that $p \neq 2$ and hence 2 is not a zero divisor. In this case, a skew-symmetric pairing is always alternating and never symmetric. Since $p \nmid \mathrm{Disc}$, we may assume that $\mathcal{O}_R = \mathrm{M}_{2k}(\mathcal{O}_{F,R}) \cong \mathrm{M}_k(\mathrm{M}_2(\mathcal{O}_{F,R}))$, that the involution ${}^\diamond$ of $\mathrm{M}_2(\mathcal{O}_{F,R})$ can be described explicitly as $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \delta & -\beta \\ \gamma & \alpha \end{smallmatrix}\right)$, and that the involution * is given explicitly by conjugating $x \mapsto {}^t x^\diamond$ by an element $c \in \mathrm{M}_{2k}(\mathcal{O}_{F,R})$ such that ${}^t c^\diamond = -c$. Therefore, the involution * can be induced by the perfect alternating pairing on $\mathcal{O}_{F,R}^{\oplus 2k}$ given by $(x, y) \mapsto \mathrm{Tr}_{\mathcal{O}_F/\mathbb{Z}}({}^t x^\diamond c^{-1} y)$ for $x, y \in \mathcal{O}_{F,R}^{\oplus 2k}$, and we see that ${}^{\maltese_N}$ has to be induced by a symmetric pairing in this case, which by definition is of orthogonal type.

There is nothing to prove in the case that B is of type A. $\square$

Let us introduce some special forms of self-dual projective $\mathcal{O}_R$-modules. Note that in what follows the notation $\mathcal{O}_R x$ stands for a rank one integrable $\mathcal{O}_R$-module with some chosen basis element x.

Definition 1.2.3.3. *Let α be any element in $\mathcal{O}_R$ such that $\alpha = \pm\alpha^\star$. The pair $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$ is defined by the rank one $\mathcal{O}_R$-module*

$$A_\alpha := (\mathcal{O}_R x)$$

spanned by some basis element x, together with the skew-Hermitian or Hermitian pairing $\llparenthesis\,\cdot\,,\,\cdot\,\rrparenthesis_\alpha$ associated to $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$ by Lemma 1.1.4.6 given by the relations $\llparenthesis x, x \rrparenthesis = \alpha$. If $p \nmid \mathrm{Disc}$, in which case $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$, this is a perfect pairing if and only if α is a unit in $\mathcal{O}_R$.

Definition 1.2.3.4. *The integrable symplectic $\mathcal{O}_R$-module $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$ is defined by the rank two $\mathcal{O}_R$-module*

$$H := (\mathcal{O}_R x) \oplus (\mathcal{O}_R y)$$

spanned by some basis elements x and y, such that the skew-Hermitian pairing $\llparenthesis\,\cdot\,,\,\cdot\,\rrparenthesis_{\mathrm{std}}$ associated to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}}$ by Lemma 1.1.4.6 is given by the relations $\llparenthesis x, x \rrparenthesis = \llparenthesis y, y \rrparenthesis = 0$ and $\llparenthesis x, y \rrparenthesis = 1$. This is always a perfect pairing when $p \nmid \mathrm{Disc}$.

For any integer $k \geq 0$, we define the integrable symplectic $\mathcal{O}_R$-module $(H_k, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},k})$ to be the orthogonal direct sum (defined as in Definition

1.1.4.15) of k copies of $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$. We say in this case that $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},k}$ (or rather the corresponding skew-Hermitian pairing) is of the form

$$\begin{pmatrix} & 1_k \\ -1_k & \end{pmatrix}.$$

Suppose that we are given a self-dual projective $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$. Let us decompose M as $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ as in (1.2.1.22). The $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ is uniquely determined by an isomorphism $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \to M^\vee$ of $\mathcal{O}_R$-modules. Let us identify $M_{\tau,R}^\vee \cong M_{\tau\circ c,R}$ by the explicit symmetric pairing defined in Lemma 1.1.5.16. Then the existence of an isomorphism $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \xrightarrow{\sim} M$ forces $m_\tau = m_{\tau\circ c}$ in the decomposition $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$. Let us define $[\tau]_c$ to be the equivalence class of τ under the action of c, and set $M_{[\tau]_c,R} = \underset{\tau\in[\tau]_c}{\oplus} M_{\tau,R}$, which is $M_{\tau,R}$ when $\tau = \tau\circ c$ and $M_{\tau,R} \oplus M_{\tau\circ c,R}$ when $\tau \neq \tau \circ c$. Accordingly, the action of $\mathcal{O}$ (resp. $\mathcal{O}_F$) on $M_{[\tau]_c,R}$ factors through $\mathcal{O}_{[\tau]_c,R}$ (resp. $\mathcal{O}_{F_{[\tau]_c},R}$), which is $\mathcal{O}_{\tau,R}$ (resp. $\mathcal{O}_{F_\tau,R}$) when $\tau = \tau \circ c$ and $\mathcal{O}_{\tau,R} \times \mathcal{O}_{\tau\circ c,R}$ (resp. $\mathcal{O}_{F_\tau,R} \times \mathcal{O}_{F_{\tau\circ c},R}$) when $\tau \neq \tau\circ c$. We shall use similar notations $\mathcal{O}_{[\tau]_c,R}$. Let $m_{[\tau]_c} := m_\tau$ for any $\tau \in [\tau]_c$. As pointed out in Remark 1.1.5.15, for the purpose of studying pairings we may decompose $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ as an orthogonal direct sum

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \overset{\perp}{\underset{[\tau]_c}{\oplus}} (M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}). \tag{1.2.3.5}$$

Note that each $[\tau]_c$ determines a unique $[\tau] : F \to \mathbb{Q}_{[\tau]}$, which corresponds to a unique simple factor $B_{[\tau]}$ of B as in (1.2.1.10).

Now let us focus on *alternating* pairings:

Definition 1.2.3.6. *A symplectic $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is called* **of standard type** *if every component $(M_{[\tau]_c,R}^{\oplus m_\tau}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ in the decomposition (1.2.3.5) can be described as follows:*

1. *If $B_{[\tau]}$ is of type* C *(defined as in Definition 1.2.1.15), then $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is isomorphic to $(H_k, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},k}) \underset{\mathcal{O}_R}{\otimes} \mathcal{O}_{[\tau]_c,R}$ for some integer $k \geq 0$.*

2. *If $B_{[\tau]}$ is of type* D*, then $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$, where α is a unit in $\mathcal{O}_{F_{[\tau]_c},R}$ satisfying $\alpha = -\alpha^\star$.*

3. *If* $B_{[\tau]}$ *is of type* A*, then* $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle)$ *is isomorphic to the orthogonal direct sum of modules of the form* $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$*, where* α *is a unit in* $\mathcal{O}_{F_{[\tau]_c},R}$ *satisfying* $\alpha = -\alpha^\star$.

Proposition 1.2.3.7. *Suppose* $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$*. Then every self-dual projective symplectic* $\mathcal{O}_R$*-module* $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ *is symplectic isomorphic to some symplectic* $\mathcal{O}_R$*-module of standard type.*

This is a generalization of [124, Lem. 3.4]. A similar result is [79, Lem. 7.2]. The proof we give here follows more closely the one in [124, Lem. 3.4]. We shall proceed by induction on the multi-ranks, based on the following basic lemmas:

Lemma 1.2.3.8. *Suppose* $p \nmid \mathrm{Disc}$*. Let* $\langle\,\cdot\,,\,\cdot\,\rangle$ *be a symmetric or skew-symmetric* $\mathcal{O}_R$*-paring on* M*, and let* $(\!|\,\cdot\,,\,\cdot\,|\!)$ *be the associated Hermitian or skew-Hermitian pairing (with value in* $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$*) as in Lemma 1.1.4.6. Suppose* x *is an element in* M *such that the projection* $(\!|x,x|\!)_\tau$ *of* $(\!|x,x|\!) \in \mathcal{O}_R$ *to* $\mathcal{O}_\tau$ *is a unit in* $\mathcal{O}_{\tau,R}$ *for some* τ*. Let* M_1 *be the* $\mathcal{O}_{\tau,R}$*-span of* x *in* M*. Then the map*

$$\phi : M \to M_1 : z \mapsto (\!|x,z|\!)(\!|x,x|\!)_\tau^{-1}x$$

is surjective with kernel $M_1^\perp$*. Consequently, the isomorphism*

$$M \xrightarrow{\sim} M_1 \oplus M_1^\perp : z \mapsto (\phi(z), z - \phi(z))$$

identifies M *as the orthogonal direct sum of* M_1 *and* $M_1^\perp$ *(with the pairings given by the restrictions of* $\langle\,\cdot\,,\,\cdot\,\rangle$*). In particular,* M_1 *and* $M_1^\perp \cong M/M_1$ *are both projective* $\mathcal{O}_R$*-submodules of* M*.*

Proof. The map ϕ is surjective because $\phi(x) = x$. If $rx = 0$ for some $r \in \mathcal{O}_{\tau,R}$, then $r(\!|x,x|\!) = (\!|x,rx|\!) = 0$, which shows that $r = 0$. Therefore, an element $z \in M$ satisfies $\phi(z) = 0$ if and only if $(\!|x,z|\!) = 0$, which shows that $\ker(\phi) = M_1^\perp$. □

Lemma 1.2.3.9. *Suppose* $p \nmid \mathrm{Disc}$*. Let* $\langle\,\cdot\,,\,\cdot\,\rangle$ *be a skew-symmetric* $\mathcal{O}_R$*-paring on* M*, and let* $(\!|\,\cdot\,,\,\cdot\,|\!)$ *be the associated skew-Hermitian pairing (with value in* $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$*) as in Lemma 1.1.4.6. Suppose* x *and* y *is a pair of elements in* M *such that* $(\!|x,x|\!) = (\!|y,y|\!) = 0$*, and such that the projection* $(\!|x,y|\!)_\tau$ *of* $(\!|x,y|\!) \in \mathcal{O}_R$ *to* $\mathcal{O}_\tau$ *is a unit in* $\mathcal{O}_{\tau,R}$ *for some* τ*. Let*

M_1 be the $\mathcal{O}_{\tau\circ c,R}$-span of x in M, let M_2 be the $\mathcal{O}_{\tau,R}$-span of y in M. Then $M_1 \cap M_2 = 0$ (and so the sum $M_1 + M_2$ is direct), and the map

$$\phi : M \to M_1 \oplus M_2 : z \mapsto -(\!| y, z |\!)(\!| x, y |\!)_\tau^{-1} x + (\!| x, z |\!)(\!| x, y |\!)_\tau^{-1} y$$

is surjective with kernel $(M_1 \oplus M_2)^\perp$. Consequently, the isomorphism

$$M \xrightarrow{\sim} (M_1 \oplus M_2) \oplus (M_1 \oplus M_2)^\perp : z \mapsto (\phi(z), z - \phi(z))$$

identifies M as the orthogonal direct sum of $(M_1 \oplus M_2)$ and $(M_1 \oplus M_2)^\perp$ (with the pairings given by the restrictions of $\langle\,\cdot\,,\,\cdot\,\rangle$). In particular, M_1, M_2, and $(M_1 \oplus M_2)^\perp \cong M/(M_1 \oplus M_2)$ are all projective $\mathcal{O}_R$-submodules of M.

Proof. If there exists $a \in \mathcal{O}_{\tau\circ c,R}$ and $b \in \mathcal{O}_{\tau,R}$ such that $ax = by \in M_1 \cap M_2$, then $a^\star(\!| x, y |\!) = (\!| ax, y |\!) = (\!| by, y |\!) = 0$ and $b(\!| x, y |\!) = (\!| x, by |\!) = (\!| x, ax |\!) = 0$ forces both $a = 0$ and $b = 0$. Hence $M_1 \cap M_2 = 0$. Moreover, the argument shows that $ax = by = 0$ is possible only when $a = 0$ and $b = 0$. Therefore, an element $z \in M$ satisfies $\phi(z) = 0$ if and only if $(\!| x, z |\!) = 0$ and $(\!| y, z |\!) = 0$, which shows that $\ker(\phi) = (M_1 \oplus M_2)^\perp$. Finally, the map ϕ is surjective simply because $\phi(x) = x$ and $\phi(y) = y$. ☐

Proof of Proposition 1.2.3.7. By Lemmas 1.2.1.11 and 1.2.1.23, and the same argument as in the proof of Proposition 1.2.2.1 based on the decomposition (1.2.2.2), we may assume that B is simple.

Let us first classify the pairings up to weak symplectic isomorphisms (defined as in Definition 1.1.5.1).

By Lemma 1.2.3.2, we may replace $\mathcal{O}_R$ by $\mathcal{O}_{F,R}$, and replace $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ by $(N, \langle\,\cdot\,,\,\cdot\,\rangle_N)$, where $\langle\,\cdot\,,\,\cdot\,\rangle_N$ is an alternating $\mathcal{O}_{F,R}$-pairing on N except when B is of type D, in which case we consider *symmetric* $\mathcal{O}_{F,R}$-pairings instead. To avoid clumsy notations, let us retain the notations $\mathcal{O}_R$, M, and $\langle\,\cdot\,,\,\cdot\,\rangle$ for the various objects. Moreover, we shall use $(\!|\,\cdot\,,\,\cdot\,|\!)$ to denote the Hermitian or skew-Hermitian pairing associated to $\langle\,\cdot\,,\,\cdot\,\rangle$ by Lemma 1.1.4.6. Since $p \nmid \mathrm{Disc}$, we have $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$ by definition. Therefore the codomain of $(\!|\,\cdot\,,\,\cdot\,|\!)$ is given by $\mathcal{O}_R = \mathcal{O}_{F,R}$. Although not logically necessary, we shall often do calculations with the Hermitian or skew-Hermitian pairings $(\!|\,\cdot\,,\,\cdot\,|\!)$ associated to the symmetric or skew-symmetric pairings $\langle\,\cdot\,,\,\cdot\,\rangle$ when it is more convenient to do so.

By replacing $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ by any of the nonzero $(M_{[\tau],R}^{\oplus m_\tau}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ in the decomposition (1.2.3.5) (which works for both alternating and symmetric pairings), we may assume from now that the multi-rank (m_τ) of M has either only one nonzero entry m_τ with $\tau = \tau \circ c$, or only two nonzero entries m_τ and $m_{\tau \circ c}$ with $\tau \neq \tau \circ c$.

In the case $\tau \neq \tau \circ c$, there exist invertible elements $f_1 \in \operatorname{End}_{\mathcal{O}_R}(M_{\tau,R}^{\oplus m_\tau})$ and $f_2 \in \operatorname{End}_{\mathcal{O}_R}(M_{\tau\circ c,R}^{\oplus m_\tau})$ such that $\langle (x_1, x_2), (y_1, y_2)\rangle = f_1(x_1)(y_2) + f_2(x_2)(y_1)$ for any $x_1, y_1 \in M_{\tau,R}^{\oplus m_\tau}$ and any $x_2, y_2 \in M_{\tau\circ c,R}^{\oplus m_\tau}$. The condition that $\langle\,\cdot\,,\,\cdot\,\rangle$ is alternating shows that $f_1(x_1)(x_2) + f_2(x_2)(x_1) = 0$ for any $(x_1, x_2) \in M$. In other words, $f_2 = -f_1^\vee$ is uniquely determined by f_1. If we conjugate the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ by the automorphism $f_1 \times \mathrm{Id}$ of M, then

$$\begin{aligned}\langle (f_1^{-1}(x_1), x_2), (f_1^{-1}(y_1), y_2)\rangle &= x_1(y_2) - f_1^\vee(x_2)(f_1^{-1}(y_1)) \\ &= x_1(y_2) - x_2((f_1 \circ f_1^{-1})(y_1)) \\ &= x_1(y_2) - x_2(y_1).\end{aligned}$$

This argument shows that any two self-dual alternating $\mathcal{O}_R$-pairings on M are isomorphic to each other. Explicitly, the pairing is isomorphic to $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)^{\overset{\perp}{\oplus} m_\tau}$ with $\alpha = (1, -1) \in \mathcal{O}_{F_\tau,R}^\times \times \mathcal{O}_{F_{\tau\circ c},R}^\times$.

In the remaining proof, let us assume that $\tau = \tau \circ c$. Then all the action of $\mathcal{O}_R = \mathcal{O}_{F,R}$ on M factors through $\overline{\mathcal{O}}_R = \mathcal{O}_{F_\tau,R}$. In this case, it makes sense to speak of *ranks* (rather than *multi-ranks*) of M and its nontrivial submodules, because there is a unique nonzero number in each multi-rank. For simplicity, let us replace $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$ and $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$ (defined in Definitions 1.2.3.4 and 1.2.3.3) by respectively $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}}) \underset{\mathcal{O}_R}{\otimes} \overline{\mathcal{O}}_R$ and $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha) \underset{\mathcal{O}_R}{\otimes} \overline{\mathcal{O}}_R$, and replace $\mathcal{O}_R$ by $\overline{\mathcal{O}}_R$.

We claim that $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is an orthogonal direct sum of submodules of the form $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$ or $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$, depending on the type of B. We shall proceed by induction on the rank of M. If the rank of M is zero, then the claim is automatic. Let us suppose that the rank of M is at least one.

Suppose there is an element x in M such that $\alpha := (\!| x, x |\!)$ is a unit. (This does not happen when B is of type C.) Let M_1 be the $\mathcal{O}_R$-span of x in M. By Lemma 1.2.3.8, M is the orthogonal direct sum of M_1 and $M_1^\perp$, with the pairings being the restrictions of $\langle\,\cdot\,,\,\cdot\,\rangle$. Hence the claim follows by induction because $(M_1, \langle\,\cdot\,,\,\cdot\,\rangle|_{M_1}) \cong (A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$.

Otherwise, we may assume that there is no element z in M such that $(\!| z, z |\!)$ is a unit. By perfectness of the pairing, there exists elements x and y in M such that $(\!| x, y |\!) = 1$. By our assumption, neither $(\!| x, x |\!)$ nor $(\!| y, y |\!)$ can be a unit in $\mathcal{O}_R = \mathcal{O}_{F_\tau,R}$.

Suppose B is of type D, in which case $\langle \cdot\,, \cdot \rangle$ is symmetric. Note that $p \nmid \mathrm{I_{bad}}\,\mathrm{Disc}$ implies $p \neq 2$ in this case. Therefore there is an element β such that $2\beta = 1$ in Λ. Set $z = x + \beta y$. Then $(\!| z, z |\!) = 1 + (\text{non-unit})$ is a unit. This is a contradiction too.

Suppose B is of type A, in which case $[F : F^+] = 2$. Let β be any element in $\mathcal{O}_{F,\Lambda}$ such that $\mathcal{O}_{F,\Lambda} = \mathcal{O}_{F^+,\Lambda} \oplus \mathcal{O}_{F^+,\Lambda}\beta$. Since $\mathrm{Tr}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}}((\beta - \beta^\star)^{-1}) = 0$ and $\mathrm{Tr}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}}((\beta - \beta^\star)^{-1}\beta) = 1$, we see that $(\beta - \beta^\star)^{-1} \in \mathrm{Diff}^{-1}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}} = \mathcal{O}_{F,\Lambda}$ (because $p \nmid \mathrm{Disc}$). Hence $\alpha := \beta - \beta^\star$ is a unit. Set $z = -\beta x + y$. Then $(\!| z, z |\!) = \beta^\star\beta(\!| x, x |\!) - \beta^\star(\!| x, y |\!) - \beta(\!| y, x |\!) + (\!| y, y |\!) = \alpha + (\text{non-unit})$, which is a unit. This is a contradiction.

Hence we may assume that B is of type C in the remaining proof of the claim, in which case $\langle \cdot\,, \cdot \rangle$ is alternating. By Lemmas 1.1.4.6 and 1.1.5.5, this implies $(\!| x, x |\!) = (\!| y, y |\!) = 0$ (under the simplified assumption that $\mathcal{O}_R = \mathcal{O}_{F,R}$). Let M_1 and M_2 be the $\mathcal{O}_R$-spans of respectively x and y. By Lemma 1.2.3.9, the sum $M_1 + M_2$ is direct, and M is the orthogonal direct sum of $(M_1 \oplus M_2)$ and $(M_1 \oplus M_2)^\perp$, with the pairings being the restrictions of $\langle \cdot\,, \cdot \rangle$. Hence the claim follows by induction because $(M_1 \oplus M_2, \langle \cdot\,, \cdot \rangle|_{M_1 \oplus M_2}) \cong (H, \langle \cdot\,, \cdot \rangle_{\mathrm{std}})$.

Summarizing what we have obtained (under the simplified assumption):

1. If B is of type C, then $(M, \langle \cdot\,, \cdot \rangle)$ is isomorphic to $(H_{\frac{m_\tau}{2}}, \langle \cdot\,, \cdot \rangle_{\mathrm{std}, \frac{m_\tau}{2}})$.

2. If B is of type D, then $(M, \langle \cdot\,, \cdot \rangle)$ (as a symmetric pairing) is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle \cdot\,, \cdot \rangle_\alpha)$. In this case, each α can be any unit in $\mathcal{O}_{F,R}$.

3. If B is of type A, then $(M, \langle \cdot\,, \cdot \rangle)$ (as an alternating pairing) is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle \cdot\,, \cdot \rangle_\alpha)$. In this case, each α can be any unit in $\mathcal{O}_{F,R}$ satisfying $\alpha = -\alpha^\star$.

The result then follows by tensoring the pairing $(M, \langle \cdot\,, \cdot \rangle)$ over $\mathcal{O}_{F,R}$ with any self-dual pairing on $\mathcal{O}$ that induces the involution $\star$ as in the proof of Lemma 1.2.3.2. Note that the difference between weak symplectic isomorphisms and symplectic isomorphisms is immaterial, because multiplying by an element in $\mathcal{O}_{F,R}^\times$ in the second factor of the pairing does not affect the classification. $\square$

Corollary 1.2.3.10. *Suppose* $p \nmid \mathrm{I_{bad}}\,\mathrm{Disc}$. *Then every two self-dual projective symplectic* $\mathcal{O}_R$*-modules* $(M_1, \langle \cdot\,, \cdot \rangle_1)$ *and* $(M_2, \langle \cdot\,, \cdot \rangle_2)$ *such that* M_1 *and*

M_2 have the same multi-rank are isomorphic after some finite étale extension $R \to R'$. Suppose B does not involve any simple factor of type D*, and suppose $p > 0$ when B involves any simple factor of type* A*. Then we may assume that $R' = R$.*

Proof. It suffices to show that, in the latter two cases in the summary at the end of the proof of Proposition 1.2.3.7, the submodules $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$ appearing in the orthogonal direct sum are all isomorphic to each other over some $R \to R'$.

Suppose B is of type D, in which case $F = F^+$. Note that $\mathcal{O}_R = \mathcal{O}_{F_\tau,R}$ is a complete local ring with residue field k, and the claim is true if we can show that any unit α in $\mathcal{O}_{F,R}$ is the square of some unit α' in $\mathcal{O}_{F,R'}$ over some finite étale extension $R \to R'$. Define a polynomial $f(X) := X^2 - \alpha$. Since α is a unit, its reduction in k is nonzero. Since $p \neq 2$, the reduction of $f'(X) = 2X$ in k has nonzero values for nonzero inputs of X. Let k' be the finite separable extension of k over which $f(X)$ has a solution, and let R' be the unique finite étale extension of R such that $R' \underset{R}{\otimes} k \cong k'$. Then it follows from Helsel's lemma (see for example [36, Thm. 7.3]) that $f(X)$ has a solution α' in $\mathcal{O}_{F,R'}$, as desired.

Suppose B is of type A, in which case $[F : F^+] = 2$. When $\mathrm{char}(k) = 0$, we replace R by a finite étale extension over which F/F^+ is split. Then the claim is true because, by Lemma 1.1.5.9 (with $\epsilon = 1$), if $\gamma := \alpha(\alpha')^{-1} \in \mathcal{O}_{F,R}^\times$ satisfies $\gamma = \gamma^\star$, then $\gamma = \delta\delta^\star$ for some unit $\delta \in \mathcal{O}_{F,R}$. ☐

Proposition 1.2.3.11. *Suppose $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$. Let $B \cong \prod\limits_{[\tau]} B_{[\tau]}$ be the decomposition of B into its simple factors as in* (*1.2.1.10*)*. Let $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ be any self-dual integrable symplectic $\mathcal{O}_R$-module. Then $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ decomposes accordingly as $(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \underset{[\tau]}{\oplus}(M_{[\tau],R}^{\oplus m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$, where $M_{[\tau],R}$ is defined as in Lemma 1.2.1.23, and where (m_τ) is the multi-rank of M. Define a group functor* H *over R by setting*

$$\mathrm{H}(R') := \left\{(g, r) \in \mathrm{GL}_{\mathcal{O}_{R'}}(M_{R'}) \times \mathbf{G}_{\mathrm{m}}(R') : \langle gx, gy\rangle = r\langle x, y\rangle, \forall x, y \in M_{R'}\right\},$$

or, when $M \neq \{0\}$, by setting equivalently

$$\mathrm{H}(R') := \{g \in \mathrm{End}_{\mathcal{O}_{R'}}(M_{R'}) : \nu(g) := g^{\maltese} g \in (R')^\times\}.$$

Let k^{sep} be a separable closure of k, and let $\tilde{R}$ be a strict Henselization of $R \twoheadrightarrow k \to k^{\mathrm{sep}}$. Then the group $\mathrm{H} \underset{R}{\otimes} \tilde{R}$ depends (*up to isomorphism*) *only* *on the multi-rank* $(m_{[\tau]})$ *of the underlying integrable* $\mathcal{O}_R$*-module* M*, and is independent of the perfect alternating* $\mathcal{O}_R$*-pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *on* M *we use. Moreover, based on the classification of the simple factors* $B_{[\tau]}$ *as in Proposition 1.2.1.14 and Definition 1.2.1.15, we have the following descriptions of simple factors of* $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$*:*

1. *If* $B_{[\tau]}$ *is of type* C*, then the existence of* $\langle\,\cdot\,,\,\cdot\,\rangle$ *forces* $m_{[\tau]}$ *to be even, and* $(M_{[\tau],R}^{\oplus\, m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ *defines simple factors of* $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ *that are isomorphic to the quotient of* $\mathrm{GSp}_{m_{[\tau]}}(k^{\mathrm{sep}})$ *by its center, which is of type* C *and rank* $\frac{m_{[\tau]}}{2}$*.*

2. *If* $B_{[\tau]}$ *is of type* D*, then* $(M_{[\tau],R}^{\oplus\, m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ *defines simple factors of* $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ *that are isomorphic to the quotient of* $\mathrm{GO}_{m_{[\tau]}}(k^{\mathrm{sep}})$ *by its center, which is either of type* B *and rank* $\frac{m_{[\tau]}-1}{2}$ *when* $m_{[\tau]}$ *is odd, or of type* D *and rank* $\frac{m_{[\tau]}}{2}$ *when* $m_{[\tau]}$ *is even.*

3. *If* $B_{[\tau]}$ *is of type* A*, then* $(M_{[\tau],R}^{\oplus\, m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ *defines simple factors of* $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ *that are isomorphic to the quotient of* $\mathrm{GL}_{m_{[\tau]}}(k^{\mathrm{sep}})$ *by its center, which is of type* A *and of rank* $m_{[\tau]} - 1$*.*

Any simple factor $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ *is contributed by some simple factor* $B_{[\tau]}$ *as described above.*

In particular, this applies to the group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda$ (defined in Definition 1.2.1.5).

Proof of Proposition 1.2.3.11. We may assume that $M \neq \{0\}$ as the result is clear when $M = \{0\}$. Since this is a question about simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ (rather than of $\mathrm{H}(k^{\mathrm{sep}})$), we can ignore the centers (and in particular the *similitude factors*) and assume as in the proof of Proposition 1.2.3.7 that B is simple. (The concern in Remark 1.2.1.12 about similitude factors is irrelevant here.)

By Lemma 1.2.3.1, there is a rank m integrable $\mathcal{O}_{F,R}$-lattice N such that $M \cong M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N$, where M_0 is the unique rank one integrable $\mathcal{O}_R$-lattice (defined as in Definition 1.2.1.20 and Lemma 1.2.1.23). As we have seen in Lemma 1.2.3.2, $\maltese$ decomposes as $\maltese = (\star) \otimes (\maltese_N)$, for some involution $\maltese_N$ of

$\mathrm{End}_{\mathcal{O}_{F,R}}(N)$ induced by some perfect pairing $\langle\,\cdot\,,\,\cdot\,\rangle_N$ on N. Since the isomorphism $\mathrm{End}_{\mathcal{O}_R}(M) \cong \mathrm{End}_{\mathcal{O}_{F,R}}(N)$ carries the restriction of the involution ${}^{\maltese}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ to an involution ${}^{\maltese_N}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(N)$, we obtain

$$\mathrm{H}(R') = \left\{ g \in \mathrm{End}_{\mathcal{O}_{F,R'}}(N_{R'}) : \nu(g) := g^{\maltese_N} g \in (R')^{\times} \right\}$$

for any R-algebra R'. For simplicity, let us assume that $\mathcal{O}_R = \mathcal{O}_{F,R}$ and $(M, \langle\,\cdot\,,\,\cdot\,\rangle) = (N, \langle\,\cdot\,,\,\cdot\,\rangle_N)$ in the remaining proof.

For our purpose, we may replace k by any sufficiently large separable extension so that $\mathrm{Frac}(\Lambda) = \mathrm{Frac}(W(k))$ contains all the images of $\tau : F \hookrightarrow \mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ (for any choice of separable closure $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ of $\mathrm{Frac}(\Lambda)$). Then we may assume that the structural morphism $\Lambda \to \mathcal{O}_{F_\tau,\Lambda}$ is an isomorphism for all τ. As in (1.2.3.5) and in the proof of Proposition 1.2.3.7, we have an orthogonal direct sum

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \overset{\perp}{\bigoplus_{[\tau]_c}} (M_{[\tau]_c,R}^{\oplus\, m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$$

of projective modules, with all the $m_{[\tau]_c}$ given by the same integer m, the rank of the integrable $\mathcal{O}_R$-module M. Let $\mathrm{H}_{[\tau]_c}$ be the algebraic group defined by $(M_{[\tau]_c,R}^{\oplus\, m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ as above. Then H is the subgroup of $\prod\limits_{[\tau]_c} \mathrm{H}_{[\tau]_c}$ consisting of elements having the same similitude factors in all $\mathrm{H}_{[\tau]_c}$. Hence it suffices to classify each of the symplectic $\mathcal{O}_R$-modules $(M_{[\tau]_c,R}^{\oplus\, m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$.

Now let us base change everything to $R \to \tilde{R}$, the strict Henselization of $R \to k \to k^{\mathrm{sep}}$. (Certainly, it is enough to work over some finite étale extension $R \to R'$ that splits everything.) By Proposition 1.2.3.7 and Corollary 1.2.3.10, the classification of $(M_{[\tau]_c,\tilde{R}}^{\oplus\, m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is completely known:

If B is of type C, then may identify $M_{[\tau]_c,\tilde{R}}$ with $\mathcal{O}_{F_\tau,\tilde{R}} \cong \tilde{R}$, and identify the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}$ explicitly with

$$\tilde{R}^{\oplus\, m} \oplus \tilde{R}^{\oplus\, m} \to \tilde{R} : (x, y) \mapsto {}^t x \begin{pmatrix} & 1_{\frac{m}{2}} \\ -1_{\frac{m}{2}} & \end{pmatrix} y.$$

This alternating pairing defines GSp_m over $\tilde{R}$.

If B is of type D, then we may identify $M_{[\tau]_c,\tilde{R}}$ with $\mathcal{O}_{F_\tau,\tilde{R}} \cong \tilde{R}$, and identify the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}$ explicitly with

$$\tilde{R}^{\oplus\, m} \oplus \tilde{R}^{\oplus\, m} \to \tilde{R} : (x, y) \mapsto {}^t x y.$$

This symmetric pairing defines the orthogonal group GO_m over $\tilde{R}$.

If B is of type A, then we may identify $M_{[\tau]_c,\tilde{R}} = M_{\tau,\tilde{R}} \times M_{\tau\circ c,\tilde{R}}$ with $\mathcal{O}_{F_{\tau,\tilde{R}}} \times \mathcal{O}_{F_{\tau\circ c,\tilde{R}}} \cong \tilde{R} \times \tilde{R}$, and identify the pairing $\langle \cdot\,,\,\cdot \rangle_{[\tau]_c}$ explicitly with

$$(\tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}) \oplus (\tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}) \to \tilde{R} : ((x_1,x_2),(y_1,y_2)) \mapsto {}^t x_1 y_2 - {}^t x_2 y_1.$$

Let $g = (g_1, g_2) \in \mathrm{End}_{\mathcal{O}_{\tilde{R}}}(M_{[\tau]_c,\tilde{R}}) \cong \mathrm{M}_m(\tilde{R}) \times \mathrm{M}_m(\tilde{R})$. Then

$$\begin{aligned}\langle (x_1,x_2), g(y_1,y_2)\rangle &= \langle (x_1,x_2),(g_1(y_1),g_2(y_2))\rangle = {}^t x_1 g_2(y_2) - {}^t x_2 g_1(y_1)\\ &= {}^t({}^t g_2 x_1) y_2 - {}^t({}^t g_1 x_2) y_1 = \langle ({}^t g_2 x_1, {}^t g_1 x_2),(y_1,y_2)\rangle\end{aligned}$$

for any $(x_1,x_2),(y_1,y_2) \in \tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}$, and hence $g^{\maltese} = ({}^t g_2, {}^t g_1)$. Therefore the condition that $\nu(g) := g^{\maltese} g \in \tilde{R}^{\times}$ implies that $g_1 \in \mathrm{GL}_m(\tilde{R})$ and $g_2 = \nu(g)^{-1}\, {}^t g_1^{-1}$, and we have an isomorphism $\mathrm{G}_{[\tau]}(\tilde{R}) \xrightarrow{\sim} \mathrm{GL}_m(\tilde{R}) \times \mathbf{G}_{\mathrm{m}}(\tilde{R}) : g = (g_1,g_2) \mapsto (g_1, \nu(g))$.

All the above identifications remain to be true if we replace $\tilde{R}$ by an $\tilde{R}$-algebra, and they are compatible in a functorial way. In each of the three cases, if we form the quotient of H by its center, then we obtain the product of the quotients of $\mathrm{H}_{[\tau]_c}$ by their centers. Hence the result follows. □

As a byproduct of these explicit identifications:

Corollary 1.2.3.12. *With assumptions on k, Λ, and R as above, the group* H *defined in Proposition 1.2.3.11 is smooth over R.*

In particular, the group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda$ (defined in Definition 1.2.1.5) is smooth.

1.2.4 Gram-Schmidt Procedures

Let us maintain the assumptions and notations of Section 1.2.3 in this section.

Definition 1.2.4.1. *An alternating $\mathcal{O}_R$-pairing $\langle \cdot\,,\,\cdot \rangle : M \times M \to R$ is called* **sufficiently alternating** *if it satisfies $\langle x, rx\rangle = 0$ for any $x \in M$ and any $r \in \mathcal{O}_\Lambda$ such that $r = r^\star$. Accordingly, a symplectic $\mathcal{O}_R$-module $(M, \langle \cdot\,,\,\cdot \rangle)$ is called* **sufficiently symplectic** *if the alternating pairing $\langle \cdot\,,\,\cdot \rangle$ is sufficiently alternating.*

Remark 1.2.4.2. The reader should be aware that the only reason to make this assumption is for technical simplicity. It is not even clear to us if there are interesting examples of alternating pairings that are not sufficiently alternating.

Lemma 1.2.4.3. *If there exists a complete noetherian local* Λ*-algebra* R' *in which* 2 *is not a zero divisor, and a symplectic* $\mathcal{O}_{R'}$ *module* $(M', \langle \,\cdot\, , \,\cdot\, \rangle')$ *such that* $(M, \langle \,\cdot\, , \,\cdot\, \rangle)$ *is the pullback of* $(M', \langle \,\cdot\, , \,\cdot\, \rangle')$ *along some morphism* $R' \hookrightarrow R$*, then* $(M, \langle \,\cdot\, , \,\cdot\, \rangle)$ *is automatically sufficiently symplectic.*

Proof. Over R', we have $\langle x, rx\rangle' = \langle r^\star x, x\rangle' = \langle rx, x\rangle' = -\langle x, rx\rangle'$ for any $x \in M'$ and any $r \in \mathcal{O}_\Lambda$, which forces $\langle x, rx\rangle' = 0$. Therefore $(M, \langle \,\cdot\, , \,\cdot\, \rangle)$ is sufficiently symplectic because it is the pullback of $(M', \langle \,\cdot\, , \,\cdot\, \rangle')$ from R' to R. ☐

Lemma 1.2.4.4. *Suppose* $p \nmid \mathrm{I}_{\mathrm{bad}} \,\mathrm{Disc}$*. Let* $\langle \,\cdot\, , \,\cdot\, \rangle : M \times M \to R$ *be an alternating* $\mathcal{O}_R$*-pairing, and let* $(\!| \,\cdot\, , \,\cdot\, |\!) : M \times M \to \mathcal{O}_R$ *be the associated skew-Hermitian pairing as in Lemma 1.1.4.6. In case that* $p = 2$ *and* B *involves any simple factor of type* C*, we assume moreover that the alternating pairing* $\langle \,\cdot\, , \,\cdot\, \rangle$ *is sufficiently alternating. Then, for any* $x \in M$*, there exists some element* $b \in \mathcal{O}_R$ *such that* $(\!| x, x |\!) = b - b^\star$*.*

Proof. Since $(\!| \,\cdot\, , \,\cdot\, |\!)$ is skew-Hermitian, we know that $a := (\!| x, x |\!)$ satisfies $a = -a^\star$. Then Proposition 1.2.2.1 implies that there exists an element β such that $a = b - b^\star$, unless we are in the case that B involves some simple factor of type C, that $p = 2$, and that 2 is a zero divisor in R.

As in the proof of Proposition 1.2.2.1, we may assume that B is simple. (The assumption now is that B is simple of type C, that $p = 2$, and that 2 is a zero divisor in R.) Moreover, we may assume that $\mathcal{O}_\Lambda \cong \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ for some integer k, and that the involution * is given by $x \mapsto c\,{}^t x c^{-1}$ for some $c \in \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ such that ${}^t c = c$. Then $a = -a^\star = c\,{}^t a c^{-1}$ implies $ac = -\,{}^t(ac)$, and we may represent ac as an element (a_{ij}) in $\mathrm{M}_k(\mathcal{O}_{F,R})$ such that $a_{ij} = -a_{ji}$ for any $1 \leq i, j \leq k$. For any $d \in \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ such that ${}^t d = d$, we have $(cd)^\star = d^\star c^\star = c\,{}^t d c^{-1} c\,{}^t c c^{-1} = cd$. Let us consider the special case that $d = \mathrm{diag}(d_i)$ is a diagonal matrix. Since $\langle \,\cdot\, , \,\cdot\, \rangle$ is sufficiently alternating, we have $\langle x, cdx\rangle = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}((cd)a) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(d(ac)) = \mathrm{Tr}_{\mathcal{O}_F/\mathbb{Z}}(\sum\limits_i d_i a_{ii}) = 0$ for any $d_i \in \mathcal{O}_{F,\Lambda}$. This forces $a_{ii} = 0$ for any i, and hence $ac = e - {}^t e$ with $e = (e_{ij})$ given by $e_{ij} = a_{ij}$ if $i < j$ and $e_{ij} = 0$ if $i \geq j$. Set $b := ec^{-1}$. Then $a = (ac)c^{-1} = (e - {}^t e)c^{-1} = ec^{-1} - c(c^{-1}\,{}^t e)c^{-1} = ec^{-1} - c\,{}^t(ec^{-1})c^{-1} = b - b^\star$, as desired. ☐

Lemma 1.2.4.5. *Suppose* $p \nmid \mathrm{I}_{\mathrm{bad}} \,\mathrm{Disc}$*. Let* $\langle \,\cdot\, , \,\cdot\, \rangle$ *be a sufficiently alternating* $\mathcal{O}_R$*-paring on* M *(defined as in Definition 1.2.4.1). Let* M_1 *be a totally isotropic projective submodule of* M *such that* M/M_1 *is projective, and such* *that the restriction of* $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \to M^\vee$ *to* M_1 *is an injection. Then* $M_1^\vee$ *is embedded as a totally isotropic projective submodule of* M*, and there is a symplectic isomorphism*

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \to (M_1 \oplus M_1^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\text{can.}}) \oplus ((M_1 \oplus M_1^\vee)^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(M_1 \oplus M_1^\vee)^\perp}).$$

Proof. Since M_1 is projective, it decomposes as $M \cong M_{\tau,R}^{\oplus m_\tau}$ for some multi-rank $m = (m_\tau)$. Let us proceed by induction on $|m| := \sum\limits_\tau m_\tau$. When $|m| = 0$, it forces $M_1 = 0$, and there is nothing to prove.

When $|m| \geq 1$, we may decompose $M_1 \cong M_{1,0} \oplus M_1'$ for some projective $\mathcal{O}_R$-modules $M_{1,0}$ and M_1' such that $M_{1,0} \cong M_{\tau,R}$ for some τ with $m_\tau \geq 1$. By assumption, M_1 is totally isotropic, and hence $M_{1,0}$ and M_1' are totally isotropic as well. By assumption that restriction of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ to M_1 is an injection, the map $M/M_{1,0}^\perp \to M_{1,0}^\vee$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle^*$ is a *surjection*. Since $M_{1,0} \cong M_{\tau,R}$, its dual $M_{1,0}^\vee \cong M_{\tau\circ c,R}$ is projective as well. Hence the surjection $M/M_{1,0}^\perp \twoheadrightarrow M_{1,0}^\vee$ splits.

Let x be any element spanning $M_{1,0}$ in the sense that $M_{1,0} = (\mathcal{O}_{\tau,R})x$. The statement above that the surjection $M/M_{1,0}^\perp \twoheadrightarrow M_{1,0}^\vee$ splits implies that there exists an element y in M such that $\langle x, y\rangle = 1_{\tau\circ c}$, the identity element in $\mathcal{O}_{\tau\circ c,R}$. By Lemma 1.2.4.4, and by the assumption that $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating, there is an element $b \in \mathcal{O}_R$ such that $b - b^\star = (\!| y, y |\!)$. Then $(\!| bx + y, bx + y |\!) = 0 - b + b^\star + (\!| y, y |\!) = 0$. Replacing y by $bx + y$, we may assume that $(\!| y, y |\!) = 0$. Then Lemma 1.2.3.9 implies that M is the orthogonal direct sum of $M_{1,0} \oplus M_{1,0}^\vee$ and its orthogonal complement $(M_{1,0} \oplus M_{1,0}^\vee)^\perp \cong M_{1,0}^\perp / M_{1,0}$.

Note that M_1' is a totally isotropic projective submodule of $M_{1,0}^\perp/M_{1,0}$ such that $(M_{1,0}^\perp/M_{1,0})/M_1' \cong M_{1,0}^\perp/M_1$, which is projective because $M_{1,0}^\vee \oplus M_{1,0}^\perp/M_1 \cong M/M_1$ is projective by assumption. By induction, we may write $M_{1,0}^\perp/M_{1,0}$ as the orthogonal direct sum of $M_1' \oplus (M_1')^\vee$ and the orthogonal complement of $M_1' \oplus (M_1')^\vee$ in $M_{1,0}^\perp/M_{1,0}$. Putting the two orthogonal direct sums together, the result follows. ☐

Proposition 1.2.4.6. *Assumptions on* k *and* Λ *as in Lemma 1.2.4.5, let* $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be a self-dual sufficiently symplectic projective* $\mathcal{O}_R$*-module, and let* M_1 *and* M_2 *be two* **totally isotropic** *projective* $\mathcal{O}_R$*-submodules of* M*, such that* $M_1 \cong M_2$ *and such that* M/M_1 *and* M/M_2 *are both projective. If* $M_1 \oplus M_1^\vee$ *has the same multi-rank as* M*, then there is a symplectic automorphism of* $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ *that sends* M_1 *to* M_2*.*

Proof. By Lemma 1.2.4.5, there exist symplectic isomorphisms

$$\psi_i : (M, \langle\,\cdot\,,\,\cdot\,\rangle) \xrightarrow{\sim} (M_i \oplus M_i^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\text{can.}}) \overset{\perp}{\oplus} ((M_i \oplus M_i^\vee)^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(M_i \oplus M_i^\vee)^\perp})$$

for $i = 1, 2$. Since $M_1 \cong M_2$, $M_i \oplus M_i^\vee$ has the same multi-rank as M for $i = 1, 2$. Hence we have $(M_i \oplus M_i^\vee)^\perp = 0$, and the result follows by taking any isomorphism $f_0 : M_1 \xrightarrow{\sim} M_2$, which induces canonically a symplectic isomorphism $(f_0 \oplus (f_0^\vee)^{-1}) : (M_1 \oplus M_1^\vee) \xrightarrow{\sim} (M_2 \oplus M_2^\vee)$. $\square$

Proposition 1.2.4.7. *Assumptions on k and Λ as above, let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ-algebras, with kernel I satisfying $I^2 = 0$. Let $(\tilde{M}, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a self-dual sufficiently symplectic projective $\mathcal{O}_{\tilde{R}}$-module, and let $(M, \langle\,\cdot\,,\,\cdot\,\rangle) := (\tilde{M}, \langle\,\cdot\,,\,\cdot\,\rangle) \underset{\tilde{R}}{\otimes} R$. Suppose M_1 is a* **totally isotropic** *projective $\mathcal{O}_R$-submodule of M, such that M/M_1 is projective. Then there is a totally isotropic projective $\mathcal{O}_{\tilde{R}}$-submodule $\tilde{M}_1$ of $\tilde{M}$ such that $\tilde{M} \underset{\tilde{R}}{\otimes} R = M$ and such that $\tilde{M}/\tilde{M}_1$ is projective.*

Proof. Since M_1 is projective, it decomposes as $M \cong M_{\tau,R}^{\oplus m_\tau}$ for some multi-rank $m = (m_\tau)$. Let us proceed by induction on $|m| := \sum\limits_{\tau} m_\tau$. When $|m| = 0$, it forces $M_1 = 0$, and there is nothing to prove.

When $|m| \geq 1$, we may decompose $M_1 \cong M_{1,0} \oplus M_1'$ for some projective $\mathcal{O}_R$-modules $M_{1,0}$ and M_1' such that $M_{1,0} \cong M_{\tau,R}$ for some τ with $m_\tau \geq 1$. By Lemma 1.2.4.5, there is an isomorphism $M \xrightarrow{\sim} (M_{1,0} \oplus M_{1,0}^\vee) \overset{\perp}{\oplus} (M_{1,0} \oplus M_{1,0}^\vee)^\perp$, so that M_1' is embedded as a projective submodule of $(M_{1,0} \oplus M_{1,0}^\vee)^\perp \cong M_{1,0}^\perp / M_{1,0}$.

Let $\llparenthesis\,\cdot\,,\,\cdot\,\rrparenthesis$ be the skew-Hermitian pairing associated to $\langle\,\cdot\,,\,\cdot\,\rangle$ by Lemma 1.1.4.6. Let x be any element spanning $M_{1,0}$, and let y be some element spanning $M_{1,0}^\vee$ such that $\llparenthesis x, y \rrparenthesis = 1_{\tau \circ c}$, the identity element of $\mathcal{O}_{\tau \circ c, R}$. Let $\tilde{x}$ and $\tilde{y}$ be any elements in $\tilde{M}$ lifting respectively x and y. Then $\llparenthesis \tilde{x}, \tilde{y} \rrparenthesis =: r$ is an element in $\mathcal{O}_{\tau \circ c, \tilde{R}}$ lifting $\llparenthesis x, y \rrparenthesis = 1_{\tau \circ c} \in \mathcal{O}_{\tau \circ c, R}$. In particular, r is a unit in $\mathcal{O}_{\tau \circ c, \tilde{R}}$. Replacing $\tilde{y}$ by $r^{-1}\tilde{y}$, we may assume that $\llparenthesis \tilde{x}, \tilde{y} \rrparenthesis = 1_{\tau \circ c} \in \mathcal{O}_{\tau \circ c, \tilde{R}}$. Let $\xi := \llparenthesis \tilde{x}, \tilde{x} \rrparenthesis \in \mathcal{O}_{\tilde{R}}$. Since $\llparenthesis x, x \rrparenthesis = 0$ by the assumption that $M_{1,0}$ is totally isotropic, we see that $\xi \in I \cdot \mathcal{O}_{\tilde{R}}$. By Lemma 1.2.4.4 and its proof, and by the assumption that $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating, we see that $\xi = \eta - \eta^\star$ for some $\eta \in I \cdot \mathcal{O}_{\tilde{R}}$. Note that $\eta^\star \in I \cdot \mathcal{O}_{\tilde{R}}$ implies $\eta\eta^\star = 0$. Then $\tilde{x}' := \tilde{x} - \eta\tilde{y}$ is another lifting of x, such that $\llparenthesis \tilde{x}', \tilde{x}' \rrparenthesis = \llparenthesis \tilde{x}, \tilde{x} \rrparenthesis - \eta^\star \llparenthesis \tilde{y}, \tilde{x} \rrparenthesis - \eta \llparenthesis \tilde{x}, \tilde{y} \rrparenthesis + \eta^\star\eta \llparenthesis \tilde{y}, \tilde{y} \rrparenthesis = \xi + \eta^\star - \eta + 0 = 0$. Similarly, we may find another lifting

$\tilde{y}' := \tilde{y} - \zeta\tilde{x}$ for some $\zeta \in I \cdot \mathcal{O}_{\tilde{R}}$ such that $\llparenthesis \tilde{y}', \tilde{y}' \rrparenthesis = 0$. For any of these choices of η and ζ, $\llparenthesis \tilde{x}', \tilde{y}' \rrparenthesis = \llparenthesis \tilde{x}, \tilde{y} \rrparenthesis - \eta \llparenthesis \tilde{y}, \tilde{y} \rrparenthesis - \zeta \llparenthesis \tilde{x}, \tilde{x} \rrparenthesis + \eta\zeta = 1$ because all terms but $\llparenthesis \tilde{x}, \tilde{y} \rrparenthesis$ at the right-hand side lie in $I^2 \cdot \mathcal{O}_{\tilde{R}} = 0$. Let $\tilde{M}_{1,0}$ be the $\mathcal{O}_{\tau,\tilde{R}}$-span of $\tilde{x}'$, and let $\tilde{M}_{1,0}^{\vee}$ be the $\mathcal{O}_{\tau\circ c,\tilde{R}}$-span of $\tilde{y}'$. Then Lemma 1.2.3.9 shows that there is a symplectic isomorphism $\tilde{M} \xrightarrow{\sim} (\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^{\vee}) \overset{\perp}{\oplus} (\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^{\vee})^{\perp}$. In particular, $\tilde{M}_{1,0}$ is a totally isotropic projective submodule lifting $M_{1,0}$, and $(\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^{\vee})^{\perp} \cong \tilde{M}_{1,0}^{\perp}/\tilde{M}_{1,0}$ is a projective self-dual symplectic $\mathcal{O}_{\tilde{R}}$-module lifting $(M_{1,0} \oplus M_{1,0}^{\vee})^{\perp}$.

Note that M_1' is a totally isotropic projective submodule of $M_{1,0}^{\perp}/M_{1,0}$ such that $(M_{1,0}^{\perp}/M_{1,0})/M_1' \cong M_{1,0}^{\perp}/M_1$ is projective. By induction, it can be lifted to a projective $\mathcal{O}_{\tilde{R}}$-submodule $\tilde{M}_1'$ of $\tilde{M}_{1,0}^{\perp}/\tilde{M}_{1,0}$ such that $(\tilde{M}_{1,0}^{\perp}/\tilde{M}_{1,0})/\tilde{M}_1' \cong \tilde{M}_{1,0}^{\perp}/\tilde{M}_1$ is projective. Let $\tilde{M}_1$ be the pre-image of $\tilde{M}_1'$ in $\tilde{M}_{1,0}^{\perp}$, which is a totally isotropic submodule of $\tilde{M}$. It is projective because it is isomorphic to $\tilde{M}_{1,0} \oplus \tilde{M}_1'$, and $\tilde{M}/\tilde{M}_1$ is projective because it is isomorphic to $\tilde{M}_{1,0}^{\vee} \oplus (\tilde{M}_{1,0}^{\perp}/\tilde{M}_1)$. Hence it satisfies all the requirements we want. $\square$

1.2.5 Reflex Fields

Recall that we have assumed (in Definition 1.2.1.3 the Condition 1.2.1.2) that there is an $\mathbb{R}$-algebra homomorphism

$$h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$$

such that

$$\langle h(z)x, y \rangle = \langle x, h(z^c)y \rangle$$

for any $z \in \mathbb{C}$ and $x, y \in L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, and such that the $\mathbb{R}$-bilinear pairing

$$\frac{1}{\sqrt{-1}} \circ \langle \,\cdot\,, h(\sqrt{-1})\,\cdot\, \rangle : (L\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \times (L\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \to \mathbb{R}$$

is symmetric and positive-definite for any choice of $\sqrt{-1}$ in $\mathbb{C}$.

The natural $\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-action on $L\underset{\mathbb{Z}}{\otimes}\mathbb{C}$ may differ from the $\mathbb{C}$-action given by h by a complex conjugation. Let us denote the complex conjugation by $c : \mathbb{C} \to \mathbb{C} : z \mapsto z^c$. Then we can decompose

$$L\underset{\mathbb{Z}}{\otimes}\mathbb{C} = V_0 \oplus V_0^c,$$

where $h(z)$ acts by $1 \otimes z$ on V_0, and by $1 \otimes z^c$ on V_0^c.

Note that both V_0 and V_0^c are totally isotropic submodules of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, because

$$\begin{aligned}\sqrt{-1}\langle x, y\rangle &= \langle (1 \otimes \sqrt{-1})x, y\rangle = \langle h(\sqrt{-1})x, y\rangle = \langle x, h(-\sqrt{-1})y\rangle \\ &= \langle x, (1 \otimes (-\sqrt{-1}))y\rangle = -\sqrt{-1}\langle x, y\rangle\end{aligned}$$

for any $x, y \in V_0$. (The case for any $x, y \in V_0^c$ is similar.) Therefore, since $\langle \,\cdot\,, \,\cdot\,\rangle$ is nondegenerate, it induces a perfect pairing between V_0 and V_0^c, or equivalently an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-linear isomorphism $V_0^c \overset{\sim}{\to} \mathrm{Hom}_{\mathbb{C}}(V_0, \mathbb{C}(1))$. This determines an isomorphism $V_0^c \overset{\sim}{\to} V_0^\vee := \mathrm{Hom}_{\mathbb{C}}(V_0, \mathbb{C})$ for each choice of $\sqrt{-1}$.

Let us denote the unique irreducible representation of $B \underset{F,\tau}{\otimes} \mathbb{C}$ by the notation W_τ as $\mathbb{C}$ is separably closed. Applying Corollary 1.1.2.5 to the $B \underset{\mathbb{Q}}{\otimes} \mathbb{C}$-modules V_0 and V_0^c, we obtain decompositions

$$V_0 \cong \underset{\tau: F \to \mathbb{C}}{\oplus} W_\tau^{\oplus p_\tau}$$

and

$$V_0^c \cong \underset{\tau: F \to \mathbb{C}}{\oplus} W_\tau^{\oplus q_\tau}.$$

Definition 1.2.5.1. *We shall say that the numbers (p_τ) and (q_τ) are respectively the* **signatures** *of V_0 and V_0^c, and we shall say that the pairs of numbers (p_τ, q_τ) are the* **signatures** *of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \,\cdot\,, \,\cdot\,\rangle)$.*

Remark 1.2.5.2. The signatures of V_0 and V_0^c are simply the multi-ranks of respectively V_0 and V_0^c when $\Lambda = k = \mathbb{C}$ in Definition 1.1.3.5.

By construction, the decomposition is determined by the action of the center F and its mappings into $\mathbb{C}$ (as in Corollary 1.1.2.5). Therefore, $p_\tau = q_{\tau \circ c}$, where $c : \mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation. Suppose that $(m_{[\tau]})$ is the multi-rank of L as in Definition 1.2.1.20, then we have the decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong \underset{\tau: F \to \mathbb{C}}{\oplus} W_\tau^{\oplus m_{[\tau]}}$$

with $[\tau]$ determined canonically by τ. In particular, we must have $p_\tau + q_\tau = p_\tau + p_{\tau \circ c} = m_{[\tau]}$ for any τ. Consequently:

Lemma 1.2.5.3. *Let $(L, \langle \cdot , \cdot \rangle)$ be a symplectic $\mathcal{O}$-lattice of multi-rank $(m_{[\tau]})$ (defined as in Definition 1.2.1.20) that satisfies Condition 1.2.1.2. Then $m_{[\tau]}$ is even for every $[\tau]$ such that $F_{[\tau]} = F^+_{[\tau]}$.*

Definition 1.2.5.4. *The* **reflex field** F_0 *of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ *is the* **field of definition** *of V_0 as a complex representation of $B = \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Equivalently, F_0 is the fixed field of $\mathbb{C}$ by the elements σ in* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ *such that V_0 and $V_0 \underset{\mathbb{C},\sigma}{\otimes} \mathbb{C}$ are isomorphic as $B \underset{\mathbb{Q}}{\otimes} \mathbb{C}$-modules.*

Remark 1.2.5.5. The reflex is always defined as a subfield of $\mathbb{C}$.

Let us state the following special cases of results in Section 1.1.2, applied with $C = B$, $E = F$, $k = \mathbb{Q}$, and $K = K^{\mathrm{sep}} = \mathbb{C}$.

Corollary 1.2.5.6. *The reflex field F_0 defined by V_0 (as in Definition 1.2.5.4) can be given by*

$$F_0 = \mathbb{Q}(\mathrm{Tr}_{\mathbb{Q}}(b|V_0) : b \in B) = \mathbb{Q}(\mathrm{Tr}_{\mathbb{Q}}(b|V_0) : b \in \mathcal{O}).$$

Proof. This is a special case of Corollary 1.1.2.12, which is applicable because $\mathrm{char}(\mathbb{Q}) = 0$. ☐

Corollary 1.2.5.7. *If a rational prime number p is unramified in F, then it is unramified in F_0.*

Proof. Note that discriminant of F over $\mathbb{Q}$ is the same as any of its Galois conjugates. Therefore, if p does not divide the discriminant of F over $\mathbb{Q}$, it does not divide the discriminant of F^{Gal} over $\mathbb{Q}$ either. By Corollary 1.1.2.8, F_0 is contained in F^{Gal}. Hence p does not divide the discriminant of F_0 over $\mathbb{Q}$ and the result follows. ☐

Now let us write the decomposition of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ as a *symplectic* isomorphism

$$(L \underset{\mathbb{Z}}{\otimes} \mathbb{C}, \langle \cdot , \cdot \rangle) \overset{\sim}{\to} (V_0 \oplus V_0^\vee, \langle \cdot , \cdot \rangle_{\mathrm{can.}}). \tag{1.2.5.8}$$

(See Definition 1.1.4.11 and Lemma 1.1.4.16.) Note that hidden in this symplectic isomorphism is the isomorphism $\mathbb{C}(1) \overset{\sim}{\to} \mathbb{C}$ and the isomorphism $V_0^c \overset{\sim}{\to} V_0^\vee$. These two isomorphisms determine each other, and are both determined by the same choice of $\sqrt{-1}$. The actual choice of them is immaterial for our purpose.

Now let us consider the surjection

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \twoheadrightarrow V_0 \tag{1.2.5.9}$$

of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules defined by (1.2.5.8) (and the first projection on the right-hand side). Let V_{0,F_0} denote the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0$-module spanned by the image of the $\mathcal{O}$-lattice L under the surjection (1.2.5.9). Then:

Lemma 1.2.5.10. *The $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-span of the image of L under the surjection (1.2.5.9) gives an $\mathcal{O}_{F_0}$-lattice L_0 in V_{0,F_0} invariant under the $\mathcal{O}$-action induced from the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module structure of V_0, together with a natural surjection*

$$L \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0} \twoheadrightarrow L_0, \tag{1.2.5.11}$$

of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-modules inducing the surjection (1.2.5.9) over $\mathbb{C}$.

Corollary 1.2.5.12. *The L_0 defined in Lemma 1.2.5.10 satisfies $\mathrm{Det}_{\mathcal{O}|L_0} = \mathrm{Det}_{\mathcal{O}|V_0}$ as elements in $\mathbb{C}[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. As a result, we can view $\mathrm{Det}_{\mathcal{O}|V_0}$ as an element in $\mathcal{O}_{F_0}[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$ too.*

Proof. This is true because $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \mathbb{C} \cong V_0$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules. ☐

Note that here $\mathrm{Det}_{\mathcal{O}|L_0}$ is defined in the context of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0$-modules, while $\mathrm{Det}_{\mathcal{O}|V_0}$ is defined in the context of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.

Suppose we have a map $\mathcal{O}_{F_0} \to k$, and suppose k is either characteristic zero or a finite field. Suppose $p = \mathrm{char}(k)$ satisfies $p \nmid \mathrm{Disc}$. Set $\Lambda = k$ when $\mathrm{char}(k) = p = 0$, and $\Lambda = W(k)$ when $\mathrm{char}(k) = p > 0$.

Lemma 1.2.5.13. *With assumptions as above, let R be a noetherian local Λ-algebra with residue field k. Let L_0 be the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-module defined in Lemma 1.2.5.10, and let M be an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module that is projective as an R-module. Then the following are equivalent:*

1. *$M \cong L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R$.*

2. *M satisfies $\mathrm{Det}_{\mathcal{O}|M} = \mathrm{Det}_{\mathcal{O}|L_0}$ (as elements in $R[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} R$).*

3. *M satisfies* $\mathrm{Det}_{\mathcal{O}|M} = \mathrm{Det}_{\mathcal{O}|V_0}$ *(as elements in* $R[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} R$*).*

4. $M_0 := M \underset{R}{\otimes} k$ *satisfies* $\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|V_0}$ *(as elements in* $k[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} k$*).*

In statements 3 and 4, we interpret $\mathrm{Det}_{\mathcal{O}|V_0}$ *as push-forwards of an element in* $\mathcal{O}_{F_0}[\mathcal{O}^\vee]$ *by Corollary 1.2.5.12.*

Proof. The implications from statements 1 to 2 and from 3 to 4 are clear. The equivalence between statements 2 and 3 is Corollary 1.2.5.12. It remains to justify the implication from statements 4 to 1. Note that L_0 is an $\mathcal{O}_{F_0}$-lattice by assumption, and since $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$ is maximal by assumption on Λ, we see that $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \Lambda$ is projective as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$-module. By base extension to R, we see that $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R = (L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \Lambda) \underset{\Lambda}{\otimes} R$ is projective as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module. If $\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|V_0}$, then

$$\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|V_0} = \mathrm{Det}_{\mathcal{O}|L_0} = \mathrm{Det}_{\mathcal{O}|(L_0 \underset{\mathcal{O}_{F_0}}{\otimes} k)}$$

over k. Since $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} k$ is separable over k by assumption, we may apply Proposition 1.1.2.16 and conclude that $M_0 \cong L_0 \underset{\mathcal{O}_{F_0}}{\otimes} k$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-modules. Then Lemma 1.1.3.1 implies that $M \cong L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R$, as desired. ☐

Remark 1.2.5.14. Assumptions on k and Λ as above. Let k^{sep} denote a separable closure of k. We have seen in Lemma 1.1.3.4 that projective $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$-modules M admit decompositions $M \cong M_\tau^{\oplus m_\tau}$. If we tensor with k^{sep}, then we obtain a decomposition compatible with the classification of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-modules as in Proposition 1.1.2.16. If we tensor with a separable closure $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ of $\mathrm{Frac}(\Lambda)$, we obtain a decomposition compatible with the classification of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules if we embed $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ into $\mathbb{C}$. This can be achieved by choosing (once and for all) an auxiliary isomorphism $\mathbb{C} \cong \mathbb{C}_p$ if necessary (when $p > 0$), which has the effect of matching the isomorphisms $\mathrm{Hom}_{\mathbb{C}}(F, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{F}_p}(\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p, \bar{\mathbb{F}}_p)$, because F is unramified at p (when $p > 0$). With such a choice in mind, we shall also talk about the decomposition of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-modules using the same set of signatures for $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.

For example, if we take the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathcal{O}_{F_0}$-module L_0 defined in Lemma 1.2.5.10, then we may arrange that $L_0\underset{\mathcal{O}_{F_0}}{\otimes}k$ has the same signatures (p_τ) when considered as an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}k^{\mathrm{sep}}$-module. (This will be useful later when we study the degeneration data for Lie algebra conditions. See Proposition 5.1.2.2 below.)

Let k and Λ be as above, and let $p \geq 0$ be a prime number such that $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[L^{\#} : L]$. Let L_0 be the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathcal{O}_{F_0}$-module defined in Lemma 1.2.5.10. By construction, we have a canonical surjection

$$L\underset{\mathbb{Z}}{\otimes}\Lambda \twoheadrightarrow L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda \tag{1.2.5.15}$$

Since $p \nmid [L^{\#} : L]$, the symplectic $\mathcal{O}_\Lambda$-module $(L\underset{\mathbb{Z}}{\otimes}\Lambda, \langle\,\cdot\,,\,\cdot\,\rangle)$ is self-dual. Since $p \nmid \mathrm{Disc}$, both sides of (1.2.5.15) are projective $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\Lambda$-modules. In particular, the surjection (1.2.5.15) splits, and $L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda$ can be embedded (non-canonically) as a totally isotropic projective submodule of $L\underset{\mathbb{Z}}{\otimes}\Lambda$. The kernel of (1.2.5.15) is projective by definition, and it can be identified canonically with the quotient $(L\underset{\mathbb{Z}}{\otimes}\Lambda)/(L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda) \cong (L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda)^\vee$. Therefore, we have a symplectic isomorphism

$$(L\underset{\mathbb{Z}}{\otimes}\Lambda, \langle\,\cdot\,,\,\cdot\,\rangle) \cong ((L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda)\oplus(L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda)^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}), \tag{1.2.5.16}$$

an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\Lambda$-module analogue of (1.2.5.8).

Let us make use of the isotropic submodule $(L_0\underset{\mathcal{O}_{F_0}}{\otimes}\Lambda)^\vee$ of $L\underset{\mathbb{Z}}{\otimes}\Lambda$ and define a *parabolic subgroup* of the group $\mathrm{G}\underset{\mathbb{Z}}{\otimes}\Lambda$ as follows:

Definition 1.2.5.17. *For any Λ-algebra R, define a subgroup functor* P_0 *of* $\mathrm{G}_\Lambda := \mathrm{G}\underset{\mathbb{Z}}{\otimes}\Lambda$ *by*

$$\mathrm{P}_{0,\Lambda}(R) := \{g \in \mathrm{G}_\Lambda(R) : g((L_0\underset{\mathcal{O}_{F_0}}{\otimes}R)^\vee) = (L_0\underset{\mathcal{O}_{F_0}}{\otimes}R)^\vee\}.$$

Remark 1.2.5.18. This terminology of a parabolic subgroup is not fully justified until we prove Lemma 1.2.5.22 and Corollary 1.2.5.23 below.

Lemma 1.2.5.19. *For any complete noetherian local Λ-algebra R, then the association*

$$g \mapsto g((L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee)$$

for $g \in \mathrm{G}_\Lambda(R)$ induces a **bijection** *from the set of R-valued points $(\mathrm{G}_\Lambda/\mathrm{P}_{0,\Lambda})(R)$ of the flag variety $\mathrm{G}_\Lambda/\mathrm{P}_{0,\Lambda}$ (defined as a quotient of functors) to the set of totally isotropic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-submodules M of $(L\underset{\mathbb{Z}}{\otimes}R, \langle\,\cdot\,,\,\cdot\,\rangle)$ such that $M \cong (L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee$ and such that $(L\underset{\mathbb{Z}}{\otimes}R)/M$ is projective as an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module.*

Proof. By Lemma 1.2.4.3, $(L\underset{\mathbb{Z}}{\otimes}R, \langle\,\cdot\,,\,\cdot\,\rangle)$ is sufficiently symplectic (defined as in Definition 1.2.4.1) because it is the pullback of the symplectic module $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ defined over $\mathbb{Z}$. Then Proposition 1.2.4.6 is applicable, and the result follows because $(L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee$ is a totally isotropic submodule of $L\underset{\mathbb{Z}}{\otimes}R$, because $(L\underset{\mathbb{Z}}{\otimes}R)/(L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee \cong (L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)$ is projective as an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module, and because their multi-ranks sum up to the multi-rank of $L\underset{\mathbb{Z}}{\otimes}R$. ☐

Proposition 1.2.5.20. *The flag variety $\mathrm{G}_\Lambda/\mathrm{P}_{0,\Lambda}$ is formally smooth over Λ.*

Proof. Let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ-algebras, with kernel I satisfying $I^2 = 0$. Suppose we have a translate $g((L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee)$ of $L\underset{\mathbb{Z}}{\otimes}R$ by some element $g \in \mathrm{G}_\Lambda(R)$. Proposition 1.2.4.7 applies because $(L\underset{\mathbb{Z}}{\otimes}\tilde{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$ is sufficiently symplectic (as explained in the proof of Lemma 1.2.5.19 above). Hence there is a totally isotropic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\tilde{R}$-submodule M of $L\underset{\mathbb{Z}}{\otimes}\tilde{R}$ lifting $g((L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee)$, such that $(L\underset{\mathbb{Z}}{\otimes}\tilde{R})/M$ is projective as an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module. Now the result follows from Lemma 1.2.5.19. ☐

Remark 1.2.5.21. We shall see in Remark 2.2.4.12 and in the proof of Proposition 2.2.4.11 that Lemma 1.2.5.19 and Proposition 1.2.5.20 will imply the formal smoothness of our moduli problems (to be defined later in Section 1.4).

Proposition 1.2.5.22. *With assumptions on k and Λ as above, the group $\mathrm{P}_{0,\Lambda}$ is smooth.*

Proof. If $L = \{0\}$, then $\mathrm{P}_{0,\Lambda} = \mathrm{G}_\Lambda = \mathbf{G}_{\mathrm{m},\Lambda}$ and the result is clear. Hence we may assume that $L \neq \{0\}$.

Let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ-algebras, with kernel I satisfying $I^2 = 0$. Let us denote by $\tilde{M} := L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \tilde{R}$, and $M := \tilde{M} \underset{\tilde{R}}{\otimes} R$. Then $\tilde{M}^\vee$ and $M^\vee$ embed as totally isotropic submodules of respectively $L \underset{\mathbb{Z}}{\otimes} \tilde{R}$ and $L \underset{\mathbb{Z}}{\otimes} R$, and we have canonical isomorphisms $(L \underset{\mathbb{Z}}{\otimes} \tilde{R})/\tilde{M}^\vee \cong \tilde{M}$ and $(L \underset{\mathbb{Z}}{\otimes} R)/M^\vee \cong M$. Let us take any isomorphism $\tilde{\psi} : (\tilde{M} \oplus \tilde{M}^\vee, \langle \,\cdot\,, \,\cdot\, \rangle_{\mathrm{can.}}) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} \tilde{R}, \langle \,\cdot\,, \,\cdot\, \rangle)$, and let $\psi := \tilde{\psi} \underset{\tilde{R}}{\otimes} R$.

Let $g \in \mathrm{P}_{0,\Lambda}(R)$. Using ψ, we see that there are three maps $\alpha \in \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M)$, $\beta \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M^\vee, M)$, and $\gamma \in \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M^\vee)$ such that $\psi(g(\psi^{-1}(x, f))) = (\alpha(x) + \beta(f), \gamma(f))$ for any $x \in M$ and any $f \in M^\vee$. For simplicity, let us suppress ψ in the following notations. Then it is convenient to express the above relation in matrix form as $g = \left(\begin{smallmatrix} \alpha & \beta \\ & \gamma \end{smallmatrix}\right)$. In this case, the relation

$$\begin{aligned} \langle (x_1, f_1), g(x_2, f_2) \rangle &= \langle (x_1, f_1), (\alpha x_2 + \beta f_2, \gamma f_2) \rangle \\ &= (\gamma f_2)(x_1) - f_1(\alpha x_2 + \beta f_2) \\ &= f_2(\gamma^\vee x_1) - (\alpha^\vee f_1)(x_2) - (\beta^\vee f_1)(f_2) \\ &= \langle (\gamma^\vee x_1 - \beta^\vee f_1, \alpha^\vee f_1), (x_2, f_2) \rangle \\ &= \langle g^{\maltese}(x_1, f_1), (x_2, f_2) \rangle \end{aligned}$$

shows that we have $g^{\maltese} = \left(\begin{smallmatrix} \gamma^\vee & -\beta^\vee \\ & \alpha^\vee \end{smallmatrix}\right)$, and the relation $g^{\maltese} g = r \in R^\times$ becomes

$$\begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix} \begin{pmatrix} \gamma^\vee & -\beta^\vee \\ & \alpha^\vee \end{pmatrix} = \begin{pmatrix} \alpha\gamma^\vee & -\alpha\beta^\vee + \beta\alpha^\vee \\ & \gamma\alpha^\vee \end{pmatrix} = \begin{pmatrix} r & \\ & r \end{pmatrix}.$$

Any such g can be decomposed as

$$\begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ & (\alpha^\vee)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ & 1 \end{pmatrix},$$

where β is symmetric in the sense that $\beta^\vee = \beta$. Then each of the three terms in the product is also an element of $\mathrm{P}_{0,\Lambda}(R)$. Therefore it suffices to show that we can lift each of the three kinds of elements.

If $g = \left(\begin{smallmatrix} 1 & 0 \\ & r \end{smallmatrix}\right)$ for some $r \in R^\times$, then any lifting $\tilde{r} \in \tilde{R}$ of r is a unit, and hence defines a lifting $\tilde{g} := \left(\begin{smallmatrix} 1 & 0 \\ & \tilde{r} \end{smallmatrix}\right)$ of g.

If $g = \left(\begin{smallmatrix} \alpha & 0 \\ & (\alpha^\vee)^{-1} \end{smallmatrix}\right)$ for some invertible $\alpha \in \operatorname{End}_{\mathcal{O}_R}(M)$, which is a product of matrix algebras over $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R$, then any lifting $\tilde{\alpha}$ of α in $\operatorname{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R}}(\tilde{M})$ is necessarily invertible. (This is basically NAK (Nakayama's lemma).) Then $\tilde{g} := \left(\begin{smallmatrix} \tilde{\alpha} & 0 \\ & (\tilde{A}^\vee)^{-1} \end{smallmatrix}\right)$ defines a lifting of g.

Suppose that $g = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$, where $\beta^\vee = \beta \in \operatorname{Hom}_{\mathcal{O}_R}(M^\vee, M)$. By Lemma 1.2.3.1, we may replace $\tilde{M}$ by some projective $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$-module $\tilde{N}$, and assume that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R} = \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$. Then elements in $\operatorname{Hom}_{\mathcal{O}_{\tilde{R}}}(\tilde{M}^\vee, \tilde{M})$ can be represented in block matrix form with entries in $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$, so that the formation of dual is simply $X \mapsto {}^tX^c$. (Here c is the restriction of * to $\mathcal{O}_F$.) Hence the question of lifting β is simply a question of lifting a matrix with the condition ${}^tX^c = X$. For an entry of the matrix above the diagonal, any lifting would do. Then they determine the liftings of the entries below the diagonal. For an entry along the diagonal, being invariant implies that they lie in $\mathcal{O}_{F^+} \underset{\mathbb{Z}}{\otimes} R$, which can certainly be lifted to $\mathcal{O}_{F^+} \underset{\mathbb{Z}}{\otimes} \tilde{R}$. Hence there is an element $\tilde{\beta}$ in $\operatorname{Hom}_{\mathcal{O}_{\tilde{R}}}(\tilde{M}^\vee, \tilde{M})$ such that $\tilde{\beta}^\vee = \tilde{B}$, and $\tilde{g} := \left(\begin{smallmatrix} 1 & \tilde{B} \\ & 1 \end{smallmatrix}\right)$ defines a lifting of g, as desired. $\square$

Corollary 1.2.5.23. *With assumptions on k and Λ as above, the group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda$ is smooth.*

Proof. This follows from Propositions 1.2.5.20 and 1.2.5.22. $\square$

Remark 1.2.5.24. This is a special case of Corollary 1.2.3.12, with an alternative reasoning.

1.2.6 Filtrations

The notions of filtrations that we are about to define will not be needed before Section 5.2.2.

Let R be any noetherian $\mathbb{Z}$-algebra. In this section we define the filtrations that we shall consider on an integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module (defined as in Definition 1.2.1.19).

Let M be an $\mathcal{O}$-lattice and let $M \underset{\mathbb{Z}}{\otimes} R$ be the integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module it defines. Suppose we are given an increasing filtration $\mathtt{F} = \mathtt{F}_{-i}$ indexed by non-positive integers $-i$ consisting of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$-sublattices $\mathtt{F}_{-i}$ of $\mathtt{F}_0 := M\underset{\mathbb{Z}}{\otimes}R$. For our purpose we shall never need to consider arbitrary filtrations of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$-modules.

The first property we should require is that the graded pieces $\mathrm{Gr}^{\mathtt{F}}_{-i}$ are also integrable for each i.

Definition 1.2.6.1. *We say that a filtration* $\mathtt{F} := \{\mathtt{F}_{-i}\}$ *of* $M\underset{\mathbb{Z}}{\otimes}R$ *is* **integrable** *if* $\mathrm{Gr}^{\mathtt{F}}_{-i}$ *is integrable as an* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$*-module* (*defined as in Definition 1.2.1.19*) *for any* i.

A similar definition is:

Definition 1.2.6.2. *We say that a filtration* $\mathtt{F} := \{\mathtt{F}_{-i}\}$ *of* $M\underset{\mathbb{Z}}{\otimes}R$ *is* **projective** *if* $\mathrm{Gr}^{\mathtt{F}}_{-i}$ *is projective as an* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$*-module for any* i.

Definition 1.2.6.3. *We say that a filtration* $\mathtt{F}$ *of* $M\underset{\mathbb{Z}}{\otimes}R$ *is* **split** *if there exists* (*noncanonically*) *some isomorphism* $\mathrm{Gr}^{\mathtt{F}} := \oplus\,\mathrm{Gr}^{\mathtt{F}}_{-i} \xrightarrow{\sim} \mathtt{Z}_0$.

If $\mathcal{O}$ is hereditary, which means every $\mathcal{O}$-lattice is projective, then every integrable filtration $\mathtt{F}$ is *projective*. Note that $\mathcal{O}$ is hereditary, for example, when $\mathcal{O}$ is maximal. (See Proposition 1.1.1.20.) Then it is obvious that:

Lemma 1.2.6.4. *Any projective filtration is automatically split.*

Proof. Simply split the exact sequences $0 \to \mathtt{Z}_{-i-1} \to \mathtt{Z}_{-i} \to \mathrm{Gr}^{\mathtt{F}}_{-i} \to 0$ one by one in increasing order of i. $\square$

Corollary 1.2.6.5. *If* $\mathcal{O}$ *is maximal, in which case* $\mathcal{O}$ *is hereditary by Proposition 1.1.1.20, then any integrable filtration* $\mathtt{F}$ *of an integrable* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$*-module* $M\underset{\mathbb{Z}}{\otimes}R$ *is automatically admissible.*

In general, when $\mathcal{O}$ is not necessarily hereditary, we do not expect all filtrations to split. It is not impossible to handel those situations, but it will not be necessary for our purse of compactifications. We will see later (in Lemma 5.2.2.4) that for our purpose of compactifications it suffices to study those filtrations that do split.

Definition 1.2.6.6. *A filtration* $\mathtt{F}$ *of an integrable* $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$*-module* $M\underset{\mathbb{Z}}{\otimes} R$ *as above is called* **admissible** *if it is integrable and split* (*defined as in Definitions 1.2.6.1 and 1.2.6.3*).

When $\mathcal{O}$ is maximal, this just means the filtration is integrable.

It is convenient to define the following related notions as well:

Definition 1.2.6.7. *Let* R *be any noetherian* $\mathbb{Z}$*-algebra. We say that a surjection* $\mathtt{F} \twoheadrightarrow \mathtt{F}'$ (*resp. a submodule* $\mathtt{F}''$ *of* $\mathtt{F}$*, resp. an embedding* $\mathtt{F}'' \hookrightarrow \mathtt{F}$) *of integrable* $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$*-modules is* **admissible** *if* $0 \subset \ker(\mathtt{F} \twoheadrightarrow \mathtt{F}') \subset \mathtt{F}$ (*resp.* $0 \subset \mathtt{F}'' \subset \mathtt{F}$*, resp.* $0 \subset \mathrm{image}(\mathtt{F}'' \hookrightarrow \mathtt{F}) \subset \mathtt{F}$) *is an admissible filtration of* $\mathtt{F}$*.*

When M is equipped with a pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ so that $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ defines a symplectic $\mathcal{O}$-lattice in the sense of Definition 1.2.1.3, we would like to consider also those filtrations that respect this pairing.

Definition 1.2.6.8. *Given a PEL-type* $\mathcal{O}$*-lattice* $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$*, a filtration* $\mathtt{F} := \{\mathtt{F}_{-i}\}$ *of* $\mathtt{F}_0$ *is called* **symplectic** *if there is an integer* $-k \leq 0$ *such that* $\mathtt{Z}_{-k+i}$ *and* $\mathtt{Z}_{-i}$ *are annihilators of each other under the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *on* $\mathtt{F}_0 = M\underset{\mathbb{Z}}{\otimes} R$*.*

Definition 1.2.6.9. *In our work, the choice of the integer* k *should be clear from the context and never arbitrary. For example, when we consider the notion of symplectic admissible filtrations in Chapter 5 later, they will be always of the form* $0 = \mathtt{F}_{-3} \subset \mathtt{F}_{-2} \subset \mathtt{F}_{-1} \subset \mathtt{F}_0$*. That is, we shall take* $k = 3$ *in Definition 1.2.6.8, in which case the condition is that* $\mathtt{F}_{-2}$ *and* $\mathtt{F}_{-1}$ *are the annihilators of each other under the pairing.*

Remark 1.2.6.10. An arbitrary splitting given for example by Lemma 1.2.6.4 has no reason to respect the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, and in general there might be no splitting that does respect the pairing. The definition of a pairing on $\mathrm{Gr}^{\mathtt{F}}$ will require some additional work. (See for example Section 5.2.2.)

1.3 Geometric Structures

1.3.1 Abelian Schemes and Quasi-Isogenies

Definition 1.3.1.1. *An* **abelian scheme** *over a base scheme* S *is a group scheme* $\pi : A \to S$ *that is proper, smooth, and with geometrically connected fibers.*

Remark 1.3.1.2. Since properness implies being of finite type by definition ([48, II, 5.4.1]), and since being of finite type means quasi-compactness and being locally of finite type ([46, I, 6.6.3]), we see that proper morphisms are in particular quasi-compact. On the other hand, smoothness implies locally of finite presentation, which can be taken as part of the definition ([50, IV, 6.8.1]): A morphism is *smooth* if it is locally of finite presentation, flat, and has fibers that are regular at each point. As a result, we see that an abelian scheme is automatically of finite presentation over its base scheme. Hence the technique of reduction to the noetherian case of [51, IV, §8] can be applied. Let us include in particular the following important theorem:

Theorem 1.3.1.3 (see [51, IV, 8.2.2, 8.9.1, 8.9.5, 8.10.5, and 17.7.8])**.** *Suppose S_0 is a quasi-compact scheme, and $S = \varprojlim_{i \in I} S_i$ is a projective system of schemes S_i that are affine over S_0, labeled by some partially ordered index set I.*

1. *Suppose X is a* **scheme** *of finite presentation over S, then there exists an index $i \in I$ and a scheme X_i of finite presentation over S_i such that $X \cong X_i \underset{S_i}{\times} S$. In this case, we can define for any $j \geq i$ in I a scheme $X_j := X_i \underset{S_i}{\times} S_j$.*

2. *Suppose X, i, and X_i are as above and $\mathscr{M}$ is a* **quasi-coherent sheaf** *of modules of finite presentation over $\mathscr{O}_X$. Then there exists an index $j \geq i$ in J and a quasi-coherent sheaf $\mathscr{M}_j$ of modules of finite presentation over $\mathscr{O}_{X_j}$ such that $\mathscr{M} \cong \mathscr{M}_j \underset{\mathscr{O}_{X_j}}{\otimes} \mathscr{O}_X$.*

3. *Suppose Y is* **another scheme** *of finite presentation over S, with some index $i \in I$ and schemes X_i and Y_i of finite presentation such that $X \cong X_i \underset{S_i}{\times} S$ and $Y \cong Y_i \underset{S_i}{\times} S$. Define for any $j \geq i$ in I schemes $X_j := X_i \underset{S_i}{\times} S_j$ and $Y_j := Y_i \underset{S_i}{\times} S_j$. Then the canonical morphism*
$$\varinjlim_{j \in I, j \geq i} \underline{\mathrm{Hom}}_{S_j}(X_j, Y_j) \to \underline{\mathrm{Hom}}_S(X, Y)$$
is a bijection.

4. *In the context above, suppose $f : X \to Y$ is a* **morphism** *satisfying any of the following properties:*

(a) an isomorphism, a monomorphisms, an immersion, an open immersion, a closed immersion, or surjective;

(b) finite, quasi-finite, or proper;

(c) projective, or quasi-projective;

(d) flat (for some quasi-coherent sheaf of finite presentation); or

(e) unramified, étale, or smooth.

Then there exist some $j \geq i$ *and some morphism* $f_j := X_j \to Y_j$ *over* S_j *with the same property such that* $f = f_j \underset{S_j}{\times} S$. *(In the case of being flat for some quasi-coherent sheaf, it means the quasi-coherent sheaf is the pullback of a quasi-coherent sheaf over* X_j *for which* f_j *is flat.)*

We say for simplicity that the properties above are **of finite presentation**.

As a result, we may reduce problems concerning schemes, modules, and/or morphisms of finite presentation to the case where the base scheme is locally noetherian.

The underlying schemes of abelian schemes enjoy a rather strong rigidity property, which can be described by a special case of the following:

Proposition 1.3.1.4 (rigidity lemma; see [97, Prop. 6.1])**.** *Suppose we are given a diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"] & & Y \arrow[dl, "q"] \\
& S \arrow[ul, bend left, "e"] &
\end{tikzcd}$$

of locally noetherian schemes such that the base scheme S *is connected, such that the structural morphism* $p : X \to S$ *is closed and flat with a section* e*, and such that the natural map* $\mathscr{O}_S \to p_*\mathscr{O}_X$ *induced by* p *is an isomorphism. Suppose there exists a point* $s \in S$ *such that* $f(X_s)$ *is set-theoretically a single point. Then* $f = f \circ e \circ p = \eta \circ p$ *for the section* $\eta : S \to Y$ *of* q *defined by* $\eta = f \circ e$.

For the convenient of the reader, let us include a proof. The proofs we shall present for Proposition 1.3.1.4 and for the following corollaries are originally due to Mumford in [97], with slight rewording by us.

Proof of Proposition 1.3.1.4. First suppose that S has only one point s. Then $f = f \circ p \circ e$ holds as a *topological map* by the assumption that $f(X_s)$ is set-theoretically just one point, and we have identifications $f_* \mathscr{O}_X = f_* e_* \mathscr{O}_S$ and $f_* \mathscr{O}_X = f_* e_* p_* \mathscr{O}_X$ as the push-forward is defined by topological maps. To show $f = f \circ p \circ e$ as a scheme-theoretic map, we need to show that $f_\# : \mathscr{O}_Y \to f_* \mathscr{O}_X$ and $(f \circ e \circ p)_\# : \mathscr{O}_Y \to f_* e_* p_* \mathscr{O}_X$ agree as morphisms of sheaves under the identification $f_* \mathscr{O}_X = f_* e_* p_* \mathscr{O}_X$. By assumption, $p_\# : \mathscr{O}_S \to p_* \mathscr{O}_X$ is an isomorphism, therefore we only need to show that $f_\# : \mathscr{O}_Y \to f_* \mathscr{O}_X$ and $(f \circ e)_\# : \mathscr{O}_Y \to f_* e_* \mathscr{O}_S$ agree under the identification $f_* \mathscr{O}_X = f_* e_* \mathscr{O}_S$. But this is trivially true by assumption, as we only need to compare the stalks at the image of e under f. Hence the proposition holds when s has only one point.

In general, let Z be the largest closed subscheme of X over which $f = \eta \circ p$. Note that Z is closed as it is the pullback of the diagonal of $Y \underset{S}{\times} Y$ via $(f, \eta \circ p) : X \to Y \underset{S}{\times} Y$. By assumption it contains the fiber $p^{-1}(s)$. We claim that $Z = X$. By the first part of the proof, we know that, for any Artinian local subscheme $T \subset S$ containing s, Z contains $p^{-1}(T)$ as a subscheme. Since p is a closed map, by taking the image under p of a closed subset of X complementing $p^{-1}(T)$, we see that Z contains $p^{-1}(U_0)$ for some open neighborhood U_0 of s. Let U_1 be the maximal subscheme of S such that $p^{-1}(U_1)$ is a subscheme of Z. Then the above argument applied to an arbitrary point t of U_1 shows that U_1 is open. On the other hand, Since Z is closed, and p is an open map as it is flat, we see that $p(X - Z)$ is also open. Now simply note that $p(X - Z)$ and U_1 cover the underlying topological space of S, and U_1 is nonempty. Hence by connectedness of S we know that $p(X - Z)$ is empty, and the proposition follows. □

Note that we do not need the group structure of X in this proposition.

Corollary 1.3.1.5 ([97, Cor. 6.2])**.** *Let A be an abelian scheme and G a group scheme of finite presentation over a connected scheme S. Let f and g be two morphisms of schemes making the following diagram commutative:*

$$\begin{tikzcd}
A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

Let $m_G : G \underset{S}{\times} G \to G$ *denote the multiplication map of* G*. Suppose that, for some point* $s \in S$*, the morphisms* f_s *and* g_s *from* A_s *to* G_s *are equal. Then there is a section* $\eta : S \to G$ *such that*

$$f = m_G \circ ((\eta \circ p) \times g).$$

Proof. By Theorem 1.3.1.3, we may assume that all A, G, and S are locally noetherian. Then we apply Proposition 1.3.1.4 to $m_G \circ (f \times ([-1]_G \circ g))$, where $[-1]_G : G \to G$ is the inverse morphism of the group scheme G. ☐

Corollary 1.3.1.6 ([97, Cor. 6.4])**.** *Let* A *be an abelian scheme and* G *any group scheme of finite presentation over a base scheme* S*. If* $f : A \to G$ *is a morphism over* S *taking the identity* e_A *of* A *to the identity* e_G *of* G*, then* f *is a homomorphism.*

Proof. By Theorem 1.3.1.3, we may assume that all A, G, and S are locally noetherian. Let us denote the multiplication map of A by m_A, and let us define two morphisms $g, h : A \underset{S}{\times} A \to A \underset{S}{\times} G$ by respectively $g(x_1, x_2) = (x_1, f(m_A(x_1, x_2)))$ and $h(x_1, x_2) = (x_1, f(x_2))$ for any functorial points x_1 and x_2 of A. By Corollary 1.3.1.5, applied to the commutative diagram

$$\begin{tikzcd}
A \underset{S}{\times} A \arrow[rr, shift left, "g"] \arrow[rr, shift right, "h"'] \arrow[dr, "\mathrm{pr}_1"'] & & A \underset{S}{\times} G \arrow[dl, "\mathrm{pr}_1"] \\
& A &
\end{tikzcd},$$

there exists some morphism $\eta : A \to G$ such that $f(m_A(x_1, x_2)) = m_G(\eta(x_1), f(x_2))$. By putting $x_2 = e_A$, we get $f(x_1) = \eta(x_1)$, and hence $f(m_A(x_1, x_2)) = m_G(f(x_1), f(x_2))$. This shows f is a group homomorphism. ☐

Corollary 1.3.1.7 ([97, Cor. 6.5])**.** *If* A *is an abelian scheme over a scheme* S*, then* A *is a commutative group scheme.*

Proof. Apply Corollary 1.3.1.6 to the inverse morphism $[-1]_A : A \to A$. ☐

Corollary 1.3.1.8 ([97, Cor. 6.6])**.** *If* A *is an abelian scheme over* S*, then there is only one structure of group scheme on* A *over* S *with the given identity* $e_A : S \to A$*.*

Proof. Applying Corollary 1.3.1.6 to the identity map from A to A, we see that any two possible group structures are identical. □

Definition 1.3.1.9. *An* **isogeny** $f : G \to G'$ *of smooth group schemes over* S *is a group scheme homomorphism over* S *that is* **surjective** *and* **with quasi-finite kernel**.

Definition 1.3.1.10. *An* **isogeny** $f : A \to A'$ *of abelian schemes over* S *is an isogeny of smooth group schemes from* A *to* A'.

Remark 1.3.1.11. In Definition 1.3.1.10, since the kernel $\ker(f)$ is proper flat, it is automatically finite flat. As a result, its rank is well-defined as a constant on each connected component of S.

Definition 1.3.1.12. *Two abelian schemes* A *and* A' *are* **isogenous** *if there exists an isogeny from* A *to* A'.

In what follows, unless otherwise specified (or clear from the context), we shall consider only isogenies between abelian schemes.

Now let $\Box$ be an arbitrary set of finite rational primes.

Definition 1.3.1.13. *An isogeny* $f : A \to A'$ *is* **prime-to-$\Box$** *if the rank of* $\ker(f)$ (*as a finite flat group scheme*) *is prime-to-*$\Box$.

Definition 1.3.1.14. *A* **quasi-isogeny** $f : A \to A'$ *of abelian schemes over* S *is an equivalence class of triples* (B, g, h)*, where* $g : B \to A$ *and* $h : B \to A'$ *are isogenies between abelian schemes over* S *as in the diagram*

$$\begin{tikzcd}
& B \arrow[ld, "g"'] \arrow[rd, "h"] & \\
A \arrow[rr, dotted, "f"'] & & A'
\end{tikzcd},$$

and where two triples (B, g, h) *and* (B', g', h') *are considered equivalent if there is a third triple* (C, i, j) *fitting into the diagram*

$$\begin{tikzcd}
& & C \arrow[ld, "i"'] \arrow[rd, "j"] & & \\
& B \arrow[ld, "g"'] \arrow[rrrd, "h"] & & B' \arrow[llld, "g'"'] \arrow[rd, "h'"] & \\
A \arrow[rrrr, dotted, "f"'] & & & & A'
\end{tikzcd}$$

and making all the solid arrows commute.

Definition 1.3.1.15. *A quasi-isogeny $f : A \to A'$ of abelian schemes over S is* **prime-to-□** *if it can be represented by a triple (B, g, h) as in Definition 1.3.1.14 such that g and h are both prime-to-□ isogenies. We shall often call a prime-to-□ isogeny a* **$\mathbb{Z}^{\times}_{(\square)}$-isogeny**. (*In particular, we shall call a quasi-isogeny a $\mathbb{Q}^{\times}$-isogenies.*)

Lemma 1.3.1.16. *The natural map from the category of isogenies (resp. prime-to-□ isogenies) to the category of quasi-isogenies (resp. prime-to-□ quasi-isogenies), defined by sending an isogeny $f : A \to A'$ to the class containing a triple (A, Id_A, f), is fully faithful.*

Therefore we have the natural picture for any □:

$$\begin{tikzcd}
\{\text{prime-to-}\square\text{ isogenies}\} \arrow[r, hook] \arrow[d, hook] & \{\text{isogenies}\} \arrow[d, hook] \\
\{\mathbb{Z}^{\times}_{(\square)}\text{-isogenies}\} \arrow[r, hook] & \{\mathbb{Q}^{\times}\text{-isogenies}\}
\end{tikzcd}$$

In particular, it makes sense to say whether a $\mathbb{Z}^{\times}_{(\square)}$-isogeny is an isogeny, or in particular whether a $\mathbb{Z}^{\times}_{(\square)}$-isogeny is an isomorphism.

Remark 1.3.1.17. The composition of two $\mathbb{Z}^{\times}_{(\square)}$-isogenies can be defined as in the diagram:

$$\begin{tikzcd}
 & & B \underset{A'}{\times} B' \arrow[dl] \arrow[dr] & & \\
 & B \arrow[dl] \arrow[dr] & & B' \arrow[dl] \arrow[dr] & \\
A \arrow[rr, dotted] & & A' \arrow[rr, dotted] & & A''
\end{tikzcd}$$

Remark 1.3.1.18. Every $\mathbb{Z}^{\times}_{(\square)}$-isogeny is invertible: The inverse of (B, g, h) is simply by (B, h, g), in the sense that their composition is (equivalent to) an isomorphism.

Remark 1.3.1.19. In Definition 1.3.1.14, we may assume that g is the multiplication by N on A, for some integer N. Equivalently, this means that $f \circ [N] = [N] \circ f$ is an isogeny $h : A \to A'$ for some integer N. We shall write $f = N^{-1}h$ in this case. This makes sense because $[N]$ is invertible in the category of $\mathbb{Q}^{\times}$-isogenies.

Remark 1.3.1.20. In Definition 1.3.1.15, we may assume that g is the multiplication by N on A for some integer N prime-to-$\Box$, and that h is a prime-to-$\Box$ isogeny. Equivalently, this means that $f \circ [N] = [N] \circ f$ is a prime-to-$\Box$ isogeny h for some integer N prime-to-$\Box$. We shall write $f = N^{-1}h$ in this case. This makes sense because $[N]$ is invertible in the category of $\mathbb{Z}^\times_{(\Box)}$-isogenies.

Remark 1.3.1.21. The isogenies and prime-to-$\Box$ isogenies from A to A' lie in $\mathrm{Hom}(A, A')$. The $\mathbb{Q}^\times$-isogenies from A to A' lie in $\mathrm{Hom}(A, A') \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. The $\mathbb{Z}^\times_{(\Box)}$-isogenies from A to A' lie in $\mathrm{Hom}(A, A') \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. But the converse is certainly not true in general even when we exclude the zero homomorphism. (Think about homomorphisms that are not surjective, which exist in abundance when we are not just talking about simple abelian schemes.)

1.3.2 Polarizations

Definition 1.3.2.1. *Let A be an abelian scheme over S with unit-section $e : S \to A$.*

1. *For any invertible sheaf $\mathcal{L}$, a* **rigidification** *of $\mathcal{L}$ is an isomorphism $\xi : \mathscr{O}_S \xrightarrow{\sim} e^*\mathcal{L}$.*

2. *The* **relative Picard functor** *is defined by*

$$\underline{\mathrm{Pic}}(A/S) : T/S \mapsto \frac{\{\text{invertible sheaves } \mathcal{L} \text{ on } A \underset{S}{\times} T\}}{\left\{\begin{array}{c}\text{invertible sheaves of the form } \mathrm{pr}_2^*(\mathcal{M}),\\ \text{for some } \mathcal{M} \text{ on } S\end{array}\right\}},$$

which is canonically isomorphic to

$$\underline{\mathrm{Pic}}_e(A/S) : T/S \mapsto \left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of invertible sheaves } \mathcal{L} \text{ on } A \underset{S}{\times} T\\ \text{with rigidifications along } e_T = e \underset{S}{\times} T\end{array}\right\}.$$

3. The subfunctor $\underline{\mathrm{Pic}}^0$ *of* $\underline{\mathrm{Pic}}$ *is defined by*

$$\underline{\mathrm{Pic}}^0(A/S) : T/S \mapsto \frac{\left\{\begin{array}{c}\text{invertible sheaves } \mathcal{L} \text{ on } A \underset{S}{\times} T \\ \text{such that for all } t \in T,\ \mathcal{L}_t \text{ is} \\ \text{algebraically equivalent to zero on } A_t\end{array}\right\}}{\left\{\begin{array}{c}\text{invertible sheaves as above} \\ \text{that are of the form } \mathrm{pr}_2^*(\mathcal{M}), \\ \text{for some } \mathcal{M} \text{ on } S\end{array}\right\}},$$

which is canonically isomorphic to

$$\underline{\mathrm{Pic}}^0_e(A/S) : T/S \mapsto \left\{\begin{array}{c}\text{isomorphism classes} \\ \text{of invertible sheaves } \mathcal{L} \text{ on } A \underset{S}{\times} T \\ \text{with rigidifications along } e_T \text{ such that} \\ \text{for all } t \in T,\ \mathcal{L}_t \text{ is} \\ \text{algebraically equivalent to zero on } A_t\end{array}\right\}.$$

Remark 1.3.2.2. Since A has a section e, the sheaf $\underline{\mathrm{Pic}}(A/S)$ is canonically isomorphic to $\underline{\mathrm{Pic}}_e(A/S)$, and the sheaf $\underline{\mathrm{Pic}}^0(A/S)$ is canonically isomorphic to $\underline{\mathrm{Pic}}^0_e(A/S)$. (See [97, Ch. 0, §5].)

Remark 1.3.2.3. Let $f : A \to S$ be the structural morphism. Then it follows from general theory (which we shall include in Theorem 3.1.2.5 below) that $\underline{\mathrm{Pic}}(A/S) \cong R^1 f_{*\mathrm{fppf}}\mathbf{G}_{\mathrm{m}}$. Therefore, we may interpret $\underline{\mathrm{Pic}}(A/S)$ (resp. $\underline{\mathrm{Pic}}^0(A/S)$) as the fppf sheaf associated to the fppf presheaf $T/S \mapsto \mathrm{Pic}(A \underset{S}{\times} T)$ (resp. $T/S \mapsto \mathrm{Pic}^0(A \underset{S}{\times} T)$).

Remark 1.3.2.4. By definition, $\underline{\mathrm{Pic}}^0(A/S)$ is an open subfunctor of $\underline{\mathrm{Pic}}(A/S)$.

Remark 1.3.2.5. By [97, Prop. 6.7], $\underline{\mathrm{Pic}}^0(A/S)$ is formally smooth over. However, since the Picard numbers might jump, $\underline{\mathrm{Pic}}(A/S)$ is not smooth over S in general.

Theorem 1.3.2.6. *For any abelian scheme* $A \to S$*, the relative Picard functors* $\underline{\mathrm{Pic}}_e(A/S)$ *and* $\underline{\mathrm{Pic}}^0_e(A/S)$ *are representable over* S*. In particular,* $\underline{\mathrm{Pic}}^0_e(A/S)$ *is representable by an abelian scheme over* S*, which we call the* **dual abelian scheme** *of* A*. We shall denote the dual abelian scheme of* A *by* $A^\vee$*. (Note that by definition the identity section* $e_{A^\vee}$ *of* $A^\vee$ *corresponds to the trivial invertible sheaf on* A*.)*

For more details, see [37, Ch. I, §1], in which they mentioned the results of Artin, Raynaud, and Deligne, and outlined a sketch of the proof.

Remark 1.3.2.7. By the reduction to the locally noetherian case by Theorem 1.3.1.3, and by the result of Hilbert schemes as in [97, Ch. 0, §5, (d), and p. 117], the functors $\underline{\mathrm{Pic}}_e(A/S)$ and $\underline{\mathrm{Pic}}_e^0(A/S)$ are representable when A is locally projective over S. This local projectivity assumption is automatically satisfied when we study our moduli problems, because we will only need those abelian schemes $A \to S$ that are *polarized* in the sense that there exist relatively ample invertible sheaves on them. (See Construction 1.3.2.10, Proposition 1.3.2.18, and Definition 1.3.2.20 below.) If the readers are unwilling to make use of the strong result of Theorem 1.3.2.6, then they may safely add the assumption of locally projectivity to all abelian schemes in what follows.

Definition 1.3.2.8. *The universal rigidified invertible sheaf $\mathcal{P}_A$ on $A \underset{S}{\times} A^\vee$ is called the* **Poincaré invertible sheaf** *of A.*

Remark 1.3.2.9. The invertible sheaf $\mathcal{P}_A$ is rigidified along $(e_A, \mathrm{Id}_{A^\vee}) : A^\vee \to A \underset{S}{\times} A^\vee$ because it is the universal object, and hence in particular an object of $\underline{\mathrm{Pic}}_e^0(A/S)(A^\vee)$. On the other hand, it is rigidified along $(\mathrm{Id}_A, e_{A^\vee}) : A \to A \underset{S}{\times} A^\vee$ by the definition of $e_{A^\vee}$ and the universal property of $A^\vee$. That is, the pullback by $e_{A^\vee}$ should correspond to giving the parameter of the trivial invertible sheaf.

Construction 1.3.2.10. Let $m, \mathrm{pr}_1, \mathrm{pr}_2 : A \underset{S}{\times} A \to A$ denote respectively the multiplication map and the two projections. For any rigidified invertible sheaf $\mathcal{L}$ on $A \to S$, define

$$\mathcal{D}_2(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^* \mathcal{L}^{\otimes -1} \otimes \mathrm{pr}_2^* \mathcal{L}^{\otimes -1}.$$

Then $\mathcal{D}_2(\mathcal{L})$ is an invertible sheaf on $A \underset{S}{\times} A \to A$, which is rigidified along the identity section $(e_A, \mathrm{Id}_A) : A \to A \underset{S}{\times} A$ if we view $A \underset{S}{\times} A$ as an abelian scheme over the second factor A. By universal property of $A^\vee$, which represents $\underline{\mathrm{Pic}}_e^0(A/S)$, we obtain a unique morphism

$$\lambda_{\mathcal{L}} : A \to A^\vee,$$

which sends e_A to $e_{A^\vee}$. This is then automatically a group scheme homomorphism by Corollary 1.3.1.6.

Lemma 1.3.2.11. *There is a canonical isomorphism* $A \xrightarrow{\sim} (A^\vee)^\vee$.

Proof. By the universal property of $(A^\vee)^\vee$ applied to the Poincaré invertible sheaf $\mathcal{P}_A$ on $A \underset{S}{\times} A^\vee$ as the pullback of $A^\vee$ to the base scheme A over S, we obtain a canonical morphism $A \to (A^\vee)^\vee$, which is an isomorphism because it is so over any geometric point, by the usual theory (such as the one in [99, §13]) of abelian varieties over algebraically closed fields. ☐

Definition 1.3.2.12. *For any group scheme homomorphism* $f : A \to A'$ *between abelian schemes over* S*, the pullback map*

$$f^* : \underline{\mathrm{Pic}}^0_e(A'/S) \to \underline{\mathrm{Pic}}^0_e(A/S)$$

induces a group scheme homomorphism

$$(A')^\vee \to A^\vee,$$

which we denote by $f^\vee$*. This is called the* **dual isogeny** *of* f *is* f *is an isogeny.*

Lemma 1.3.2.13. *For any group scheme homomorphism* $f : A \to A'$ *between abelian schemes over* S*, we have a canonical isomorphism* $(\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A \cong (f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'}$ *over* $A \underset{S}{\times} (A')^\vee$ *by the universal properties of* $\mathcal{P}_A$ *and* $\mathcal{P}_{A'}$.

Lemma 1.3.2.14. *If* f *is an isogeny, then the rank of* f *is the same as the rank of* $f^\vee$*. (See the discussion in Section 5.2.4 below.) Hence it makes sense to say that the dual of a prime-to-*$\Box$ *isogeny is again a prime-to-*$\Box$ *isogeny.*

Definition 1.3.2.15. *For any group scheme homomorphism* $\lambda : A \to A^\vee$*, consider the composition*

$$A \xrightarrow{\sim} (A^\vee)^\vee \xrightarrow{\lambda^\vee} A^\vee,$$

where the isomorphism is the canonical one given by Lemma 1.3.2.11. By abuse of notation, we denote this composition also by $\lambda^\vee$*. We say* λ *is* **symmetric** *if* $\lambda = \lambda^\vee$.

Lemma 1.3.2.16. *The morphism* $\lambda_{\mathcal{L}}$ *constructed above from an invertible sheaf* $\mathcal{L}$ *is symmetric.*

Proof. By Lemma 1.3.2.13, we have $(\mathrm{Id}_A \times {\lambda_{\mathcal{L}}}^\vee)^* \mathcal{P}_A \cong (\lambda_{\mathcal{L}} \times \mathrm{Id}_{(A^\vee)^\vee})^* \mathcal{P}_{A^\vee}$ over $A \underset{S}{\times} (A^\vee)^\vee$. By pulling back using $(\mathrm{Id}_A \times \mathrm{can.})$, and by remembering our abuse of notation in Definition 1.3.2.15, we get $(\mathrm{Id}_A \times {\lambda_{\mathcal{L}}}^\vee)^* \mathcal{P}_A \cong (\lambda_{\mathcal{L}} \times \mathrm{can.})^* \mathcal{P}_{A^\vee}$ over $A \underset{S}{\times} A$. On the other hand, by the construction of the canonical $A \xrightarrow{\sim} (A^\vee)^\vee$, we have $(\mathrm{Id}_{A^\vee} \times \mathrm{can.})^* \mathcal{P}_{A^\vee} \cong s^* \mathcal{P}_A$, where $s : A^\vee \underset{S}{\times} A \to A \underset{S}{\times} A^\vee$ is the map switching the two components. Therefore, we have $(\lambda_{\mathcal{L}} \times \mathrm{can.})^* \mathcal{P}_{A^\vee} \cong (\lambda_{\mathcal{L}} \times \mathrm{Id}_A)^* s^* \mathcal{P}_A \cong \mathcal{D}_2(\mathcal{L}) \cong (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^* \mathcal{P}_A$. Note that the second last isomorphism is valid because $\mathcal{D}_2(\mathcal{L})$ is isomorphic to its pullback under the map switching the two factors of $A \underset{S}{\times} A$. As a result, we have $(\mathrm{Id}_A \times {\lambda_{\mathcal{L}}}^\vee)^* \mathcal{P}_A \cong (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^* \mathcal{P}_A$. Now the universal property of $\mathcal{P}_A$ shows that we must have ${\lambda_{\mathcal{L}}}^\vee = \lambda_{\mathcal{L}}$. ☐

Following [29, 1.2, 1.3, 1.4], we have the following converse:

Proposition 1.3.2.17. *Locally for the étale topology, every symmetric homomorphism from A to $A^\vee$ is of the form $\lambda_{\mathcal{L}}$ for some invertible sheaf $\mathcal{L}$.*

Proof. Note that if we set $\mathcal{M} := (\mathrm{Id}_A, \lambda)^* \mathcal{P}_A$, then $\lambda_{\mathcal{M}} = \lambda + \lambda^\vee = 2\lambda$. (This follows essentially from the universal property of $\mathcal{P}_A$ and its symmetric bilinear properties; that is, from the theorem of cube.) Therefore the question is whether we can find some $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} = \mathcal{M}$.

Note that, locally in the fppf topology, any group extension of a commutative finite flat group scheme by $\mathbf{G}_{\mathrm{m}}$ splits. This can be shown by the arguments in [95, Thm. 2.3(ii)]. Therefore, using the representation theory of theta group schemes, the argument in the proof of [99, §23, Thm. 3] generalizes and shows that fppf locally $\mathcal{M} = \mathcal{L}^{\otimes m}$ for some $\mathcal{L}$ if $\ker(\lambda_{\mathcal{L}}) \supset A[m]$ for some integer m. In particular, this is true for $m = 2$. Therefore, the map of functors

$$\underline{\mathrm{Pic}}_e(A/S) \to \underline{\mathrm{Hom}}^{\mathrm{sym.}}(A, A^\vee) : \mathcal{L} \mapsto \lambda_{\mathcal{L}}$$

(with kernel $\underline{\mathrm{Pic}}_e^0(A/S) \cong A^\vee$) is surjective and smooth, as we have verified locally in the fppf topology. Note that $\underline{\mathrm{Hom}}^{\mathrm{sym.}}(A, A^\vee)$ is representable by a scheme, as it is an algebraic space and as it is unramified over S by rigidity (given by Corollary 1.3.1.5). As a result, λ is also étale locally of the form $\lambda_{\mathcal{L}}$, as smooth morphisms between schemes has a section étale locally (by [20, §2.2, Prop. 14]). ☐

As relative ampleness over A can be checked over the fibers, because we can assume that the base is locally noetherian (by Theorem 1.3.1.3) and because generation by global sections over any noetherian local ring can be checked over the special fiber by NAK (Nakayama's lemma), this leads to:

Proposition 1.3.2.18. *Let λ be a symmetric homomorphism from A to $A^\vee$. The following conditions are equivalent:*

1. *After base change to any geometric point $\bar{s}$ of S, $\lambda_{\bar{s}}$ is of the form $\lambda_{\mathcal{L}}$ for some ample invertible sheaf $\mathcal{L}$ over $A_{\bar{s}}$.*
2. *Locally for the étale topology, λ is of the form $\lambda_{\mathcal{L}}$ for some relatively ample invertible sheaf $\mathcal{L}$ over A.*
3. *The invertible sheaf $(\mathrm{Id}_A, \lambda)^* \mathcal{P}_A$ is relatively ample over A.*

Definition 1.3.2.19. *A homomorphism from A to $A^\vee$ is* **positive** *if it is symmetric and satisfies either of the equivalent conditions in Proposition 1.3.2.18.*

Definition 1.3.2.20. *A* **polarization** *λ of A is a positive homomorphism from A to $A^\vee$. A* **principal polarization** *is a polarization that is an isomorphism. A* **prime-to-□ polarization** *is a polarization that is a prime-to-□ isogeny.*

Remark 1.3.2.21. A positive homomorphism is necessarily an isogeny, because $\lambda_{\mathcal{L}}$ is quasi-finite by the usual theory of abelian varieties over fields. (See for example [99].)

Corollary 1.3.2.22. *An isogeny $\lambda : A \to A^\vee$ is positive if and only if $[N] \circ \lambda$ is positive for some positive integer N.*

Motivated by this, we can extend the notion of positivity to $\mathbb{Z}^\times_{(\Box)}$-isogenies as well:

Definition 1.3.2.23. *A $\mathbb{Z}^\times_{(\Box)}$-isogeny λ is* **positive** *if $[N] \circ \lambda$ is a positive isogeny for some positive integer N.*

Note that we do not have to assume N is prime-to-□.

Definition 1.3.2.24. *The* **dual $\mathbb{Z}^{\times}_{(\Box)}$-isogeny** $f^{\vee} : (A')^{\vee} \to A^{\vee}$ *of a $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $f : A \to A'$ represented by some triple (B, g, h) is the $\mathbb{Z}^{\times}_{(\Box)}$-isogeny represented by $((A')^{\vee} \underset{h^{\vee},B^{\vee},g^{\vee}}{\times} A^{\vee}, \mathrm{pr}_1, \mathrm{pr}_2)$, as in the diagram:*

$$\begin{tikzcd}
& (A')^{\vee} \underset{h^{\vee},B^{\vee},g^{\vee}}{\times} A^{\vee} \arrow[ld, "\mathrm{pr}_1"'] \arrow[rd, "\mathrm{pr}_2"] & \\
(A')^{\vee} \arrow[rr, dotted] \arrow[rd, "h^{\vee}"'] & & A^{\vee} \arrow[ld, "g^{\vee}"] \\
& B^{\vee} &
\end{tikzcd}$$

This definition makes sense because of Lemma 1.3.2.14.

Corollary 1.3.2.25. *If λ is a positive $\mathbb{Z}^{\times}_{(\Box)}$-isogeny from A to $A^{\vee}$, then $f^{\vee} \circ \lambda \circ f$ is a positive $\mathbb{Z}^{\times}_{(\Box)}$-isogeny for any $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $f : A' \to A$. Moreover, λ^{-1} is also a positive $\mathbb{Z}^{\times}_{(\Box)}$-isogeny from $A^{\vee}$ to A.*

Proof. To show these, we use 3 in Proposition 1.3.2.18, and the fact that the pullback of an invertible sheaf under an isogeny is relatively ample if and only if the original invertible sheaf is relatively ample. ☐

Definition 1.3.2.26. *A* **$\mathbb{Z}^{\times}_{(\Box)}$-isogeny** $f : (A, \lambda) \to (A', \lambda')$ *is a $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $f : A \to A'$ such that $\lambda = f^{\vee} \circ \lambda' \circ f$.*

Definition 1.3.2.27. *A* **$\mathbb{Z}^{\times}_{(\Box)}$-polarization** λ *of A is a positive $\mathbb{Z}^{\times}_{(\Box)}$-isogeny from A to $A^{\vee}$.*

1.3.3 Endomorphisms Structures

Assume as in Section 1.2.1 that $\mathcal{O} \subset B$ is a fixed order in a finite-dimensional semisimple algebra B over $\mathbb{Q}$ with a positive involution $\star$.

Definition 1.3.3.1. *Let A be an abelian scheme with a $\mathbb{Z}^{\times}_{(\Box)}$-polarization λ over S. Let R be any $\mathbb{Z}$-subalgebra of $\mathbb{Q}$. An* **$\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-endomorphism structure** (*or simply an* **$\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-structure**) *of (A, λ) is a ring homomorphism*

$$i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} R \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} R \qquad (1.3.3.2)$$

satisfying the **Rosati condition** *that the restriction of the* λ*-Rosati involution of* $\mathrm{End}_S(A)\underset{\mathbb{Z}}{\otimes} R$ *on the image of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$ *agrees with the one induced by the involution* * *of* $\mathcal{O}$*. (In particular, it has to fix* $\mathcal{O}$*.) Equivalently, this means the commutativity of the diagram*

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "i(b^\star)"'] & A^\vee \arrow[d, "i(b)^\vee"] \\ A \arrow[r, "\lambda"'] & A^\vee \end{tikzcd} .$$

In symbolic notations, this means the equality

$$i(b)^\vee \circ \lambda = \lambda \circ i(b^\star)$$

for any $b \in \mathcal{O}\underset{\mathbb{Z}}{\otimes} R$*.*

Remark 1.3.3.3. If $R = \mathbb{Z}$, then we are given a ring homomorphism $i : \mathcal{O} \to \mathrm{End}_S(A)$ called an *$\mathcal{O}$-structure* by definition. In this case, we shall be thinking of A as a *left* $\mathcal{O}$-module via i.

Remark 1.3.3.4. For any $\mathbb{Z}$-subalgebra R of $\mathbb{Q}$, if i is an $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$-structure of (A, λ) as in Definition 1.3.3.1, then i is an $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R$-structure of $(A, r\lambda)$ for any $r \in \mathbb{Q}^\times_{>0}$.

Definition 1.3.3.5. *Let* i *(resp.* i'*) be an* $\mathcal{O}\underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-structure of* (A, λ) *(resp.* (A', λ')*). A* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda) \to (A', \lambda')$ *is* **$\mathcal{O}$-equivariant** *if* $f \circ i(b) = i'(b) \circ f$ *for all* $b \in \mathcal{O}\underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*. We say in this case that we have a* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda, i) \to (A', \lambda', i')$*.*

Remark 1.3.3.6. If f and (A, λ, i) are prescribed in a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i) \to (A', \lambda', i')$, then λ' and i' are determined necessarily by $\lambda' = (f^\vee)^{-1} \circ \lambda \circ f^{-1}$ and $i'(b) = f \circ i(b) \circ f^{-1}$ for all $b \in \mathcal{O}\underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$.

1.3.4 Conditions on Lie Algebras

We will use the information of the real symplectic vector space $(L\underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ to impose a condition on the Lie algebras of the abelian schemes we consider.

Let us first consider the complex analytic story as a motivation. The Hermitian pairing $(\!| \cdot , \cdot |\!)$ associated to the symplectic pairing $\langle \cdot , \cdot \rangle$ via Lemma

1.1.4.6 (and via a twist making a skew-Hermitian form Hermitian) is not necessarily definite. Thinking of how the polarized abelian varieties we consider over characteristic zero should be uniformized over the complex numbers, we need to modify the complex structure, so that the corresponding Hermitian pairing could be positive definite.

What we need is the following condition: There is an $\mathbb{R}$-algebra homomorphism

$$h : \mathbb{C} \to \operatorname{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$$

such that

$$\langle h(z)x, y\rangle = \langle x, h(z^c)y\rangle$$

for any $z \in \mathbb{C}$ and $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and such that the $\mathbb{R}$-bilinear pairing

$$\frac{1}{\sqrt{-1}} \circ \langle \,\cdot\,, h(\sqrt{-1})\,\cdot\,\rangle : (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$$

is symmetric and positive-definite for any choice of $\sqrt{-1}$. This is exactly Condition 1.2.1.2, which was used as part of the definition of our PEL-type $\mathcal{O}$-lattice (as in Definition 1.2.1.3).

Remark 1.3.4.1. The map h is essentially determined by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \,\cdot\,, \,\cdot\,\rangle)$ in the sense that any possible different choice differ by a conjugation of an element in $\mathrm{G}^+(\mathbb{R})$, namely those elements in $\mathrm{G}(\mathbb{R})$ with positive similitudes. Any different choice of h will produce the same conditions in Definitions 1.2.5.4 and 1.3.4.2 below. Therefore the choice that we have to make is only of auxiliary purpose.

Recall that (in Sections 1.2.1 and 1.2.5) we have a decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c,$$

where $h(z)$ acts by $1 \otimes z$ on V_0, and by $1 \otimes z^c$ on V_0^c. Moreover, both V_0 and V_0^c are totally isotropic under the pairing $\langle \,\cdot\,, \,\cdot\,\rangle$.

The reason to consider V_0 is that, according to the Hodge decomposition, it is natural to compare V_0 with the Lie algebra of an abelian variety. Corollary 1.1.2.12 above and its proof suggest that we may use the trace to determine if two representations are the same, at least in characteristic zero. Moreover, by the technique developed in Sections 1.1.2, 1.1.3, and 1.2.5, we know that we can classify representations over certain integral bases as well. (See in particular Proposition 1.1.2.16, Lemma 1.1.3.1, and Remark 1.2.5.14.)

By Definitions 1.1.2.14, the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-module defines an element $\mathrm{Det}_{\mathcal{O}|V_0}$ in $\mathbb{C}[\mathcal{O}^\vee]$. By Corollary 1.2.5.12, we may view $\mathrm{Det}_{\mathcal{O}|V_0}$ as an element in $\mathcal{O}_{F_0}[\mathcal{O}^\vee]$.

On the other hand, suppose A is an abelian scheme over a locally noetherian base scheme S over $\mathcal{O}_{F_0,(\Box)}$, together with a $\mathbb{Z}_{(\Box)}^\times$-polarization λ and an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$-endomorphism structure $i : \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A)\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ giving $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$-action on $\underline{\mathrm{Lie}}_{A/S}$. Since the Lie algebra $\underline{\mathrm{Lie}}_{A/S}$ is a locally free $\mathscr{O}_S$-module with $\mathcal{O}$-action, we may define in Definition 1.1.2.17 an element $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$ in $\mathscr{O}_S[\mathcal{O}^\vee]$. Here we can define the determinant for an element in $\mathrm{End}_S(A)\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ because nonzero integers that are prime-to-$\Box$ are invertible in $\mathscr{O}_S$.

Since S is an $\mathcal{O}_{F_0,(\Box)}$-scheme, we have the structural map from $\mathcal{O}_{F_0,(\Box)}$ to $\mathscr{O}_S$. Therefore it makes sense to compare the image of $\mathrm{Det}_{\mathcal{O}|V_0}$ with $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$:

Definition 1.3.4.2. *The* (**Kottwitz**) **determinantal condition** *on* $\underline{\mathrm{Lie}}_{A/S}$ (*given by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle)$) *is that* $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$ *agrees with the image of* $\mathrm{Det}_{\mathcal{O}|V_0}$ *under the morphism from* $\mathcal{O}_{F_0,(\Box)}$ *to* $\mathscr{O}_S$.

Although we can define this condition for all $\mathcal{O}_{F_0,(\Box)}$-schemes, the module V_0 is defined by objects over $\mathbb{C}$. Therefore one may wonder if this condition is also good for fields or complete local rings over $\mathcal{O}_{F_0,(\Box)}$ when the residue characteristic is positive. We shall see in the proof of Proposition 2.2.2.9 that this is indeed the case if we assume that $\Box$ does not divide Disc.

1.3.5 Tate Modules

The Tate modules we consider can be defined formally as follows:

Definition 1.3.5.1. *Let G be any abelian group. Let Δ be any semi-subgroup of $\mathbb{N}$, the semi-group of* (*positive*) *natural numbers. Suppose that G is* **Δ-divisible** *in the sense that the multiplication by N in G is* **surjective** *for any $N \in \Delta$.*

1. *We define $\mathrm{V}_\Delta(G)$ to be the group of sequences $(\alpha_i)_{i\in\Delta}$ such that:*
    (a) *$N\alpha_{Ni} = \alpha_i$ for any $i, N \in \Delta$.*

(b) For any $i \in \Delta$, there exists $N \in \Delta$ such that $N\alpha_i = 0$.

If $1 \in \Delta$, $\mathrm{T}_\Delta(G)$ is defined to be the subgroup of $\mathrm{V}_\Delta(G)$ with $\alpha_1 = 0$.

2. *We define $\mathrm{V}(G)$ to be $\mathrm{V}_{\mathbb{N}}(G)$, and $\mathrm{T}(G)$ to be $\mathrm{T}_{\mathbb{N}}(G)$.*
3. *For any prime number $l > 0$, we set $\Delta_l = \{l^{\mathbb{N}}\}$, and define $\mathrm{V}_l(G)$ to be $\mathrm{V}_{\Delta_l}(G)$, and $\mathrm{T}_l(G)$ to be $\mathrm{T}_{\Delta_l}(G)$.*
4. *For any set of prime numbers $\Box$, we set $\Delta^\Box = \mathbb{Z}_{(\Box)}^\times \cap \mathbb{N}$, and define $\mathrm{V}^\Box(G)$ to be $\mathrm{V}_{\Delta^\Box}(G)$, and $\mathrm{T}^\Box(G)$ to be $\mathrm{T}_{\Delta^\Box}(G)$.*

Let us denote by G_{tors} the torsion subgroup of G, and denote by $G_{\mathrm{tors}}^\Box$ the prime-to-$\Box$ part of the torsion subgroup of G_{tors}. Then we have the canonical exact sequences

$$0 \to \mathrm{T}(G) \to \mathrm{V}(G) \to G_{\mathrm{tors}} \to 0$$

and

$$0 \to \mathrm{T}^\Box(G) \to \mathrm{V}^\Box(G) \to G_{\mathrm{tors}}^\Box \to 0.$$

Note that the surjectivity onto G_{tors} in the first sequence (resp. onto $G_{\mathrm{tors}}^\Box$ in the second sequence) requires the assumption that multiplication by N in G is *surjective* for any $N \in \mathbb{N}$ (resp. any $N \in \mathbb{Z}_{(\Box)}^\times \cap \mathbb{N}$).

Let A be any abelian variety over an algebraic closed field k. Let $p := \mathrm{char}(k)$. Consider $G := A(k)$, the k-points of A. Let $A_{\mathrm{tors}}^\Box$ denote the subgroup of all prime-to-$\Box$ torsion points of A.

If $p = 0$, then it makes sense to talk about all of $\mathrm{V}(G)$, $\mathrm{V}_l(G)$, $\mathrm{V}^\Box(G)$, $\mathrm{T}(G)$, $\mathrm{T}_l(G)$, and $\mathrm{T}^\Box(G)$. If $p > 0$, then we shall consider only $\mathrm{V}_l(G)$, $\mathrm{V}^\Box(G)$, $\mathrm{T}_l(G)$, and $\mathrm{T}^\Box(G)$, for $l \neq p$, because we want to study only separable isogenies. We shall denote $\mathrm{V}(G)$, $\mathrm{V}_l(G)$, $\mathrm{V}^\Box(G)$, $\mathrm{T}(G)$, $\mathrm{T}_l(G)$, and $\mathrm{T}^\Box(G)$, respectively by $\mathrm{V}\,A$, $\mathrm{V}_l\,A$, $\mathrm{V}^\Box\,A$, $\mathrm{T}\,A$, $\mathrm{T}_l\,A$, and $\mathrm{T}^\Box\,A$.

Assume that $\Box$ contains $p = \mathrm{char}(k)$ if $p > 0$. Then we always have the exact sequence

$$0 \to \mathrm{T}^\Box\,A \to \mathrm{V}^\Box\,A \to A_{\mathrm{tors}}^\Box \to 0.$$

Any group scheme homomorphism $f : A \to A'$ sends A_{tors} to A'_{tors} and induces a map $\mathrm{V}^\Box(f) : \mathrm{V}^\Box\,A \to \mathrm{V}^\Box\,A'$. The map $\mathrm{V}^\Box(f)$ is an isomorphism when f is an isogeny, and we can extend the definition of $\mathrm{V}^\Box(f)$ to the case that f is a $\mathbb{Z}_{(\Box)}^\times$-isogeny by setting $\mathrm{V}^\Box(f) = \mathrm{V}^\Box(g)^{-1} \circ \mathrm{V}^\Box(h)$ if f is represented by some triple (B, g, h) as in Definition 1.3.1.15. In particular, for any $\mathbb{Z}^\times_{(\Box)}$-isogeny of the form $N^{-1}f$ where N is an integer prime-to-$\Box$ and f is an isogeny from A to A', we can define $\mathrm{V}^\Box(f)$ by setting

$$\mathrm{V}^\Box(N^{-1}f)((\alpha_i)) = (f(\alpha_{Ni}))$$

for any $\alpha = (\alpha_i) \in \mathrm{V}^\Box A$.

Lemma 1.3.5.2. *Fix a triple (A, λ, i), where A is an abelian variety over an algebraically closed field k, where λ is a $\mathbb{Z}^\times_{(\Box)}$-polarization of A, and where i is an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-structure of (A, λ). Then there is a one-one correspondence*

$$\left\{\begin{matrix}\text{equivalence classes of} \\ \mathbb{Z}^\times_{(\Box)}\text{-isogenies} \\ f : (A, \lambda, i) \to (A', \lambda', i')\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{open compact} \\ \text{subgroups of } \mathrm{V}^\Box A\end{matrix}\right\}$$

given by sending a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i) \to (A', \lambda', i')$ to $\mathrm{V}(f)^{-1}(A')$. The $\mathcal{O}$-invariant open compact subgroups of $\mathrm{V}^\Box A$ correspond to $\mathbb{Z}^\times_{(\Box)}$-isogenies $f : (A, \lambda, i) \to (A', \lambda', i')$ such that i' has image in $\mathrm{End}_S(A')$.

Here two $\mathbb{Z}^\times_{(\Box)}$-isogenies $f_1 : (A, \lambda, i) \to (A_1, \lambda_1, i_1)$ and $f_2 : (A, \lambda, i) \to (A_2, \lambda_2, i_2)$ are equivalent if there exists an isomorphism $h : A_1 \xrightarrow{\sim} A_2$ such that $h \circ f_1 = f_2$.

Since prime-to-$\Box$ isogenies are characterized by their kernels, which are commutative groups schemes finite étale over the base field, it is useful to have the following:

Proposition 1.3.5.3. *Let S be a connected locally noetherian scheme, and $\bar{s}$ any fixed geometric point on S. Then there is an équivalence of categories between*

$$\left\{\begin{matrix}\text{commutative group schemes} \\ \text{finite étale over } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite continuous} \\ \pi_1(S, \bar{s})\text{-modules}\end{matrix}\right\}$$

given by sending a group scheme H finite étale over S to its geometric fiber $H_{\bar{s}}$ over $\bar{s}$.

Combining the above two propositions, we have:

Corollary 1.3.5.4. *Let S be a connected locally noetherian scheme, with residual characteristics either 0 or a prime number in $\Box$. Let $\bar{s}$ be any fixed geometric point of S.*

Fix a triple (A, λ, i)*, where* A *is an abelian scheme over* S*, where* λ *is a* $\mathbb{Z}^\times_{(\Box)}$*-polarization of* A*, and where* i *is an* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$*-structure of* (A, λ)*.* (*Here we need* λ *for the purpose of defining* i*.*) *Then there is a one-one correspondence*

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \mathbb{Z}^\times_{(\Box)}\text{-isogenies}\\ f:(A,\lambda,i)\to(A',\lambda',i')\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\pi_1(S,\bar{s})\text{-invariant}\\ \text{open compact}\\ \text{subgroups of } \mathrm{V}^\Box A_{\bar{s}}\end{matrix}\right\}$$

given by sending a $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda, i) \to (A', \lambda', i')$ *to* $\mathrm{V}(f_{\bar{s}})^{-1}(A'_{\bar{s}})$*. The* $\pi_1(S, \bar{s})$*-invariant* $\mathcal{O}$*-invariant open compact subgroups of* $\mathrm{V}^\Box A_{\bar{s}}$ *correspond to* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f : (A, \lambda, i) \to (A', \lambda', i')$ *such that* i' *defines an* $\mathcal{O}$*-structure* (*in the sense that it maps* $\mathcal{O}$ *to* $\mathrm{End}_S(A')$*, as in Definition 1.3.3.1*)*.*

Here two $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f_1 : (A, \lambda, i) \to (A_1, \lambda_1, i_1)$ *and* $f_2 : (A, \lambda, i) \to (A_2, \lambda_2, i_2)$ *are equivalent if there exists an isomorphism* $h : A_1 \xrightarrow{\sim} A_2$ *such that* $h \circ f_1 = f_2$*.*

Remark 1.3.5.5. In Corollary 1.3.5.4, if (A, λ, i) is a triple such that λ is not prime-to-$\Box$, then there is no triple (A', λ', i') in the equivalence class of (A, λ, i) such that λ' is a principal polarization.

1.3.6 Principal Level Structures

Let us fix a choice of a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in Definition 1.2.1.3 and a positive integer n prime to some particular set $\Box$ of rational prime numbers.

For defining the moduli problems we consider, we shall explain our notion of principal level-n structures.

Let (A, λ, i) be a triple such that:

1. A is an abelian scheme over a locally noetherian base scheme S over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.
2. $\lambda : A \to A^\vee$ is a prime-to-$\Box$ polarization of A.
3. $i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ).

A naive way to define a level structure of (A, λ, i) when $L \neq \{0\}$ is to consider an $\mathcal{O}$-equivariant *symplectic* isomorphism $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$, where being symplectic here means the existence of an isomorphism $((\mathbb{Z}/n\mathbb{Z})(1))_S \xrightarrow{\sim} \boldsymbol{\mu}_{n,S}$ making the following diagram commute:

$$\begin{tikzcd}
(L/nL)_S \underset{S}{\times} (L/nL)_S \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\alpha_n\times\alpha_n"', "\wr"] & ((\mathbb{Z}/n\mathbb{Z})(1))_S \arrow[d, dotted, "\wr"] \\
A[n] \underset{S}{\times} A[n] \arrow[r, "\lambda\text{-Weil}"'] & \boldsymbol{\mu}_{n,S}
\end{tikzcd}$$

This works perfectly for studying the moduli of *principally polarized* abelian schemes as in [37].

However, since the lattice L is not necessarily self-dual, the pairing on L/nL induced from $\langle\,\cdot\,,\,\cdot\,\rangle$ may be *degenerate*, or even trivial. Moreover, the $\mathcal{O}$-equivariance relation might be too weak to be verified by modules mod n. Therefore, for a geometric point $\bar{s}$ of S, the isomorphism $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ may not be *liftable* to an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$, where being symplectic means the existence of an isomorphism $\hat{\mathbb{Z}}^\square(1) \xrightarrow{\sim} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar{s}}$ making the following diagram commute:

$$\begin{tikzcd}
(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \times (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}\times\hat{\alpha}"', "\wr"] & \hat{\mathbb{Z}}^\square(1) \arrow[d, dotted, "\wr"] \\
\mathrm{T}^\square A_{\bar{s}} \times \mathrm{T}^\square A_{\bar{s}} \arrow[r, "\lambda\text{-Weil}"'] & \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd} .$$

Roughly speaking, the problem is that the isomorphism α_n may carry too little information to be qualified as a level structure. Since level structures are important for defining the Hecke actions of $\mathrm{G}(\mathbb{A}^{\infty,\square})$, we have to take into account this problem in our definitions.

Definition 1.3.6.1. *Let S be a locally noetherian scheme over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$. *Let (A, λ, i) be a triple such that:*

1. *A is an abelian scheme over S.*

2. *$\lambda : A \to A^\vee$ is a prime-to-$\square$ polarization of A.*

3. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ).*

Let $\bar{s}$ be any geometric point of S. Recall that an $\mathcal{O}$-equivariant **symplectic isomorphism**

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{s}}$$

(*defined as in Definition 1.1.4.11*) *is an isomorphism of the underlying modules together with an isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^\Box(1) \xrightarrow{\sim} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{s}}$ making the following diagram commute:*

$$\begin{tikzcd}
(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \arrow[r, "\langle \cdot , \cdot \rangle"] \arrow[d, "\hat{\alpha}\times\hat{\alpha}"', "\wr"] & \hat{\mathbb{Z}}^\Box(1) \arrow[d, dotted, "\wr"', "\nu(\hat{\alpha})"] \\
\mathrm{T}^\Box A_{\bar{s}} \times \mathrm{T}^\Box A_{\bar{s}} \arrow[r, "\lambda\text{-Weil}"'] & \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd} .$$

An (**integral**) **principal level-n structure of** (A, λ, i) **of type** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot , \cdot \rangle)$ *is an $\mathcal{O}$-equivariant* **symplectic-liftable isomorphism**

$$\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n],$$

which is an isomorphism of the underlying modules together with an isomorphism $\nu(\alpha_n) : ((\mathbb{Z}/n\mathbb{Z})(1))_S \xrightarrow{\sim} \boldsymbol{\mu}_{n,S}$ such that, for any geometric point $\bar{s}$ of S, there exists (*noncanonically*) *an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{s}}$ lifting $\alpha_{n,\bar{s}} : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$ as a reduction mod n of $\hat{\alpha}$, which comes together with an isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^\Box(1) \xrightarrow{\sim} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{s}}$ lifting $\nu(\alpha_n)$ as a reduction mod n and making the following diagram commute:*

$$\begin{tikzcd}
(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \arrow[r, "\langle \cdot , \cdot \rangle"] \arrow[d, "\hat{\alpha}\times\hat{\alpha}"', "\wr"] & \hat{\mathbb{Z}}^\Box(1) \arrow[d, dotted, "\wr"', "\nu(\hat{\alpha})"] \\
\mathrm{T}^\Box A_{\bar{s}} \times \mathrm{T}^\Box A_{\bar{s}} \arrow[r, "\lambda\text{-Weil}"'] & \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

If $L \neq \{0\}$, then $\nu(\hat{\alpha})$ is uniquely determined by $\hat{\alpha}$, and $\nu(\alpha_n)$ is uniquely determined by α_n. If $L = \{0\}$, then $\nu(\alpha_n)$ is the essential nontrivial information.

Remark 1.3.6.2. The condition is nontrivial even when $n = 1$. Moreover, it forces the kernel of the prime-to-$\Box$ polarization λ to be isomorphic to $(L^\# \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box)/(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box)$.

Remark 1.3.6.3. If there is no risk of confusion, we shall simply speak of a level structure without the term *integral* and other modifiers.

When we talk about $\mathbb{Z}_{(\Box)}^{\times}$-isogeny classes of similar triples, a better notion of principal level-n structures is given as follows:

Definition 1.3.6.4. *Let S be a locally noetherian scheme over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and let $\bar{s}$ be a geometric point of S. Let (A, λ, i) be a triple such that:*

1. *A is an abelian scheme over S.*
2. *$\lambda : A \to A^{\vee}$ is a $\mathbb{Z}_{(\Box)}^{\times}$-polarization of A.*
3. *$i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ defines an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-structure of (A, λ).*

An $\mathcal{O}$-equivariant **symplectic isomorphism**

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$$

is an isomorphism of the underlying modules together with an isomorphism $\nu(\hat{\alpha}) : \mathbb{A}^{\infty,\Box}(1) \xrightarrow{\sim} \mathrm{V}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}$ making the following diagram commute:

$$\begin{tikzcd}
(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) \arrow[r, "\langle \cdot , \cdot \rangle"] \arrow[d, "\hat{\alpha}\times\hat{\alpha}"', "\wr"] & \mathbb{A}^{\infty,\Box}(1) \arrow[d, dotted, "\wr"', "\nu(\hat{\alpha})"] \\
\mathrm{V}^{\Box} A_{\bar{s}} \times \mathrm{V}^{\Box} A_{\bar{s}} \arrow[r, "\lambda\text{-Weil}"'] & \mathrm{V}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

The group $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ has a natural right action on the set of such symplectic isomorphisms, which is defined by the composition $\hat{\alpha} \mapsto \hat{\alpha} \circ g$ and $\nu(\hat{\alpha}) \mapsto \nu(\hat{\alpha}) \circ \nu(g)$ for any $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$. Then a **rational principal level-n structure of** *(A, λ, i)* **of type** *$(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$* **based at** *$\bar{s}$ is a $\pi_1(S, \bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_n$ of $\mathcal{O}$-equivariant symplectic isomorphisms*

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}.$$

When the context is clear, we shall abbreviate a *rational principal level-n structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$ based at $\bar{s}$* as a *rational level-n structure based at $\bar{s}$*.

Construction 1.3.6.5. Suppose we have a triple (A, λ, i) as in Definition 1.3.6.1 over a connected locally noetherian base scheme S over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Let $\bar{s}$ be any geometric point $\bar{s}$. If $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ is any (integral) principal level-n structure of (A, λ, i), then by definition there exists non-canonically an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}$ lifting $\alpha_{n,\bar{s}} : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$. Note that the $\mathcal{U}^{\Box}(n)$-orbit of this lifting is unique, and (by Proposition 1.3.5.3) this orbit is invariant under the action of $\pi_1(S, \bar{s})$ because $A[n]$ is a locally constant étale sheaf on S. In other words, we have a $\pi_1(S, \bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_n$ of $\mathcal{O}$-equivariant symplectic isomorphisms

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}.$$

By base extension and abuse of notation, we get a $\pi_1(S, \bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_n$ of $\mathcal{O}$-equivariant symplectic isomorphisms

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}},$$

namely a rational principal level-n structure $[\hat{\alpha}]_n$ based at $\bar{s}$, which sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}}$.

Lemma 1.3.6.6. *With the setting as in Definition 1.3.6.1, suppose moreover that S is connected. Then the liftability condition is true at a geometric point $\bar{s}$ of S if and only if the following is true:*

There is a tower $(S_m \twoheadrightarrow S)_{n|m,\Box\nmid m}$ of finite étale surjections such that:

1. *$S_n = S$.*
2. *For any l such that $n|l$ and $l|m$, there is a finite étale surjection $S_m \twoheadrightarrow S_l$ whose composition with $S_l \to S$ is the finite étale surjection $S_m \to S$.*
3. *Over each S_m, there is an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_{m,S_m} : (L/mL)_{S_m} \xrightarrow{\sim} A[m]_{S_m}$ in the sense that there is an isomorphism $((\mathbb{Z}/m\mathbb{Z})(1))_{S_m} \xrightarrow{\sim} \boldsymbol{\mu}_{m,S_m}$ making the diagram commute:*

$$\begin{tikzcd}
(L/mL)_{S_m} \underset{S_m}{\times} (L/mL)_{S_m} \arrow[r, "\langle\cdot,\cdot\rangle"] \arrow[d, "\alpha_{m,S_m}\times\alpha_{m,S_m}"', "\wr"] & ((\mathbb{Z}/m\mathbb{Z})(1))_{S_m} \arrow[d, dotted, "\wr"] \\
A[m]_{S_m} \underset{S_m}{\times} A[m]_{S_m} \arrow[r, "\lambda\text{-Weil}"'] & \boldsymbol{\mu}_{m,S_m}
\end{tikzcd}$$

4. *For each l such that $n|l$ and $l|m$, the pullback of α_{l,S_l} to S_m is the reduction mod l of α_{m,S_m}.*

Proof. Let us start with an integral level-n structure $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ for (A, λ, i) as in Definition 1.3.6.1, and fix a choice of a geometric point $\bar{s}$ of S and a choice of an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}$ whose reduction mod n is $\alpha_{n,\bar{s}} : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$. The $\pi_1(S, \bar{s})$-invariance of the $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_n$ of $\hat{\alpha}$ (in Construction 1.3.6.5) expresses the fact that we have a continuous map $\pi_1(S, \bar{s}) \to \mathcal{U}^{\Box}(n)$, and by taking the pre-image of $\mathcal{U}^{\Box}(m)$ under this continuous map, we obtain an open compact subgroup $\pi_1(S_m, \bar{s})$ of $\pi_1(S, \bar{s})$ corresponding to some finite étale surjection $S_m \to S$ and some lifting $\bar{s} \to S_m$ of $\bar{s} \to S$. The $\mathcal{U}^{\Box}(m)$-orbit of $\hat{\alpha}$ is therefore invariant under $\pi_1(S_m, \bar{s})$ when we pass to the finite étale surjection $S_m \to S$, and we obtain a $\pi_1(S_m, \bar{s})$-equivariant isomorphism $L/mL \xrightarrow{\sim} A[m]_{\bar{s}}$. By Proposition 1.3.5.3, this is equivalent to an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_{m,S_m} : (L/mL)_{S_m} \xrightarrow{\sim} A[m]_{S_m}$ between finite étale group schemes. The compatibility between different m and l follows from the natural containment relation between different $\mathcal{U}^{\Box}(m) \subset \mathcal{U}^{\Box}(n)$ and $\mathcal{U}^{\Box}(l) \subset \mathcal{U}^{\Box}(n)$.

Conversely, if a tower as above exists, then by specializing the compatible tower $(\alpha_{m,S_m})_{n|m,\Box\nmid m}$ defined on $(S_m \to S)_{n|m,\Box\nmid m}$ to any geometric point $\bar{s}'$ of S (with a compatible choice of liftings to each $S_m \to S$), we obtain an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}'}$ satisfying the requirements in Definition 1.3.6.1. $\Box$

Corollary 1.3.6.7 (of the proof of Lemma 1.3.6.6)**.** *In Definition 1.3.6.1, if the liftability condition is true at a geometric point $\bar{s}$ of S, then it is true at any geometric point $\bar{s}'$ of the same connected component of S (to which $\bar{s}$ belongs).*

Corollary 1.3.6.8. *Let S be a connected locally noetherian scheme over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*, and let $\bar{s}$ be a geometric point of S. Let (A, λ, i) be a triple as in Definition 1.3.6.1. Then a rational principal level-n structure $[\hat{\alpha}]_n$ of (A, λ, i) based at $\bar{s}$ comes from a (necessarily unique) integral principal level-n structure α_n as in Construction 1.3.6.5 if and only if the following condition is satisfied: Any symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$ in the $\pi_1(S, \bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_n$ induces an $\mathcal{O}$-equivariant symplectic isomorphism $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}$ (such that $\nu(\hat{\alpha}) : \mathbb{A}^{\infty,\Box}(1) \xrightarrow{\sim} \mathrm{V}^{\Box} \mathbf{G}_{\mathrm{m},\bar{s}}$ sends $\hat{\mathbb{Z}}^{\Box}(1)$ to $\mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{s}}$).*

We shall postpone the proof as Corollary 1.3.6.8 is a special case of Corollary 1.3.7.11 below.

Now we are ready to show that the choice of the base point $\bar{s}$ is immaterial in practice, because we have the following:

Corollary 1.3.6.9. *Let S be a locally noetherian scheme over $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$, and let $\bar{s}$ and $\bar{s}'$ be two geometric points of S lying on the same connected component. Let (A, λ, i) be a triple as in Definition 1.3.6.10. Then the rational principal level-n structures of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,,\,\cdot \rangle)$ based at $\bar{s}$ are canonically in bijection with the rational principal level-n structures based at $\bar{s}'$. Under this bijection, those rational principal level-n structures $[\hat{\alpha}]_n$ based at $\bar{s}$ that are represented by symplectic isomorphisms $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$ sending $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}}$ correspond to rational principal level-n structures $[\hat{\alpha}']_n$ based at $\bar{s}$ that are represented by symplectic isomorphisms $\hat{\alpha}' : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}'}$ sending $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}'}$.*

We shall postpone the proof as Corollary 1.3.6.9 is a special case of Corollary 1.3.7.13 below.

Definition 1.3.6.10. *Let S be a locally noetherian scheme over $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$, then a* **rational principal level-n structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,,\,\cdot \rangle)$** *is an assignment to each geometric $\bar{s}$ on S a rational principal level-n structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,,\,\cdot \rangle)$ based at $\bar{s}$, so that the assignments to two geometric points of S lying on the same connected component are corresponded to each other under the canonical bijection in Corollary 1.3.6.9.*

Convention 1.3.6.11. *By abuse of notations, we shall still denote the rational principal level-n structures of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,,\,\cdot \rangle)$ by the same notation $[\hat{\alpha}]_n$ we use for the structures based at a particular geometric point $\bar{s}$ of S. This is reasonable because we have to take a particular choice of a geometric point $\bar{s}$ of S only when we take a representative $\hat{\alpha}$ of $[\hat{\alpha}]_n$.*

Remark 1.3.6.12. By Corollary 1.3.6.9, Construction 1.3.6.5 determines a well-defined rational principal level-n structure $[\hat{\alpha}]_n$ whose choice is independent of the geometric point $\bar{s}$ at which it is based.

1.3.7 General Level Structures

Let us fix a choice of a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle)$ as in Definition 1.2.1.3 and a particular set $\Box$ of rational prime numbers. For the purpose of studying automorphic forms, it is often useful to have the notion of level structures other than the principal level structures introduced in Section 1.3.6.

It is easier to begin with the rational version:

Definition 1.3.7.1. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\mathbb{A}^{\infty,\Box})$. Let S be a locally noetherian scheme over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and let $\bar{s}$ be a geometric point of S. Let (A, λ, i) be a triple such that:*

1. *A is an abelian scheme over S.*
2. *$\lambda : A \to A^\vee$ is a $\mathbb{Z}_{(\Box)}^\times$-polarization of A.*
3. *$i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ defines an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-structure of (A, λ).*

Then a **rational level-$\mathcal{H}$ structure of** (A, λ, i) **of type** $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$ **based at** $\bar{s}$ *is a $\pi_1(S, \bar{s})$-invariant $\mathcal{H}$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ of $\mathcal{O}$-equivariant symplectic isomorphisms*

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^\Box A_{\bar{s}}.$$

On the other hand, the integral version can be defined only for open compact subgroups $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^\Box)$.

Definition 1.3.7.2. *For any $\hat{\mathbb{Z}}^\Box$-algebra R, set* $\mathrm{G}^{\mathrm{ess}}(R) := \mathrm{image}(\mathrm{G}(\hat{\mathbb{Z}}^\Box) \to \mathrm{G}(R))$.

Lemma 1.3.7.3. *Let $n \geq 1$ be an integer such that $\Box \nmid n$. With the setting as in Definition 1.3.6.1, any two level-n structures $\alpha_n, \alpha'_n : (L/nL)_S \xrightarrow{\sim} A[n]$ (as in Definition 1.3.6.1) are related by $\alpha'_n = \alpha_n \circ g_n$ and $\nu(\alpha'_n) = \nu(\alpha_n) \circ \nu(g_n)$ for a unique element $g_n \in \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$.*

In what follows, we shall often suppress the expression $\nu(\alpha'_n) = \nu(\alpha_n) \circ \nu(g_n)$ from the context, although it is an essential ingredient when we use the expression $\alpha'_n = \alpha_n \circ g_n$ to mean we are relating two level structures.

Definition 1.3.7.4. *Let $n \geq 1$ be an integer such that $\Box \nmid n$. Let (A, λ, i) and S be as in Definition 1.3.6.1. Let $\mathcal{H}_n$ be a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. By an* **$\mathcal{H}_n$-orbit of étale-locally-defined level-n structures**, *we mean a subscheme $\alpha_{\mathcal{H}_n}$ of*

$$\underline{\mathrm{Isom}}_S((L/nL)_S, A[n]) \underset{S}{\times} \underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$$

that is étale locally (over S) the disjoint union of elements in some $\mathcal{H}_n$-orbit of level-n structures. (Note that we need to include the second factor $\underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ because we do not exclude the possibility that $L = \{0\}$.) In this case, we denote by $\nu(\alpha_{\mathcal{H}_n})$ the projection of $\alpha_{\mathcal{H}_n}$ to $\underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$, which is an $\nu(\mathcal{H})$-orbit of étale-locally-defined isomorphisms with its natural interpretation.

Remark 1.3.7.5. In Definition 1.3.7.4, the statement that $\alpha_{\mathcal{H}_n}$ is *étale locally* over S a disjoint union can be replaced by the more restrictive statement that $\alpha_{\mathcal{H}_n}$ is a disjoint union over a *finite étale* surjection over S.

Lemma 1.3.7.6. *Let $n, m \geq 1$ be two integers prime-to-$\Box$ such that $n|m$. Let (A, λ, i) and S be as in Definition 1.3.6.1. Then there is a bijection from the set of $\mathcal{U}^{\Box}(n)/\mathcal{U}^{\Box}(m)$-orbits of étale-locally-defined level-m structures for (A, λ, i) to the set of level-n structures for (A, λ, i), induced étale locally over S by the procedure of taking the reduction mod n of an level-m structure.*

Proof. This is actually part of Lemma 1.3.6.6. □

Corollary 1.3.7.7. *Let (A, λ, i) and S be as in Definition 1.3.6.1. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. For any integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, set $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$, the image of $\mathcal{H}$ under $\mathrm{G}(\hat{\mathbb{Z}}^{\Box}) \twoheadrightarrow \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(n)$. Then there is a canonical bijection from the set of $\mathcal{H}_m$-orbits of étale-locally-defined level-m structures for (A, λ, i) to the set of $\mathcal{H}_n$-orbits of étale-locally-defined level-n structures for (A, λ, i), induced (étale locally over S) by the procedure of taking the reduction mod n of an level-m structure.*

Proof. Simply because $\mathcal{H}_m/(\mathcal{U}^{\Box}(n)/\mathcal{U}^{\Box}(m))$ is identified with $\mathcal{H}_n$ under the canonical isomorphism $(\mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(m))/(\mathcal{U}^{\Box}(n)/\mathcal{U}^{\Box}(m)) \cong \mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(n)$. □

Definition 1.3.7.8. *Let* (A, λ, i) *and* S *be as in Definition 1.3.6.1. Let* $\mathcal{H}$ *be any open compact subgroup of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. *For any integer* $n \geq 1$ *such that* $\Box \nmid n$ *and* $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, *set* $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$ *as in Corollary 1.3.7.7. Then an* (**integral**) **level-**$\mathcal{H}$ **structure of** (A, λ, i) **of type** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot\,, \cdot \rangle)$ *is a collection* $\alpha_{\mathcal{H}} = \{\alpha_{\mathcal{H}_n}\}$ *labeled by integers* $n \geq 1$ *such that* $\Box \nmid n$ *and* $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, *with elements* $\alpha_{\mathcal{H}_n}$ *described as follows:*

1. *For any* n *in the index set,* $\alpha_{\mathcal{H}_n}$ *is an* $\mathcal{H}_n$*-orbit of étale-locally-defined level-*n *structures as in Definition 1.3.7.4.*

2. *For any* $n|m$ *in the index set, the* $\mathcal{H}_m$*-orbit* $\alpha_{\mathcal{H}_m}$ *corresponds to the* $\mathcal{H}_n$*-orbit* $\alpha_{\mathcal{H}_n}$ *under Corollary 1.3.7.7.*

Remark 1.3.7.9. According to Corollary 1.3.7.7, the collection $\alpha_{\mathcal{H}} = \{\alpha_{\mathcal{H}_n}\}$ is determined by any element $\alpha_{\mathcal{H}_n}$ in it. Therefore, by abuse of notations, it is often convenient to identify the level-$\mathcal{H}$ structure with any $\alpha_{\mathcal{H}_n}$ in $\alpha_{\mathcal{H}}$.

Construction 1.3.7.10 (analogue of Construction 1.3.6.5). With the setting as in Definition 1.3.7.8, let $\alpha_{\mathcal{H}} = \{\alpha_{\mathcal{H}_n}\}$ be any level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot\,, \cdot \rangle)$. Let $\bar{s} \to S$ be any geometric point of S. Let us choose any particular integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. Let $\tilde{S} \twoheadrightarrow S$ be a *finite étale* surjection such that the pullback of $\alpha_{\mathcal{H}_n}$ to $\tilde{S}$ is a disjoint union of elements in the $\mathcal{H}_n$-orbit of some level-n structure $\alpha_n : (L/nL)_{\tilde{S}} \xrightarrow{\sim} A[n]_{\tilde{S}}$ over $\tilde{S}$. (See Remark 1.3.7.5.) Let us lift $\bar{s} \to S$ to some $\bar{s} \to \tilde{S}$ and view $\bar{s}$ as a geometric point of $\tilde{S}$ by this particular lifting. Then we obtain definition some symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}$ lifting α_n. By Construction, we see that the $\mathcal{H}$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ of $\hat{\alpha}$ is independent of the choice of n, $\tilde{S}$, and α_n, as it is the $\mathcal{H}_n$-orbit of the $\mathcal{U}^{\Box}(n)$-orbit. Moreover, $[\hat{\alpha}]_{\mathcal{H}}$ is invariant under $\pi_1(\tilde{S}, \bar{s})$ for any choice of liftings $\bar{s} \to \tilde{S}$, which means that it is invariant under $\pi_1(S, \bar{s})$. As a result, we obtain a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,, \cdot \rangle)$, such that any $\hat{\alpha}$ in $[\hat{\alpha}]_{\mathcal{H}}$ sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}}$.

Corollary 1.3.7.11 (analogue of Corollary 1.3.6.8)**.** *Let* S *be a connected locally noetherian scheme over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, *and let* $\bar{s}$ *be a geometric point of* S. *Let* (A, λ, i) *be a triple as in Definition 1.3.7.8. Let* $\mathcal{H}$ *be any open compact subgroup of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. *Then a rational level-*$\mathcal{H}$ *structure* $[\hat{\alpha}]_{\mathcal{H}}$ *of* (A, λ, i) *based at*

$\bar{s}$ comes from a (necessarily unique) integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ as in Construction 1.3.7.10 if and only if the following condition is satisfied: Any isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$ in the $\pi_1(S, \bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ induces an $\mathcal{O}$-equivariant symplectic isomorphism $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{s}}$ (such that $\nu(\hat{\alpha}) : \mathbb{A}^{\infty,\Box}(1) \xrightarrow{\sim} \mathrm{V}^{\Box}\,\mathbf{G}_{\mathrm{m},\bar{s}}$ sends $\hat{\mathbb{Z}}^{\Box}(1)$ to $\mathrm{T}^{\Box}\,\mathbf{G}_{\mathrm{m},\bar{s}}$).

Proof. Let us take any integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. Under the assumption, we recover an $\pi_1(S, \bar{s})$-equivariant $\mathcal{H}_n$-orbit of some $\mathcal{O}$-invariant isomorphisms $\alpha_{n,\bar{s}} : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$. Let $\tilde{S} \twoheadrightarrow S$ be the finite étale surjection corresponding to the open compact subgroup of $\pi_1(S, \bar{s})$ that leaves $\alpha_{n,\bar{s}}$ invariant. By Proposition 1.3.5.3, $\alpha_{n,\bar{s}}$ is the specialization of an $\mathcal{O}$-equivariant isomorphism $\alpha_n : (L/nL)_{\tilde{S}} \xrightarrow{\sim} A[n]_{\tilde{S}}$ over $\tilde{S}$. Moreover, by Corollary 1.3.6.7, the liftability condition of α_n at $\bar{s}$ implies the liftability condition at any other geometric point of S. In other words, α_n is a level-n structure over $\tilde{S}$. The $\pi_1(S, \bar{s})$-invariance of the $\mathcal{H}_n$-orbit of $\alpha_{n,\bar{s}}$ implies that the disjoint union of the elements in the $\mathcal{H}_n$-orbit of α_n is the pullback of a subscheme $\alpha_{\mathcal{H}_n}$ of $\underline{\mathrm{Isom}}_S((L/nL)_S, A[n]) \times \underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ over S. As a result, $\alpha_{\mathcal{H}_n}$ is an $\mathcal{H}_n$-orbit of étale-locally-defined (integral) level-n structures (defined as in Definition 1.3.7.4), which defines an integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ as in Definition 1.3.7.8, as desired. ☐

Remark 1.3.7.12. This implies in particular Corollary 1.3.6.8 if we have $\mathcal{H} = \mathcal{U}^{\Box}(n)$.

Corollary 1.3.7.13 (analogue of Corollary 1.3.6.9)**.** *Let S be a locally noetherian scheme over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and let $\bar{s}$ and $\bar{s}'$ be two geometric points of S lying on the same connected component. Let (A, λ, i) be a triple as in Definition 1.3.7.15. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. Then the rational level-$\mathcal{H}$ structures of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle\,\cdot\,,\,\cdot\,\rangle)$ based at $\bar{s}$ are canonically in bijection with the rational level-$\mathcal{H}$ structures based at $\bar{s}'$. Under this bijection, those rational level-$\mathcal{H}$ structures $[\hat{\alpha}]_{\mathcal{H}}$ based at $\bar{s}$ that are represented by symplectic isomorphisms $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$ sending $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}}$ correspond to rational level-$\mathcal{H}$ structures $[\hat{\alpha}']_{\mathcal{H}}$ based at $\bar{s}$ that are represented by symplectic isomorphisms $\hat{\alpha}' : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}'}$ sending $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{\bar{s}'}$.*

Proof. Let us first describe how the bijection is constructed. Start with a rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ based at $\bar{s}$, and let $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$ be a representative of $[\hat{\alpha}]_{\mathcal{H}}$. By Corollary 1.3.5.4, there is a $\mathbb{Z}_{(\Box)}{}^{\times}$-isogeny $f : (A, \lambda, i) \to (A_1, \lambda_1, i_1)$ such that the composition $\hat{\alpha}_1 := \mathrm{V}^{\Box}(f) \circ \hat{\alpha}$ sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} A_{1,\bar{s}}$. Then the $\mathcal{H}$-orbit of $\hat{\alpha}_1$ gives a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}_1]_{\mathcal{H}}$ of (A_1, λ_1, i_1) based at $\bar{s}$. By Corollary 1.3.7.11, $[\hat{\alpha}_1]_{\mathcal{H}}$ comes from a (necessarily unique) integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ of (A_1, λ_1, i_1) under Construction 1.3.7.10. Apply Construction 1.3.7.10 to $\alpha_{\mathcal{H}}$ with a different base point $\bar{s}'$, we obtain a rational level-$\mathcal{H}$ structure $[\hat{\alpha}'_1]_{\mathcal{H}}$ of (A_1, λ_1, i_1) based at $\bar{s}'$. Let $\hat{\alpha}'_1 : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{1,\bar{s}'}$ be any representative of $[\hat{\alpha}'_1]_{\mathcal{H}}$. Then the $\mathcal{H}$-orbit of $\hat{\alpha}' := \mathrm{V}^{\Box}(f)^{-1} \circ \hat{\alpha}'_1 : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}'}$ gives a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}']_{\mathcal{H}}$ of (A, λ, i) based at $\bar{s}'$. This procedure gives a bijection because it is reversible (by switching the roles of $\bar{s}$ and $\bar{s}'$). It is clear from the construction that it satisfies the remaining properties described in the lemma. $\Box$

Remark 1.3.7.14. This implies in particular Corollary 1.3.6.9 if we have $\mathcal{H} = \mathcal{U}^{\Box}(n)$.

Definition 1.3.7.15. *Let S be a locally noetherian scheme over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*, then a* **rational level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$** *is an assignment to each geometric $\bar{s}$ on S a rational level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$ based at $\bar{s}$, so that the assignments to two geometric points of S lying on the same connected component are corresponded to each other under the canonical bijection in Corollary 1.3.7.13.*

Convention 1.3.7.16. *By abuse of notations, we shall still denote the rational level-$\mathcal{H}$ structures of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$ by the same notation $[\hat{\alpha}]_{\mathcal{H}}$ we use for the structures based at a particular geometric point $\bar{s}$ of S. This is reasonable because we have to take a particular choice of a geometric point $\bar{s}$ of S only when we take a representative $\hat{\alpha}$ of $[\hat{\alpha}]_{\mathcal{H}}$.*

Remark 1.3.7.17. By Corollary 1.3.7.13, Construction 1.3.7.10 determines a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ whose choice is independent of the geometric point $\bar{s}$ at which it is based.

1.4 Definitions of the Moduli Problems

Assume as in Section 1.2.1 that B is a finite-dimensional semisimple algebra over $\mathbb{Q}$ with a positive involution $\star$, and $\mathcal{O}$ is a $\mathbb{Z}$-order invariant under $\star$. Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$. In particular, by Proposition 1.1.1.17, its prime factors include those rational prime numbers that are ramified in B and F, and those primes at which $\mathcal{O}$ is not maximal and not isomorphic to a product of matrix algebra(s).

1.4.1 Definition by Isomorphism Classes

Let us fix a choice of a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ (defined as in Definition 1.2.1.3) and a positive integer n. Let Disc be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (defined as in Definition 1.1.1.6; see also Proposition 1.1.1.12). Closely related to Disc is the invariant $\mathrm{I}_{\mathrm{bad}}$ for $\mathcal{O}$ defined in Definition 1.2.1.17, which is either 2 or 1.

Definition 1.4.1.1. *We say that a prime number p is* **bad** *if $p|n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[L^{\#} : L]$. We say a prime number p is* **good** *if it is not bad. We say that $\Box$ is a set of good primes if it does not contain any bad primes.*

Let us fix a choice of a set $\Box$ of good primes. Let $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(\mathrm{LNSch}/\mathsf{S}_0)$ be the category whose objects are *locally noetherian schemes* over S_0. We shall define our moduli problem in the language of category fibred in groupoids. (See Appendix A for more details.)

Definition 1.4.1.2. *The moduli problem M_n is defined by the category fibred in groupoids over $(\mathrm{LNSch}/\mathsf{S}_0)$ whose fiber over each S is the groupoid $\mathsf{M}_n(S)$ described as follows: The objects of $\mathsf{M}_n(S)$ are tuples $(A, \lambda, i, \alpha_n)$, where:*

1. *A is an abelian scheme over S.*
2. *$\lambda : A \to A^\vee$ is a $\mathbb{Z}_{(\Box)}^\times$-polarization of A.*
3. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ).*
4. *$\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.2 given by $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$.*

5. *$\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ is an (integral) principal level-nstructure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as defined in Definition 1.3.6.1.*

The isomorphisms

$$(A, \lambda, i, \alpha_n) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_n)$$

of $\mathsf{M}_n(S)$ are given by isomorphisms $f : A \xrightarrow{\sim} A'$ such that:

1. *$\lambda = f^\vee \circ \lambda' \circ f$.*
2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$.*
3. *$f|_{A[n]} : A[n] \xrightarrow{\sim} A'[n]$ satisfies $\alpha'_n = (f|_{A[n]}) \circ \alpha_n$.*

Definition 1.4.1.3. *If we have two tuples $(A, \lambda, i, \alpha_n) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_n)$ as in Definition 1.4.1.2 under an isomorphism $f : A \xrightarrow{\sim} A'$, then we say in this case that we have an isomorphism $f : (A, \lambda, i, \alpha_n) \xrightarrow{\sim} (A', \lambda', i', \alpha'_n)$.*

The definition for general level structures is as follows:

Definition 1.4.1.4. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. The moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined by the category fibred in groupoids over $(\mathrm{LNSch}/\mathsf{S}_0)$ whose fiber over each S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$, where:*

1. *A is an abelian scheme over S.*
2. *$\lambda : A \to A^\vee$ is a $\mathbb{Z}^\times_{(\Box)}$-polarization of A.*
3. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ).*
4. *$\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.2 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$.*
5. *$\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as in Definition 1.3.7.8.*

The isomorphisms

$$(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$$

of $\mathsf{M}_{\mathcal{H}}(S)$ are given by isomorphisms $f : A \xrightarrow{\sim} A'$ such that:

1. *$\lambda = f^\vee \circ \lambda' \circ f$.*

2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$.*

3. *We have the symbolical relation $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ defined in the following sense: For any integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^\Box(n) \subset \mathcal{H}$, let $\alpha_{\mathcal{H}_n} \subset \underline{\mathrm{Hom}}_S((L/nL)_S, A[n]) \underset{S}{\times} \underline{\mathrm{Hom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ and $\alpha'_{\mathcal{H}_n} \subset \underline{\mathrm{Hom}}_S((L/nL)_S, A'[n]) \underset{S}{\times} \underline{\mathrm{Hom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ be the subschemes defining respectively $\alpha_{\mathcal{H}}$ and $\alpha'_{\mathcal{H}}$ as in Definition 1.3.7.8. Then $\alpha_{\mathcal{H}_n}$ is the pullback of $\alpha'_{\mathcal{H}_n}$ under the morphism $f|_{A[n]} \times \mathrm{Id}$. (It is clear that it suffices to verify this condition for one n.)*

Definition 1.4.1.5. *If we have two tuples $(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$ as in Definition 1.4.1.4 under an isomorphism $f : A \xrightarrow{\sim} A'$, then we say in this case that we have an isomorphism $f : (A, \lambda, i, \alpha_{\mathcal{H}}) \xrightarrow{\sim} (A', \lambda', i', \alpha'_{\mathcal{H}})$.*

Remark 1.4.1.6. We have a canonical identification $\mathsf{M}_n = \mathsf{M}_{\mathcal{U}^\Box(n)}$ because (integral) level-$\mathcal{H}$ structures are just (integral) principal level-n structures when $\mathcal{H} = \mathcal{U}^\Box(n)$.

Remark 1.4.1.7. For readers not familiar with the language of category fibred in groupoids, they can pretend that the moduli problem M_n is given by the association

$$S \mapsto \mathsf{M}_{\mathcal{H}}(S) := \{\text{tuples } (A, \lambda, i, \alpha_{\mathcal{H}})\}/\sim_{\text{isom.}},$$

where each $\mathsf{M}_{\mathcal{H}}(S)$ are now given by *sets*. This gives essentially the same information when the objects of $\mathsf{M}_{\mathcal{H}}(S)$ have no nontrivial automorphism. For example, this is the case when $\mathcal{H}$ is *neat* (defined as in Definition 1.4.1.8; see also Corollary 1.4.1.11 below.)

Following Pink [107, 0.6], we define the neatness of open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as follows: Let us view $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as a *subgroup* of $\mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times \mathbf{G}_{\mathrm{m}}(\hat{\mathbb{Z}}^\Box)$ (as in Definition 1.2.1.5). (Alternatively, we may take any faithful linear algebraic representation of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$.) Then, for each rational prime $p > 0$ not in $\Box$, it makes sense to talk about *eigenvalues* of elements g_p in $\mathrm{G}(\mathbb{Z}_p)$, which are elements in $\bar{\mathbb{Q}}_p^\times$. Let $g = (g_p) \in \mathrm{G}(\hat{\mathbb{Z}}^\Box)$, with p running through rational primes such that $\Box \nmid p$. For each such p, let Γ_p be the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by eigenvalues of g_p. For any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, consider the subgroup $(\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\text{tors}}$ of torsion elements of $\bar{\mathbb{Q}}^\times \cap \Gamma_p$, which is independent of the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Definition 1.4.1.8. *We say that* $g = (g_p)$ *is* **neat** *if* $\bigcap\limits_{p \notin \Box} (\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\mathrm{tors}} = \{1\}$*. We say that an open compact subgroup* $\mathcal{H}$ *of* $\mathrm{Z}(\hat{\mathbb{Z}}^\Box)$ *is* **neat** *if all its elements are neat.*

Remark 1.4.1.9. The usual Serre's lemma that no nontrivial root of unity can be congruent to 1 mod n if $n \geq 3$ shows that $\mathcal{H}$ is neat if $\mathcal{H} \subset \mathcal{U}^\Box(n)$ for some $n \geq 3$ such that $\Box \nmid n$.

The usefulness of neatness is that it eliminates the possible automorphisms of tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ parameterized by $\mathsf{M}_{\mathcal{H}}$. To show this, let us quote the following result of Mumford:

Lemma 1.4.1.10 ([99, §21, Thm. 5 and its proof]). *Let* A *be an abelian scheme over a base scheme* S*. For any* $n \geq 3$*, and any polarization* $\lambda : A \to A^\vee$ *on* A*, the restriction homomorphism*

$$\mathrm{Aut}_S(A, \lambda) := \{f \in \mathrm{Aut}_S(A) : f^\vee \circ \lambda \circ f = \lambda\} \to \mathrm{Aut}_S(A[n])$$

is injective, and its image acts via a subgroup of the roots of unity.

Corollary 1.4.1.11. *Let* S *be a scheme over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ *and let* $(A, \lambda, i, \alpha_{\mathcal{H}})$ *be an object of* $\mathsf{M}_{\mathcal{H}}(S)$*. Then* $(A, \lambda, i, \alpha_{\mathcal{H}})$ *has no nontrivial automorphism if* $\mathcal{H}$ *is* **neat** (*defined as in Definition 1.4.1.8*).

Proof. Let $\bar{s}$ be any geometric point of S. It suffices to show that restriction of any automorphism of $(A, \lambda, i, \alpha_{\mathcal{H}})$ to $\bar{s}$ is trivial. Let f be any automorphism of $(A_{\bar{s}}, \lambda_{\bar{s}}, i_{\bar{s}})$, the pullback of (A, λ, i) to $\bar{s}$. By Lemma 1.4.1.10, the restriction

$$\mathrm{Aut}_{\bar{s}}(A_{\bar{s}}, \lambda_{\bar{s}}, i_{\bar{s}}) \to \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box}(\mathrm{T}^\Box A_{\bar{s}})$$

is an injection, and its image acts via a subgroup of the roots of unity. If the image of f also preserves any symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{s}}$ lifting $\alpha_{\mathcal{H}}$ over $\bar{s}$, then it lies in the intersection of $\mathcal{H}$ and a subgroup of the roots of unity, which is $\{1\}$ by neatness of $\mathcal{H}$. ☐

Theorem 1.4.1.12. *Let* $\mathcal{H}$ *be an open compact subgroup of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$*. The moduli problem* $\mathsf{M}_{\mathcal{H}}$ *is representable by a smooth separated algebraic stack of finite type over* S_0*. It is representable by an algebraic space if the objects it parameterizes have no nontrivial automorphism, which is in particular the case when* $\mathcal{H}$ *is neat* (*defined as in Definition 1.4.1.8*).

As a special case:

Corollary 1.4.1.13. *The moduli problem* M_n *is representable by a smooth separated algebraic stack of finite type over* S_0*. It is representable by an algebraic space if* $n \geq 3$*.* (*See Remark 1.4.1.9.*)

The proof of Theorem 1.4.1.12 will be given in Chapter 2. We shall denote the algebraic stacks (or algebraic spaces) representing $\mathsf{M}_{\mathcal{H}}$ and M_n by respectively the same notations $\mathsf{M}_{\mathcal{H}}$ and M_n.

Remark 1.4.1.14. We shall see in Corollary 7.2.3.10, which is a byproduct of an intermediate construction in the proof of Theorem 7.2.4.1, that the $\mathsf{M}_{\mathcal{H}}$ is actually a quasi-projective scheme when $\mathcal{H}$ is neat. Therefore it is not necessary to argue that it is a scheme at this moment.

1.4.2 Definition by $\mathbb{Z}_{(\Box)}^\times$-Isogeny Classes

Let $V := L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Then we may write $(V \underset{\mathbb{Q}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)$ and $(V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,, \cdot \rangle)$ instead of respectively $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)$ and $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot\,, \cdot \rangle)$.

Definition 1.4.2.1. *The moduli problem* $\mathsf{M}_n^{\mathrm{rat}}$ *is defined by the category fibred in groupoids over* $(\mathrm{LNSch}/\mathsf{S}_0)$ *whose fiber over each* S *is the groupoid* $\mathsf{M}_n(S)$ *described as follows: The objects of* $\mathsf{M}_n^{\mathrm{rat}}(S)$ *are tuples* $(A, \lambda, i, [\hat{\alpha}]_n)$*, where:*

1. A *is an abelian scheme over* S*.*
2. $\lambda : A \to A^\vee$ *is a* $\mathbb{Z}_{(\Box)}^\times$*-polarization of* A*.*
3. $i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ *defines an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-structure of* (A, λ)*.*
4. $\underline{\mathrm{Lie}}_{A/S}$ *with its* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-module structure given by* i *satisfies the determinantal condition as in Definition 1.3.4.2 given by* $(V \underset{\mathbb{Q}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)$*.*
5. $[\hat{\alpha}]_n$ *is a rational principal level-*n *structure of* (A, λ, i) *of type* $(V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot\,, \cdot \rangle)$ *as defined in Definition 1.3.6.10.*

The isomorphisms

$$(A, \lambda, i, [\hat{\alpha}]_n) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_n)$$

of $\mathsf{M}_n^{\mathrm{rat}}(S)$ *are given by* $\mathbb{Z}_{(\Box)}^\times$*-isogenies* $f : A \to A'$ *such that:*

1. *$\lambda = rf^\vee \circ \lambda' \circ f$ for some $r \in \mathbb{Z}^\times_{(\Box),>0}$.*

2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$.*

3. *For each geometric point $\bar{s}$ of S, the map $\mathrm{V}^\Box : \mathrm{V}^\Box A_{\bar{s}} \xrightarrow{\sim} \mathrm{V}^\Box A'_{\bar{s}}$ induced by f satisfies the condition that, for any representatives $\hat{\alpha}$ and $\hat{\alpha}'$ representing respectively $[\hat{\alpha}]_n$ and $[\hat{\alpha}']_n$ at $\bar{s}$ (see Convention 1.3.6.11), $(\hat{\alpha}')^{-1} \circ \mathrm{V}^\Box(f) \circ \hat{\alpha}$ lies in the $\mathcal{U}^\Box(n)$-orbit of the identity on $V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}$, and $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ lies in the $\nu(\mathcal{U}^\Box(n))$-orbit of the $r \in \mathbb{Z}^\times_{(\Box),>0}$ above (such that $\lambda = rf^\vee \circ \lambda' \circ f$).*

Definition 1.4.2.2. *If we have two tuples $(A, \lambda, i, [\hat{\alpha}]_n) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_n)$ as in Definition 1.4.2.1 under a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : A \xrightarrow{\sim} A'$, then we say in this case that we have a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i, [\hat{\alpha}]_n) \xrightarrow{\sim} (A', \lambda', i', [\hat{\alpha}']_n)$.*

Remark 1.4.2.3. Suppose $L \neq 0$. Let $\bar{s}$ be any geometric point of S. Then the $r \in \mathbb{Z}^\times_{(\Box),>0}$ above (in the definition of a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i, [\hat{\alpha}]_n) \xrightarrow{\sim} (A', \lambda', i', [\hat{\alpha}']_n)$) such that $\lambda = rf^\vee \circ \lambda' \circ f$ implies that

$$\mathrm{e}^{\lambda'}(\mathrm{V}^\Box(f)(x), \mathrm{V}^\Box(f)(y)) = \mathrm{e}^{\lambda}(x, r^{-1}y)$$

for any $x, y \in \mathrm{V}^\Box A_{\bar{s}}$. In this case, we may interpret r^{-1} as some *similitude factor* for $\mathrm{V}^\Box(f)$. Then the condition that $(\hat{\alpha}')^{-1} \circ \mathrm{V}^\Box(f) \circ \hat{\alpha}$ lies in the $\mathcal{U}^\Box(n)$-orbit of the identity on $V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}$ forces $\nu(\hat{\alpha}')^{-1} \circ r^{-1} \circ \nu(\hat{\alpha})$ to lie in the $\nu(\mathcal{U}^\Box(n))$-orbit of the identity on $\mathbb{A}^{\infty,\Box}$, and hence the equivalent condition that $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ lies in the $\nu(\mathcal{U}^\Box(n))$-orbit of the $r \in \mathbb{Z}^\times_{(\Box),>0}$.

The definition for general level structures is as follows:

Definition 1.4.2.4. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\mathbb{A}^{\infty,\Box})$. The moduli problem $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ is defined by the category fibred in groupoids over $(\mathrm{LNSch}/\mathsf{S}_0)$ whose fiber over each S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$ are tuples $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}})$, where:*

1. *A is an abelian scheme over S.*

2. *$\lambda : A \to A^\vee$ is a $\mathbb{Z}^\times_{(\Box)}$-polarization of A.*

3. $i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ *defines an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-structure of* (A, λ).

4. $\underline{\mathrm{Lie}}_{A/S}$ *with its* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-module structure given by* i *satisfies the determinantal condition as in Definition 1.3.4.2 given by* $(V \underset{\mathbb{Q}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$.

5. $[\hat{\alpha}]_{\mathcal{H}}$ *is a rational principal level-*$\mathcal{H}$ *structure of* (A, λ, i) *of type* $(V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$ *as defined in Definition 1.3.7.15.*

The isomorphisms

$$(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$$

of $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$ *are given by* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f : A \to A'$ *such that:*

1. $\lambda = r f^\vee \circ \lambda' \circ f$ *for some* $r \in \mathbb{Z}^\times_{(\Box),>0}$.

2. $f \circ i(b) = i'(b) \circ f$ *for all* $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$.

3. *For each geometric point* $\bar{s}$ *of* S*, the map* $\mathrm{V}^\Box f : \mathrm{V}^\Box A_{\bar{s}} \xrightarrow{\sim} \mathrm{V}^\Box A'_{\bar{s}}$ *induced by* f *satisfies the condition that, for any representatives* $\hat{\alpha}$ *and* $\hat{\alpha}'$ *representing respectively* $[\hat{\alpha}]_{\mathcal{H}}$ *and* $[\hat{\alpha}']_{\mathcal{H}}$ *at* $\bar{s}$ *(see Convention 1.3.7.16),* $(\hat{\alpha}')^{-1} \circ \mathrm{V}^\Box(f) \circ \hat{\alpha}$ *lies in the* $\mathcal{H}$*-orbit of the identity on* $V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}$*, and* $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ *lies in the* $\nu(\mathcal{H})$*-orbit of the* $r \in \mathbb{Z}^\times_{(\Box),>0}$ *above (such that* $\lambda = r f^\vee \circ \lambda' \circ f$*).*

Definition 1.4.2.5. *If we have two tuples* $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$ *as in Definition 1.4.2.4 under a* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : A \xrightarrow{\sim} A'$*, then we say in this case that we have a* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \xrightarrow{\sim} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$.

Remark 1.4.2.6. We have a canonical identification $\mathsf{M}^{\mathrm{rat}}_n = \mathsf{M}^{\mathrm{rat}}_{\mathcal{U}^\Box(n)}$ because rational level-$\mathcal{H}$ structures are just rational principal level-n structures when $\mathcal{H} = \mathcal{U}^\Box(n)$.

Remark 1.4.2.7. Definition 1.4.2.4 uses only the existence of some $(L, \langle \cdot , \cdot \rangle)$ inside $(V \underset{\mathbb{Q}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ and $(V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\Box}, \langle \cdot , \cdot \rangle)$, or rather $(V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\Box}, \langle \cdot , \cdot \rangle)$. Moreover, it uses only the properties of the maximal $\mathbb{Z}_{(\Box)}$-order $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$, rather than the possibly non-maximal $\mathbb{Z}$-order $\mathcal{O}$. We could have started with a maximal order $\mathcal{O}$ and a symplectic vector space $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ (with suitable conditions) containing some $\mathcal{O}$-lattice L on which $\langle\,\cdot\,,\,\cdot\,\rangle$ takes integral values and such that $(L^{\#})\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} = L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, without changing the definition at all. (The subgroups $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ and $\mathcal{U}^{\Box}(n)$ depend on the choice of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$, anyway.)

1.4.3 Relation Between Two Definitions

Let $\mathcal{H}$ be a fixed choice of an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$.

Construction 1.4.3.1. Note that we can define a canonical morphism

$$\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{rat}} \tag{1.4.3.2}$$

as follows: Over each connected locally noetherian base scheme S over S_0, with any choice of a geometric point $\bar{s}$ of S, and given a tuple $(A, \lambda, i, \alpha_n)$ representing a class in $\mathsf{M}_{\mathcal{H}}(S)$, we can associate a tuple $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}})$ representing a class in $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S)$ by using the same A, λ, i, and by using the $\mathcal{H}$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ associated to $\alpha_{\mathcal{H}}$ as described in Construction 1.3.7.10. This definition is independent of the choice of the geometric point $\bar{s}$, and extends to general local noetherian base schemes by the formulae

$$\mathsf{M}_{\mathcal{H}}(S) \cong \prod_i \mathsf{M}_{\mathcal{H}}(S_i)$$

and

$$\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}} \cong \prod_i \mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S_i)$$

when $S = \coprod_i S_i$ is the decomposition of S into its (not necessarily finitely many) connected components S_i.

Proposition 1.4.3.3. *The map (1.4.3.2) is an isomorphism.*

Remark 1.4.3.4. This is in the sense of 1-isomorphisms between 2-categories. In particular, this only requires the map (1.4.3.2) to induce *equivalences of categories* $\mathsf{M}_{\mathcal{H}}(S) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S)$ for each S. (See Remark A.1.2.7 and Definition A.4.4 for details.)

Proof of Proposition 1.4.3.3. Without loss of generality, we shall assume that S is connected locally noetherian, and fix a particular choice $\bar{s}$ of a geometric point on S. All rational level-$\mathcal{H}$ structures we consider will be based at $\bar{s}$, without further explanations. (See Remark 1.3.7.17.)

Suppose $(A, \lambda, i, \alpha_{\mathcal{H}})$ and $(A', \lambda', i', \alpha'_{\mathcal{H}})$ are rewritten respectively as $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}})$ and $(A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$ as in Construction 1.4.3.1. Let us take any choices of $\hat{\alpha}$ and $\hat{\alpha}'$ that represent respectively $[\hat{\alpha}]_{\mathcal{H}}$ and $[\hat{\alpha}']_{\mathcal{H}}$.

Suppose $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$. By definition, this means there is a $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $f : (A, \lambda, i) \to (A', r\lambda', i')$ for some $r \in \mathbb{Z}^{\times}_{(\Box),>0}$, such that $(\hat{\alpha}')^{-1} \circ \mathrm{V}^{\Box}(f) \circ \hat{\alpha}$ lies in the $\mathcal{H}$-orbit of the identity on $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$, and such that $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ lies in the $\nu(\mathcal{H})$-orbit of r.

By Construction 1.4.3.1, we have $\mathrm{T}^{\Box} A_{\bar{s}} = \hat{\alpha}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$, and $\mathrm{T}^{\Box} A'_{\bar{s}} = \hat{\alpha}'(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$. Therefore we have $\mathrm{V}^{\Box}(f)(\mathrm{T}^{\Box} A_{\bar{s}}) = \mathrm{T}^{\Box} A'_{\bar{s}}$, which by Corollary 1.3.5.4 implies that $f : A \to A'$ is an isomorphism. Since $\mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{s}} = \nu(\hat{\alpha})(\hat{\mathbb{Z}}^{\Box}(1)) = \nu(\hat{\alpha}')(\hat{\mathbb{Z}}^{\Box}(1))$ by construction, $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ lies in $\hat{\mathbb{Z}}^{\Box,\times}$, which by assumption has to contain the $\nu(\mathcal{H})$-orbit of r. Since $\nu(\mathcal{H}) \subset \hat{\mathbb{Z}}^{\Box,\times}$ (because $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$), the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}^{\times}_{(\Box),>0} \cdot \hat{\mathbb{Z}}^{\Box,\times}$ forces $r = 1$. Furthermore, it is clear that we have $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ in the sense of Definition 1.4.1.4. Therefore $(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$, and we can conclude the injectivity of (1.4.3.2).

On the other hand, suppose $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}})$ is any tuple representing a class in $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$. We must show that there exists a tuple $(A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$, satisfying $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$, such that $(A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$ comes from a tuple representing a class in $\mathsf{M}_{\mathcal{H}}(S)$. That is, the tuple $(A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$ has the following properties:

1. λ' is a polarization (instead of a $\mathbb{Z}^{\times}_{(\Box)}$-polarization).

2. i' defines an $\mathcal{O}$-structure, in the sense that it maps $\mathcal{O}$ to $\mathrm{End}_S(A')$.

3. For any representative $\hat{\alpha}'$ of $[\hat{\alpha}']_{\mathcal{H}}$, $\hat{\alpha}'$ induces (by Corollary 1.3.7.11) an $\mathcal{O}$-equivariant symplectic isomorphism $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} A'_{\bar{s}}$, so that $\nu(\hat{\alpha}')$ induces an isomorphism $\hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{s}}$ making the following

diagram commutative:

$$\begin{tikzcd} (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\times(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}'\times\hat{\alpha}'"', "\wr"] & \hat{\mathbb{Z}}^\Box(1) \arrow[d, dotted, "\wr"', "\nu(\hat{\alpha}')"] \\ \mathrm{T}^\Box A'_{\bar{s}}\times\mathrm{T}^\Box A'_{\bar{s}} \arrow[r, "\lambda'\text{-Weil}"'] & \mathrm{T}^\Box\mathbf{G}_{\mathrm{m},\bar{s}} \end{tikzcd} \tag{1.4.3.5}$$

Note that this is also a condition on λ': If we replace λ' by a positive multiple different from itself, then this will not hold.

Let us fix any representative $\hat{\alpha}$ of $[\hat{\alpha}]_{\mathcal{H}}$. By Corollary 1.3.5.4, the $\mathcal{O}$-invariant open compact subgroup $\hat{\alpha}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ of $\mathrm{V}^\Box A_{\bar{s}}$ corresponds to some $\mathbb{Z}^\times_{(\Box)}$-isogeny $f:(A,\lambda,i)\to(A',\lambda'',i')$ such that

$$\mathrm{V}^\Box(f)^{-1}(\mathrm{T}^\Box A'_{\bar{s}})=\hat{\alpha}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$$

and such that i' has image in $\mathrm{End}_S(A')$. The conditions for $\sim_{\mathbb{Z}_{(\Box)}\text{-isog.}}$ suggests that we should define $\hat{\alpha}'$ as the composition of $\mathrm{V}^\Box(f)$ and $\hat{\alpha}$, and define $[\hat{\alpha}']_{\mathcal{H}}$ as the $\mathcal{H}$-orbit of $\hat{\alpha}'$. Then

$$\hat{\alpha}'(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)=\mathrm{V}^\Box(f)(\hat{\alpha}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box))=\mathrm{T}^\Box A'_{\bar{s}},$$

and we do have $(A,\lambda,i,[\hat{\alpha}]_{\mathcal{H}})\sim_{\mathbb{Z}_{(\Box)}\text{-isog.}}(A',r\lambda'',i',[\hat{\alpha}']_{\mathcal{H}})$ for any $r\in\mathbb{Z}^\times_{(\Box),>0}$. It only remains to show the existence of a (necessarily unique) number $r\in\mathbb{Z}^\times_{(\Box),>0}$ so that $\lambda'=r\lambda''$ is a polarization, and so that there exists an isomorphism $\hat{\mathbb{Z}}^\Box(1)\xrightarrow{\sim}\mathrm{T}^\Box\mathbf{G}_{\mathrm{m},\bar{s}}$ making the diagram (1.4.3.5) commute.

Let us denote the λ''-Weil (resp. λ'-Weil) pairing by $\mathrm{e}^{\lambda''}(\,\cdot\,,\,\cdot\,)$ (resp. $\mathrm{e}^{\lambda'}(\,\cdot\,,\,\cdot\,)$). Let $\gamma:\mathbb{A}^{\infty,\Box}(1)\xrightarrow{\sim}\mathrm{V}^\Box\mathbf{G}_{\mathrm{m},\bar{s}}$ be the unique isomorphism making the diagram

$$\begin{tikzcd} (L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\times(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}'\times\hat{\alpha}'"'] & \mathbb{A}^{\infty,\Box}(1) \arrow[d, dotted, "\wr"', "\gamma"] \\ \mathrm{V}^\Box A'_{\bar{s}}\times\mathrm{V}^\Box A'_{\bar{s}} \arrow[r, "\mathrm{e}^{\lambda''}(\,\cdot\,,\,\cdot\,)"'] & \mathrm{V}^\Box\mathbf{G}_{\mathrm{m},\bar{s}} \end{tikzcd}$$

commute. If we replace λ'' by $r\lambda''$ for some $r\in\mathbb{Z}^\times_{(\Box),>0}$, then we have to replace γ by $r\gamma$. The approximation $\mathbb{A}^{\infty,\Box,\times}=\mathbb{Z}^\times_{(\Box),>0}\cdot\hat{\mathbb{Z}}^{\Box,\times}$ shows the existence of a unique $r \in \mathbb{Z}^{\times}_{(\Box),>0}$ so that $r\gamma(\hat{\mathbb{Z}}^{\Box}(1)) = \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}$. Then we have by definitions of the Weil-pairing and of $L^{\#}$ the commutative diagram with surjective horizontal maps defining perfect pairings:

$$\begin{tikzcd}
(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}) \times (L^{\#}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}'\times\hat{\alpha}'"'] & \hat{\mathbb{Z}}^{\Box}(1) \arrow[d, dotted, "r\gamma", "\wr"'] \\
\mathrm{T}^{\Box}\, A'_{\bar{s}} \times \mathrm{T}^{\Box}((A'_{\bar{s}})^{\vee}) \arrow[r, "\mathrm{e}^{r\lambda''}(\,\cdot\,,\,\cdot\,)"'] & \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

In particular, the inclusion $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}) \subset (L^{\#}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ now corresponds to the inclusion $\mathrm{V}^{\Box}(r\lambda'')(\mathrm{T}^{\Box}\, A'_{\bar{s}}) \subset \mathrm{T}^{\Box}((A'_{\bar{s}})^{\vee})$. In other words, $\lambda' = r\lambda''$ is a polarization (instead of a $\mathbb{Z}^{\times}_{(\Box)}$-polarization), and it makes the diagram (1.4.3.5) commute with the isomorphism $r\gamma : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}$ above, which is now the $\nu(\hat{\alpha}')$ we want. This concludes the proof of surjectivity of (1.4.3.2). $\Box$

As a trivial consequence:

Corollary 1.4.3.6. $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ *enjoys the same representability properties as* $\mathsf{M}_{\mathcal{H}}$ *does in Theorem 1.4.1.12.*

Moreover, we obtain the following *exotic* isomorphism between moduli problems defined by reasonably different choices of PEL-type $\mathcal{O}$-lattices:

Corollary 1.4.3.7. *Let* $\mathcal{O}$ *and* $\mathcal{O}'$ *be two* $\mathbb{Z}$*-orders invariant under the involution* $\star$ *of* B*, and let* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *and* $(L', \langle\,\cdot\,,\,\cdot\,\rangle')$ *be respectively a PEL-type* $\mathcal{O}$*-lattice and a PEL-type* $\mathcal{O}'$*-lattice. Suppose* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \cong \mathcal{O}'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$*, and* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (L'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}, \langle\,\cdot\,,\,\cdot\,\rangle)$ *(respecting the two module structures), so that* $\Box$ *is a set of good primes for both of them. Then the two moduli problems* $\mathsf{M}_{\mathcal{H}}$ *and* $\mathsf{M}'_{\mathcal{H}}$ *over* $\mathsf{S} = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ *defined respectively by them are isomorphic to each other.*

Proof. This follows from Proposition 1.4.3.3 and Remark 1.4.2.7. $\Box$

Remark 1.4.3.8. Let $\mathcal{O}'$ be any fixed choice of maximal order in B containing $\mathcal{O}$. Let L' denote the $\mathcal{O}'$-span of L in $V = L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. It is an $\mathcal{O}'$-lattice because it is $\mathbb{Z}$-torsion-free and finitely generated over $\mathcal{O}'$. By Proposition 1.1.1.17, we have $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} = \mathcal{O}'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, and hence $L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} = (L')\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ have the same self-dual property. Therefore, by Corollary 1.4.3.7, the isomorphism class of the moduli problem $\mathsf{M}_{\mathcal{H}}$ we define is unchanged if we replace $\mathcal{O}$ by $\mathcal{O}'$, and L by L'. (Certainly, it might no longer make sense to say that $\mathcal{H} = \mathcal{U}^{\Box}(n)$ for a particular integer $n \geq 1$ because we have modified $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$. This is also one reason that it is more natural to work with general level structures.)

Condition 1.4.3.9. *The PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ is chosen so that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$.*

As explained in Remark 1.4.3.8, this is harmless for our purpose of compactifications. We will make essential use of this technical condition only when we study the degeneration of objects in M_n. (See Lemma 5.2.2.4 below.)

Remark 1.4.3.10. If we form the tower $\mathsf{M}^{\Box} := \varprojlim_{n, \Box \nmid n} \mathsf{M}_n^{\mathrm{rat}}$ or $\mathsf{M}^{\Box} := \varprojlim_{\mathcal{H}, \mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})} \mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$, then the objects of this tower can be represented by tuples of the form $(A, \lambda, i, [\hat{\alpha}])$ in the obvious sense, and there is a natural right action of elements $g \in \mathrm{G}(\mathbb{A}^{\infty, \Box})$ on $\mathsf{M}^{\Box}$ defined by sending a representative $(A, \lambda, i, \hat{\alpha})$ to $(A, \lambda, i, \hat{\alpha} \circ g)$. At finite levels, the action can be defined more precisely either by sending $(A, \lambda, i, [\hat{\alpha}]_m)$ at level m to $(A, \lambda, i, [\hat{\alpha} \circ g]_n)$ at level n, if $n|m$, $\Box \nmid m$, and $g^{-1}\mathcal{U}^{\Box}(m)g \subset \mathcal{U}^{\Box}(n)$, or by sending $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}'})$ at level $\mathcal{H}'$ to $(A, \lambda, i, [\hat{\alpha} \circ g]_{\mathcal{H}})$ at level $\mathcal{H}$, if $\mathcal{H}' \subset \mathcal{H} \cap (g\mathcal{H}g^{-1})$. (Note that here we use $g^{-1}\mathcal{U}^{\Box}(m)g$ rather than $g\mathcal{U}^{\Box}(m)g^{-1}$, or $\mathcal{H} \cap (g\mathcal{H}g^{-1})$ rather than $\mathcal{H} \cap (g^{-1}\mathcal{H}g)$, because we are using a right action.) (We will elaborate more on this idea in Sections 5.4.3 and 6.4.3.)

Remark 1.4.3.11. In the theory of integral models of Shimura varieties, it is not easy to refine the definition of $\mathsf{M}_{\mathcal{H}}$ so that its characteristic zero fiber does not contain unnecessary components other than the canonical model of the Shimura variety we want. By first identifying $\mathsf{M}_{\mathcal{H}}$ with $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ by Proposition 1.4.3.3, we see from Definition 1.4.2.4 that the definition involves only $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\Box}, \langle\,\cdot\,,\,\cdot\,\rangle)$, but not $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$. Therefore any non-isomorphic pair $(L_1, \langle\,\cdot\,,\,\cdot\,\rangle_1)$ and $(L_2, \langle\,\cdot\,,\,\cdot\,\rangle_2)$ define the same moduli problem $\mathsf{M}_{\mathcal{H}}$ if they become isomorphic after tensoring with $\mathbb{A}^{\Box}$. (This is exactly the problem of the *failure of Hasse's principle* mentioned in Remark 1.2.1.9.) In some cases there does exist such a pair, and therefore we obtain more than one Shimura varieties in the components of the characteristic zero fiber of $\mathsf{M}_{\mathcal{H}}$. Note that the characteristic zero fiber of $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ can be identified as a submoduli problem of the one defined by Kottwitz in [79, §5]. As explained by Kottwitz in [79, §8], the canonical models of Shimura varieties appearing in the characteristic zero fiber of $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ are all isomorphic to each other (even as canonical models). Therefore, the failure of Hasse's principle in the definition of our moduli problems is harmless for our purpose.

Remark 1.4.3.12. Even if the failure of Hasse's principle does not occur, the algebraic stack (or algebraic space) $\mathsf{M}_{\mathcal{H}}$ (or $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$) is not geometrically connected in general.

Remark 1.4.3.13. Although Definition 1.4.2.4 involves only $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\Box},\langle\,\cdot\,,\,\cdot\,\rangle)$ but not $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$, the existence of the lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ is important. Suppose we have defined the moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ using only some adelic pairings $(M_{\mathbb{R}},\langle\ ,\ \rangle)$ (together with the existence of some $h:\mathbb{C}\to\mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ satisfying Condition 1.2.1.2) and $(M_{\mathbb{A}^{\infty,\Box}},\langle\ ,\ \rangle)$, without assuring that the pairings come from some particular integral pairing $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ (or some rational analogue). Then the proofs of Theorem 1.4.1.12 and Proposition 1.4.3.3 still work, and they show that $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ is smooth over S_0. However, it is not clear that it is nonempty! In fact, the existence of any geometric point will force the existence of some complex point by smoothness, and the H_1 (with pairing) of the corresponding polarized complex abelian variety will force the existence of some PEL-type $\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ inducing both $(M_{\mathbb{R}},\langle\ ,\ \rangle)$ and $(M_{\mathbb{A}^{\infty,\Box}},\langle\ ,\ \rangle)$. Conversely, if we have some PEL-type PEL$\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$, then we can define a complex abelian variety by taking the real torus $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})/L$ with complex structure given by any map $h:\mathbb{C}\to\mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ as in Condition 1.2.1.2. In particular, the moduli problem defined by $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ is nonempty. This justifies our use of PEL-type $\mathcal{O}$-lattices (integral versions rather than adelic versions) in the definition of moduli problems.

1.4.4 Definition By Different Set of Primes

Let $\Box_1$ and $\Box_2$ be two choices of sets of good primes (defined as in Definition 1.4.1.1). Let $\Box:=\Box_1\cap\Box_2$. For $i=1,2$, let

$$\mathcal{U}_{\Box_i-\Box}:=\prod_{p\in\Box_i-\Box}\mathrm{G}(\mathbb{Z}_p).$$

Suppose $\mathcal{H}$ is an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, such that there exists an open compact subgroup $\mathcal{H}_i \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box_i})$ such that $\mathcal{H} = \mathcal{H}_i \times \mathcal{U}_{\Box_i - \Box}$, for each $i = 1, 2$. Let $\mathsf{M}_{\mathcal{H}_1}$ and $\mathsf{M}_{\mathcal{H}_2}$ be defined respectively over $\mathsf{S}_1 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box_1)})$ and $\mathsf{S}_2 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box_2)})$ as in Definition 1.4.1.2. Then we have the following important fact:

Proposition 1.4.4.1. *Notations and assumptions as above, let* $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*, and let* $\mathsf{M}_{\mathcal{H}}$ *be the moduli problem defined over* S_0 *as in Definition 1.4.1.2. Suppose there is a unique isomorphism class of self-dual* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$*-modules of each multi-rank for any* $p \in \Box_1 \cup \Box_2$*. Then the two canonical morphisms*

$$\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_1} \underset{\mathsf{S}_1}{\times} \mathsf{S}_0$$

and

$$\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_2} \underset{\mathsf{S}_2}{\times} \mathsf{S}_0$$

are canonical isomorphisms.

Proof. It suffices to show that $\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}_1} \underset{\mathsf{S}_1}{\times} \mathsf{S}_0$. Let S be any locally noetherian scheme over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. We claim that we can identify canonically $\mathsf{M}_{\mathcal{H}}(S) = \mathsf{M}_{\mathcal{H}_1}(S)$, which will suffice for our result. The definitions of the two moduli problems $\Box_1$ and $\Box$ are the same except that the level structures (defined as in Definition 1.3.7.8) of objects of $\mathsf{M}_{\mathcal{H}}$ require the lifting conditions at those primes $p > 0$ such that $\Box_1 | p$ but $\Box \nmid p$. In particular, any such $\Box_1 | p$ still satisfies $p \nmid \mathrm{I}_{\mathrm{bad}} \,\mathrm{Disc}[L^\# : L]$. Let $(A, \lambda, i, \alpha_{\mathcal{H}_1})$ be an object of $\mathsf{M}_{\mathcal{H}_1}(S)$, and let $\bar{s}$ be any geometric point of S. The question is whether we can extend $\alpha_{\mathcal{H}_1}$ to some level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$. At any $p > 0$ such that $\Box_1 | p$ but $\Box \nmid p$, the λ-Weil pairing defines a *perfect* alternating pairing on $\mathrm{T}_p \, A_{\bar{s}}$ with values in $\mathrm{T}_p \, \mathbf{G}_{\mathrm{m},\bar{s}}$. By choosing any isomorphism $\mathbb{Z}_p(1) \xrightarrow{\sim} \mathrm{T}_p \, \mathbf{G}_{\mathrm{m},\bar{s}}$, which is possible because $\Box \nmid p$, we obtain a perfect alternating $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-pairing with values in $\mathbb{Z}_p$. By assumption, there exists a symplectic isomorphism from $(L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \langle \cdot \, , \, \cdot \rangle)$ to this self-dual integrable symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattice, simply because they have the same multi-rank. Hence the liftability condition is automatic at any prime $p > 0$ such that $\Box_1 | 1$ but $\Box \nmid p$, and our claim follows. □

Remark 1.4.4.2. Proposition 1.4.4.1 suggests that we may focus on just one single good prime $p > 0$ when studying certain integral models, if in the context there is a unique symplectic isomorphism class of each multi-rank over the good primes. This is indeed the approach adopted in [79] and many other works. According to [124, Lem. 3.4] and [79, Lem. 7.2], or according to Proposition 1.2.3.7 and Corollary 1.2.3.10, this is the case except when the semisimple algebra B involves simple factors of type D (defined as in Definition 1.2.1.15).

Remark 1.4.4.3. The case that $\Box_1 = \{p\}$ for some good prime $p > 0$ and $\Box_2 = \emptyset$ is already significant when the semisimple algebra B involves any simple factors of type D (defined as in Definition 1.2.1.15). In this case, $\mathsf{S}_1 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ and $\mathsf{S}_0 = \mathrm{Spec}(F_0)$, but the statement that

$$\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}_1} \underset{\mathsf{S}_1}{\times} \mathsf{S}_0,$$

or equivalently that

$$\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}_1} \underset{\mathrm{Spec}(\mathbb{Z}_{(p)})}{\times} \mathrm{Spec}(\mathbb{Q}),$$

might not necessarily be true!

Chapter 2

Representability of Moduli Problems

Let us assume the same setting as in Section 1.4 in this chapter. Let us fix a choice of an open compact subgroup $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\square})$.

Our main objective is to prove Theorem 1.4.1.12, with Proposition 2.3.4.2 as a byproduct. Technical results worth noting are Proposition 2.1.6.8 in Section 2.1 and Corollary 2.2.4.15 in Section 2.2. The proof of Theorem 1.4.1.12 is carried out by verifying Artin's criterion in Section 2.3.3. (See in particular Theorems B.3.8, B.3.10, and B.3.12.) For those readers who might have wondered, let us make it clear that we will not need Condition 1.4.3.9 in this chapter.

Let us outline the strategy of our proof before we begin. (Those readers who are willing to believe the representability statement as explained in [79, §5] should feel free to skip this section.)

There exist at least two different methods for proving the representability of the moduli problem $\mathsf{M}_{\mathcal{H}}$ defined in Definition 1.4.1.4 (in the category of algebraic stacks).

The first one is given in [97, Ch. 7] using geometric invariant theory. The advantage of this method is that it is then clear that $\mathsf{M}_{\mathcal{H}}$ is a scheme when the objects it parameterizes have no nontrivial automorphism (as it always works in the category of schemes). Indeed, there is always a map from $\mathsf{M}_{\mathcal{H}}$ to the Siegel moduli schemes (of some polarization degree possibly greater than one), which is relatively representable by a scheme of finite type over its image. The image is closed in the Siegel moduli schemes, as the existence of additional structures are described by closed conditions. Therefore the general result using geometric invariant theory in [97, Ch. 7] for the Siegel moduli schemes implies that $\mathsf{M}_{\mathcal{H}}$ is of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and is actually a scheme when the objects it parameterizes have no nontrivial automorphism. This gives a (sketchy) proof of Theorem 1.4.1.12 modulo the understanding of the local moduli, in particular the smoothness.

The second one is Artin's criterion (for algebraic spaces or algebraic stacks), which has the advantage that it requires little more than showing the pro-representability of local moduli. Note that to prove the claim of smoothness in Theorem 1.4.1.12 we have to understand the local moduli anyway. Therefore it seems justified to us that our point of view should be biased toward the second method: Following the well-explained arguments in [113] and [104, §2] we will show that the local moduli is pro-representable and formally smooth at each point of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. The tricky input is the endomorphism structure and the Lie algebra condition that require some elaboration. Since the moduli problem can be shown to be of finite type by theory of Hilbert schemes, we conclude by Artin's criterion that $\mathsf{M}_{\mathcal{H}}$ is representable by a smooth separated algebraic stack of finite type over the base scheme. (See Appendices A and B for more details.) Note that Corollary 1.4.1.11 implies that the existence of a level-$\mathcal{H}$ structure kills all possible automorphisms of objects parameterized by $\mathsf{M}_{\mathcal{H}}$. Hence $\mathsf{M}_{\mathcal{H}}$ is necessarily an algebraic space in this case.

As already mentioned in Remark 1.4.1.14, the fact that $\mathsf{M}_{\mathcal{H}}$ is actually a scheme when $\mathcal{H}$ is neat will be a byproduct of our later work, and therefore could be suppressed at this moment.

Note that in this argument we do not need the Serre-Tate theory of local moduli (as in [87] or [69]), and hence we do not need Barsotti-Tate groups and any kind of Cartier-Dieudonné theory. They would be important for the study of integral models of Shimura varieties that are not smooth. As explained in the introduction, we have chosen not to discuss them in this work.

2.1 Theory of Obstructions for Smooth Schemes

Let us introduce some basic terminologies for the deformation of smooth schemes. Unless otherwise specified, all schemes in this section will be assumed to be separated and of finite type. (The readers might want to take a look at Section B.1 before reading this section.)

The idea of deforming smooth objects originated from the fundamental works of Kodaira and Spencer in [77] and [78], and the algebraic version dates back to Grothendieck's fundamental works in [54] and [58]. Some of the formulations we adopt here follow closely the presentation of [104, §2]. We hope it will not be unpleasant for the reader to see a lot of repetitions here.

In this section, we shall assume that all schemes are of finite type and separated over a common noetherian base scheme.

A special remark: We shall use notations such as $\underline{H}^0$ or $\underline{H}^i$ to denote the global sections and its higher direct images in the relative setting. The only reason to do this, instead of using notations like f_* or R^if_*, is because we do not want to introduce the structural map f every time we talk about a scheme in the relative setting. We hope this is acceptable for the readers. As soon as we arrive at the case that the base scheme is affine, this should incur no confusion.

2.1.1 Preliminaries

Lemma 2.1.1.1 (cf. [58, III, Lemma 4.2], or [104, Lemma 2.2.2])**.** *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$, and let $\tilde{f} : \tilde{Y} \to \tilde{X}$ be a morphism of schemes over $\tilde{S}$ such that $f := \tilde{f} \underset{\tilde{S}}{\times} S$ is an isomorphism. Suppose $\tilde{Y}$ is flat over $\tilde{S}$. Then $\tilde{f}$ is an isomorphism.*

Proof. Since $\tilde{f}$ induces a homeomorphism on the underlying topological spaces, it suffices to treat the following affine case: Let $I := \ker(\tilde{R} \twoheadrightarrow R)$ be a nilpotent ideal in $\tilde{R}$. Let $u : M \to N$ be a morphism of $\tilde{R}$-modules such that N is flat over $\tilde{R}$, and such that $u := u \underset{\tilde{R}}{\otimes} R : M/I \cdot M \to N/I \cdot N$ is an isomorphism. Then u is an isomorphism.

To show this, let $K := \ker(u)$, and $Q := N/u(M)$. By assumption, we have $Q/I \cdot Q = 0$, and so $Q = I \cdot Q = I^2 \cdot Q = \ldots = I^n \cdot Q = 0$ for some n, because I is nilpotent. Thus

$$0 = \operatorname{Tor}_1^{\tilde{R}}(N, \tilde{R}/I) \to K/I \cdot K \to M/I \cdot M \to N/I \cdot N \to 0,$$

because N is $\tilde{R}$-flat. Hence $K/I \cdot K = 0$, and $K = 0$ as before. □

Lemma 2.1.1.2 (cf. [58, III, §5])**.** *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X} \to \tilde{S}$ and $\tilde{Y} \to \tilde{S}$ be schemes over $\tilde{S}$, $X = \tilde{X} \underset{\tilde{S}}{\times} S$, $Y = \tilde{Y} \underset{\tilde{S}}{\times} S$, and denote by $\underline{\mathrm{Der}}_{Y/S}$ the sheaf of germs of $\mathscr{O}_S$-derivations from $\mathscr{O}_Y$ into itself. Let $f : X \to Y$ be an S-morphism of schemes. Let us denote by $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ the set of $\tilde{S}$-morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\tilde{f} \underset{\tilde{S}}{\times} S = f$. Suppose moreover that $\tilde{X}$ is flat over $\tilde{S}$. Then $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ is either empty or a torsor under*

$$\underline{H}^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

Proof. Note that the map $\tilde{f} : \tilde{X} \to \tilde{Y}$ on the underlying topology spaces was already determined by $f : X \to Y$. Therefore it suffices to treat the affine case. Let us assume $\tilde{S} = \mathrm{Spec}(\tilde{R})$, $S = \mathrm{Spec}(R)$, with $S \hookrightarrow \tilde{S}$ given by $\tilde{R} \twoheadrightarrow R$ with kernel I such that $I^2 = 0$. Let $\tilde{X} = \mathrm{Spec}(\tilde{M})$, $\tilde{Y} = \mathrm{Spec}(\tilde{N})$. Let us denote $\tilde{M}/I \cdot \tilde{M}$ (resp. $\tilde{N}/I \cdot \tilde{N}$) by M (resp. N), and let $u : N \to M$ denote the map given by $f : X \to Y$. Suppose we have a map $\tilde{u} : \tilde{N} \to \tilde{M}$ lifting u.

If $\tilde{u}' : \tilde{N} \to \tilde{M}$ is any other lifting, then $D := \tilde{u}' - \tilde{u}$ maps $\tilde{N}$ to $I \cdot \tilde{M}$. By [91, Thm. 7.7], the flatness of $\tilde{M}$ implies that the canonical morphism $\tilde{M} \underset{\tilde{R}}{\otimes} I \to \tilde{M} \underset{\tilde{R}}{\otimes} \tilde{R} \cong \tilde{M}$ is injective. Hence $I \cdot \tilde{M} \cong \tilde{M} \underset{\tilde{R}}{\otimes} I \cong M \underset{R}{\otimes} I$ because $I^2 = 0$. Moreover, the kernel of D contains $I \cdot \tilde{N}$. Therefore we may identify D as an R-linear map $D : N \to M \underset{R}{\otimes} I$.

Let n_1 and n_2 be elements in $\tilde{N}$. Then the comparison between

$$u'(n_1 n_2) - u(n_1 n_2) = D(n_1 n_2)$$

and

$$\begin{aligned} u'(n_1)u'(n_2) - u(n_1)u(n_2) &= (u'(n_1) - u(n_1))u'(n_2) + (u'(n_2) - u(n_2))u'(n_1) \\ &= D(n_1)u(n_2) + D(n_2)u(n_1) \end{aligned}$$

shows that

$$u'(n_1 n_2) = u'(n_1)u'(n_2)$$

if and only if

$$D(n_1 n_2) = D(n_1)u(n_2) + D(n_2)u(n_1).$$

Combining with other more trivial relations, this shows that u' is an algebra homomorphism if and only if D is an R-derivative from N to $M \underset{R}{\otimes} I$, where the N-module structure of M is given by $u : N \to M$. Note that we have canonical isomorphisms

$$\mathrm{Der}_R(N, M \underset{R}{\otimes} I) \cong \mathrm{Hom}_N(\Omega^1_{N/R}, M \underset{R}{\otimes} I) \cong \mathrm{Hom}_M(\Omega^1_{N/R} \underset{N,u}{\otimes} M, M \underset{R}{\otimes} I).$$

Written globally, this is the group of global sections of

$$\underline{\mathrm{Der}}_{\mathscr{O}_S}(\mathscr{O}_Y, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

which is precisely the group $\underline{H}^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ in the statement of the lemma in this affine local case. □

Corollary 2.1.1.3 (cf. [104, Lemma 2.2.3])**.** *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{Z} \to \tilde{S}$ be a flat morphism, $Z = \tilde{Z} \underset{\tilde{S}}{\times} S$, and denote by $\underline{\mathrm{Der}}_{Z/S}$ the sheaf of germs of $\mathscr{O}_S$-derivations from $\mathscr{O}_Z$ into itself. Let us denote by $\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S)$ the set of $\tilde{S}$-automorphisms of $\tilde{Z}$ inducing the identity on Z. Then there is a canonical isomorphism*

$$\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S) \xrightarrow{\sim} \underline{H}^0(Z, \underline{\mathrm{Der}}_{Z/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

Proof. Take $\tilde{X} = \tilde{Y} = \tilde{Z}$ in Lemma 2.1.1.2, and take $f : Z \to Z$ to be the identity. It is known that the identity map $\tilde{Z} \to \tilde{Z}$ is a lifting of f, so we have an isomorphism

$$\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S) = \mathrm{Mor}_{\tilde{S}}(\tilde{Z}, \tilde{Z}, f) \xrightarrow{\sim} \underline{H}^0(Z, \underline{\mathrm{Der}}_{Z/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

the last equality follows from Lemma 2.1.1.1 and the flatness of $\tilde{Z}$ over $\tilde{S}$. □

Corollary 2.1.1.4. *With the setting as in Lemma 2.1.1.2, assume moreover that $\tilde{Y}$ is also flat over $\tilde{S}$. Consider the natural maps*

$$df : \underline{H}^0(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \underline{H}^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

and

$$f^* : \underline{H}^0(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \underline{H}^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

The addition on the target space makes these two maps actions of the source spaces on the target space. Then, by Lemma 2.1.1.2 and Corollary 2.1.1.4, these two actions are transformed into the natural actions of $\mathrm{Aut}_{\tilde{S}}(\tilde{X}, S)$ *and* $\mathrm{Aut}_{\tilde{S}}(\tilde{Y}, S)$ *on* $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ *given respectively by pre- and post-compositions.*

Proof. Recall that the proof of Lemma 2.1.1.2 is achieved by identifying $\mathrm{Aut}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ in the affine case as a torsor under the global sections of

$$\underline{\mathrm{Der}}_{\mathscr{O}_S}(\mathscr{O}_X, \mathscr{O}_Y \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

The commutative diagram

$$\begin{tikzcd}
f^*\mathscr{O}_Y \arrow[r, "d"] \arrow[d, "f^*"'] & f^*\Omega^1_{Y/k} \arrow[d, "df"] \\
\mathscr{O}_X \arrow[r, "d"'] & \Omega^1_{X/S}
\end{tikzcd}$$

of canonical morphisms induces

$$df^* : \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X) \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X)$$

and

$$f^* : f^*\underline{\mathrm{Hom}}_{\mathscr{O}_Y}(\Omega^1_{Y/S}, \mathscr{O}_Y) \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X),$$

whose global versions are the maps df and f^* we see in the statement of the corollary. Now it suffices to observe from the proof of Lemma 2.1.1.2 that the addition of the images of these maps is compatible with pre- and post-compositions of isomorphisms. □

Lemma 2.1.1.5 (cf. [52, IV, 17.11.4] or [20, §2.2, Prop. 11])**.** *A morphism* $\tilde{Y} \to \tilde{S}$ *is smooth at* $y \in \tilde{Y}$ *if and only if there exists an open neighborhood* $\tilde{U} \subset \tilde{Y}$ *of* y*, an integer* r*, and an étale* $\tilde{S}$*-morphism* $\tilde{U} \to \mathbb{A}^r_{\tilde{S}}$*, where* $\mathbb{A}^r_{\tilde{S}}$ *is the affine* r*-space over* $\tilde{S}$*.*

Lemma 2.1.1.6 ([52, IV, 18.1.2])**.** *For any closed immersion* $S \hookrightarrow \tilde{S}$ *defined by a sheaf of ideals* $\mathscr{I}$ *such that* $\mathscr{I}^2 = 0$ *as in the case of Lemma 2.1.1.7, the functor* $\tilde{Y} \mapsto \tilde{Y} \underset{\tilde{S}}{\times} S$ *is an equivalence of categories between schemes étale over the bases (respectively* $\tilde{S}$ *and* S*).*

Combining Lemmas 2.1.1.5 and 2.1.1.6, we obtain:

Lemma 2.1.1.7 ([58, III, Thm. 4.1] or [104, Lemma 2.2.4])**.** *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $X \to S$ be a smooth scheme. For every $x \in X$, there exists an affine open neighborhood $U \subset X$ of x, and a smooth morphism $\tilde{U} \to \tilde{S}$, such that $\tilde{U} \underset{\tilde{S}}{\times} S \cong U$. Moreover, if V is another such affine open neighborhood, with $\tilde{V} \to \tilde{S}$ smooth and $\tilde{V} \underset{\tilde{S}}{\times} S \cong V$, then for every affine neighborhood W of x in $U \cap V$, there exists an isomorphism $\tilde{U}|_W \xrightarrow{\sim} \tilde{V}|_W$ making the following diagram commute:*

$$\begin{tikzcd}
\tilde{U}|_W \arrow[rr, dotted, "\sim"] & & \tilde{V}|_W . \\
& W \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Remark 2.1.1.8. Here $\tilde{U}|_W$ has a meaning because $\tilde{U}$ is a scheme define over the underlying topological space of U, and the underlying topological space of W is an open subset.

2.1.2 Deformation of Smooth Schemes

Definition 2.1.2.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let X be a smooth scheme over S. Then we denote by $\mathsf{Lift} = \mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ the isomorphism classes of pairs $(\tilde{X}, \varphi)$ such that $\tilde{X} \to \tilde{S}$ is smooth and such that $\varphi : \tilde{X} \underset{\tilde{S}}{\times} S \to X$ is an isomorphism.*

Proposition 2.1.2.2 (cf. [54], or [58, III, Thm. 6.3, Prop. 5.1], or [104, Prop. 2.2.5])**.** *Suppose we have the same setting as in Definition 2.1.2.1. Then the following are true:*

1. *There exists a unique element*

$$\mathsf{o} = \mathsf{o}(X; S \hookrightarrow \tilde{S}) \in \underline{H}^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

called the **obstruction** *to $\mathsf{Lift} = \mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, such that $\mathsf{o} \neq 0$ if and only if Lift is nonempty.*

2. *If $\mathsf{o} = 0$, then Lift is a torsor under the group $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

3. *Let $f : X \xrightarrow{\sim} Y$ be any isomorphism of smooth S-schemes. Then the two natural isomorphisms*

$$df : \underline{H}^2(X, \underline{\mathrm{Der}}_{X/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \xrightarrow{\sim} \underline{H}^2(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

and

$$f^* : \underline{H}^2(Y, \underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \xrightarrow{\sim} \underline{H}^2(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

induce an identification

$$df(\mathsf{o}(X)) = f^*(\mathsf{o}(Y)).$$

Proof. By Lemma 2.1.1.7, there is an affine open covering $\{U_\alpha\}$ of X such that each $U_\alpha \to S$ can be lifted to a smooth affine scheme $\tilde{U}_\alpha \to \tilde{S}$, with an isomorphism $\varphi_\alpha : \tilde{U}_\alpha \underset{\tilde{S}}{\times} S \xrightarrow{\sim} U_\alpha$. Let us write $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and similarly for more indices. Since $X \to S$ is separated, we know $U_{\alpha\beta}$ is affine, and hence there exists a morphism

$$\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \to \tilde{U}_\beta|_{U_{\alpha\beta}}$$

inducing $\varphi_\beta^{-1} \circ \varphi_\alpha$ over $U_{\alpha\beta} \to S$. By Lemma 2.1.1.1, $\xi_{\alpha\beta}$ is an isomorphism.

Let us denote the restrictions of $\xi_{\alpha\beta}$ to $U_{\alpha\beta\gamma}$ by the same notation. For these $\tilde{U}_\alpha$ to glue together and form a scheme $\tilde{X}$ lifting X, it is desirable if these morphisms $\xi_{\alpha\beta}$ could satisfy the so-called cocycle condition

$$\xi_{\alpha\gamma} = \xi_{\beta\gamma} \circ \xi_{\alpha\beta} \tag{2.1.2.3}$$

over $U_{\alpha\beta\gamma}$. Let us measure the failure of this by defining

$$c_{\alpha\beta\gamma} := \xi_{\alpha\gamma}^{-1} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}).$$

(We do not need to know if $\xi_{\alpha\gamma}^{-1} = \xi_{\alpha\gamma}$.) Since $c_{\alpha\beta\gamma} \underset{S}{\times} S$ is the identity on $U_{\alpha\beta\gamma}$, by Corollary 2.1.1.3, the automorphism $c_{\alpha\beta\gamma}$ defines an element of $\underline{H}^0(U_{\alpha\beta\gamma}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, which we also denote by $c_{\alpha\beta\gamma}$. The identification of Corollary 2.1.1.3 suggests that we have an identification

$$\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\beta|_{U_{\alpha\beta}}, S) \xrightarrow{\sim} \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \tag{2.1.2.4}$$

by sending a to $\xi_{\alpha\beta}^{-1} \circ a \circ \xi_{\alpha\beta}$. Since the group $\operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha, S)$ is commutative for every $\tilde{U}_\alpha$ (and their intersections), it does not really depend on the isomorphism $\xi_{\alpha\beta}$ we choose.

We claim that $c = \{c_{\alpha\beta\gamma}\}$ is a 2-cocycle with respect to the open covering $\{U_\alpha\}$. By definition, its coboundary is given by

$$(\partial c)_{\alpha\beta\gamma\delta} := c_{\beta\gamma\delta} \circ c_{\alpha\gamma\delta}^{-1} \circ c_{\alpha\beta\delta} \circ c_{\alpha\beta\gamma}^{-1}. \tag{2.1.2.5}$$

We would like to represent all four elements at the right-hand side by elements of $\operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma\delta}}, S)$, via (2.1.2.4) if necessary. Since this group is commutative, we may switch the orders of elements in the right-hand side of (2.1.2.5), and therefore

$$\begin{aligned}(\partial c)_{\alpha\beta\gamma\delta} = c_{\alpha\gamma\delta}^{-1} \circ c_{\alpha\beta\delta} \circ c_{\beta\gamma\delta} \circ c_{\alpha\beta\gamma}^{-1} &= [\xi_{\alpha\gamma}^{-1} \circ \xi_{\gamma\delta}^{-1} \circ \xi_{\alpha\delta}] \circ [\xi_{\alpha\delta}^{-1} \circ \xi_{\beta\delta} \circ \xi_{\alpha\beta}] \circ \\ &\quad [\xi_{\alpha\beta}^{-1} \circ (\xi_{\beta\delta}^{-1} \circ \xi_{\gamma\delta} \circ \xi_{\beta\gamma}) \circ \xi_{\alpha\beta}] \circ [\xi_{\alpha\beta}^{-1} \circ \xi_{\beta\gamma}^{-1} \circ \xi_{\alpha\gamma}] \\ &= \xi_{\alpha\gamma}^{-1} \circ [\xi_{\gamma\delta}^{-1} \circ [\xi_{\alpha\delta} \circ \xi_{\alpha\delta}^{-1}] \circ [\xi_{\beta\delta} \circ [\xi_{\alpha\beta} \circ \xi_{\alpha\beta}^{-1}] \circ \xi_{\beta\delta}^{-1}] \circ \xi_{\gamma\delta}] \\ &\quad \circ [\xi_{\beta\gamma} \circ [\xi_{\alpha\beta} \circ \xi_{\alpha\beta}^{-1}] \circ \xi_{\beta\gamma}^{-1}] \circ \xi_{\alpha\gamma} = \operatorname{Id}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma\delta}}}.\end{aligned}$$

Suppose we have chosen isomorphisms $\xi'_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ that differ from $\xi_{\alpha\beta}$ by

$$\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta},$$

for some $\eta_{\alpha\beta} \in \operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S)$. Then we may identify $\eta = \{\eta_{\alpha\beta}\}$ with a 1-cochain in $\underline{C}^1(\{U_\alpha\}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Its coboundary can be represented by

$$(\partial\eta)_{\alpha\beta\gamma} := \eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\beta}^{-1} \circ \eta_{\beta\gamma} \circ \xi_{\alpha\beta}) \circ \eta_{\alpha\beta}$$

in $\operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S)$, where we have used the identification (2.1.2.4) again. Then $c_{\alpha\beta\gamma}$ becomes

$$\begin{aligned} c'_{\alpha\beta\gamma} :=& [\eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\gamma})^{-1}] \circ [\xi_{\beta\gamma} \circ \eta_{\beta\gamma}] \circ [\xi_{\alpha\beta} \circ \eta_{\alpha\beta}] \\ &= [\xi_{\alpha\gamma}^{-1} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta}] \circ [\eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\beta}^{-1} \circ \eta_{\beta\gamma} \circ \xi_{\alpha\beta}) \circ \eta_{\alpha\beta}] \\ &= c_{\alpha\beta\gamma} \circ (\partial\eta)_{\alpha\beta\gamma}\end{aligned}$$

over $U_{\alpha\beta\gamma}$, where we can move $\eta_{\alpha\gamma}^{-1}$ because $\operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta,S}})$ is commutative. Hence we obtain a class $[c]$ in $\underline{H}^2(\{U_\alpha\}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{H}^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that does not depend on the choices of $\xi_{\alpha\beta}$.

To show that this is independent of the choices of $\tilde{U}_\alpha$ over U_α, note that any different choice of $\tilde{U}'_\alpha$ over a particular U_α is (noncanonically) isomorphic to $\tilde{U}_\alpha$ by Lemma 2.1.1.7. Hence the class $[c]$ does not depend on the choice of $\{\tilde{U}_\alpha\}$ over any particular $\{U_\alpha\}$. For the same reason, the class $[c]$ defined in $\underline{H}^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ by a particular open covering $\{U_\alpha\}$ remain unchanged if we refine the open covering. Thus we have shown that the definition of $[c]$ is independent of all choices.

Now, if $[c]$ is trivial, then it means there exists a particular choice of $\{(U_\alpha, \tilde{U}_\alpha, \varphi_\alpha)\}$ and $\{\xi_{\alpha\beta}\}$ such that, up to a modification of $\{\xi_{\alpha\beta}\}$ by some $\{\eta_{\alpha\beta}\}$ as above, the cocycle condition (2.1.2.3) can be satisfied. By gluing the $\{\tilde{U}_\alpha\}$ together using the modified $\{\xi_{\alpha\beta}\}$, we obtain a smooth scheme $\tilde{X}$ lifting X, together with an isomorphism $\varphi : \tilde{X} \underset{\tilde{S}}{\times} S \xrightarrow{\sim} X$ as desired. Conversely, if any such smooth $\tilde{X}$ exists, then there exists an affine open covering $\{U_\alpha\}$ such that each $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ is smooth and affine. Then $[c]$ is necessarily trivial because we can compute it by this choice of $\{\tilde{U}_\alpha\}$, and the isomorphisms $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ coming from the identify maps on $\tilde{X}|_{U_{\alpha\beta}}$ certainly glue. This proves the first statement of the proposition if we set $\mathsf{o} = [c]$.

For the second statement, let us suppose there exists an element $(\tilde{X}, \varphi)$ in Lift. Let $\{U_\alpha\}$ be an affine open covering of X such that each of $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ is affine.

Suppose we are given a 1-cocycle $d = \{d_{\alpha\beta}\}$ in $\underline{C}^1(\{U_\alpha\}, \underline{\mathrm{Der}}_{X/S})$, where $d_{\alpha\beta} \in \underline{H}^0(\{U_{\alpha\beta}\}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \mathrm{Aut}_{\tilde{S}}(\tilde{X}|_{U_{\alpha\beta}}, S)$. We can interpret each $d_{\alpha\beta}$ as an isomorphism $\tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ as both the source and target are canonically identified with $\tilde{X}|_{U_{\alpha\beta}}$. Since d is a 1-cocycle, these maps glue the affine schemes $\tilde{U}_\alpha$ together and define an object $(\tilde{X}^d, \varphi^d)$ in Lift. Suppose we take another 1-cocycle d' that differs from d by a 1-coboundary. This means there exists a 0-cochain $e = \{e_\alpha\}$ with $e_\alpha \in \underline{H}^0(U_\alpha, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha, S)$. As in (2.1.2.4), its coboundary $(\partial e)_{\alpha\beta}$ can be represented in $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S)$ by

$$(d_{\alpha\beta}^{-1} \circ e_\beta^{-1} \circ d_{\alpha\beta}) \circ e_\alpha,$$

and therefore

$$d'_{\alpha\beta} = d_{\alpha\beta} \circ (\partial e)_{\alpha\beta} = d_{\alpha\beta} \circ [(d_{\alpha\beta}^{-1} \circ e_\beta^{-1} \circ d_{\alpha\beta}) \circ e_\alpha] = e_\beta^{-1} \circ d_{\alpha\beta} \circ e_\alpha$$

implies we have the following commutative diagram

$$\begin{tikzcd}
\tilde{U}_\alpha|_{U_{\alpha\beta}} \arrow[r, "d_{\alpha\beta}"] \arrow[d, "e_\alpha^{-1}"'] & \tilde{U}_\beta|_{U_{\alpha\beta}} \\
\tilde{U}_\alpha|_{U_{\alpha\beta}} \arrow[r, "d'_{\alpha\beta}"'] & \tilde{U}_\beta|_{U_{\alpha\beta}} \arrow[u, "e_\beta"']
\end{tikzcd}$$

that glues together the isomorphisms $e_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} \tilde{U}_\alpha$ into an isomorphism $\tilde{X}^{d'} \xrightarrow{\sim} \tilde{X}^{d}$. Hence there is a well-defined map sending the $[d]$ in $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ to the isomorphism class of $(\tilde{X}^d, \varphi^d)$ in Lift. This map is injective because any isomorphism $\tilde{X}^{d'} \xrightarrow{\sim} \tilde{X}^d$ restricts to isomorphisms $e_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} \tilde{U}_\alpha$ that necessarily defines a 1-cochain giving the difference between d' and d.

Let us show that it is also surjective. Suppose there is any other element $(\tilde{X}', \varphi')$ in Lift. Then by smoothness of $\tilde{X}' \to S$, the maps $U_\alpha \to \tilde{U}'_\alpha := \tilde{X}'|_{U_\alpha}$ can be lifted to maps $\tilde{U}_\alpha \to \tilde{U}'_\alpha$, which are necessarily isomorphisms by Lemma 2.1.1.1. The isomorphisms $\xi'_{\alpha\beta} : \tilde{U}'_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}'_\beta|_{U_{\alpha\beta}}$ coming from the identity maps on $\tilde{X}'|_{U_{\alpha\beta}}$ pull back to isomorphisms $\tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ that differ from $\xi_{\alpha\beta}$ by automorphisms $d_{\alpha\beta}$ of $\tilde{U}_\alpha|_{U_{\alpha\beta}}$, which we identify as elements in $\underline{H}^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. The cochain $d = \{d_{\alpha\beta}\}$ necessarily satisfies the cocycle condition, as both of $\xi_{\alpha\beta}$ and $\xi'_{\alpha\beta}$ do (in a slightly different context). This gives a class $[d]$ of d in $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and shows the surjectivity.

The third statement of the proposition is the consequence that o can be computed by any affine open covering $\{U_\alpha\}$ such that each U_α is lifted to a smooth scheme over $\tilde{S}$. By abuse of notation, let $\tilde{f}^{-1}(\tilde{U}) := \tilde{U} \underset{U,f}{\times} f^{-1}(U)$ be the pullback of open subscheme $\tilde{U}$ of $\tilde{Y}$, and let us denote the map $\tilde{f}^{-1}(\tilde{U}) \xrightarrow{\sim} \tilde{U}$ by $\tilde{f}$. By Corollary 2.1.1.4, the restriction of the composition $df^{-1} \circ f^*$ to U is nothing but the isomorphism

$$\underline{H}^2(U, \underline{\mathrm{Der}}_{U/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \xrightarrow{\sim} \underline{H}^2(f^{-1}(U), \underline{\mathrm{Der}}_{f^{-1}(U)/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

which corresponds to

$$\mathrm{Aut}_{\tilde{S}}(\tilde{U}, S) \xrightarrow{\sim} \mathrm{Aut}_{\tilde{S}}(\tilde{f}^{-1}(\tilde{U}), S) : a \mapsto \tilde{f}^{-1} \circ a \circ \tilde{f}$$

under Corollary 2.1.1.3. Suppose we have an explicit open covering $\{U_\alpha\}$ of Y that is lifted to some $\{\tilde{U}_\alpha\}$, and $c_{\alpha\beta\gamma} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S) \cong \underline{H}^0(U_{\alpha\beta\gamma}, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ representing the class $\mathsf{o}(Y) \in \underline{H}^2(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Then $\{f^{-1}(U_\alpha)\}$ is an open covering of X that is lifted to $\{\tilde{f}^{-1}(\tilde{U}_\alpha)\}$ in our above notation, which makes $(df^{-1} \circ f^*)(c_{\alpha\beta\gamma})$ a valid representative of $\mathsf{o}(X) \in \underline{H}^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and proves the result. □

2.1.3 Deformation of Morphisms

Definition 2.1.3.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let X be a smooth scheme over S. Suppose $f : X \to Y$ is a map between smooth schemes over S such that the source X and the target Y lift respectively to smooth schemes $\tilde{X}$ and $\tilde{Y}$ over $\tilde{S}$. Then we denote by $\mathsf{Lift} = \mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S})$ the set of morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\tilde{f} \underset{\tilde{S}}{\times} S = f$.*

Proposition 2.1.3.2. *Suppose that we have the same setting as in Definition 2.1.3.1. Then the following are true:*

1. *There exists a unique element*

$$\mathsf{o} = \mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) \in \underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

   called the **obstruction** *to $\mathsf{Lift} = \mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S})$, such that $\mathsf{o} \neq 0$ if and only if Lift is nonempty.*

2. *If $\mathsf{o} = 0$, then Lift is a torsor under the group $\underline{H}^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

3. *By Proposition 2.1.2.2, the sets $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ and $\mathsf{Lift}(Y; S \hookrightarrow \tilde{S})$ are torsors under respectively the groups $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and $\underline{H}^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Hence it makes sense to write*

$$\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$$

   and

$$\tilde{Y}' = \mathsf{m}_{\tilde{Y}} + \tilde{Y}$$

for elements $\mathsf{m}_{\tilde{X}} \in \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ *and* $\mathsf{m}_{\tilde{Y}} \in \underline{H}^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$*. Now consider the natural morphisms*

$$df : \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

and

$$f^* : \underline{H}^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

Then we have a relation

$$\begin{aligned} &\mathsf{o}(f; \mathsf{m}_{\tilde{X}} + \tilde{X}, \mathsf{m}_{\tilde{Y}} + \tilde{Y}, S \hookrightarrow \tilde{S}) \\ &= \mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) - df(\mathsf{m}_{\tilde{X}}) + f^*(\mathsf{m}_{\tilde{Y}}). \end{aligned}$$

Proof. Let $\{U_\alpha\}$ and $\{V_\alpha\}$ be affine open coverings of X and Y indexed by the same set, such that $f(U_\alpha) \subset V_\alpha$ for each index α. Let $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ and $\tilde{V}_\alpha := \tilde{Y}|_{U_\alpha}$. Then $\{\tilde{U}_\alpha\}$ and $\{\tilde{V}_\alpha\}$ are respectively open coverings of $\tilde{X}$ and $\tilde{Y}$. By smoothness of f, each of the map $f|_{U_\alpha} : U_\alpha \to V_\alpha$ can be lifted to a map $\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha$. By Lemma 2.1.1.2, the different choices of the liftings over any open subscheme U of U_α is a torsor under the group $\underline{H}^0(U, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Let us write the group action by addition. Comparing the restrictions to $U_{\alpha\beta}$, there exists elements $c_{\alpha\beta} \in \underline{H}^0(U_{\alpha\beta}, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ such that

$$\tilde{f}_\alpha|_{U_{\alpha\beta}} = c_{\alpha\beta} + \tilde{f}_\beta|_{U_{\alpha\beta}}. \tag{2.1.3.3}$$

Comparing the relations over $U_{\alpha\beta\gamma}$, we obtain the cocycle relation

$$c_{\alpha\gamma} = c_{\beta\gamma} + c_{\alpha\beta}.$$

If we modify each choice of $\tilde{f}_\alpha$ by $\tilde{f}'_\alpha = \tilde{f}_\alpha + e_\alpha$ for some $e_\alpha \in \underline{H}^0(\tilde{U}_\alpha, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, then we obtain

$$c'_{\alpha\beta} = c_{\alpha\beta} + (\partial e)_{\alpha\beta},$$

where $(\partial e)_{\alpha\beta} := -e_\beta + e_\alpha$. Thus there is a well-defined class $[c]$ for $c = \{c_{\alpha\beta}\}$ in $\underline{H}^1(\{U_\alpha\}, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that is independent of the choice of $\{\tilde{f}_\alpha\}$. The class $[c]$ in $\underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ does not depend on the choices of $\{U_\alpha\}$ and $\{V_\alpha\}$ either, as we can always pass to refinements.

If $[c]$ is trivial, then there exists $e = \{e_\alpha\}$ as above such that $c_{\alpha\beta} = e_\beta - e_\alpha$. Hence $e_\alpha + \tilde{f}_\alpha = e_\beta + \tilde{f}_\beta$ defines a global morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$. Conversely, the existence of any global morphism forces $[c]$ to be trivial. Hence we can conclude the proof of the first statement by setting $\mathsf{o} = [c]$.

For the second statement, simply note that the existence of any global lifting $\tilde{f}$ gives a choice of all the $\tilde{f}_\alpha$ as above over each U_α, and hence any other global choice $\tilde{f}'$ must differ by some e_α that patches together to some e describing the difference between $\tilde{f}$ and $\tilde{f}'$.

For the third statement, simply note that in the proof of Proposition 2.1.2.2, the group actions of $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and $\underline{H}^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ are given by modifying respectively the gluing isomorphisms between $\tilde{U}_\alpha|_{U_\alpha}$ and $\tilde{V}_\alpha|_{V_\alpha}$. Suppose $\mathsf{m}_{\tilde{X}}$ (resp. $\mathsf{m}_{\tilde{Y}}$) is represented by some 1-cochain $m_{\tilde{X}} = \{m_{\tilde{X},\alpha\beta}\}$ (resp. $m_{\tilde{Y}} = \{m_{\tilde{Y},\alpha\beta}\}$), where $m_{\tilde{X},\alpha\beta} \in \underline{H}^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and $m_{\tilde{Y},\alpha\beta} \in \underline{H}^0(V_{\alpha\beta}, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. By Corollary 2.1.1.4, the relation (2.1.3.3) is transformed into

$$df(m_{\tilde{X},\alpha\beta}) + \tilde{f}_\alpha|_{U_{\alpha\beta}} = c'_{\alpha\beta} + f^*(m_{\tilde{Y},\alpha\beta}) + \tilde{f}_\beta|_{U_{\alpha\beta}},$$

which implies

$$c'_{\alpha\beta} = c_{\alpha\beta} - df(m_{\tilde{X},\alpha\beta}) + f^*(m_{\tilde{Y},\alpha\beta}).$$

Hence the result follows. □

2.1.4 Change of Bases

The arguments used in the proofs in Sections 2.1.2 and 2.1.3 above have functorial implications in the following situation: Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, and let $T \hookrightarrow \tilde{T}$ be a closed immersion defined a sheaf of ideals $\mathscr{J}$ such that $\mathscr{J}^2 = 0$. Suppose there is a commutative diagram of closed embeddings

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook] & \tilde{S} \arrow[d, hook, "\iota"] \\ T \arrow[r, hook] & \tilde{T} \end{tikzcd} .$$

(We do not assume that this is Cartesian.) The commutativity shows that $\mathscr{J}$ is mapped to $\mathscr{I}$ by the pullback $\iota^* : \mathscr{O}_{\tilde{T}} \to \mathscr{O}_{\tilde{S}}$. We shall denote the induced map simply by $\iota^* : \mathscr{J} \to \mathscr{I}$.

If X is any smooth scheme over T, then $X \underset{T}{\times} S$ is a smooth scheme over S. Therefore it makes sense to compare the two sets

$$\mathsf{Lift}(X; T \hookrightarrow \tilde{T})$$

and

$$\mathsf{Lift}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S}).$$

Moreover, suppose we have a map $f : X \to Y$ between smooth schemes over T such that the source X and the target Y are lifted respectively to smooth schemes $\tilde{X}$ and $\tilde{Y}$ over $\tilde{T}$. Then $f \underset{T}{\times} S$ is a map from $X \underset{T}{\times} S$ to $Y \underset{T}{\times} S$, and it also makes sense to compare the two sets

$$\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T})$$

and

$$\mathsf{Lift}(f \underset{T}{\times} S; \tilde{X} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{Y} \underset{\tilde{T}}{\times} \tilde{S}, S \hookrightarrow \tilde{S}).$$

Note that if $\tilde{U} \to \tilde{T}$ is any smooth scheme lifting an affine open subscheme U of X, then $\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}$ is a lifting of the affine open subscheme $U \underset{T}{\times} S$ of $X \underset{T}{\times} S$. Moreover, if $\tilde{V}$ is any smooth schemes lifting an affine open subscheme V of Y such that $f(U) \subset V$, then $\tilde{V} \underset{\tilde{T}}{\times} \tilde{S}$ is a lifting of the affine open subscheme $V \underset{T}{\times} S$ of $Y \underset{T}{\times} S$ such that $(f \underset{T}{\times} S)(U \underset{T}{\times} S) \subset V \underset{T}{\times} S$. Therefore, if $\tilde{f}_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is a map lifting $f|_U : U \to V$, then $\tilde{f}_U \underset{\tilde{T}}{\times} \tilde{S} : \tilde{U} \underset{\tilde{T}}{\times} \tilde{S} \to \tilde{V} \underset{\tilde{T}}{\times} \tilde{S}$ is a map lifting $(f \underset{T}{\times} S)|_{U \underset{T}{\times} S} : U \underset{T}{\times} S \to V \underset{T}{\times} S$.

Lemma 2.1.4.1. *The diagram*

$$\begin{tikzcd}
\mathrm{Mor}_{\tilde{T}}(\tilde{U}, \tilde{V}, f|_U) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\underset{\tilde{T}}{\times} \tilde{S}"'] & \underline{H}^0(U, f^*(\underline{\mathrm{Der}}_{Y/T}) \underset{\mathscr{O}_T}{\otimes} \mathscr{J}) \arrow[d, "\text{can.} \otimes \iota^*"] \\
\mathrm{Mor}_{\tilde{S}}(\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{V} \underset{\tilde{T}}{\times} \tilde{S}, f|_{U \underset{T}{\times} S}) \arrow[r, "\sim", "\text{can.}"'] & \underline{H}^0(U \underset{T}{\times} S, (f \underset{T}{\times} S)^*(\underline{\mathrm{Der}}_{Y \underset{T}{\times} S/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})
\end{tikzcd} ,$$

with horizontal canonical isomorphisms given by Lemma 2.1.1.2, is commutative.

Proof. This follows if we note that the proof of Lemma 2.1.1.2 is functorial with respect to base changes of rings. The upper canonical isomorphism is given there by relating the two dotted vertical rows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{J} \cdot \mathscr{O}_{\tilde{V}} \arrow[r] \arrow[d, dotted] & \mathscr{O}_{\tilde{V}} \arrow[r] \arrow[d, dotted] & \mathscr{O}_{V} \arrow[r] \arrow[d] & 0 \, , \\
0 \arrow[r] & \mathscr{J} \cdot \mathscr{O}_{\tilde{U}} \arrow[r] & \mathscr{O}_{\tilde{U}} \arrow[r] & \mathscr{O}_{U} \arrow[r] & 0
\end{tikzcd}$$

while the lower canonical isomorphism is given by relating the two dotted vertical rows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{I} \cdot \mathscr{O}_{\tilde{V} \underset{\tilde{T}}{\times} \tilde{S}} \arrow[r] \arrow[d, dotted] & \mathscr{O}_{\tilde{V} \underset{\tilde{T}}{\times} \tilde{S}} \arrow[r] \arrow[d, dotted] & \mathscr{O}_{V \underset{T}{\times} S} \arrow[r] \arrow[d] & 0 \, . \\
0 \arrow[r] & \mathscr{I} \cdot \mathscr{O}_{\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}} \arrow[r] & \mathscr{O}_{\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}} \arrow[r] & \mathscr{O}_{U \underset{T}{\times} S} \arrow[r] & 0
\end{tikzcd}$$

Here the second diagram is obtained from the first one by the tensor operation $- \underset{\mathscr{O}_{\tilde{T}}}{\otimes} \mathscr{O}_{\tilde{S}}$ given by $\iota : S \hookrightarrow T$, which induces $\iota^* : \mathscr{J} \to \mathscr{I}$. ☐

Corollary 2.1.4.2. *The diagram*

$$\begin{tikzcd}
\mathrm{Aut}_{\tilde{T}}(\tilde{U}, T) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\underset{\tilde{T}}{\times} \tilde{S}"'] & \underline{H}^0(U, \underline{\mathrm{Der}}_{X/T} \underset{\mathscr{O}_T}{\otimes} \mathscr{J}) \arrow[d, "\text{can.} \otimes \iota^*"] & , \\
\mathrm{Aut}_{\tilde{S}}(\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}, S) \arrow[r, "\sim", "\text{can.}"'] & \underline{H}^0(U \underset{T}{\times} S, \underline{\mathrm{Der}}_{X \underset{T}{\times} S/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) &
\end{tikzcd}$$

with the horizontal canonical isomorphisms given by Corollary 2.1.1.3, is commutative.

Corollary 2.1.4.3. *1. The two obstructions*

$$\mathsf{o}(X; T \hookrightarrow \tilde{T}) \in \underline{H}^2(X, \underline{\mathrm{Der}}_{X/T} \underset{\mathscr{O}_T}{\otimes} \mathscr{J})$$

and

$$\mathsf{o}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S}) \in \underline{H}^2(X \underset{T}{\times} S, \underline{\mathrm{Der}}_{X \underset{T}{\times} S/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

are related by

$$(\mathrm{can.} \otimes \iota^*)(\mathsf{o}(X; T \hookrightarrow \tilde{T})) = \mathsf{o}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S})$$

under the natural map

$$\mathrm{can.} \otimes \iota^* : \underline{H}^2(X, \underline{\mathrm{Der}}_{X/T} \underset{\mathscr{O}_T}{\otimes} \mathscr{J}) \to \underline{H}^2(X \underset{T}{\times} S, \underline{\mathrm{Der}}_{X \underset{T}{\times} S/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

2. *If* $\mathsf{o}(X; T \hookrightarrow \tilde{T}) = \mathsf{o}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S}) = 0$, *then the natural map*

$$- \underset{\tilde{T}}{\times} \tilde{S} : \mathsf{Lift}(X; T \hookrightarrow \tilde{T}) \to \mathsf{Lift}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S})$$

of torsors is equivariant under the natural map

$$\mathrm{can.} \otimes \iota^* : \underline{H}^1(X, \underline{\mathrm{Der}}_{X/T} \underset{\mathscr{O}_T}{\otimes} \mathscr{J}) \to \underline{H}^1(X \underset{T}{\times} S, \underline{\mathrm{Der}}_{X \underset{T}{\times} S/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

of groups.

Proof. The statements follow from the proof of Proposition 2.1.2.2, as any affine open covering $\{U_\alpha\}$ of X that defines the obstructions and torsor structures also define the corresponding objects for $X \underset{T}{\times} S$ by the base change operation $- \underset{\tilde{T}}{\times} \tilde{S}$ corresponding to the tensor operation $- \underset{\mathscr{O}_{\tilde{T}}}{\otimes} \mathscr{O}_{\tilde{S}}$ on sheaves. □

Similarly, following the proof of Proposition 2.1.3.2, we obtain:

Corollary 2.1.4.4. *The two obstructions*

$$\mathsf{o}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T}) \in \underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{X/T}) \underset{\mathscr{O}_T}{\otimes} \mathscr{J})$$

and

$$\mathsf{o}(f \underset{T}{\times} S; \tilde{X} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{Y} \underset{\tilde{T}}{\times} \tilde{S}, S \hookrightarrow \tilde{S}) \in \underline{H}^1(X \underset{T}{\times} S, (f \underset{T}{\times} S)^* \underline{\mathrm{Der}}_{X \underset{T}{\times} S/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

are related by

$$(\mathrm{can.} \otimes \iota^*)\mathsf{o}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T}) = \mathsf{o}(f \underset{T}{\times} S; \tilde{X} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{Y} \underset{\tilde{T}}{\times} \tilde{S}, S \hookrightarrow \tilde{S})$$

under the natural map

$$\begin{aligned}\text{can.}\otimes\iota^* :&\underline{H}^1(X\underset{T}{\times}S,(f\underset{T}{\times}S)^*\underline{\mathrm{Der}}_{X\underset{T}{\times}S/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\\ &\qquad\qquad\to\underline{H}^1(X\underset{T}{\times}S,(f\underset{T}{\times}S)^*\underline{\mathrm{Der}}_{X\underset{T}{\times}S/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\end{aligned}$$

Remark 2.1.4.5. Corollaries 2.1.4.3 and 2.1.4.4 are especially useful when $\iota^* : \mathscr{J} \to \mathscr{I}$ is an isomorphism. This will be the case when we apply Schlessinger's criterions (given by Theorem 2.2.1.4) in Section 2.2.3.

2.1.5 Deformation of Invertible Sheaves

Let us review the definition of cup-products (and in particular the sign convention we use) before state the results. Suppose we have a scheme Z over S, and two invertible sheafs $\mathscr{F}$ and $\mathscr{G}$ on Z. We shall define a linear map

$$\cup : \underline{H}^i(Z,\mathscr{F})\underset{\mathscr{O}_S}{\otimes}\underline{H}^j(Z,\mathscr{G})\to\underline{H}^{i+j}(Z,\mathscr{F}\underset{\mathscr{O}_Z}{\otimes}\mathscr{G})$$

as follows. Given two cohomology classes $[a]\in\underline{H}^i(Z,\mathscr{F})$ and $[b]\in\underline{H}^j(Z,\mathscr{G})$. Suppose there is an affine open covering $\{Z_\alpha\}$ of Z, with the convention that $Z_{\alpha\beta}=Z_\alpha\cap Z_\beta$ etc as before, such that $[a]$ and $[b]$ can be represented respectively by cocycles

$$a=\{a_{\alpha_0\ldots\alpha_i}\}\in\underline{H}^0(Z_{\alpha_0\ldots\alpha_i},\mathscr{F})\}$$

and

$$b=\{b_{\alpha_0\ldots\alpha_j}\}\in\underline{H}^0(Z_{\alpha_0\ldots\alpha_j},\mathscr{G})\}.$$

By definition, this means

$$(\partial a)_{\alpha_0\ldots\alpha_{i+1}}:=\sum_{k=0}^{i+1}(-1)^k a_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_{i+1}}=0 \tag{2.1.5.1}$$

and

$$(\partial b)_{\alpha_0\ldots\alpha_{j+1}}:=\sum_{k=0}^{j+1}(-1)^k b_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_{i+1}}=0, \tag{2.1.5.2}$$

where $\hat{\alpha}_k$ means the removal of the term α_k. Then we define a $(i+j)$-cochain $a\cup b$ by setting

$$(a\cup b)_{\alpha_0\ldots\alpha_{i+j}}=a_{\alpha_0\ldots\alpha_i}b_{\alpha_i\ldots\alpha_{i+j}},$$

where the notation $a_{\alpha_0\ldots\alpha_i} b_{\alpha_i\ldots\alpha_{i+j}}$ means the image of $a_{\alpha_0\ldots\alpha_i} \otimes b_{\alpha_i\ldots\alpha_{i+j}}$ under the canonical morphism

$$\underline{H}^0(Z_{\alpha_0\ldots\alpha_{i+j}}, \mathscr{F}) \underset{\mathscr{O}_S}{\otimes} \underline{H}^0(Z_{\alpha_0\ldots\alpha_{i+j}}, \mathscr{G}) \overset{\text{can.}}{\to} \underline{H}^0(Z_{\alpha_0\ldots\alpha_{i+j}}, \mathscr{F} \underset{\mathscr{O}_Z}{\otimes} \mathscr{G}).$$

Then, by (2.1.5.1) and (2.1.5.2), we have

$$\begin{aligned}
(\partial(a \cup b))_{\alpha_0\ldots\alpha_{i+j+1}} &:= \sum_{k=0}^{i+j+1} (-1)^k c_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_{i+j+1}} \\
&= \sum_{k=0}^{i} (-1)^k a_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_{i+1}} b_{\alpha_{i+1}\ldots\alpha_{i+j+1}} \\
&\quad + \sum_{k=i+1}^{i+j+1} (-1)^k a_{\alpha_0\ldots\alpha_{i+1}} b_{\alpha_{i+1}\ldots\hat{\alpha}_k\ldots\alpha_{i+j+1}} \\
&= [(\partial a)_{\alpha_0\ldots\alpha_{i+1}} - (-1)^{i+1} a_{\alpha_0\ldots\alpha_i\hat{\alpha}_{i+1}}] b_{\alpha_{i+1}\ldots\alpha_{i+j+1}} \\
&\quad + (-1)^i a_{\alpha_0\ldots\alpha_i} [-b_{\hat{\alpha}_i\alpha_{i+1}\ldots\alpha_{i+j+1}} + (\partial b)_{\alpha_i\ldots\alpha_{i+j+1}}] \\
&= 0,
\end{aligned}$$

which means $a \cup b$ is a cocycle. Note that above calculation actually shows the stronger fact that, for any i'-cochain a' and j'-cochain b', we have

$$\partial(a' \cup b') = (\partial a') \cup b' + (-1)^{i'} a' \cup (\partial b').$$

In particular, $(\partial a') \cup b = \partial(a' \cup b)$ and $a \cup (\partial b') = \partial(a \cup b')$, which imply that the cohomology class of $[a \cup b]$ is independent of the cocycles a and b representing respectively $[a]$ and $[b]$. Hence the *cup-product* operation $([a], [b]) \mapsto [a] \cup [b] := [a \cup b]$ is well-defined.

Definition 2.1.5.3. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let X be a smooth scheme over S that is lifted to some smooth scheme $\tilde{X} \to \tilde{S}$. Suppose $\mathcal{L}$ is an invertible sheaf on X. Then we denote by $\mathsf{Lift} = \mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$ the isomorphism classes of invertible sheaves $\tilde{\mathcal{L}}$ on $\tilde{X}$ such that $\tilde{\mathcal{L}} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S \cong \mathcal{L}$.*

Proposition 2.1.5.4. *Suppose that we have the same setting as in Definition 2.1.5.3. Then the following are true:*

1. *There exists a unique element*

$$\mathsf{o} = \mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S}) \in \underline{H}^2(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

   called the **obstruction** *to* $\mathsf{Lift} = \mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$*, such that* $\mathsf{o} \neq 0$ *if and only if* Lift *is nonempty.*

2. *If* $\mathsf{o} = 0$ *and the canonical morphism*

$$\underline{H}^0(\tilde{X}, \mathscr{O}_{\tilde{X}}^{\times}) \to \underline{H}^0(X, \mathscr{O}_X^{\times}) \tag{2.1.5.5}$$

   is surjective, then Lift *is a torsor under the group* $\underline{H}^1(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$*.*

3. *By Proposition 2.1.2.2, the set* $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ *is a torsor under the group* $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$*. Hence it makes sense to write*

$$\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$$

   for any element $\mathsf{m}_{\tilde{X}} \in \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$*. Let*

$$d\log : \underline{\mathrm{Pic}}(X/S) \cong \underline{H}^1(X, \mathscr{O}_X^{\times}) \to \underline{H}^1(X, \Omega^1_{X/S})$$

   be the map induced by

$$d\log : \mathscr{O}_X^{\times} \to \Omega^1_{X/S} : a \mapsto d\log(a) := a^{-1}da.$$

   Then cup-product with $d\log(\mathcal{L})$ *defines a natural morphism*

$$\begin{aligned} \mathsf{d}_{\mathcal{L}} : {} & \underline{H}^i(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \\ & \to \underline{H}^{i+1}(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_X}{\otimes} \Omega^1_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \overset{\text{can.}}{\to} \underline{H}^{i+1}(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}), \end{aligned} \tag{2.1.5.6}$$

   which in the case $i = 1$ *makes the following identity hold:*

$$\mathsf{o}(\mathcal{L}; \mathsf{m}_{\tilde{X}} + \tilde{X}, S \hookrightarrow S) = \mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow S) + \mathsf{d}_{\mathcal{L}}(\mathsf{m}_{\tilde{X}}).$$

Proof. First let us take any smooth affine open covering $\{U_\alpha\}$ of X such that $\mathcal{L}$ is given by a cohomology class $[l] \in \underline{H}^1(X, \mathscr{O}_X^\times)$ represented by some $l = \{l_{\alpha\beta} \in \mathscr{O}_{U_{\alpha\beta}}^\times\}$. Note that we have the cocycle condition

$$l_{\alpha\gamma}^{-1} \cdot l_{\beta\gamma} \cdot l_{\alpha\beta} = 1 \tag{2.1.5.7}$$

over $U_{\alpha\beta}$. Let $\tilde{l}_{\alpha\beta}$ be any element in $\mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}}^\times$ lifting $l_{\alpha\beta}$. Let $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ denote the isomorphism giving the gluing of the lifting $\tilde{X}$ of X. If $\tilde{l} = \{\tilde{l}_{\alpha\beta}\}$ comes from some invertible sheaf $\tilde{\mathcal{L}}$ on X lifting $\mathcal{L}$, then we must have

$$\tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} = 1. \tag{2.1.5.8}$$

(Here we prefer to put $\xi_{\alpha\beta}^*$ to signify the choice of $\tilde{X}$ in this study.) Let us measure this failure by defining

$$h_{\alpha\beta\gamma} := \tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} - 1 \in \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}}$$

Note that by (2.1.5.7),

$$h_{\alpha\beta\gamma} \in \mathscr{I} \cdot \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}} \cong \mathscr{O}_{U_{\alpha\beta\gamma}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}.$$

Moreover, as one checks easily

$$(1 + h_{\beta\gamma\delta})(1 - h_{\alpha\gamma\delta})(1 + h_{\alpha\beta\delta})(1 + h_{\alpha\beta\gamma}) = 1 + (\partial h)_{\alpha\beta\gamma\delta} = 1,$$

which means $h = \{h_{\alpha\beta\gamma}\}$ is a 2-cocycle. If we replace $\tilde{l}$ by any other choice lifting l, then we arrive at a 2-cocycle that differs from h by a coboundary. This shows that $[h]$ is a cohomology class in $\underline{H}^2(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that does not depend on the choice of $\tilde{l}$. Moreover, $[h]$ is trivial if and only if we can find some choice $\tilde{l}$ such that the cocycle condition (2.1.5.7) is satisfied by $\tilde{l}$. This shows that we can define $\mathsf{o} = \mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$ by $\mathsf{o} = [h]$. Note that this is simply the image of the class of $\mathcal{L} \in \underline{\mathrm{Pic}}(X/S) \cong \underline{H}^1(X, \mathscr{O}_X^\times)$ under the connecting homomorphism in the long exact sequence

$$\underline{H}^1(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}}^\times) \to H^1(X, \mathscr{O}_X^\times) \to \underline{H}^2(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \dots \tag{2.1.5.9}$$

associated to

$$0 \to \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \to \mathscr{O}_{\tilde{X}}^\times \to \mathscr{O}_X^\times \to 0.$$

This proves the first statement of the proposition.

The second statement also follows by observing that the first map in (2.1.5.9) is injective under the assumption of the statement.

To prove the third statement, let us investigate what happens when we replace the $\{\xi_{\alpha\beta}\}$ representing $\tilde{X}$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ by some different element $\mathsf{m}_{\tilde{X}} + \tilde{X}$ with $\mathsf{m}_{\tilde{X}} \in \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. By refining the open covering $\{U_\alpha\}$ if necessary, let us suppose $\mathsf{m}_{\tilde{X}}$ is represented by some

$$\eta_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \cong \underline{H}^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

Then

$$\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta}$$

defines $\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$ that gives a possibly different lifting of X, and we have to revise our definition of $h_{\alpha\beta\gamma}$ by

$$h'_{\alpha\beta\gamma} := \tilde{l}_{\alpha\gamma}^{-1} \cdot [(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} - 1 \in \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}}.$$

Let us write

$$\eta^*_{\alpha\beta} : \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}} \to \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}}$$

as

$$\eta^*_{\alpha\beta} = \mathrm{Id} + T_{\alpha\beta} \circ d$$

for $T_{\alpha\beta} \circ d \in \underline{H}^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, as in Lemma 2.1.1.2 and Corollary 2.1.1.3. We use the notation $T_{\alpha\beta} \circ d$ to signify the fact that it is a composition of the universal differentiation $d : \mathscr{O}_{U_{\alpha\beta}/S} \to \Omega^1_{U_{\alpha\beta}/S}$ and some homomorphism $T_{\alpha\beta} \in \underline{\mathrm{Hom}}_{\mathscr{O}_{U_{\alpha\beta}}}(\Omega^1_{U_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Note that $\{\eta_{\alpha\beta}\}$ and $\{T_{\alpha\beta}\}$ are simply different ways of representing the same class $\mathsf{m}_{\tilde{X}}$. Note that both $\xi^*_{\alpha\beta}(\tilde{l}_{\alpha\beta})$ and $(\xi'_{\alpha\beta})^*(\tilde{l}_{\alpha\beta})$ become the same $l_{\alpha\beta}$ modulo $\mathscr{I}$. Since

$$\eta^*_{\alpha\beta} = (\xi^{-1}_{\alpha\beta} \circ \xi'_{\alpha\beta})^* = (\xi'_{\alpha\beta})^* \circ (\xi^*_{\alpha\beta})^{-1},$$

we have

$$\begin{aligned}(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma}) &= \eta^*_{\alpha\beta}\xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma}) = (\mathrm{Id} + T_{\alpha\beta} \circ d)(\xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma})) \\ &= \xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma}) + (T_{\alpha\beta} \circ d)(l_{\beta\gamma}) \\ &= (\xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma}))(1 + (\xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma}))^{-1} T_{\alpha\beta}(dl_{\beta\gamma})) \\ &= (\xi^*_{\alpha\beta}(\tilde{l}_{\beta\gamma}))(1 + T_{\alpha\beta}(d\log(l_{\beta\gamma}))),\end{aligned}$$

where we have used the usual convention of log differentiation. As a result, we have

$$\begin{aligned} h'_{\alpha\beta\gamma} - h_{\alpha\beta\gamma} &= \tilde{l}_{\alpha\gamma}^{-1}[(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma}) - \xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})]\tilde{l}_{\beta\gamma} \\ &= (\tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\beta\gamma}) \cdot T_{\alpha\beta}(d\log(l_{\beta\gamma})) \\ &= (1 + h_{\alpha\beta\gamma}) \cdot T_{\alpha\beta}(d\log(l_{\beta\gamma})) = T_{\alpha\beta}(d\log(l_{\beta\gamma})). \end{aligned}$$

This is just the cup-product of the class $\mathsf{m}_{\tilde{X}}$ represented by $T = \{T_{\alpha\beta}\}$ and the class $d\log(\mathcal{L})$ represented by $\{d\log(l_{\alpha\beta})\}$. This proves the third statement. □

Corollary 2.1.5.10. *Suppose moreover that X is a flat group scheme over S. Then we have a canonical isomorphism*

$$\underline{H}^1(X, \mathscr{O}_X) \cong \underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S} := \underline{\mathrm{Pic}}(X/S)(\mathscr{O}_S[\varepsilon]/(\varepsilon^2)). \tag{2.1.5.11}$$

Proof. Note that $\underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S} = \underline{\mathrm{Pic}}(X/S)(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$ is by definition the set of liftings of the trivial invertible sheaf $\mathscr{O}_X$ on X. Set $\tilde{S} := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$. Then, by abuse of notations, $\mathscr{O}_{\tilde{S}} \twoheadrightarrow \mathscr{O}_S$ has kernel $\mathscr{I} := \varepsilon\mathscr{O}_S$ satisfying $\mathscr{I}^2 = 0$. As $\mathscr{O}_S$-modules, we have $\mathscr{I} \cong \mathscr{O}_S$ as it is generated by the single element ε. Note that this surjection has a canonical section given by $a \mapsto a \in \mathscr{O}_S[\varepsilon]/(\varepsilon^2)$ for any $a \in \mathscr{O}_S$. Therefore we may pullback X to a family $\tilde{X}$ over $\tilde{S}$, together with the trivial invertible sheaf lifting the trivial invertible sheaf on X. This forces the obstruction to vanish and the second statement of Proposition 2.1.5.4 applies. The torsor $\underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S}$ under $\underline{H}^1(X, \mathscr{O}_X)$ can be canonically trivialized by the section of the surjection above. □

Lemma 2.1.5.12. *Suppose X is a smooth group scheme over S, then $\underline{\mathrm{Der}}_{X/S}$ is trivialized by the group translation of sections of the tangent space at the identity, and therefore there is a canonical isomorphism*

$$\underline{H}^0(X, \underline{\mathrm{Der}}_{X/S}) \cong \underline{H}^0(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S}. \tag{2.1.5.13}$$

In this case, $\underline{\mathrm{Der}}_{X/S}$ is the pullback of the sheaf $\underline{\mathrm{Lie}}_{X/S}$ on S. The analogous statement is true if we replace $\underline{\mathrm{Der}}_{X/S}$ by $\Omega^1_{X/S}$.

In the case that S is the spectrum of a field, this is obvious, and its proof can be seen in, for example, [115, III, §3, Lem. 9].

When X is an abelian scheme over S. Then by definition $\underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S} = \underline{\mathrm{Lie}}_{X^\vee/S}$. By the two identifications (2.1.5.13) and (2.1.5.11), we can interpret the map (2.1.5.6) defined by the cup-product with $d\log(\mathcal{L})$ as a map

$$\mathsf{d}_{\mathcal{L}} : \underline{\mathrm{Lie}}_{X/S} \to \underline{\mathrm{Lie}}_{X^\vee/S}.$$

Proposition 2.1.5.14. *This map agrees with the differential* $d\lambda_{\mathcal{L}}$ *of the map* $\lambda_{\mathcal{L}} : X \to X^\vee$ *defined by* $\mathcal{L}$ *(in Construction 1.3.2.10).*

Proof. This is just a matter of identifications.

Any vector in $\underline{\mathrm{Lie}}_{X/S}$ can be realized as a map $T : \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2)) \to X$ extending the identity section $e_X : S \to X$. Its relation to sections of $\underline{H}^0(X, \underline{\mathrm{Der}}_{X/S})$ can be described explicitly as follows: Suppose a function f on X is evaluated as some $a + b\varepsilon$ under T, for $a, b \in \mathscr{O}_S$. Then $b = T(df)$. Once this identification is made, we may also regard T as a differentiation.

On the other hand, the pullback $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ of $\mathcal{D}_2(\mathcal{L})$ along

$$(\mathrm{Id}_X \underset{S}{\times} T) : X \underset{S}{\times} \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2)) \to X \underset{S}{\times} X$$

gives a deformation of $\mathcal{D}_2(\mathcal{L})|_{X \underset{S}{\times} e_X} \cong \mathscr{O}_X$ over $\underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$. By the universal property of Poincaré invertible sheaf, this invertible sheaf $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ is the pullback of $\mathcal{P}_X$ along some unique map $(\mathrm{Id}_X \times T') : X \underset{S}{\times} \underline{\mathrm{Spec}}_{\mathscr{O}_S}[\varepsilon]/(\varepsilon^2)) \to X \underset{S}{\times} X^\vee$. Since $\lambda_{\mathcal{L}}$ is by definition the unique map such that $\mathcal{D}_2(\mathcal{L})$ is the pullback of $\mathcal{P}_X$ along $(\mathrm{Id}_X \times \lambda_{\mathcal{L}})$, the map $T' : \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2)) \to X^\vee$ is nothing but the vector $d\lambda_{\mathcal{L}}(T)$ in $\underline{\mathrm{Lie}}_{X^\vee/S}$.

Let us interpret T' as a deformation of $\mathscr{O}_X$. Let $\mathcal{L}$ be defined by some cocycle represented by some $\{l_{\alpha\beta}\}$ in $\underline{H}^1(X, \mathscr{O}_X^\times)$. If we interpret $X \underset{S}{\times} \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$ as an abelian scheme over $\underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$ lifting X over S, then $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ is an invertible sheaf lifting the trivial invertible sheaf $\mathscr{O}_X$ over S. The cocycle for $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ can be given explicitly by $m_{\alpha\beta}$, where

$$m_{\alpha\beta} := [l_{\alpha\beta,0} + T(dl_{\alpha\beta,0})\varepsilon]l_{\alpha\beta,0}^{-1} = 1 + T(d\log(l_{\alpha\beta,0}))\varepsilon.$$

By looking at the coefficient ε, we see that the deformation $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ corresponds to $\{T(d\log(l_{\alpha\beta}))\}$ in $\underline{H}^1(X, \mathscr{O}_X)$ under the isomorphism (2.1.5.11) given by Corollary 2.1.5.10. This is exactly the map $\mathsf{d}_{\mathcal{L}}$ defined by cup-product with $d\log(\mathcal{L})$. $\square$

Proposition 2.1.5.15. *When X is an abelian scheme over S, the cup-products*

$$\underline{H}^i(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{H}^j(X, \mathscr{O}_X) \to \underline{H}^{i+j}(X, \mathscr{O}_X) \tag{2.1.5.16}$$

gives $\underline{H}^*(X, \mathscr{O}_X)$ a structure of an exterior algebra. In particular, (2.1.5.16) is a surjection when $i = j = 1$, and we have an isomorphism

$$\wedge^2 \underline{H}^1(X, \mathscr{O}_X) \cong \underline{H}^2(X, \mathscr{O}_X).$$

See [115, VII, §4, Thm. 10] for a proof when S is the spectrum of a field. (Note that our explicit definition is sufficient for the application of [115, VII, §4, Prop. 16] there.) The general case follows since all the sheaves are locally free.

Corollary 2.1.5.17. *Let X be an abelian scheme over S. Then the following diagram is commutative:*

$$\begin{tikzcd}
\underline{\mathrm{Lie}}_{X^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S} \arrow[r, "\mathrm{Id}_{X^\vee} \otimes d\lambda_{\mathcal{L}}"] \arrow[d, "\mathrm{can.} \otimes \mathrm{can.}"', "\wr"] & \underline{\mathrm{Lie}}_{X^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S} \arrow[d, "\wr"', "\mathrm{can.} \otimes \mathrm{can.}"] \\
\underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{H}^0(X, \underline{\mathrm{Der}}_{X/S}) \arrow[r, "\mathrm{Id} \otimes \mathsf{d}_{\mathcal{L}}"] \arrow[d, "\mathrm{can.}"', "\wr"] & \underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{H}^1(X, \mathscr{O}_X) \arrow[d, "\cup"] \\
\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S}) \arrow[r, "\mathsf{d}_{\mathcal{L}}"'] & \underline{H}^2(X, \mathscr{O}_X)
\end{tikzcd}$$

2.1.6 De Rham Cohomology

Let $\pi : X \to S$ a morphism of schemes. Let $\Omega^\bullet_{X/S}$ be the complex on X whose differentials $d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ are given by the canonical $d : \mathscr{O}_{X/S} \to \Omega^1_{X/S}$ of the Kähler differentials. Then one can define the *de Rham cohomology* $\underline{H}^i_{\mathrm{dR}}(X/S)$ to be the hypercohomology $R^i\pi_*\Omega^\bullet_{X/S}$ on S. In practice, this can be calculated if we have an explicit quasi-isomorphism from $\Omega^\bullet_{X/S}$ to some complex whose cohomology is especially easy to describe.

Let us suppose from now that X is smooth. By Lemma 2.1.1.5, we may find an affine open covering $\{U_\alpha\}$ of X, such that each U_α is étale over some affine r-space $\mathbb{A}^r_S$ over S. In this case, locally over the base scheme S, the sheaf of differentials on U_α has a basis $dx_1, \ldots dx_r$ given by the coordinates $x_1, \ldots, x_r$ of $\mathbb{A}^r_S$. In particular $R^i\pi_*\Omega^\bullet_{U_\alpha/S}$ is trivial for all $i > 0$. This gives us an explicit way to compute the de Rham cohomology of X: Let $\underline{C}^{\bullet,\bullet}$ be the double complex of sheaves with terms $\underline{C}^{p,q} := \underline{C}^p(\{U_\alpha\}, \Omega^q_{X/S})$ and differentials

$$\partial : \underline{C}^p(\{U_\alpha\}, \Omega^q_{X/S}) \to \underline{C}^{p+1}(\{U_\alpha\}, \Omega^q_{X/S})$$
$$(x_{\alpha_1\ldots\alpha_p}) \mapsto ((\partial x)_{\alpha_1\ldots\alpha_{p+1}}) := (\sum_{k=0}^{p+1}(-1)^k x_{\alpha_1\ldots\hat{\alpha}_k\ldots\alpha_{p+1}})$$

and

$$d : \underline{C}^p(\{U_\alpha\}, \Omega^q_{X/S}) \to \underline{C}^p(\{U_\alpha\}, \Omega^{q+1}_{X/S})$$
$$(x_{\alpha_1\ldots\alpha_p}) \mapsto (dx_{\alpha_1\ldots\alpha_p}).$$

Let $\underline{C}^\bullet$ be the total complex of $\underline{C}^{\bullet,\bullet}$ with terms $\underline{C}^n := \underset{p+q=n}{\oplus} \underline{C}^{p,q}$ and differentials

$$D = D_{p,q} = \partial \oplus (-1)^p d : \underline{C}^{p,q} \to \underline{C}^{p+1,q} \oplus \underline{C}^{p,q+1}.$$

Note that $\partial d = d\partial$, and hence $D^2 = 0$ by the sign twist we have specified. Let us define a map $\Omega^\bullet_{X/S} \to \underline{C}^\bullet$ by assign over each open subscheme U of X the map $\Omega^n_{X/S}(U) \to \underline{C}^{0,n}(U) \subset \underline{C}^n(U)$ given by simply sending a differential over U to its restrictions over $U \cap \tilde{U}_\alpha$. Then the following result is well-known:

Proposition 2.1.6.1. *The map $\Omega^\bullet_{X/S} \to \underline{C}^\bullet$ defined above is a quasi-isomorphism.*

Now suppose $S \hookrightarrow \tilde{S}$ is a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X}$ be a smooth scheme over $\tilde{S}$, and let $X := \tilde{X} \underset{\tilde{S}}{\times} S$. By Proposition 2.1.3.2 and its proof, we know that $\tilde{X}$ is glued from liftings $\{\tilde{U}_\alpha\}$ of affine open smooth subschemes $\{U_\alpha\}$ forming an open covering of X. By refining the open covering if necessary, we may assume that each of the affine open subscheme $\tilde{U}_\alpha$ is étale over some affine r-space $\mathbb{A}^r_{\tilde{S}}$ as above.

Then we know that the de Rham cohomology $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ can be computed as the cohomology of the total complex $\underline{C}^\bullet$ of $\underline{C}^{p,q} = \underline{C}^p(\{\tilde{U}_\alpha\}, \Omega^q_{\tilde{X}/\tilde{S}})$. Note that the groups $\underline{C}^{p,q}$ are defined only using the information on each $\tilde{U}_\alpha$. Namely, we just need to know the groups $\underline{H}^0(U_{\alpha_0\ldots\alpha_p}, \Omega^q_{\tilde{U}_{\alpha_1\ldots\alpha_p}/S})$. On the other, the differential D of the complex $\underline{C}^\bullet$ does depend on the gluing maps $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$. Nevertheless, we have:

Proposition 2.1.6.2. *Let* $\mathsf{m}_{\tilde{X}} \in \underline{H}^1(X, \underline{\mathrm{Der}}X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$*, and let* $\tilde{X}' := \mathsf{m}_{\tilde{X}} + \tilde{X} \in \mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ *denote the object given by* $\mathsf{m}_{\tilde{X}}$ *and* $\tilde{X}$ *under the action of* $\underline{H}^1(X, \underline{\mathrm{Der}}X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ *on* $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$*. Then there is a canonical isomorphism*

$$\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$$

(*lifting the identity map on* $\underline{H}^1_{\mathrm{dR}}(X/S)$) *that is unique up to unique isomorphisms.*

Although this is quite well-known, we would like to give a proof here to show its relation to the theory of obstruction we have studied so far.

Proof of Proposition 2.1.6.2. Take an affine open covering $\{\tilde{U}_\alpha\}$ of $\tilde{X}$ such that, if we set $U_\alpha := \tilde{U}_\alpha \underset{\tilde{S}}{\times} S$, and set $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ to the isomorphism identifying the open subscheme $\tilde{U}_{\alpha\beta}$ with itself, then $\tilde{X}'$ is obtain by replacing this gluing isomorphism $\xi_{\alpha\beta}$ by $\xi'_{\alpha\beta} \circ \eta_{\alpha\beta}$, where $\eta_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \cong \underline{H}^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ represents the class of $\mathsf{m}_{\tilde{X}} \in \underline{H}^1(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. As in the proof of Proposition 2.1.5.4, we identify $\eta^*_{\alpha\beta} = \mathrm{Id} + T_{\alpha\beta} \circ d$ with $T_{\alpha\beta} \in \underline{\mathrm{Hom}}_{\mathscr{O}_{U_{\alpha\beta}}}(\Omega^1_{U_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Then $(\xi'_{\alpha\beta})* = (\eta_{\alpha\beta})^*(\xi_{\alpha\beta})^* = (\mathrm{Id} + T_{\alpha\beta})(\xi_{\alpha\beta})^*$ and $T = \{T_{\alpha\beta}\}$ defines a class of $\underline{H}^1(X, \Omega^1_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.

Let $\underline{C}^\bullet$ and $(\underline{C}')^\bullet$ be respectively the complexes computing $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ and $\underline{H}^i_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$ as explained above, with the same affine open schemes $\tilde{U}_\alpha$ covering both $\tilde{X}$ and $\tilde{X}'$ (with different gluing maps identifying the overlaps). More precisely, we have $\underline{C}^n = \underset{p+q=n}{\oplus} \underline{C}^{p,q}$, $(\underline{C}')^n = \underset{p+q=n}{\oplus} (\underline{C}')^{p,q}$, $\underline{C}^{p,q} = \underline{H}^0(\tilde{U}_{\alpha_0}|_{U_{\alpha_0\cdots\alpha_p}}, \Omega^q_{\tilde{X}/\tilde{S}})$, and $(\underline{C}')^{p,q} = \underline{H}^0(\tilde{U}_{\alpha_0}|_{U_{\alpha_0\cdots\alpha_p}}, \Omega^q_{\tilde{X}'/\tilde{S}})$. Note that we have a natural identification $\underline{C}^{p,q} \cong (\underline{C}')^{p,q}$, which we shall automatically assume. Any element in $\underline{C}^n$ is represented by a tuple $(x^{(p,q)})_{p+q=n}$ with $x^{(p,q)} = \{x^{(p,q)}_{\alpha_0\dots\alpha_p}\}$ defining an element in $\underline{C}^p(\{\tilde{U}_\alpha\}, \Omega^q_{\tilde{X}/\tilde{S}}) = \underline{C}^p(\{\tilde{U}_\alpha\}, \Omega^q_{\tilde{X}'/\tilde{S}})$. Note that $T = \{T_{\alpha\beta}\}$ defines a *cup-product* map

$$\underline{C}^{p,q} \to \underline{C}^{p+1,q-1} : x^{(p,q)} = (x^{(p,q)}_{\alpha_0\dots\alpha_p}) \mapsto (T \cup x^{(p,q)})$$

defined by

$$(T \cup x^{(p,q)})_{\alpha_0\dots\alpha_{p+1}} := T_{\alpha_0\alpha_1} x^{(p,q)}_{\alpha_1\dots\alpha_{p+1}}.$$

Naturally

$$\partial(T \cup x^{(p,q)}) = (\partial T \cup x^{(p,q)}) + (-1)^1 (T \cup \partial x^{(p,q)})$$

because $\partial T = 0$ (as it is a cocycle). The differential

$$\partial' : (\underline{C}')^{p,q} \to (\underline{C}')^{p+1,q} : x^{(p,q)} = (x^{(p,q)}_{\alpha_0\dots\alpha_p}) \mapsto \partial' x^{(p,q)}$$

is defined by

$$\begin{aligned}(\partial' x^{(p,q)})_{\alpha_0\dots\alpha_{p+1}} &:= (\xi'_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1\dots\alpha_{p+1}}) + \sum_{k=1}^{p+1} (-1)^k x^{(p,q)}_{\alpha_0\dots\hat{\alpha}_k\dots\alpha_{p+1}} \\ &= (\xi_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1\dots\alpha_{p+1}}) + (T_{\alpha_0\alpha_1} \circ d)((\xi_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1\dots\alpha_{p+1}})) \\ &\quad + \sum_{k=1}^{p+1} (-1)^k x^{(p,q)}_{\alpha_0\dots\hat{\alpha}_k\dots\alpha_{p+1}} \\ &= (\partial x^{(p,q)})_{\alpha_0\dots\alpha_{p+1}} + (T_{\alpha_0\alpha_1} \circ d)(x^{(p,q)}_{\alpha_1\dots\alpha_{p+1}}).\end{aligned}$$

In other words, we have simply

$$\partial' = \partial + (T \cup d).$$

On the other hand, although the map $d : \mathscr{O}_{\tilde{U}_\alpha} \to \Omega^1_{\tilde{U}_\alpha/S}$ coming from the restriction of $d : \mathscr{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}$ is unique up to a canonical isomorphism, it does not mean that it is the *same* map as the map d' coming from $\Omega^1_{\tilde{X}'/\tilde{S}}$. Let us measure this difference over $\tilde{U}_\alpha$ by $d' = (\mathrm{Id} + E_\alpha) \circ d$, for some map $E_\alpha : \Omega^1_{\tilde{U}_\alpha/\tilde{S}} \to \mathscr{I} \cdot \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$, or rather a map $E_\alpha : \Omega^1_{U_\alpha/S} \to \Omega^1_{U_\alpha/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$. This E is canonical because of the universal properties of d and d'. Note that we need $(E_\alpha \circ d + d \circ E_\alpha) \circ d = 0$ in order to make $(d')^2 = 0$. Then we have $d \circ E_\alpha \circ d = 0 = E_\alpha \circ d \circ d$, which means $d \circ E_\alpha = 0 = E_\alpha \circ d$ as everything in $\Omega^1_{\tilde{U}_\alpha/\tilde{S}}$ is in the image of $d : \mathscr{O}_{\tilde{U}_\alpha} \to \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$. Moreover, we need to glue d (resp. d') as well using $\xi_{\alpha\beta}$ (resp. $\xi'_{\alpha\beta}$). Therefore, we need both the relations

$$d(\xi^*_{\alpha\beta}(x)) = \xi^*_{\alpha\beta}(dx)$$

and

$$d'((\xi'_{\alpha\beta})^*(x)) = (\xi'_{\alpha\beta})^*(dx).$$

If we expand all the terms in the second relation and substitute the first relation into the it, then we get

$$E_\alpha(dx) + dT_{\alpha\beta}(dx) = \xi^*_{\alpha\beta}(E_\beta)(dx),$$

$$dT_{\alpha\beta} = \xi^*_{\alpha\beta}(E_\beta) - E_\alpha =: -(\partial E)_{\alpha\beta},$$

or simply

$$dT = -\partial E.$$

Now let us specialize to $n = 1$ and consider the map

$$\underline{C}^1 \to (\underline{C}')^1 : x = (x^{(1,0)}, x^{(0,1)}) \mapsto x' = ((x')^{(1,0)}, (x')^{(0,1)})$$

given explicitly by

$$x' := (x^{(1,0)} + (T \cup x^{(0,1)}), x^{(0,1)} + Ex^{(0,1)}) \tag{2.1.6.3}$$

The differential D on $\underline{C}^1$ sends

$$x = (x^{(1,0)}, x^{(0,1)}) \mapsto Dx = (\partial x^{(1,0)}, -dx^{(1,0)} + \partial x^{(0,1)}, dx^{(0,1)}).$$

On the other hand, we have a similar formula for D' on x', whose components are given by:

$$\begin{aligned}\partial'(x')^{(1,0)} &= (\partial + (T \cup d))(x^{(1,0)} + (T \cup x^{(0,1)}))\\ &= \partial x^{(1,0)} + (T \cup dx^{(1,0)}) + \partial(T \cup \partial x^{(0,1)})\\ &= \partial x^{(1,0)} - (T \cup (-dx^{(1,0)} + \partial x^{(0,1)})),\end{aligned}$$

$$\begin{aligned} -d'(x')^{(1,0)} + \partial'(x')^{(0,1)} &= -(d+Ed)(x^{(1,0)} + (T \cup x^{(0,1)})) \\ &\quad + (\partial + (T \cup d))(x^{(0,1)} + Ex^{(0,1)}) \\ &= (-dx^{(1,0)} + \partial x^{(0,1)}) - Edx^{(1,0)} - d(T \cup x^{(0,1)}) \\ &\quad + T \cup dx^{(0,1)} - \partial(Ex^{(0,1)}) \\ &= (-dx^{(1,0)} + \partial x^{(0,1)}) - Edx^{(1,0)} - dT \cup x^{(0,1)} \\ &\quad - (\partial E) \cup x^{(0,1)} - E\partial x^{(0,1)} \\ &= (\mathrm{Id} + E)(-dx^{(1,0)} + \partial x^{(0,1)}), \\ d'(x')^{(0,1)} &= (d+Ed)(x^{(0,1)} + Ex^{(0,1)}) \\ &= dx^{(0,1)} + Edx^{(0,1)} + dEx^{(0,1)} = dx^{(0,1)}, \end{aligned}$$

where we have used $dT = -\partial E$ in the second relation. Since $\mathrm{Id} + E$ is an automorphism, we see that $D'x' = 0$ in and only if $Dx = 0$. On the other hand, if $x^{(1,0)} = \partial x^{(0,0)}$ and $x^{(0,1)} = dx^{(0,0)}$ for some $(x^(0,0)) \in \underline{C}^0$, then

$$\begin{gathered} x^{(1,0)} + (T \cup x^{(0,0)}) = \partial x^{(0,0)} + T \cup dx^{(0,0)} = \partial' x^{(0,0)} \\ x^{(0,1)} + Ex^{(0,1)} = dx^{(0,0)} + Edx^{(0,0)} = d'x^{(0,0)}. \end{gathered}$$

As a result, we have show that there is a unique way to associate to each representative of $\mathsf{m}_X \in \underline{H}^1(X, \underline{\mathrm{Der}} X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ an isomorphism from the first cohomology of $(\underline{C}^\bullet, D)$ to the first cohomology of $((\underline{C}')^\bullet, D')$. If we modify the representative $T = \{T_{\alpha\beta}\}$ of $\mathsf{m}_{\tilde{X}}$ by a coboundary, then all the E_α are also modified in a uniquely determined way. This shows that we have constructed a canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$ that is unique up to unique isomorphisms. □

Now suppose that $\tilde{Y}$ is another scheme over $\tilde{S}$ such that there is a map f from X to $Y := \tilde{Y} \underset{\tilde{S}}{\times} S$. Then:

Proposition 2.1.6.4. *There is a canonical morphism*

$$f^* : \underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$$

(lifting the natural functorial morphism $\underline{H}^1_{\mathrm{dR}}(Y/S) \to \underline{H}^1_{\mathrm{dR}}(X/S)$ induced by f) that is unique up to unique isomorphisms.

Proof. We shall denote by respectively $d_{\tilde{X}} : \mathscr{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}$ and $d_{\tilde{Y}} : \mathscr{O}_{\tilde{Y}} \to \Omega^1_{\tilde{Y}/\tilde{S}}$ the map of universal differentials. Take affine open coverings $\{\tilde{U}_\alpha\}$ of

$\tilde{X}$ and $\{\tilde{V}_\alpha\}$ of $\tilde{Y}$ as in the proof of Proposition 2.1.3.2 such that $f(U_\alpha) \subset V_\alpha$. Let $\tilde{f}_\alpha$ be any morphism lifting the restriction of f to U_α. Then we know that the obstruction of lifting f globally to some morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a cohomology class in $\underline{H}^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ represented by some $T \circ d_{\tilde{Y}} = \{T_{\alpha\beta} \circ d_{\tilde{Y}}\}$, where the notation $T_{\alpha\beta} \circ d_{\tilde{Y}}$ means it is the composition of $d_{\tilde{Y}}$ with $T_{\alpha\beta} \in \underline{\mathrm{Hom}}_{\mathscr{O}_{U_{\alpha\beta}}}(f^*\Omega^1_{V_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. By Lemma 2.1.1.2, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\tilde{V}_{\alpha\beta}} \arrow[r, "d_{\tilde{Y}}"] \arrow[d, "\tilde{f}^*_\alpha - \tilde{f}^*_\beta"'] & \Omega^1_{\tilde{V}_{\alpha\beta}/\tilde{S}} \arrow[d, "\tilde{f}^*_\alpha - \tilde{f}^*_\beta"] \arrow[dl, dotted, "T_{\alpha\beta}"] \\
\mathscr{O}_{\tilde{U}_{\alpha\beta}} \arrow[r, "d_{\tilde{X}}"'] & \Omega^1_{\tilde{U}_{\alpha\beta}/\tilde{S}}
\end{tikzcd}$$

measuring the difference between the different liftings. Note that here the two maps $\tilde{f}^*_\alpha$ and $\tilde{f}^*_\beta$ from $\Omega^1_{\tilde{V}_{\alpha\beta}/\tilde{S}}$ to $\Omega^1_{\tilde{U}_{\alpha\beta}/\tilde{S}}$ are determined by the universal property of $\Omega^1_{\tilde{V}_{\alpha\beta}/\tilde{S}}$. (This was the tricky part in the proof of Proposition 2.1.6.2.)

Let $\underline{C}^\bullet_{\tilde{X}}$ and $\underline{C}^\bullet_{\tilde{Y}}$ be respectively the complexes computing $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ and $\underline{H}^i_{\mathrm{dR}}(\tilde{Y}/\tilde{S})$ as above, with the affine open schemes $\tilde{U}_\alpha$ (resp. $\tilde{V}_\alpha$) covering $\tilde{X}$ (resp. $\tilde{Y}$). Let us define a map

$$\underline{C}^1_{\tilde{Y}} \to \underline{C}^1_{\tilde{X}} : y = (y^{(1,0)}, y^{(0,1)}) \mapsto x = (x^{(1,0)}, x^{(0,1)})$$

by

$$\begin{aligned} x^{(1,0)}_{\alpha\beta} &:= \tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(1,0)}_\beta) \\ x^{(0,1)}_\alpha &:= \tilde{f}^*_\alpha(y^{(0,1)}_\alpha). \end{aligned}$$

The differential $D_{\tilde{Y}}$ on $\underline{C}^1_{\tilde{Y}}$ sends

$$y = (y^{(1,0)}, y^{(0,1)}) \mapsto D_{\tilde{Y}} y = (\partial_{\tilde{Y}} y^{(1,0)}, -d_{\tilde{Y}} y^{(1,0)} + \partial_{\tilde{Y}} y^{(0,1)}, d_{\tilde{Y}} y^{(0,1)}).$$

On the other hand, we have a similar formula for $D_{\tilde{X}}$ on x, whose components are given by:

$$\begin{aligned}
(\partial_{\tilde{X}} x^{(1,0)})_{\alpha\beta\gamma} &= [\tilde{f}^*_\beta(y^{(1,0)}_{\beta\gamma}) + T_{\beta\gamma}(y^{(0,1)}_\gamma)] - [\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\gamma}) + T_{\alpha\gamma}(y^{(0,1)}_\gamma)] \\
&\quad + [\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(0,1)}_\beta)] \\
&= \tilde{f}^*_\alpha((\partial_{\tilde{Y}} y^{(1,0)})_{\alpha\beta\gamma}) + (\tilde{f}^*_\beta - \tilde{f}^*_\alpha)(y^{(1,0)}_{\beta\gamma}) \\
&\quad + (\partial T)_{\alpha\beta\gamma}(y^{(0,1)}_\gamma) + T_{\alpha\beta}((\partial y^{(0,1)})_{\beta\gamma}) \\
&= \tilde{f}^*_\alpha((\partial_{\tilde{Y}} y^{(1,0)})_{\alpha\beta\gamma}) + T_{\alpha\beta}(-d_{\tilde{Y}} y^{(1,0)}_{\beta\gamma} + (\partial_{\tilde{Y}} y^{(0,1)})_{\beta\gamma}), \\
-d_{\tilde{X}} x^{(1,0)} + \partial_{\tilde{X}} x^{(0,1)} &= -d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta})) - (d_{\tilde{X}} \circ T_{\alpha\beta})(y^{(0,1)}_\beta) \\
&\quad - \tilde{f}^*_\beta(y^{(0,1)}_\beta) + \tilde{f}^*_\alpha(y^{(0.1)}_\alpha) \\
&= -d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta})) - (\tilde{f}^*_\alpha - \tilde{f}^*_\beta)(y^{(0,1)}_\beta) \\
&\quad - \tilde{f}^*_\beta(y^{(0,1)}_\beta) + \tilde{f}^*_\alpha(y^{(0.1)}_\alpha) \\
&= \tilde{f}^*_\alpha(-d_{\tilde{Y}} y^{(1,0)}_{\alpha\beta} + (-y^{(0,1)}_\beta + y^{(0,1)}_\alpha)) \\
&= \tilde{f}^*_\alpha(-d_{\tilde{Y}} y^{(1,0)}_{\alpha\beta} + (\partial_{\tilde{Y}} y^{(0,1)})_{\alpha\beta}), \\
d_{\tilde{X}} x^{(0,1)} &= d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(0,1)}_\alpha) = \tilde{f}^*_\alpha(d_{\tilde{Y}} y^{(0,1)}_\alpha).
\end{aligned}$$

Therefore, if $D_{\tilde{Y}} y = 0$, then $D_{\tilde{X}} x = 0$ as well. On the other hand, if $y^{(1,0)} = \partial_{\tilde{Y}} y^{(0,0)}$ and $y^{(0,1)} = d_{\tilde{Y}} y^{(0,0)}$ for some $(y^{(0,0)}) \in \underline{C}^0_{\tilde{Y}}$, then

$$\begin{aligned}
x^{(1,0)}_{\alpha\beta} &= \tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(0,1)}_\beta) = \tilde{f}^*_\alpha(-y^{(0,0)}_\beta + y^{(0,0)}_\alpha) + T_{\alpha\beta}(d_{\tilde{Y}} y^{(0,0)}_\beta) \\
&= \tilde{f}^*_\alpha(-y^{(0,0)}_\beta + y^{(0,0)}_\alpha) + (\tilde{f}^*_\alpha - \tilde{f}^*_\beta)(y^{(0,0)}_\beta) = \tilde{f}^*_\alpha(y^{(0,0)}_\alpha) - \tilde{f}^*_\beta(y^{(0,0)}_\beta), \\
x^{(0,1)}_\alpha &= \tilde{f}^*_\alpha(d_{\tilde{Y}} y^{(0,0)}_\alpha) = d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(0,0)}_\alpha)),
\end{aligned}$$

which shows that $x = (x^{(1,0)}, x^{(0,1)}) = D_{\tilde{X}}(x^{(0,0)})$ for $x^{(0,0)} = (x^{(0,0)}_\alpha) := (\tilde{f}^*_\alpha(y^{(0,0)}_\alpha))$. As a result, we have show that there is a unique way to associate to each representative of $\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) \in \underline{H}^1(X, f^*(\underline{\mathrm{Der}} Y/S) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ an isomorphism from first cohomology of $(\underline{C}^\bullet_{\tilde{Y}}, D_{\tilde{Y}})$ to the one of $(\underline{C}^\bullet_{\tilde{X}}, D_{\tilde{X}})$. If we modify the representative $T = \{T_{\alpha\beta}\}$ of by a coboundary, then all the maps are also modified in a uniquely determined way that does not affect the result. This shows that we have constructed a canonical morphism $\underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$. $\square$

Note that by construction of $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ using $\underline{C}^\bullet$, we see that there is a map from $\underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}})$ to $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ and a map from $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ to

$\underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}})$. Indeed, they correspond to respectively $\underline{C}^{0,1}$ and $\underline{C}^{1,0}$ in $\underline{C}^1$. Alternatively, consider the truncated subcomplex $\Omega^{\bullet \geq 1}_{\tilde{X}/\tilde{S}}$ of $\Omega^{\bullet}_{\tilde{X}/\tilde{S}}$ and the exact sequence

$$0 \to \Omega^{\bullet \geq 1}_{\tilde{X}/\tilde{S}} \to \Omega^{\bullet}_{\tilde{X}/\tilde{S}} \to \mathscr{O}_{\tilde{X}} \to 0,$$

where $\mathscr{O}_{\tilde{X}}$ is considered as a complex with only one nonzero term in degree 0. Then the first hypercohomology gives an exact sequence

$$0 \to \underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \to \underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}}). \tag{2.1.6.5}$$

When $\tilde{X}$ is an abelian scheme over $\tilde{S}$, it is known that $\underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}) \cong e^*_{\tilde{X}}\Omega^1_{\tilde{X}/\tilde{S}} =: \underline{\mathrm{Lie}}^{\vee}_{\tilde{X}/\tilde{S}} \cong (\underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}})^{\vee}$, that $\underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) \cong \underline{\mathrm{Lie}}_{\tilde{X}^{\vee}/\tilde{S}}$, and that these are all sheaves of locally free $\mathscr{O}_{\tilde{S}}$-modules. Moreover, by [17, Lem. 2.5.3], the last morphism in (2.1.6.5) is actually surjective:

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{\tilde{X}/\tilde{S}} \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \to \underline{\mathrm{Lie}}_{\tilde{X}^{\vee}/\tilde{S}} \to 0. \tag{2.1.6.6}$$

If we dualize this exact sequence (2.1.6.6), then we obtain

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{\tilde{X}^{\vee}/\tilde{S}} \to \underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S}) \to \underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}} \to 0. \tag{2.1.6.7}$$

Here $\underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S})$ is the dual of $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$, formally defined to be the *de Rham homology* of $\tilde{X}$.

If $\tilde{X}'$ is a different lifting in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, then we have a similar exact sequence for $\tilde{X}'$. Note that the dual of the canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$ does not map $\underline{\mathrm{Lie}}^{\vee}_{\tilde{X}^{\vee}/\tilde{S}}$ to $\underline{\mathrm{Lie}}^{\vee}_{(\tilde{X}')^{\vee}/\tilde{S}}$: We have seen in the proof of Proposition 2.1.6.2, in particular the explicit map (2.1.6.3), that if $\tilde{X}'$ is a different lifting in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, then the part that maps onto $\underline{\mathrm{Lie}}_{\tilde{X}^{\vee}/\tilde{S}}$ is mapped, under the map $\underline{C}^1 \to (\underline{C}')^1$ defining the canonical isomorphism, to a subspace that is different from the part that maps onto $\underline{\mathrm{Lie}}_{(\tilde{X}')^{\vee}/\tilde{S}}$. On the other hand, since $\tilde{X}$ and $\tilde{X}'$ are both liftings of X, all their corresponding objects are identical after base change from $\tilde{S}$ to S. Therefore, we have two submodules $\underline{\mathrm{Lie}}^{\vee}_{\tilde{X}^{\vee}/\tilde{S}}$ and $\underline{\mathrm{Lie}}^{\vee}_{(\tilde{X}')^{\vee}/\tilde{S}}$ in $\underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ such that

$$\underline{\mathrm{Lie}}^{\vee}_{\tilde{X}^{\vee}/\tilde{S}} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S = \underline{\mathrm{Lie}}^{\vee}_{(\tilde{X}')^{\vee}/\tilde{S}} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S = \underline{\mathrm{Lie}}^{\vee}_{X^{\vee}/S}$$

in the *same* space $\underline{H}^{\mathrm{dR}}_1(X/S)$. We say in this case that these two modules are the same modulo $\mathscr{I}$. Using the exact sequence (2.1.6.7), sheaves of projective submodules $\mathscr{M}$ in $\underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S})$ that become the same as $\underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ after modulo $\mathscr{I}$ and that $\underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S})/\mathscr{M}$ is also projective are parameterized by

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathscr{O}_{\tilde{S}}}(\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}, \mathscr{I}\cdot\underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}}) \cong \underline{\mathrm{Hom}}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{X^\vee/S}, \underline{\mathrm{Lie}}_{X/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\\ &\cong \underline{\mathrm{Lie}}_{X^\vee/S}\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{X/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I} \cong \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I}),\end{aligned}$$

which is the same set that parameterizes different liftings in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. Thus we see that they must coincide. Now we can conclude with the following analogue of a weaker form of the *Grothendieck-Messing theory* (see [93] and [60]):

Proposition 2.1.6.8. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X}$ be an abelian scheme over $\tilde{S}$. Consider the exact sequence (2.1.6.7) associated to $\tilde{X}$. Then the objects in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ are in bijection with sheaves of projective submodules $\mathscr{M}$ of $\underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S})$ that become the same as $\underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ after modulo $\mathscr{I}$ and that $\underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S})/\mathscr{M}$ is also projective. In other words, objects in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ are in bijection with exact sequences*

$$0 \to \mathscr{M} \to \underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S}) \to \mathscr{N} \to 0$$

of **sheaves of projective $\mathscr{O}_{\tilde{S}}$-modules** *such that $\mathscr{M}\underset{\mathscr{O}_{\tilde{S}}}{\otimes}\mathscr{O}_S = \underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ in $\underline{H}_1^{\mathrm{dR}}(X/S) = \underline{H}_1^{\mathrm{dR}}(\tilde{X}/\tilde{S})\underset{\mathscr{O}_{\tilde{S}}}{\otimes}\mathscr{O}_S$.*

Similar analysis for the case of lifting morphisms (following the explicit construction in the proof of Proposition 2.1.6.4) shows the following:

Proposition 2.1.6.9. *With the setting as above, if $\tilde{Y}$ is another abelian scheme over $\tilde{S}$ and $f : X := \tilde{X}\underset{\tilde{S}}{\times} S \to Y := \tilde{Y}\underset{\tilde{S}}{\times} S$ is a map of underlying schemes defined only over S. Then f can be lifted to a map of schemes $\tilde{f} : \tilde{X} \to \tilde{Y}$ if and only if $\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}$ is mapped to $\underline{\mathrm{Lie}}^\vee_{\tilde{Y}^\vee/\tilde{S}}$ under the dual of the canonical morphism $\underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ (whose reduction modulo $\mathscr{I}$ maps $\underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ to $\underline{\mathrm{Lie}}^\vee_{Y^\vee/S}$).*

In other words, the dictionary in Proposition 2.1.6.8 is *functorial* in nature.

Remark 2.1.6.10. We say this is a weaker form because it only cares about thickenings defined by sheaves of ideals whose square is zero. If we are working over p-nilpotent bases, the general theory should include the introduction of *crystals* and *crystalline cohomology*. Then the first de Rham cohomology of an abelian scheme is canonical because the crystalline cohomology is.

Remark 2.1.6.11. These results (of the theory of obstruction as we prove it) are convenient (as we shall see) for the proof of Proposition 2.2.4.11.

2.1.7 Kodaira-Spencer Maps

So far we have been studying closed immersions defined by a sheaf of ideals whose square is zero. Let S be a scheme over some fixed choice of universal base scheme U. Then there is a canonical situation in which our theory applies: Let $\tilde{S}$ be the first infinitesimal neighborhood of the image of the closed immersion $\Delta : S \hookrightarrow S \underset{\mathsf{U}}{\times} S$. (We are assuming the convention that the scheme S is separated here.) More precisely, let $\mathscr{I}$ be the ideal defining the image of the closed immersion $\Delta : S \hookrightarrow S \underset{\mathsf{U}}{\times} S$, and let $\tilde{S}$ be the subscheme of $S \underset{\mathsf{U}}{\times} S$ defined by $\mathscr{I}^2 = 0$. By abuse of notations, we shall also denote the sheaf of ideals defining S as a subscheme of $\tilde{S}$ by $\mathscr{I}$. Then $S \hookrightarrow \tilde{S}$ is a closed immersion defined by the sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Note that the two projections of $S \underset{\mathsf{U}}{\times} S \to S$ induce two canonical sections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{S} \to S$ of $S \hookrightarrow \tilde{S}$. By definition, $\Omega^1_{S/\mathsf{U}} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$, and there is a universal differential $d : \mathscr{O}_S \to \Omega^1_{S/\mathsf{U}}$ given by $a \mapsto a \otimes 1 - 1 \otimes a = \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(a)$ for any $a \in \mathscr{O}_S$. In other words, $\Omega^1_{S/\mathsf{U}}$ is simply the sheaf of ideals $\mathscr{I}$ over $\mathscr{O}_{\tilde{S}}$ considered as a sheaf of ideals over $\mathscr{O}_S$.

Now suppose $X \to S$ is a smooth scheme. Then $\tilde{X}_1 := \mathrm{pr}_1^*(X)$ and $\tilde{X}_2 := \mathrm{pr}_2^*(X)$ are two elements of $\mathsf{Lift}(X, S \hookrightarrow \tilde{S})$. By Proposition 2.1.3.2, there is an element

$$\mathsf{m} \in \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

such that

$$\tilde{X}_1 = \mathsf{m} + \tilde{X}_2$$

by the torsor structure of $\mathsf{Lift}(X, S \hookrightarrow \tilde{S})$. Since $\mathscr{I}$ is identified with $\Omega^1_{X/S}$ in this situation, we have obtained an element m in

$$\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$$

describing the difference between $\tilde{X}_1$ and $\tilde{X}_2$.

Definition 2.1.7.1. *This element* m *is called the* **Kodaira-Spencer class** *of* X *over* S (*over the universal base scheme* U). *We shall denote* m *by the symbol* $\mathsf{KS}_{X/S/\mathsf{U}}$ *to signify its meaning as a Kodaira-Spencer class.*

Let f denote the structural map $X \to S$, which is smooth by our assumption. Then the *first exact sequence* for $X \to S \to \mathsf{U}$ is of the form

$$0 \to f^*(\Omega^1_{S/\mathsf{U}}) \to \Omega^1_{X/\mathsf{U}} \to \Omega^1_{X/S} \to 0. \tag{2.1.7.2}$$

By [52, IV, 17.2.3], smoothness of f implies that (2.1.7.2) is exact and locally split. By splitting this exact sequence locally over affine open subschemes, the extension class of this exact sequence in $\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{X/S}, f^*(\Omega^1_{S/\mathsf{U}}))$ is described by a cohomology class of

$$\begin{aligned}\underline{H}^1(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\Omega^1_{X/S}, f^*\Omega^1_{S/\mathsf{U}})) &\cong \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_X}{\otimes} f^*(\Omega^1_{S/\mathsf{U}})) \\ &\cong \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}).\end{aligned}$$

Proposition 2.1.7.3. *The extension class of* (*2.1.7.2*), *when represented in* $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$, *is* (*up to a difference in sign*) *the same Kodaira-Spencer class* $\mathsf{KS}_{X/S/\mathsf{U}}$ *defined above* (*as in Definition 2.1.7.1*).

Proof. Take an affine open covering $\{U_\alpha\}$ of X such that U_α is étale over the affine r-space $\mathbb{A}^r_S$ over S for some integer $r \geq 0$. Note that this open covering can be lifted to open coverings of $\tilde{X}_i := \mathrm{pr}_i^*(X)$ over $\tilde{S}$, $i = 1, 2$, via the two projections from $\tilde{S}$ to S splitting $S \hookrightarrow \tilde{S}$, and therefore it suffices to compare the gluing maps for $\tilde{X}_1$ and $\tilde{X}_2$. As in the proof of Proposition 2.1.3.2, the comparison are given by morphisms $T_{\alpha\beta} : \Omega^1_{U_{\alpha\beta}/S} \to \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$, and $\mathsf{KS}_{X/S/\mathsf{U}}$ is represented by the 1-cocycle formed by these $T_{\alpha\beta}$.

Suppose $\Omega^1_{U_\alpha/S}$ has $\mathscr{O}_S$-basis elements $dx_1, \ldots dx_r$ given by the coordinates of $\mathbb{A}^r_S$, and suppose $\Omega^1_{U_\beta/S}$ has $\mathscr{O}_S$-basis elements $dy_1, \ldots dy_r$ given by the coordinates of $\mathbb{A}^r_S$ (for the same r), then there is a change of coordinates $dx_i = \sum\limits_j a_{ij} dy_j$ for some $a_{ij} \in \mathscr{O}_S$ as both of the two induce bases for $\Omega^1_{U_{\alpha\beta}/S}$ by restriction. This form an invertible matrix $a = (a_{ij})$ over $\mathscr{O}_S$, and for convenience let us denote its inverse matrix by $a^{-1} = (a^{ij})$. Note that the comparison of two pullbacks by projections over $\tilde{S}$ gives the universal differentiation $d : \mathscr{O}_S \to \Omega^1_{S/\mathsf{U}}$ for $\mathscr{O}_S$. Therefore we have a comparison between the two maps $dx_i \mapsto \sum\limits_j \mathrm{pr}_1^*(a_{ij})dy_j$ and $dx_i \mapsto \sum\limits_j \mathrm{pr}_2^*(a_{ij})dy_j$, and their different is that the first is given by multiplying the corresponding matrix entries of $\mathrm{Id} + (da)a^{-1}$ to the second one, or more explicitly by multiplying $\mathrm{Id} + \sum\limits_k da_{ik}a^{kj}$ to $\sum\limits_j a_{ij}dx_j$. This shows that (up to a difference in sign) $T_{\alpha\beta}$ is given by the matrix $(da)a^{-1}$.

On the other hand, the statement that U_α is étale over $\mathbb{A}^r_S$ with $\mathscr{O}_S$ coordinates $x_1, \ldots, x_r$ also shows that we may split (2.1.7.2) over U_α by taking the basis elements $dx_1, \ldots, dx_r$ of $\Omega^1_{U_\alpha/S}$ as part of a basis of $\Omega^1_{U_\alpha/\mathsf{U}}$. If we split similarly (2.1.7.2) over U_β by the basis elements $dy_1, \ldots, dy_r$ of $\Omega^1_{U_\beta/S}$, then the difference of the two splittings is again measured by $(da)a^{-1}$. This gives exactly the 1-cocycle representing the extension class of (2.1.7.2) in $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$. $\square$

Remark 2.1.7.4. In the context of Section 2.1.6, there is a canonical isomorphism between the first de Rham cohomologies of the two liftings $\tilde{X}_1$ and $\tilde{X}_2$ of X given by the Kodaira-Spencer class $\mathsf{KS}_{X/S/\mathsf{U}}$. On the other hand, since the two liftings come from pullback by projections, they are naturally isomorphic if we identify the two projections by a flipping map. The difference of the two canonical isomorphisms gives a map

$$\underline{H}^1_{\mathrm{dR}}(\tilde{X}_1/\tilde{S}) \to \mathscr{I} \cdot \underline{H}^1_{\mathrm{dR}}(\tilde{X}_2/\tilde{S}), \tag{2.1.7.5}$$

which is essentially determined by the map

$$\underline{H}^0(X, \Omega^1_{X/S}) \to \underline{H}^1(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}) \tag{2.1.7.6}$$

induced on the filtrations. This is nothing but the map defined by the cup-product with the Kodaira-Spencer class $\mathsf{KS}_{X/S/\mathsf{U}}$. When $\Omega^1_{S/\mathsf{U}}$ is locally free over $\mathscr{O}_S$ of finite rank (which is in particular the case when S is smooth over U), we can rewrite the map (2.1.7.6) as

$$\underline{H}^0(X, \Omega^1_{X/S}) \to \underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}. \tag{2.1.7.7}$$

(by the projection formula, see for example [64, Ch. III, Exer. 8.3]). Hence we can rewrite the map (2.1.7.5) as

$$\underline{H}^1_{\mathrm{dR}}(X/S) \to \underline{H}^1_{\mathrm{dR}}(X/S) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}},$$

which is the so-called *Gauss-Manin connection* of $\underline{H}^1_{\mathrm{dR}}(X/S)$.

Now let us assume that $\Omega^1_{S/\mathsf{U}}$ is locally free of finite rank over $\mathscr{O}_S$. When X is an abelian scheme over S, then we have canonical identifications

$$\underline{H}^0(X,\Omega^1_{X/S}) \cong \underline{H}^0(X,\mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{X/S} \cong \underline{\mathrm{Lie}}^\vee_{X/S} \cong (\underline{\mathrm{Lie}}_{X/S})^\vee,$$

and

$$\underline{H}^1(X,\mathscr{O}_X) \cong \underline{\mathrm{Lie}}_{X^\vee/S}$$

(given by Lemma 2.1.5.12 and Corollary 2.1.5.10). By the canonical identification $\underline{\mathrm{Lie}}^\vee_{X^\vee/S} \cong (\underline{\mathrm{Lie}}_{X^\vee/S})^\vee$ as well, we may reinterpret the map (2.1.7.7) as a map

$$\mathrm{KS}_{X/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{X/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{X^\vee/S} \to \Omega^1_{S/\mathsf{U}}.$$

Definition 2.1.7.8. *The map* $\mathrm{KS}_{X/S/\mathsf{U}}$ *above is called the* **Kodaira-Spencer map** *for the abelian scheme* X *over* S (*over the base scheme* U).

The map $\mathrm{KS}_{X/S/\mathsf{U}}$ and the class $\mathsf{KS}_{X/S/\mathsf{U}}$ determine each other by the various canonical identifications. Hence we can treat them equally when working under the assumption that $\Omega^1_{S/\mathsf{U}}$ is locally free over $\mathscr{O}_S$. The essential difference is that the class $\mathsf{KS}_{X/S/\mathsf{U}}$ makes sense without such an assumption.

2.2 Formal Theory

2.2.1 Local Moduli Functors and Schlessinger's Criterion

Let us make precise the meaning of the local moduli problems, or rather infinitesimal deformations, associated to $\mathsf{M}_{\mathcal{H}}$.

Let $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and let s be a point of finite type over S_0. Let k be a finite field extension of $k(s)$. Let $p = \mathrm{char}(k)$. Suppose $p > 0$, then $p \in \Box$ and $k(s)$ and k are necessarily finite fields of characteristic p. By assumption, p is ramified in F, and by Corollary 1.2.5.7, p is unramified in F_0. Therefore the completion $\hat{\mathscr{O}}_{\mathsf{S}_0,s}$ is simply the Witt vectors $W(k(s))$ (by Theorem B.1.1.14). According to Lemma B.1.1.16, there is a complete local ring Λ_k, which is given by $W(k)$ in this case, such that a complete noetherian local ring R with residue field k fits into the diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[dr] & \mathrm{Spec}(R) \arrow[d, dotted] \\
& \mathsf{S}_0
\end{tikzcd}$$

with solids maps given by the natural residue maps if and only if it is a Λ_k-algebra. Then we see that $\hat{\mathsf{C}}_{\Lambda_k}$ can be viewed as the infinitesimal neighborhoods of $\mathrm{Spec}(k) \to \mathsf{S}_0$, and C_{Λ_k} can be viewed as those in which $\mathrm{Spec}(k)$ is defined by nilpotent ideals. If $\mathrm{char}(k) = 0$, then k is a field extension of F_0, and the same is true if we replace $W(k)$ by k itself. For simplicity, let us abbreviate C_{Λ_k} by C and $\hat{\mathsf{C}}_{\Lambda_k}$ by $\hat{\mathsf{C}}$.

Let us denote by $\xi_0 : \mathrm{Spec}(k) \to \mathsf{M}_{\mathcal{H}}$ a point of $\mathsf{M}_{\mathcal{H}}$ corresponding to an object $\xi_0 = (A_0, \lambda_0, i_0, \alpha_{n,0})$ in $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(k))$. Let us denote by Def_{ξ_0} the functor from $\hat{\mathsf{C}}$ to (Sets) defined by the assignment

$$R \mapsto \{\text{isomorphism classes of pairs } (\xi, f_0)\ \},$$

where $\xi = (A, \lambda, i, \alpha_{\mathcal{H}})$ is an object in $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(R))$, and where

$$f_0 : \xi \underset{R}{\otimes} k := \mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(R) \to \mathrm{Spec}(k))(\xi) \xrightarrow{\sim} \xi_0$$

an isomorphism (in the sense of Definition 1.4.1.3) identifying $\xi \underset{R}{\otimes} k$ with ξ_0. (See Appendix B, especially Theorem B.3.10 for the reasoning behind this definition.)

Our first main objective in this chapter is to show that Def_{ξ_0} is effectively prorepresentable and formally smooth, and to show that Theorems B.3.8, B.3.10, and B.3.12 can be applied. (Note that, without the effectiveness, the prorepresentability is a condition on $\mathsf{Def}_{\xi_0}|_{\mathsf{C}}$ only.)

To achieve this, it is helpful to introduce some other functors having fewer conditions to control, and hence easier to understand than $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$:

1. Let us denote by Def_{A_0} the functor from $\hat{\mathsf{C}}$ to (Sets) defined by

   $$R \mapsto \{\text{isomorphism classes of pairs } (A, f_0) \text{ over } R\},$$

   where:

   (a) A is an abelian scheme over R.
   (b) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism.

2. Let us denote by $\mathsf{Def}_{(A_0,\lambda_0)}$ the functor from $\hat{\mathsf{C}}$ to (Sets) defined by

$$R \mapsto \{\text{isomorphism classes of triples } (A, \lambda, f_0) \text{ over } R\},$$

   where:

   (a) A is an abelian scheme over R.
   (b) $\lambda : A \to A^\vee$ is a polarization of A.
   (c) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism that pulls λ_0 back to $\lambda \underset{R}{\otimes} k$.

3. Let us denote by $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ the functor from $\hat{\mathsf{C}}$ to (Sets) defined by

$$R \mapsto \{\text{isomorphism classes of tuples } (A, \lambda, i, f_0) \text{ over } R\},$$

   where:

   (a) A is an abelian scheme over R.
   (b) $\lambda : A \to A^\vee$ is a polarization of A.
   (c) $i : \mathcal{O} \to \mathrm{End}_R(A)$ is an endomorphism structure.
   (d) $\underline{\mathrm{Lie}}_{A/\operatorname{Spec}(R)}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.2 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle)$.
   (e) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism that pulls λ_0 back to $\lambda \underset{R}{\otimes} k$ and pulls i_0 back to $i \underset{R}{\otimes} k$.

We will show the prorepresentability of formal smoothness one by one.

Let us record Schlessinger's general criterion of prorepresentability before we proceed. Following Schlessinger's fundamental paper [113]:

Definition 2.2.1.1. *A surjection* $p : \tilde{R} \twoheadrightarrow R$ *in* C *is a* **small surjection** *if the kernel of* p *is an ideal* I *such that* $I \cdot \mathfrak{m}_{\tilde{R}} = 0$*, where* $\mathfrak{m}_{\tilde{R}}$ *is the maximal ideal of* $\tilde{R}$*.*

Remark 2.2.1.2. Any surjection in C is the composition of a finite number of small surjections.

Note that when a functor $F : \mathsf{C} \to (\mathrm{Sets})$ is prorepresentable, then the following two conditions necessarily hold:

1. $F(k)$ has exactly one object. (Here k is understood as the final object of C.)

2. For any surjections $\tilde{R} \twoheadrightarrow R$ and $Q \twoheadrightarrow R$ in C, the functoriality of F gives an isomorphisms
$$F(Q \underset{R}{\times} \tilde{R}) \xrightarrow{\sim} F(Q) \underset{F(R)}{\times} F(\tilde{R}). \tag{2.2.1.3}$$

We may ask how much of the converse is true. The answer is provided by the following theorem of Schlessinger:

Theorem 2.2.1.4. *A covariant functor* $F : \mathsf{C} \to (\mathrm{Sets})$ *is prorepresentable if and only if* F *satisfies* (*2.2.1.3*) *for any surjections* $\tilde{R} \twoheadrightarrow R$ *and* $Q \twoheadrightarrow R$ *and*
$$\dim_k(F(k[\varepsilon]/(\varepsilon^2))) < \infty.$$
It suffices to check (*2.2.1.3*) *in the case that the surjection* $\tilde{R} \twoheadrightarrow R$ *is a small surjection* (*as in Definition 2.2.1.1*).

Moreover, suppose F *is prorepresentable by some* $R^{\mathrm{univ}} \in \hat{\mathsf{C}}$, *formally smooth,* (*namely* $F(\tilde{R}) \to F(R)$ *is surjective for any surjection* $\tilde{R} \twoheadrightarrow R$ *in* C), *and* $\dim_k(F(k[\varepsilon]/(\varepsilon^2))) = m$. *Then there is an isomorphism* $R^{\mathrm{univ}} \cong \Lambda[[t_1, \ldots, t_m]]$.

Proof. The first half is a weakened form of [113, Thm. 2.11]. The second half is just [113, Prop. 2.5(i)]. $\square$

Remark 2.2.1.5. For ease of notations, when $\tilde{S} = \mathrm{Spec}(\tilde{R})$ and $S = \mathrm{Spec}(R)$, we shall write $\mathsf{o}(X; \tilde{R} \twoheadrightarrow R)$ etc in place of the notations $\mathsf{o}(X; S \hookrightarrow \tilde{S})$ etc in Section 2.1. When $\tilde{R} \twoheadrightarrow R$ is a *small surjection* with kernel I (as in Definition 2.2.1.1), then $\tilde{M} \underset{R}{\otimes} I \cong (\tilde{M}/\mathfrak{m}_{\tilde{R}} \cdot \tilde{M}) \underset{k}{\otimes} I$ for any $\tilde{R}$-module $\tilde{M}$, because $I \cdot \mathfrak{m}_{\tilde{R}} = 0$. Hence we shall define $S_0 := \mathrm{Spec}(k)$, $X_0 := X \underset{R}{\otimes} k$, etc, and write $H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0}) \underset{k}{\otimes} I$ in place of $\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{R}{\otimes} I)$, as it is more precise. Note that we are allowed to write $H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0}) \underset{k}{\otimes} I$ instead of

$H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0} \underset{k}{\otimes} I)$ (by the projection formula, see for example [64, Ch. III, Exer. 8.3]) because I is always free of finite rank over the residue field $k = \tilde{R}/\mathfrak{m}_{\tilde{R}}$. We do not need the relative notations $\underline{H}$ in this case as the base schemes are now affine (with only one point).

2.2.2 Rigidity of Structures

Let us assume the notations in Section 2.2.1 and the conventions mentioned in Remark 2.2.1.5. Apart from Proposition 2.2.2.9 and consequently Corollary 2.2.2.10, where we do need more refined assumptions on k and hence on Λ (to make Proposition 1.1.2.16 work), the remaining results work for arbitrary choices of k and Λ as in the beginning of Section 2.2.1 (or in Section B.1).

Let us first show that the Def_{A_0} can be understood by the deformation of the underlying smooth scheme structures. The rigidity of abelian schemes has the following implication:

Lemma 2.2.2.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ and $\tilde{A}'$ be two abelian schemes over $\tilde{S}$, and let $A := \tilde{A} \underset{\tilde{S}}{\times} S$ and $A' := \tilde{A}' \underset{\tilde{S}}{\times} S$. Then the natural map from the group homomorphisms $\underline{\mathrm{Hom}}_{\tilde{S}}(\tilde{A}, \tilde{A}')$ to $\underline{\mathrm{Hom}}_S(A, A')$ is* **injective**. *Similarly, the natural map from the group isomorphism $\underline{\mathrm{Isom}}_{\tilde{S}}(\tilde{A}, \tilde{A}')$ to $\underline{\mathrm{Isom}}_S(A, A')$ is* **injective**.

Proof. Suppose f and g are two group homomorphisms from $\tilde{A}$ to $\tilde{A}'$ such that $f \underset{\tilde{S}}{\times} S = g \underset{\tilde{S}}{\times} S$. Then by Corollary 1.3.1.5, there exists a section $\eta : \tilde{S} \to \tilde{A}'$ such that $g = f + \eta$. Since f and g are group homomorphisms, both of them send the identify section $e_{\tilde{A}}$ of $\tilde{A}$ to the identity section $e_{\tilde{A}'}$ of $\tilde{A}'$. This forces η must be the identity section $e_{\tilde{A}'}$, and hence $f = g$. The argument for group isomorphisms is identical. □

In particular, there are no *infinitesimal automorphisms* for A as an *abelian scheme*. Note that we might have infinitesimal automorphisms if we only consider A as a *smooth scheme*. For example, when $R = k[\varepsilon]/(\varepsilon^2)$, we know they are parameterized by $\Gamma(A_0, \underline{\mathrm{Der}}_{A_0/S_0})$, the tangent space of $A_0 := A \underset{R}{\otimes} k$ over $S_0 := \mathrm{Spec}(k)$. The essential extra freedom is controlled by the identity sections:

Corollary 2.2.2.2. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Suppose $\tilde{A}$ and $\tilde{A}'$ are abelian schemes over $\tilde{S}$, $A :=$*

$\tilde{A} \underset{\tilde{S}}{\times} S$, $A' := \tilde{A}' \underset{\tilde{S}}{\times} S$, and $f : A \to A'$ is a group homomorphism that is lifted to some morphism $\tilde{f} : \tilde{A} \to \tilde{A}'$ (not necessarily a group homomorphism) in the sense that $\tilde{f} \underset{\tilde{S}}{\times} S = f$. Then, by replacing $\tilde{f}$ by $\tilde{f} - \tilde{f}(e_{\tilde{A}})$, we obtain the unique group homomorphism lifting f.

Proof. The replacement works because of Corollary 1.3.1.6. The claim of uniqueness is just Lemma 2.2.2.1. ☐

Now we can state the following important fact:

Proposition 2.2.2.3 ([104, Prop. 2.27])**.** *Suppose $p : \tilde{R} \twoheadrightarrow R$ be a small surjection in $\hat{\mathsf{C}}$ with kernel I, and suppose (A, f_0) represents an object of $\mathsf{Def}_{A_0}(R)$. Let*

$$\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$$

be the isomorphism classes of objects $(\tilde{A}, \tilde{f}_0)$ in $\mathsf{Def}_{A_0}(\tilde{R})$ that are mapped to the class of (A, f_0). Then forgetting the structure of abelian schemes induces a well-defined bijection

$$\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)]) \to \mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R). \tag{2.2.2.4}$$

Proof. Let us first show that this map is well-defined. If $\mathsf{Def}_{A_0}(p)$ maps the class of $(\tilde{A}, \tilde{f}_0)$ to (A, f_0), then we have some group isomorphism $\psi : \tilde{A} \underset{\tilde{R}}{\otimes} R \xrightarrow{\sim} A$ reducing to $f_0^{-1} \circ \tilde{f}_0 : \tilde{A} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A \underset{R}{\otimes} k$. Suppose $(\tilde{A}', \tilde{f}_0')$ is also a representative of some object in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$. Let $(\tilde{A}', \psi')$ be associated to $(\tilde{A}', \tilde{f}_0')$ by the above recipe. If there is an isomorphism $h : \tilde{A} \xrightarrow{\sim} \tilde{A}'$ such that $h \underset{\tilde{R}}{\otimes} k = (\tilde{f}_0')^{-1} \circ \tilde{f}_0$ defines an isomorphism $(\tilde{A}, \tilde{f}_0) \xrightarrow{\sim} (\tilde{A}', \tilde{f}_0')$, then both ψ and $\psi' \circ (h \underset{\tilde{R}}{\otimes} R)$ are isomorphisms from $\tilde{A} \underset{\tilde{R}}{\otimes} R$ to A reducing to $\tilde{f}_0 : \tilde{A} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A_0$. Therefore, by Lemma 2.2.2.1, we have $\psi = \psi' \circ (h \underset{\tilde{R}}{\otimes} R)$. This shows that the isomorphism h defines an isomorphism $(\tilde{A}, \psi) \xrightarrow{\sim} (\tilde{A}', \psi')$, and that the map (2.2.2.4) sending the isomorphism class of $(\tilde{A}, f_0)$ in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$ to the isomorphism of $(\tilde{A}, \psi)$ in $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ is well-defined and independent of the choice of the ψ we have made.

Now suppose that we have two pairs $(\tilde{A}, \tilde{f}_0)$ and $(\tilde{A}', \tilde{f}_0')$ defining classes in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$. Let $(\tilde{A}, \psi)$ and $(\tilde{A}', \psi')$ be any two choices of pairs associated respective to $(\tilde{A}, \tilde{f}_0)$ and $(\tilde{A}', \tilde{f}_0')$ as above. Note that ψ and ψ'

are chosen to be group isomorphisms. Suppose that there is an isomorphism $h : \tilde{A} \to \tilde{A}'$ of the underlying schemes that induces an isomorphism $(\tilde{A}, \psi) \xrightarrow{\sim} (\tilde{A}', \psi')$. This means that $\psi = \psi' \circ (h \underset{\tilde{R}}{\otimes} R)$, and implies that $h \underset{\tilde{R}}{\otimes} R$ is a group isomorphism, as both ψ and ψ' are. Then, by Corollaries 2.2.2.2, we may assume that h is a group isomorphism by replacing h by $h - h(e_{\tilde{A}})$. Since $\psi \underset{R}{\otimes} k = f_0^{-1} \circ \tilde{f}_0$ and $\psi' \underset{R}{\otimes} k = f_0^{-1} \circ \tilde{f}'_0$ by construction, we have $h \underset{\tilde{R}}{\otimes} k = (\psi' \underset{R}{\otimes} k)^{-1} \circ (\psi \underset{R}{\otimes} k) = (\tilde{f}'_0)^{-1} \circ \tilde{f}_0$, and hence h defines an isomorphism $(\tilde{A}, \tilde{f}_0) \xrightarrow{\sim} (\tilde{A}', \tilde{f}'_0)$. This shows the injectivity of (2.2.2.4).

Finally, the surjectivity of (2.2.2.4) follows from Proposition 2.2.2.5 below. (The existence of the identity section needed is automatic by smoothness.) □

Proposition 2.2.2.5 (cf. [97, Prop. 6.15]). *Let $p : \tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I. Let $\tilde{S} := \mathrm{Spec}(\tilde{R})$, and $S := \mathrm{Spec}(R)$. Let $\pi : \tilde{A} \to \tilde{S}$ be a proper smooth morphism with a section $e : \tilde{S} \to \tilde{A}$. Suppose $\tilde{A} \underset{\tilde{S}}{\times} S \to S$ is an abelian scheme with identity section $e \underset{\tilde{S}}{\times} S$. Then $\tilde{A} \to \tilde{S}$ is an abelian scheme with identity section e.*

Proof. Let $g : A \underset{S}{\times} A \to A$ be the morphism $g(x, y) = x - y$ defined for any functorial points x and y of A. To show that $\tilde{A}$ is an abelian scheme, our first task is to lift g to some morphism $\tilde{g} : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$. Let $S_0 := \mathrm{Spec}(k)$, $A_0 := \tilde{A} \underset{\tilde{S}}{\times} S_0$, and $g_0 := \tilde{g} \underset{\tilde{S}}{\times} S_0$. By Proposition 2.1.3.2, there is an element

$$\mathsf{o}_0 := \mathsf{o}(g; \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$$

whose vanishing is equivalent to the existence of a morphism $\tilde{g}$ lifting g. Let $j_1, j_2 : \tilde{A} \to \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}$ be the maps defined respectively by $j_1(x) = (x, e)$ and $j_2(x) = (x, x)$ for any functorial point x of $\tilde{A}$, and let $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$ be the two projection maps. By abuse of notations, we shall use the same symbols for their pullbacks to S and S_0. Then, by repeating the arguments in the proof of Proposition 2.1.3.2 if necessary, the obstructions

$$\mathsf{o}_i := \mathsf{o}(g \circ j_i; \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^1(A_0, j_i^* g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$$

to lifting $g \circ j_i$, $i = 1, 2$, are related to the obstruction of lifting g by

$$\mathsf{o}_i = j_i^*(\mathsf{o}_0)$$

under the natural maps

$$j_i^* : H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I \to H^1(A_0, j_i^* g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I.$$

Since $\tilde{g} \circ j_1 = \mathrm{Id}_{\tilde{A}}$ and $\tilde{g} \circ j_2 = e_{\tilde{A}} \circ \pi$ do lift the maps $g \circ j_1$ and $g \circ j_2$, we must have

$$\mathsf{o}_i = 0$$

for $i = 1, 2$. On the other hand, as A_0 is an abelian variety, the structure of $H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0}))$ can be completely understood by the Künneth formula. Moreover, by Lemma 2.1.5.12, $\underline{\mathrm{Der}}_{A_0/S_0}$ is a (relatively) constant sheaf, which we may identify as a k-vector space. Explicitly,

$$\begin{aligned}
&H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I \\
&\cong H^1(A_0 \underset{S_0}{\times} A_0, \mathscr{O}_{A_0 \underset{S_0}{\times} A_0}) \underset{k}{\otimes} \underline{\mathrm{Lie}}_{A_0/S_0} \underset{k}{\otimes} I \\
&\cong [\mathrm{pr}_1^* \, H^1(A_0, \mathscr{O}_{A_0}) \oplus \mathrm{pr}_2^* \, \underline{H}^1(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} \underline{\mathrm{Lie}}_{A_0/S_0} \underset{k}{\otimes} I.
\end{aligned}$$

As a result, every element in $H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$ can be described as the pullback from one of the two factors $\mathrm{pr}_i^* \, H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$, $i = 1, 2$. As $j_1^*(\mathsf{o}_0) = \mathsf{o}_1 = 0)$ and $\mathrm{pr}_1 \circ j_1 = \mathrm{Id}_{\tilde{A}}$, we see that the first factor of o_0 is trivial. On the other hand, as $j_2^*(\mathsf{o}_0) = \mathsf{o}_2 = 0$ and $\mathrm{pr}_2 \circ j_2 = \mathrm{Id}_{\tilde{A}}$, we see that the second factor of o_0 is trivial as well. Hence we must have $\mathsf{o}_0 = 0$, and the existence of some morphism $\tilde{g} : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$ lifting g.

Note that this morphism is not unique. By Proposition 2.1.3.2, $\mathsf{Lift}(g; \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R)$ is a torsor under the group

$$H^0(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I,$$

which is canonically isomorphic to

$$\underline{\mathrm{Lie}}_{A_0/S_0} \underset{k}{\otimes} I$$

by the constancy of $\underline{\mathrm{Der}}_{A_0/S_0}$ (by Lemma 2.1.5.12). Similarly, the restrictions of the liftings $\tilde{g}$ to $(e_{\tilde{A}}, e_{\tilde{A}}) : \tilde{S} \to \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}$ form a torsor under the group

$$H^0(S_0, (g_0|_{(e_{A_0}, e_{A_0})})^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I,$$

which is also canonically isomorphic to $\underline{\mathrm{Lie}}_{A_0/S_0} \underset{k}{\otimes} I$. As the restriction to $(e_{\tilde{A}}, e_{\tilde{A}})$ defines a natural map between the two torsors equivariant under the same group, we see that there exists a unique lifting $\tilde{g}$ of g that sends $(e_{\tilde{A}}, e_{\tilde{A}})$ to $e_{\tilde{A}}$.

It remains to prove that $\tilde{g}$ determines a group structure of $\tilde{A}$. The existence of $\tilde{g}$ gives formal definitions of the inverse and multiplication maps, and it only remains to check the various compatibility relations given by morphisms of the form

$$h : \tilde{A} \underset{\tilde{S}}{\times} \ldots \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A},$$

which sends $(e_{\tilde{A}}, \ldots, e_{\tilde{A}})$ to $e_{\tilde{A}}$, and sends everything to e_A over S. Certainly, the condition to check is that h sends everything to $e_{\tilde{A}}$ over $\tilde{S}$. By Proposition 1.3.1.4, there is necessarily a section $\eta : \tilde{S} \to \tilde{A}$ such that h is the composition of the structural projection

$$\tilde{A} \underset{\tilde{S}}{\times} \ldots \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{S}$$

with η. Since h sends $(e_{\tilde{A}}, \ldots, e_{\tilde{A}})$ to $e_{\tilde{A}}$, this η must be the identity section $e_{\tilde{A}}$. □

Let us state similar rigidity results for some other structures as corollaries of Lemma 2.2.2.1:

Corollary 2.2.2.6. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ be an abelian scheme over $\tilde{S}$, $A := \tilde{A} \underset{\tilde{S}}{\times} S$, and $\lambda : A \to A^\vee$ a* **polarization** *of A (defined as in Definition 1.3.2.20). Suppose $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ is any group homomorphism such that $\tilde{\lambda} \underset{\tilde{S}}{\times} S = \lambda$. Then $\tilde{\lambda}$ is necessarily a polarization of $\tilde{A}$.*

Proof. Note that both $\tilde{\lambda}$ and $\tilde{\lambda}^\vee$ lift $\lambda = \lambda^\vee$, where the symmetry follows because λ is a polarization. Hence $\tilde{\lambda} = \tilde{\lambda}^\vee$ by Lemma 2.2.2.1. Now that $\tilde{\lambda}$ is symmetric, by Proposition 1.3.2.18, it suffices to know that $\tilde{\lambda}$ is a polarization over any geometric point of $\tilde{S}$, which is true because λ is a polarization over any geometric point of S. □

Corollary 2.2.2.7. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ be an abelian scheme over $\tilde{S}$ and $A := \tilde{A} \underset{\tilde{S}}{\times} S$. With the setting as in Section 1.3.3, there exists at most one unique way to lift an $\mathcal{O}$-endomorphism structure from A to $\tilde{A}$, if any of them exist. Moreover, to check the existence of liftings, we do not have to check the Rosati condition.*

Proof. It is clear from Lemma 2.2.2.1 that liftings of morphisms are uniquely, as long as they exist. Note that the Rosati condition is defined by relations of group homomorphisms that are already verified over S. Hence it is automatic by Corollary 1.3.1.5. ☐

Corollary 2.2.2.8. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ be an abelian scheme over $\tilde{S}$ and $A := \tilde{A} \underset{\tilde{S}}{\times} S$. With the setting as in Section 1.3.7, there exists at most one unique way to lift a level structure from A to $\tilde{A}$, as long as they are defined.*

Proof. By definition, level structures are orbits of étale-locally-defined morphisms between étale group schemes that are successively liftable to symplectic isomorphisms of higher levels after étale base extensions. The necessary extra structures of polarizations and endomorphisms can be liftable in at most one unique way, as we have seen in Corollaries 2.2.2.6 and 2.2.2.7. Hence there is at most one way to make sense of the definition of level structures, and the result follows because the underlying maps are uniquely liftable by Lemma 2.1.1.6. ☐

The Lie algebra condition requires some different argument. Let L_0 be the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-module defined in Lemma 1.2.5.10, which satisfies $\mathrm{Det}_{\mathcal{O}|L_0} = \mathrm{Det}_{\mathcal{O}|V_0}$ by Corollary 1.2.5.12. Let k be either characteristic zero or a finite field. Set $\Lambda = k$ when $\mathrm{char}(k) = 0$, and $\Lambda = W(k)$ when $\mathrm{char}(k) = p > 0$. Suppose we have a map $\mathcal{O}_{F_0,(\Box)} \to k$. Note that the above assumptions on k are true for any field of finite type $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ by our assumption on $\Box$ when defining $\mathsf{M}_{\mathcal{H}}$. (This is exactly the same as the setting preceding Lemma 1.2.5.13.)

Proposition 2.2.2.9. *With assumptions as above, let $\tilde{R} \twoheadrightarrow R$ be a surjection in $\hat{\mathsf{C}}$, the category of complete noetherian local Λ-algebras with residue field k. Let $\tilde{S} := \mathrm{Spec}(\tilde{R})$ and $S := \mathrm{Spec}(R)$. Suppose that $\tilde{A} \to \tilde{S}$ is an abelian scheme, that $A := \tilde{A} \underset{\tilde{S}}{\times} S$, and that both of them admit compatibly the* *necessary polarizations and endomorphism structures so that the Lie algebra condition in Definition 1.3.4.2 can be defined. Then* $\underline{\mathrm{Lie}}_{\tilde{A}/\tilde{S}}$ *satisfies the Lie algebra condition if and only if* $\underline{\mathrm{Lie}}_{A/S}$ *does.*

Proof. It is clear that it suffices to treat the case when $R = k$. Since $\underline{\mathrm{Lie}}_{\tilde{A}/\tilde{S}}$ is locally free and $\tilde{R}$ is local, we see that $\mathrm{Lie}_{\tilde{A}/\tilde{S}}$ is a free $\tilde{R}$-module. Let $A_0 := \tilde{A} \underset{\tilde{S}}{\otimes} S_0$, so that $(\mathrm{Lie}_{\tilde{A}/\tilde{S}}) \underset{\tilde{R}}{\otimes} k \cong \mathrm{Lie}_{A_0/S_0}$. By Lemma 1.2.5.13, we see that $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{\tilde{A}/\tilde{S}}} = \mathrm{Det}_{\mathcal{O}|V_0}$ if and only if $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A_0/S_0}} = \mathrm{Det}_{\mathcal{O}|V_0}$, as desired. ☐

Combining the results above, we obtain:

Corollary 2.2.2.10. *The series of forgetful functors*

$$\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})} \to \mathsf{Def}_{(A_0,\lambda_0,i_0)} \to \mathsf{Def}_{(A_0,\lambda_0)} \to \mathsf{Def}_{A_0}$$

induce the series of equivalence or embeddings

$$\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})} \cong \mathsf{Def}_{(A_0,\lambda_0,i_0)} \hookrightarrow \mathsf{Def}_{(A_0,\lambda_0)} \hookrightarrow \mathsf{Def}_{A_0}$$

of each category as a full subcategory of the next one. Moreover, we may ignore the Rosati condition and the Lie algebra condition when studying $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$.

2.2.3 Prorepresentability

Consider Cartesian diagrams

$$\begin{tikzcd}
\tilde{Q} \arrow[r, two heads, "\tilde{\pi}"] \arrow[d, two heads, "q"'] & \tilde{R} \arrow[d, two heads, "r"] \\
Q \arrow[r, two heads, "\pi"'] & R
\end{tikzcd} \tag{2.2.3.1}$$

of surjections such that r and q are small surjections with kernels respectively I and J. In this case $\tilde{\pi} : \tilde{Q} \twoheadrightarrow \tilde{R}$ induces an isomorphism $J \xrightarrow{\sim} I$, which we again denote by $\tilde{\pi}$. If we define $\tilde{S} := \mathrm{Spec}(\tilde{R})$, $S := \mathrm{Spec}(R)$, $\tilde{T} := \mathrm{Spec}(\tilde{Q})$, and $T := \mathrm{Spec}(Q)$, then we arrive at the setting of Section 2.1.4, together with the isomorphism $\tilde{\pi} : J \xrightarrow{\sim} I$ that makes the results there more powerful (cf. Remark 2.1.4.5).

According to Schlessinger's criterion (given by Theorem 2.2.1.4), a covariant functor $F : \mathsf{C} \to (\mathrm{Sets})$ is prorepresentable if and only if the following two conditions are satisfied:

1. The natural map
$$F(\tilde{Q}) \to F(Q) \underset{F(R)}{\times} F(\tilde{R}) \tag{2.2.3.2}$$
is a bijection for any Cartesian diagram as above.

2. Let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers over k, then
$$\dim_k(F(k[\varepsilon]/(\varepsilon^2))) < \infty. \tag{2.2.3.3}$$

We shall check these conditions one by one for Def_{A_0}, $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, and $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$.

Proposition 2.2.3.4. *The functor* Def_{A_0} *is prorepresentable.*

Proof. Let us first check the bijectivity of the map
$$\mathsf{Def}_{A_0}(\tilde{Q}) \to \mathsf{Def}_{A_0}(Q) \underset{\mathsf{Def}_{A_0}(R)}{\times} \mathsf{Def}_{A_0}(\tilde{R}) \tag{2.2.3.5}$$
for any diagram like (2.2.3.1). Suppose
$$\mathsf{Def}_{A_0}(Q) \underset{\mathsf{Def}_{A_0}(R)}{\times} \mathsf{Def}_{A_0}(\tilde{R}),$$
is empty, then clearly (2.2.3.5) is bijective. Therefore we may assume that it is nonempty. Let $((A_Q, f_{0,Q}), (A_{\tilde{R}}, f_{0,\tilde{R}}))$ represent any object in it. Then we have
$$\mathsf{Def}_{A_0}(\pi)([(A_Q, f_{0,Q})]) = \mathsf{Def}_{A_0}(r)([(A_{\tilde{R}}, f_{0,\tilde{R}})]) = [(A_R, f_{0,R})]$$
for some object in $\mathsf{Def}_{A_0}(R)$ represented by some $[(A_R, f_{0,R})]$. By Proposition 2.2.2.3 and Corollary 2.1.4.3, we have a commutative diagram of isomorphisms (with the dotted arrow induced by the other arrows)
$$\begin{tikzcd}
\mathsf{Def}_{A_0}(r)^{-1}([(A_R, f_{0,R})]) \arrow[r, "\sim"] & \mathsf{Lift}(A_R; \tilde{R} \twoheadrightarrow R) \\
\mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})]) \arrow[r, "\sim"'] \arrow[u, dotted, "\wr"] & \mathsf{Lift}(A_Q; \tilde{Q} \twoheadrightarrow Q) \arrow[u, "\wr", "\underset{\tilde{Q}}{\otimes} \tilde{R}"']
\end{tikzcd}, \tag{2.2.3.6}$$
which shows that there must be an isomorphism class
$$[(A_{\tilde{Q}}, f_{0,\tilde{Q}})] \in \mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})])$$

corresponding to the given

$$[(A_{\tilde{R}}, f_{0,\tilde{R}})] \in \mathsf{Def}_{A_0}(q)^{-1}([(A_R, f_{0,R})]).$$

Moreover, for any $(A_{\tilde{Q}}, f_{0,\tilde{Q}})$ representing the class, there is isomorphism

$$A_{\tilde{Q}} \underset{\tilde{Q}}{\otimes} \tilde{R} \cong A_{\tilde{R}}$$

between underlying schemes, as this is how the solid vertical arrow in (2.2.3.6) is defined. By Corollary 2.2.2.2, the existence of such an isomorphism implies the unique existence of an isomorphism of abelian schemes lifting the isomorphism $f_{0,\tilde{R}} \circ f_{0,\tilde{Q}}^{-1} : A_{\tilde{Q}} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A_{\tilde{R}} \underset{\tilde{R}}{\otimes} k$ between the special fibers. Hence we also have

$$\mathsf{Def}_{A_0}(\tilde{\pi})([(A_{\tilde{Q}}, f_{0,\tilde{Q}})]) = [(A_{\tilde{R}}, f_{0,\tilde{R}})],$$

which shows the surjectivity of (2.2.3.5). Note that this argument shows that the dotted arrow in (2.2.3.6) can be identified with $\mathsf{Def}_{A_0}(\tilde{\pi})$. As a result, no two elements in $\mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})])$ can be mapped to the same element in $\mathsf{Def}_{A_0}(r)^{-1}([(A_{\tilde{R}}, f_{0,\tilde{R}})])$ by $\mathsf{Def}_{A_0}(\tilde{\pi})$, which shows the injectivity of (2.2.3.6) as well.

Now let us compute the dimension of $\mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2))$. Let $S_0 := \mathrm{Spec}(k)$ and let $t : k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ denote the canonical surjection. Since $\mathsf{Def}_{A_0}(k)$ has only one object $[(A_0, \mathrm{Id}_{A_0})]$, and since t has a section forcing $\mathsf{o}(A_0; t) = 0$, we have

$$\mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) = \mathsf{Def}_{A_0}(t)^{-1}([(A_0, \mathrm{Id}_{A_0})]) \cong \mathsf{Lift}(A_0; t) \cong H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0})$$

by Propositions 2.2.2.3 and 2.1.2.2. Since A_0 is an abelian scheme over S_0, by Lemma 2.1.5.12, $\underline{\mathrm{Der}}_{A_0/S_0}$ is a constant sheaf and is the pullback of the tangent space Lie_{A_0/S_0} of A_0 considered as a sheaf $\underline{\mathrm{Lie}}_{A_0/S_0}$ on the base scheme $S_0 = \mathrm{Spec}(k)$. By this fact and Corollary 2.1.5.10, we see that

$$H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \cong H^1(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0},$$

where $\mathrm{Lie}_{A_0^\vee/S_0} = \mathrm{Lie}_{\mathrm{Pic}(A_0)/S_0}$. The dimensions of both Lie_{A_0/S_0} and $\mathrm{Lie}_{A_0^\vee/S_0}$ are finite and equal to the dimension of A_0 over k. Hence

$$\dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) = (\dim_k A_0)^2 < \infty$$

as desired. □

Proposition 2.2.3.7. *The functor* $\mathsf{Def}_{(A_0,\lambda_0)}$ *is prorepresentable.*

Proof. By Corollary 2.2.2.10, $\mathsf{Def}_{(A_0,\lambda_0)}$ is a subfunctor of Def_{A_0}. Hence the map

$$\mathsf{Def}_{(A_0,\lambda_0)}(\tilde{Q}) \to \mathsf{Def}_{(A_0,\lambda_0)}(Q) \underset{\mathsf{Def}_{(A_0,\lambda_0)}(R)}{\times} \mathsf{Def}_{(A_0,\lambda_0)}(\tilde{R}) \tag{2.2.3.8}$$

is always injective because (2.2.3.5) is so. It suffices to show that it is also surjective. Suppose $((A_Q, \lambda_Q, f_{0,Q}), (A_{\tilde{R}}, \lambda_{\tilde{R}}, f_{0,\tilde{R}}))$ represent any object in the right-hand side of (2.2.3.8). Then we have

$$\mathsf{Def}_{(A_0,\lambda_0)}(\pi)([(A_Q, \lambda_Q, f_{0,Q})]) = \mathsf{Def}_{A_0}(p)([(A_{\tilde{R}}, \lambda_{\tilde{R}}, f_{0,\tilde{R}})]) = [(A_R, \lambda_R, f_{0,R})]$$

for some object in $\mathsf{Def}_{(A_0,\lambda_0)}(R)$ represented by some $[(A_R, \lambda_R, f_{0,R})]$. In particular, because of the existence of the polarization $\lambda_{\tilde{R}} : A_{\tilde{R}} \to A^\vee_{\tilde{R}}$ as a map between schemes, $\mathsf{Lift}(\lambda_R; A_{\tilde{R}}, A^\vee_{\tilde{R}}, \tilde{R} \twoheadrightarrow R)$ is nonempty, and hence

$$\mathsf{o}(\lambda_Q; A_{\tilde{Q}}, A^\vee_{\tilde{Q}}, \tilde{Q} \twoheadrightarrow Q) = \mathsf{o}(\lambda_R; A_{\tilde{R}}, A^\vee_{\tilde{R}}, \tilde{R} \twoheadrightarrow R) = 0$$

by Corollary 2.1.4.4. Therefore there is some morphism $\lambda_{\tilde{Q},0} : A_{\tilde{Q}} \to A^\vee_{\tilde{Q}}$ lifting $\lambda_Q : A_Q \to A^\vee_Q$, which might not be a group homomorphism. But this is sufficient as Corollaries 2.2.2.2 and 2.2.2.6 then imply the unique existence of a polarization $\lambda_{\tilde{Q}} : A_{\tilde{Q}} \to A^\vee_{\tilde{Q}}$ lifting λ_Q.

Finally, note that

$$\dim_k \mathsf{Def}_{(A_0,\lambda_0)}(k[\varepsilon]/(\varepsilon^2)) \leq \dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) < \infty$$

again because $\mathsf{Def}_{(A_0,\lambda_0)}$ is a subfunctor of Def_{A_0}. □

Proposition 2.2.3.9. *The functor* $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ *is prorepresentable.*

Proof. By Corollary 2.2.2.10, $\mathsf{Def}_{(A_0,\lambda_0)}$ is a subfunctor of $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$. Moreover, we may ignore the Rosati condition and the Lie algebra condition when studying $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$. Therefore the prorepresentability is just a question about lifting morphisms of schemes, which can be shown by exactly the same argument as in the proof of Proposition 2.2.3.7. □

Finally:

Theorem 2.2.3.10. *The functor* $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ *is prorepresentable.*

Proof. Simply combine Corollary 2.2.2.10 and Proposition 2.2.3.9. □

2.2.4 Formal Smoothness

Proposition 2.2.4.1. *The functor* Def_{A_0} *is formally smooth.*

Proof. Let $S_0 := \mathrm{Spec}(k)$. By Proposition 2.2.2.3, this will follow if we can show that, for any small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and any (A, f_0) defining an object of $\mathsf{Def}_{A_0}(\tilde{R})$, the obstruction

$$\mathsf{o} := \mathsf{o}(A; \tilde{R} \twoheadrightarrow R) \in H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$$

to $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ vanishes. Let $\tilde{S} := \mathrm{Spec}(\tilde{R})$ and $S := \mathrm{Spec}(R)$ as usual. Let us also look at the obstruction

$$\mathsf{o}_2 := \mathsf{o}(A \underset{S}{\times} A; \tilde{R} \twoheadrightarrow R) \in H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I$$

to $\mathsf{Lift}(A \underset{S}{\times} A; \tilde{R} \twoheadrightarrow R)$. Note that by the proof of Proposition 2.1.2.2, the cohomology class o can be calculated by forming an affine open covering $\{U_\alpha\}$ of A over S such that each U_α is lifted to a smooth scheme $\tilde{U}_\alpha$ over $\tilde{S}$, and by forming $c = \{c_{\alpha\beta\gamma}\}$ with

$$c_{\alpha\beta\gamma} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S) \cong \Gamma((U_{\alpha\beta\gamma})_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$$

defining the class $\mathsf{o} = [c]$ in $H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$. Then we have an affine open covering $\{U_\alpha \underset{S}{\times} U_{\alpha'}\}$ of $A \underset{S}{\times} A$ enjoying the same smooth lifting properties, and we can calculate o_2 by forming the class of $c_2 = \{c_{\alpha\beta\gamma} \underset{S}{\times} c_{\alpha'\beta'\gamma'}\}$ in its natural sense.

Let $j_1, j_2 : A_0 \to A_0 \underset{S_0}{\times} A_0$ be the maps defined respectively by $j_1(x) = (x, e)$ and $j_2(x) = (x, x)$ for any functorial point x of A_0, and let $\mathrm{pr}_1, \mathrm{pr}_2 : A_0 \underset{S_0}{\times} A_0 \to A_0$ be the two projection maps. As in the proof of Proposition 2.2.2.5, by the Künneth formula and the fact that $\underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}$ is a constant sheaf, we know that there is a canonical isomorphism

$$\begin{aligned}
&H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I \\
&\cong H^2(A_0 \underset{S_0}{\times} A_0, \mathscr{O}_{A_0 \underset{S_0}{\times} A_0}) \underset{k}{\otimes} \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0} \underset{k}{\otimes} I \\
&\cong [\mathrm{pr}_1^* H^2(A_0, \mathscr{O}_{A_0}) \oplus (\mathrm{pr}_1^* H^1(A_0, \mathscr{O}_{A_0}) \otimes \mathrm{pr}_2^* H^1(A_0, \mathscr{O}_{A_0})) \\
&\qquad \oplus \mathrm{pr}_2^* H^2(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0} \underset{k}{\otimes} I,
\end{aligned}$$

which decomposes $H^2(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$ into three factors. By decomposing

$$\mathrm{Lie}_{A_0 \underset{S_0}{\times} A_0/S_0} \cong \mathrm{Lie}_{A_0/S_0} \oplus \mathrm{Lie}_{A_0/S_0}$$

into two factors as well (using j_1 and j_2), we obtain a projection

$$\begin{aligned} &H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I \\ &\to \mathrm{pr}_1^*[H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0}] \oplus \mathrm{pr}_2^*[H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0}] \underset{k}{\otimes} I \\ &\to \mathrm{pr}_1^* H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0})] \oplus \mathrm{pr}_2^* H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I, \end{aligned} \tag{2.2.4.2}$$

which is determined by the two pullbacks under j_1^* and j_2^*.

From the above explicit construction, it is clear that $\mathsf{o}_2 = \mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$. On the other hand, by Proposition 2.1.2.2, o_2 is preserved under the automorphism of $A \underset{S}{\times} A$ defined by $\alpha : (x, y) \mapsto (x+y, y)$ for any functorial points x and y of A. Therefore we also have

$$\mathsf{o}_2 = (\mathrm{pr}_1 \circ \alpha)^*(\mathsf{o}) + (\mathrm{pr}_2 \circ \alpha)^*(\mathsf{o}) = m^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o}),$$

where $m : A_0 \underset{S_0}{\times} A_0 \to A_0$ is the multiplication map. Since $m \circ j_1 = m \circ j_2 = \mathrm{Id}_X$, we see that $m^*(\mathsf{o})$ and $\mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$ has the same projection under (2.2.4.2). As a result, o_2 and $\mathrm{pr}_1^*(\mathsf{o}) + 2\,\mathrm{pr}_2^*(\mathsf{o})$ has the same projection under (2.2.4.2). Since $\mathsf{o}_2 = \mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$, this implies $\mathrm{pr}_2^*(\mathsf{o}) = 0$ and hence $\mathsf{o} = 0$ as pr_2^* is injective. ☐

Corollary 2.2.4.3. *The functor* Def_{A_0} *is (noncanonically) prorepresented by the smooth algebra* $\Lambda[[x_1, \ldots, x_{g^2}]]$ *over* Λ*, where* $g = \dim_k A_0$*.*

Proof. By Propositions 2.2.3.4, 2.2.4.1, and Theorem 2.2.1.4, it suffices to show that

$$\dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) = g^2.$$

But this is already seen in the proof of Proposition 2.2.3.4. ☐

Proposition 2.2.4.4. *The functor* $\mathsf{Def}_{(A_0,\lambda_0)}$ *is formally smooth.*

Proof. For any small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and any (A, λ, f_0) defining an object of $\mathsf{Def}_{(A_0,\lambda_0)}(\tilde{R})$, we know from Proposition 2.2.4.1 that there always exists some abelian scheme $(\tilde{A}, \tilde{f}_0)$ lifting (A, f_0). By Proposition 1.3.2.18, we know that, by an étale base change if necessary, we may suppose that $\lambda = \lambda_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$, where $\lambda_{\mathcal{L}}$ is associated to $\mathcal{L}$ by Construction 1.3.2.10. Then the question becomes whether we can lift $\mathcal{L}$ to some invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{A}$. Or, if not, the question becomes whether we can replace $\tilde{A}$ by a different lifting $\tilde{A}'$ of A so that $\mathcal{L}$ can lifted to an invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{A}'$.

Let $\mathcal{L}_0 := \mathcal{L} \underset{\tilde{R}}{\otimes} k$, so that $\lambda_0 = \lambda_{\mathcal{L}_0}$. By Proposition 2.1.5.4, we know that there is an element

$$\mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I,$$

such that we can lift $\mathcal{L}$ to some $\tilde{\mathcal{L}}$ if and only if $\mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) = 0$. Let $S_0 := \operatorname{Spec}(k)$. If we replace $\tilde{A}$ by $\mathsf{m} + \tilde{A}$ for some $\mathsf{m} \in H^1(A_0, \underline{\operatorname{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$, then there is a relation

$$\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, \tilde{R} \twoheadrightarrow R) = \mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) + \mathsf{d}_{\mathcal{L}_0}(\mathsf{m}).$$

By Corollary 2.1.5.17, the map

$$\mathsf{d}_{\mathcal{L}_0} : H^1(A_0, \underline{\operatorname{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \to H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I$$

is surjective when

$$(\operatorname{Id}_{A_0^\vee} \otimes d\lambda_0 \otimes \operatorname{Id}) : \operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \operatorname{Lie}_{A_0/S_0} \underset{k}{\otimes} I \to \operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I$$

is surjective. By assumption, λ_0 is prime-to-$\Box$ and hence separable. Therefore $d\lambda_0 : \operatorname{Lie}_{A_0/S_0} \to \operatorname{Lie}_{A_0^\vee/S_0}$ is an isomorphism, which is in particular surjective. This shows the surjectivity of $\mathsf{d}_{\mathcal{L}_0}$, and hence the existence of some element m such that $\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, R \twoheadrightarrow R) = 0$.

Note that the elements m making $\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, R \twoheadrightarrow R) = 0$ form a torsor under those *symmetric elements* in

$$H^1(A_0, \underline{\operatorname{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \cong \operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I,$$

namely those elements that are mapped to zero in

$$H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I \cong [\wedge^2 H^1(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} I \cong [\wedge^2 \operatorname{Lie}_{A_0^\vee/S_0}] \underset{k}{\otimes} I.$$

(See Proposition 2.1.5.15 for the first isomorphism.) $\Box$

Remark 2.2.4.5. The essential condition on (A_0, λ_0) that we have used in this proof is that λ_0 is a *separable isogeny* because it is prime-to-$\Box$. The same proof works for any such (A_0, λ_0) without assuming that (A_0, λ_0) comes from a tuple $(A_0, \lambda_0, i_0, \alpha_{\mathcal{H},0})$ parameterized by $\mathsf{M}_{\mathcal{H}}$. (The case that λ_0 can be inseparable is much more delicate and seems to be beyond the reach of this technique.)

Corollary 2.2.4.6. *Let* $\tilde{R} \twoheadrightarrow R$ *be a small surjection in* C *with kernel* I*, and let* (A, λ, f_0) *define an object in* $\mathsf{Def}_{(A_0,\lambda_0)}(R)$*. Let* $\mathsf{Lift}(A, \lambda; \tilde{R} \twoheadrightarrow R)$ *denote the subset of* $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ *consisting of those liftings* $\tilde{A}$ *of* A *that admit liftings* $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ *of* $\lambda : A \to A^\vee$*. Note that* $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ *is a torsor under the group* $H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$*. Then* $\mathsf{Lift}(A, \lambda; \tilde{R} \twoheadrightarrow R)$ *is a torsor under the group of* **symmetric elements** *in*

$$\mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I.$$

Proof. Once the we know that this set is nonempty, the statement follows from either Proposition 2.1.3.2, or simply the observation at the very end of the proof of Proposition 2.2.4.4. ☐

Corollary 2.2.4.7. *Under the assumption that* λ_0 *is separable, the functor* $\mathsf{Def}_{(A_0,\lambda_0)}$ *is (noncanonically) prorepresented by the smooth algebra* $\Lambda[[x_1, \ldots, x_{\frac{1}{2}g(g+1)}]]$ *over* Λ*, where* $g = \dim_k A_0$*.*

Proof. By applying Corollary 2.2.4.6 to the small surjection $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$, we see that

$$\dim_k \mathsf{Def}_{(A_0,\lambda_0)}(k[\varepsilon]/(\varepsilon^2))$$

is the same as the dimension of the subspace of symmetric elements in

$$\mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0},$$

which is $\frac{1}{2}g(g+1)$. Then the result follows from Propositions 2.2.3.7, 2.2.4.4, and Theorem 2.2.1.4. ☐

Let us formulate Proposition 2.2.4.4 and Corollary 2.2.4.6 in the context of Proposition 2.1.6.8. Note that any polarization λ of an abelian scheme A over S defines a canonical pairing $\langle \cdot, \cdot \rangle_\lambda$ on $\underline{H}^1_{\mathrm{dR}}(A/S)$ and hence on $\underline{H}_1^{\mathrm{dR}}(A/S)$ as well. Concretely, as mentioned in [29, 1.5], the Poincaré invertible sheaf $\mathcal{P}_A$

$A \underset{S}{\times} A^\vee$ induces an alternating pairing between $\underline{H}^1_{\mathrm{dR}}(A/S)$ and $\underline{H}^1_{\mathrm{dR}}(A^\vee/S)$, which is a perfect duality by [17, 5.1]. In particular, there is a canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(A/S) \cong \underline{H}_1^{\mathrm{dR}}(A^\vee/S)$. Thus, any polarization $\lambda : A \to A^\vee$ induces canonically a morphism $\lambda^* : H^1_{\mathrm{dR}}(A^\vee/S) \to H^1_{\mathrm{dR}}(A/S)$, and hence a morphism $H_1^{\mathrm{dR}}(A/S) \to H^1_{\mathrm{dR}}(A/S)$ giving a pairing $\langle \cdot , \cdot \rangle_\lambda$ on $H_1^{\mathrm{dR}}(A/S)$.

Under this pairing, the $\underline{\mathrm{Lie}}^\vee_{A^\vee/S}$ in the exact sequence

$$0 \to \underline{\mathrm{Lie}}^\vee_{A^\vee/S} \to \underline{H}_1^{\mathrm{dR}}(A/S) \to \underline{\mathrm{Lie}}_{A/S} \to 0 \qquad (2.2.4.8)$$

is an isotropic submodule of $\underline{H}_1^{\mathrm{dR}}(A/S)$. Moreover, when λ is separable, the pairing $\langle \cdot , \cdot \rangle_\lambda$ is a perfect pairing, and the embedding $\underline{\mathrm{Lie}}^\vee_{A^\vee/S} \hookrightarrow \underline{H}_1^{\mathrm{dR}}(A/S)$ induces an isomorphism $\underline{\mathrm{Lie}}_{A/S} \to \underline{\mathrm{Lie}}_{A^\vee/S}$, which is nothing but the differential $d\lambda$ of the separable isogeny $\lambda : A \to A^\vee$. By choosing (non-canonically) a splitting of (2.2.4.8), we obtain a corresponding isomorphism

$$(H_1^{\mathrm{dR}}(A/S), \langle \cdot , \cdot \rangle_\lambda) \cong (\underline{\mathrm{Lie}}_{A/S} \oplus (\underline{\mathrm{Lie}}_{A/S})^\vee, \langle \cdot , \cdot \rangle_{\mathrm{std}}). \qquad (2.2.4.9)$$

Suppose we have a small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and suppose that we have an abelian scheme $\tilde{A}$ over $\tilde{S} := \mathrm{Spec}(\tilde{R})$. Let $A := \tilde{A} \underset{\tilde{R}}{\otimes} R$, and let $\lambda : A \to A^\vee$ be a polarization. We claim that we can define a canonical pairing $\langle \cdot , \cdot \rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ without knowing the existence of some polarization $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ lifting λ. Indeed, the existence of λ gives a canonical morphism $\lambda_* : \underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to \underline{H}_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$ by dualizing Proposition 2.1.6.4, which necessarily maps $\underline{\mathrm{Lie}}^\vee_{(\tilde{A}')^\vee/\tilde{S}}$ to $\underline{\mathrm{Lie}}^\vee_{\tilde{A}'/\tilde{S}}$ for any lifting $\tilde{A}'$ of A that does admit a lifting $\tilde{\lambda}'$ of λ, as can be seen in the proofs of Propositions 2.1.6.2 and 2.1.6.4. This morphism is necessarily an isomorphism, because it induces an isomorphism $\underline{H}_1^{\mathrm{dR}}(A/S) \xrightarrow{\sim} \underline{H}_1^{\mathrm{dR}}(A^\vee/S)$ modulo I. (See Lemma 2.1.1.1.) Therefore it defines a canonical pairing $\langle \cdot , \cdot \rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$, which agrees with $\langle \cdot , \cdot \rangle_{\tilde{\lambda}}$ whenever there does exist a lifting $\tilde{\lambda}$ of λ. Moreover, suppose f_1 and f_2 are two endomorphism on A satisfying $\lambda \circ f_1 = f_2^\vee \circ \lambda$. (For example, suppose $f_1 = i(b^\star)$ and $f_2 = i(b)$ for some endomorphism structure $i : \mathcal{O} \to \mathrm{End}_S(A)$.) Then the canonical morphisms $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to \underline{H}_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$ defined by $\lambda \circ f_1$ and by $f_2^\vee \circ \lambda$ has to agree. This shows that we can have $\langle f_{1,*}(x), y \rangle_\lambda = \langle x, f_{2,*}(y) \rangle_\lambda$ even without lifting the endomorphisms f_1 and f_2 to $\tilde{A}$.

Let us return to the situation that $\tilde{A}$ is a lifting of A that admits a lifting $\tilde{\lambda}$ of the polarization λ. Let us work with modules rather than sheaves, as our base scheme is now affine. Starting with the isotropic submodule $\mathrm{Lie}^\vee_{\tilde{A}/\tilde{S}}$ of $H_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$, the $\tilde{R}$-submodules M as in Proposition 2.1.6.8 that becomes the same as $\mathrm{Lie}^\vee_{A^\vee/S}$ modulo I and is moreover *isotropic* under the pairing $\langle \cdot, \cdot \rangle_{\tilde{\lambda}}$, are parameterized by the subgroup of *symmetric elements* in

$$\begin{aligned}\mathrm{Hom}_{\tilde{R}}(\mathrm{Lie}^\vee_{\tilde{A}^\vee/\tilde{S}}, I \cdot \mathrm{Lie}_{\tilde{A}/\tilde{S}}) &\cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I \\ &\overset{\mathrm{Id} \otimes d\lambda}{\overset{\sim}{\to}} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I.\end{aligned}$$

According to Corollary 2.2.4.6, this is the same set that parameterizes the liftings of abelian schemes that admit liftings of the polarization λ on A. Combining with the discussion about lifting endomorphisms, we arrive at:

Corollary 2.2.4.10. *Let $\tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I, and let $\tilde{A}$ be an abelian scheme over $\tilde{S} = \mathrm{Spec}(\tilde{R})$. As explained in Proposition 2.1.6.8, the objects in $\mathsf{Lift}(\tilde{A}; \tilde{R} \twoheadrightarrow R)$ (which are necessarily abelian schemes by Proposition 2.2.2.3) are in bijection with modules M in exact sequences*

$$0 \to M \to H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to N \to 0$$

of **projective $\tilde{R}$-modules** *such that $M \underset{\tilde{R}}{\otimes} R = \mathrm{Lie}^\vee_{A^\vee/S}$ in $H_1^{\mathrm{dR}}(A/S) = H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \underset{\tilde{R}}{\otimes} R$. Let us denote the abelian scheme corresponding to a submodule M as above by $\tilde{A}_M$. Suppose moreover that $A := \tilde{A} \underset{\tilde{R}}{\otimes} R$ has a separable polarization $\lambda : A \to A^\vee$ that defines a perfect pairing on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$. Then the following are true:*

1. *The polarization λ can be lifted to some polarization $\tilde{\lambda}_M : \tilde{A}_M \to \tilde{A}_M^\vee$ on $\tilde{A}_M$ if and only if M is an isotropic submodule in $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ under $\langle \cdot, \cdot \rangle_\lambda$. (See Corollaries 2.2.2.2 and 2.2.2.6.)*

2. *Suppose that A has a collection of endomorphisms $f_i : A \to A$. Then these endomorphisms of f_i can be lifted to a collection of endomorphisms $\tilde{f}_{i,M} : \tilde{A}_M \to \tilde{A}_M$ on $\tilde{A}_M$ if and only if M is invariant under the actions of $f_{i,*}$ on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$. (See Proposition 2.1.6.4.)*

3. *Suppose that $i : \mathcal{O} \to \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure for (A, λ). Then i can be lifted to an $\mathcal{O}$-endomorphism structure $\tilde{i} : \mathcal{O} \to$*

$\mathrm{End}_{\tilde{S}}(\tilde{A}_M)$ for some $\tilde{A}_M$ that admits a lifting $\tilde{\lambda}_M$ of λ if and only if M is both isotropic under $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ and invariant under the actions on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ defined by $i(b)_*$ for any $b \in \mathcal{O}$.

Proof. This is just a combination of what we have discussed, together with Corollary 2.2.2.7, with the following additional remark: The compatibility between different morphisms (including the Rosati condition) are given by relations defined by group homomorphisms that are trivial over S. Therefore the same trick of Corollary 1.3.1.5 is applicable. □

Proposition 2.2.4.11. *The functor $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ is formally smooth.*

Proof. For any small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and any (A,λ,i,f_0) defining an object of $\mathsf{Def}_{(A_0,\lambda_0,i_0)}(R)$, we know from Proposition 2.2.4.1 that there always exists some abelian scheme $(\tilde{A},\tilde{\lambda},\tilde{f}_0)$ lifting (A,λ_0,f_0). The question is whether we can also lift i to an endomorphism structure on $(\tilde{A},\tilde{\lambda},\tilde{f}_0)$. If so, then by Proposition 2.2.2.9, the object $(\tilde{A},\tilde{\lambda},\tilde{i})$ will automatically satisfy the Lie algebra condition (defined in Definition 1.3.4.2).

It suffices to show the existence of a projective submodule M of $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ lifting $\mathrm{Lie}^\vee_{A^\vee/S}$ as in Corollary 2.2.4.10 that is both isotropic under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}$ defined by $\tilde{\lambda}$ and invariant under the action defined by i. Note that the pairing defined by $\tilde{\lambda}$ is perfect because $\tilde{\lambda}$ is a separable isogeny. Moreover, as we have already discussed before, it is determined by λ and has the Hermitian property given by the Rosati condition satisfied by λ and i. That is, $(H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}),\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}})$ is a self-dual symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R}$-module as in Definition 1.1.4.13.

Our assumption on k and Λ includes the hypothesis that there is a morphism $\mathcal{O}_{F_0,(\Box)} \to \Lambda$ of finite type, that $\Lambda = k$ when $\mathrm{char}(k) = 0$, and that $\Lambda = W(k)$ when k is a finite field of $\mathrm{char}(k) = p > 0$. By the assumption that (A_0,λ_0,i_0) satisfies the Lie algebra condition, and by Lemma 1.2.5.13, we know that there is an isomorphism $\mathrm{Lie}_{A/S} \cong L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R$, where L_0 is the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-module chosen in Lemma 1.2.5.10. By (2.2.4.9) and (1.2.5.16), and by the same argument as in the proof of Proposition 1.2.4.6, we obtain a symplectic isomorphism

$$f : (H_1^{\mathrm{dR}}(A/S),\langle\,\cdot\,,\,\cdot\,\rangle_\lambda) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} R,\langle\,\cdot\,,\,\cdot\,\rangle)$$

sending $\mathrm{Lie}^\vee_{A^\vee/S}$ to $(L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee$. By Proposition 1.2.3.7 and Corollary 1.2.3.10, and by the fact that the choices of modules of standard type (defined as in Definition 1.2.3.6) is discrete in nature, we see that there is also a symplectic isomorphism

$$\tilde{f} : (H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}), \langle \cdot, \cdot \rangle_{\tilde{\lambda}}) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} \tilde{R}, \langle \cdot, \cdot \rangle).$$

Since f maps $\mathrm{Lie}^\vee_{A^\vee/S}$ to $(L_0 \underset{\mathcal{O}_{F_0}}{\otimes} R)^\vee$ in $(L \underset{\mathbb{Z}}{\otimes} R, \langle \cdot, \cdot \rangle)$, we obtain a point of $(\mathrm{G}/\mathrm{P}_0)(R)$. Then the liftability of $\underline{\mathrm{Lie}}^\vee_{A^\vee/S}$ to a projective submodule M of $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ corresponds to the liftability of the point to a point in $(\mathrm{G}/\mathrm{P}_0)(\tilde{R})$, which follows from Proposition 1.2.5.20. □

Remark 2.2.4.12. In the proof of Proposition 2.2.4.11, the upshot is that we have related the smoothness of our moduli problem $\mathsf{M}_{\mathcal{H}}$ over Λ to the smoothness of some flag variety $\mathrm{G}_\Lambda/\mathrm{P}_{0,\Lambda}$. (See Lemma 1.2.5.19 and Proposition 1.2.5.20.) A suggestive way to understand this argument is as follows: The parabolic subgroup $\mathrm{P}_{0,\Lambda}$ of G_Λ (defined in Definition 1.2.5.17) is characterized over the complex numbers by the property that $\mathrm{P}_{0,\Lambda}(\mathbb{C})$ (for some morphism $\Lambda \to \mathbb{C}$ compatible with the choice made in Remark 1.2.5.14) is the normalizer of the maximal isotropic subspace V_0 in $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. Over the complex numbers, this flag variety gives the compact dual of the Hermitian symmetric space associated to $\mathrm{G}(\mathbb{R})$. Locally near any point, it has the same local neighborhood as its arithmetic quotient, or rather $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$. Therefore the smoothness of one implies the smoothness of the other (if we always form the arithmetic quotients in the category of algebraic stacks). Under our assumption, we have a morphism $\mathcal{O}_{F_0,(\Box)} \to \Lambda$, where $\Lambda = W(k)$ for some finite field k such that $\mathrm{char}(k) = p > 0$ is a *good prime*. In this case, the flag variety $\mathrm{G}_\Lambda/\mathrm{P}_{0,\Lambda}$ is a *smooth* integral model (over Λ) of the above-mentioned complex dual of the Hermitian symmetric space, and what we have been doing is exactly relating the smoothness of this integral model to the smoothness of the integral model $\mathsf{M}_{\mathcal{H}}$ of the Shimura variety.

Corollary 2.2.4.13. *Let $\tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I, and let (A, λ, i, f_0) define an object in $\mathsf{Def}_{(A_0,\lambda_0,i_0)}(R)$. Let $\mathsf{Lift}(A, \lambda, i; \tilde{R} \twoheadrightarrow R)$ denote the subset of $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ consisting of those liftings $\tilde{A}$ of A that admit liftings $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ of $\lambda : A \to A^\vee$ and liftings $\tilde{i} : \mathcal{O} \to \mathrm{End}_{\tilde{S}}(\tilde{A})$ of* *the $\mathcal{O}$-endomorphism structure $i : \mathcal{O} \to \operatorname{End}_S(A)$. Then $\mathsf{Lift}(A, \lambda, i; \tilde{R} \twoheadrightarrow R)$ is a torsor under the group of symmetric elements in*

$$\operatorname{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \operatorname{Lie}_{A_0/S_0} \underset{k}{\otimes} I$$

that are killed by the endomorphisms

$$(d(i(b)^\vee) \otimes \mathrm{Id} \otimes \mathrm{Id}) - (\mathrm{Id} \otimes d(i(b)) \otimes \mathrm{Id})$$

for all $b \in \mathcal{O}$, or equivalently a torsor under the group of k-linear maps

$$\operatorname{Lie}^\vee_{A_0^\vee/S_0} \underset{k}{\otimes} \operatorname{Lie}^\vee_{A_0/S_0} / \begin{pmatrix} x \otimes \lambda_0^*(x') - x' \otimes \lambda_0^*(x) \\ ((i(b)^\vee)^*(x)) \otimes y - x \otimes (i(b)^*(y)) \end{pmatrix}_{\substack{x, x' \in \underline{\operatorname{Lie}}^\vee_{A_0^\vee/S_0}, \\ y \in \underline{\operatorname{Lie}}^\vee_{A_0/S_0}, b \in \mathcal{O}}} \to I. \tag{2.2.4.14}$$

Proof. Once the we know that the set of liftings is nonempty, the statement follows simply from Proposition 2.1.3.2. (One can also try to analyze the set of possibly liftings of an isotropic sublattice in the proof of Proposition 2.2.4.11.) □

Corollary 2.2.4.15. *The functor $\mathsf{Def}_{(A_0, \lambda_0, i_0)}$ is (noncanonically) prorepresented by the smooth algebra $\Lambda[[x_1, \ldots, x_r]]$ over Λ, where r is an integer that can be calculated as follows: Let V_0 be the complex vector space defined in Section 1.3.4. Set*

$$\mathbf{Sym}_\varrho(V_0) := (V_0 \underset{\mathbb{C}}{\otimes} V_0) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x, y, z \in V_0, b \in \mathcal{O}}.$$

Then we have simply $r = \dim_{\mathbb{C}} \mathbf{Sym}_\varrho(V_0)$.

Proof. By applying Corollary 2.2.4.13 to the small surjection $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$, we see that

$$r := \dim_k \mathsf{Def}_{(A_0, \lambda_0, i_0)}(k[\varepsilon]/(\varepsilon^2))$$

is the same as the k-vector-space dimension of the domain in (2.2.4.13). Note that $\operatorname{Lie}_{A_0/S_0}$ is isomorphic to $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} k$, where L_0 is the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-module defined in Lemma 1.2.5.10. Under the polarization λ_0, we may identify $\operatorname{Lie}_{A_0^\vee/S_0}$ with $\operatorname{Lie}_{A_0/S_0}$, and hence with $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} k$ too. Let $L_1 := L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \Lambda$, and let $L_1^\vee := \operatorname{Hom}_\Lambda(L_1, \Lambda)$ denotes its dual Λ-module. Then we may identify both

$\underline{\mathrm{Lie}}^{\vee}_{A_0^{\vee}/S_0}$ and $\underline{\mathrm{Lie}}^{\vee}_{A_0/S_0}$ with $L_1^{\vee} \underset{\Lambda}{\otimes} k$. By Proposition 1.2.2.4, we may identify the dimension r above with the Λ-rank of

$$\mathbf{Sym}_{\varrho}(L_1^{\vee}) := (L_1^{\vee} \underset{\Lambda}{\otimes} L_1^{\vee}) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^{\star} z) \end{pmatrix}_{x,y,z \in L_1^{\vee}, b \in \mathcal{O}},$$

or equivalently the Λ-rank of

$$\mathbf{Sym}_{\varrho}(L_1) := (L_1 \underset{\Lambda}{\otimes} L_1) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^{\star} z) \end{pmatrix}_{x,y,z \in L_1, b \in \mathcal{O}},$$

because both of the Λ-modules have no torsion. We may compute this rank by tensoring everything with a large field containing Λ and then compute the dimension of the corresponding vector space. Afterwards we may replace the large field by a smaller field over which every object is defined. In particular, its dimension can be calculated by replacing L_1 by $V_0 := L_0 \underset{\mathcal{O}_{F_0}}{\otimes} \mathbb{C}$. Once we have calculated the dimension r of $\dim_k \mathsf{Def}_{(A_0,\lambda_0,i_0)}(k[\varepsilon]/(\varepsilon^2))$, the result follows as before from Propositions 2.2.3.9, 2.2.4.11, and Theorem 2.2.1.4. ☐

Theorem 2.2.4.16. *The functor* $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ *is formally smooth. Moreover,* Def_{ξ_0} *is (noncanonically) prorepresented by the smooth algebra* $\Lambda[[x_1, \ldots, x_r]]$ *over* Λ*, where* r *is the integer calculated in Corollary 2.2.4.13.*

Proof. Simply combine Corollary 2.2.2.10 and Proposition 2.2.4.11. ☐

2.3 Algebraic Theory

2.3.1 Grothendieck's Formal Existence Theory

Let us recall some useful results in [47, III, §5].

Here we shall assume that R is a noetherian ring I-adically complete for some ideal I of R.

If $Y = \mathrm{Spec}(R)$, the affine formal scheme $\mathrm{Spf}(R)$ is the same as the completion Y_{for} of Y along the closed subset $Y_0 := \mathrm{V}(I)$. Let $f : X \to Y$ be a morphism of schemes of finite type. We denote by X_{for} the completion X along the closed subset $X_0 := f^{-1}(Y_0)$, or equivalently the Y_{for}-formal scheme $X \underset{Y}{\times} Y_{\mathrm{for}}$, and by $f_{\mathrm{for}} : X_{\mathrm{for}} \to Y_{\mathrm{for}}$ the extension of f to the completions. Finally, for all coherent $\mathscr{O}_X$-module $\mathscr{F}$, we denote by $\mathscr{F}_{\mathrm{for}}$ its completion, $\mathscr{F}_{/X_0}$, which is a coherent $\mathscr{O}_{X_{\mathrm{for}}}$-module.

Proposition 2.3.1.1 ([47, III, 5.1.2])**.** *Assumptions and notations as above, if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module whose support is proper over Y, then the canonical homomorphisms*

$$\rho_i : H^i(X, \mathscr{F}) \to H^i(X_{\text{for}}, \mathscr{F}_{\text{for}})$$

are isomorphisms.

Theorem 2.3.1.2 ([47, III, 5.1.4])**.** *Let R be a noetherian adic ring, $Y = \operatorname{Spec}(R)$, I an ideal of definition of R, $Y_0 := \mathrm{V}(I)$, $f : X \to Y$ a separated morphism of finite type, and $X_0 := f^{-1}(Y_0)$. Let $Y_{\text{for}} = Y_{/Y_0} = \operatorname{Spf}(R)$ and $X_{\text{for}} = X_{/X_0}$ be the completions of Y and X along respectively Y_0 and X_0, and $f_{\text{for}} : X_{\text{for}} \to Y_{\text{for}}$ the extension of f to the completions. Then, the functor $\mathscr{M} \mapsto \mathscr{M}_{/X_0} = \mathscr{M}_{\text{for}}$ is an equivalence between the category of coherent $\mathscr{O}_X$-modules with proper support over* $\operatorname{Spec}(R)$*, and the category of coherent $\mathscr{O}_{X_{\text{for}}}$-modules with proper support over* $\operatorname{Spf}(R)$*.*

Corollary 2.3.1.3 ([47, III, 5.1.6])**.** *Suppose X is proper over $Y = \operatorname{Spec}(R)$. Then the functor $\mathscr{M} \mapsto \mathscr{M}_{\text{for}}$ is an equivalence between the category of coherent $\mathscr{O}_X$-modules and the category of coherent $\mathscr{O}_{X_{\text{for}}}$-modules.*

Theorem 2.3.1.4 ([47, III, 5.4.1])**.** *Let R be a noetherian adic ring, I an ideal of definition of R, $S = \operatorname{Spec}(R)$, and $S_0 := \mathrm{V}(I)$. Let $u : X \to S$ be a proper morphism, $v : Y \to S$ a separated morphism of finite type, and let S_{for}, X_{for}, and Y_{for} be the completions of S, X, and Y along S_0, $u^{-1}(S_0)$, and $v^{-1}(S_0)$ respectively. For all morphism $f : X \to Y$ over S, let f_{for} be the extension of f to the completions. Then the map $f \mapsto f_{\text{for}}$ is a bijection*

$$\operatorname{Hom}_S(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{S_{\text{for}}}(X_{\text{for}}, Y_{\text{for}}).$$

We can then say, in the language of categories, that the functor $X \mapsto X_{\text{for}}$ is fully faithful from the category of schemes proper over $\operatorname{Spec}(R)$ to the category of formal schemes proper over $\operatorname{Spf}(R)$, for all noetherian complete local ring R. It establishes therefore an equivalence between the former of these categories and a sub-category of the latter. The objects of this last category are called *algebraizable formal schemes*. For such a scheme $\mathfrak{X}$, there exists a usual scheme X, proper over $\operatorname{Spec}(R)$, determined up to unique isomorphism, such that $\mathfrak{X}$ is isomorphic to X_{for}.

Theorem 2.3.1.5 ([47, III, 5.4.5])**.** *Let R be a noetherian adic ring, I an ideal of definition of R, $S = \operatorname{Spec}(R)$, $S_{\text{for}} = \operatorname{Spf}(R)$, and $f : \mathfrak{X} \to S_{\text{for}}$ a* *proper morphism of formal schemes. Put* $S_i = \mathrm{Spec}(R/I^{i+1})$, $X_i = \mathfrak{X} \underset{S_{\mathrm{for}}}{\times} S_i$, *and for all* $\mathscr{O}_{\mathfrak{X}}$*-module* $\mathscr{M}$, $\mathscr{M}_i = \mathscr{M} \underset{\mathscr{O}_{\mathfrak{X}}}{\otimes} \mathscr{O}_{X_i} = \mathscr{M}/I^{i+1}\mathscr{M}$. *Let* $\mathscr{L}$ *be an invertible* $\mathscr{O}_{\mathfrak{X}}$*-module, and suppose that* $\mathscr{L}_0 := \mathscr{L}/I\mathscr{L}$ *is an ample* $\mathscr{O}_{X_0}$*-module. Then* $\mathfrak{X}$ *is algebraizable, and if* X *is a proper scheme over* S *such that* $\mathfrak{X}$ *is isomorphic to* X_{for}*, then there exists an ample* $\mathscr{O}_X$*-module* $\mathscr{M}$ *such that* $\mathscr{L}$ *is isomorphic to* $\mathscr{M}_{\mathrm{for}}$ (*which implies* X *is* **projective** *over* S).

2.3.2 Effectiveness of Local Moduli

With the definitions of Def_{A_0}, $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, and $\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ as in Section 2.2.1, we have shown the prorepresentability of these deformation functors in Section 2.2.3. A natural question is whether they are effectively prorepresented.

For each of these functors, the prorepresentability means there is a complete noetherian local ring R in $\hat{\mathsf{C}}$ and a compatible system of abelian schemes A_i over $R_{A_0}/\mathfrak{m}_R^{i+1}$ (with some additional structures if necessary), which induce an isomorphism h_R to Def_{A_0} via the natural isomorphism $\hat{\mathsf{Def}}_{A_0} \xrightarrow{\sim} \mathrm{Hom}(h_R, \mathsf{Def}_{A_0})$. (See Section B.1 for more details.) In particular, this means there is a formal abelian scheme $\hat{A}_R \to \mathrm{Spf}(R_{A_0})$. The question whether this moduli problem is effectively prorepresentable means whether we can find an abelian scheme A over R that induces this compatible system A_i by reductions. In other words, the question is whether this formal abelian scheme $\{A_i\}$ is *algebraizable* in the sense of Section 2.3.1. According to Grothendieck's Theorem 2.3.1.5, the answer will be affirmative if there is also a compatible system of relatively ample invertible sheaves $\mathcal{L}_i$ on A_i. This suggests:

Proposition 2.3.2.1. *All the three functors* $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, *and* $\mathsf{Def}_\xi = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ *are* **effectively** *prorepresentable.*

Proof. Let us first study the case $\mathsf{Def}_{(A_0,\lambda_0)}$. Suppose that this functor is prorepresented over a complete local ring R in $\hat{\mathsf{C}}$ and a compatible system of polarized abelian schemes (A_i, λ_i) over $R/\mathfrak{m}^{i+1}$. Over each of the A_i, we may take $\mathcal{L}_i = (\mathrm{Id}_{A_i}, \lambda_i)^* \mathcal{P}_{A_i}$, which is relatively ample by definition of polarizations. (See Definition 1.3.2.20 and Proposition 1.3.2.18). Therefore Theorem 2.3.1.2 implies $\{(A_i, \mathcal{L}_i)\}$ is algebraizable by some algebraic object $(A, \mathcal{L})$ over R. This implies that $\{A_i{}^\vee\}$ is also algebraizable. Moreover, as the algebraizable schemes form a full subcategory in the category the formal schemes by Theorem 2.3.1.4, the maps $\{\lambda_i : A_i \to A_i^\vee\}$ are algebraizable by a unique $\lambda : A \to A^\vee$, which is necessarily a polarization (by Definition 1.3.2.20 and Proposition 1.3.2.18 again). This proves that $\mathsf{Def}_{(A_0,\lambda_0)}$ is effectively prorepresentable. Since the other two moduli problems $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ and $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ only involve more algebraizations of morphisms, the same argument above implies they are effectively prorepresentable as well. □

Remark 2.3.2.2. According to [110, XI, 1.4], any abelian scheme over a normal base scheme admits a relatively ample invertible sheaf. This suggests (among other reasons) that it is much more natural to work with polarizations when studying algebraizability of smooth moduli problems such as Def_{A_0}. Certainly, there are cases when studying Def_{A_0} is actually equivalent to studying $\mathsf{Def}_{(A_0,\lambda_0')}$ for some polarization λ_0' (possibly different from λ_0). This is the case when A_0 is an elliptic curve, but rarely the case in general.

2.3.3 Proof of Representability

Let us now prove the representability of $\mathsf{M}_{\mathcal{H}}$ using the Artin's criterion in Appendix B. Although the criterion is literally formulated for categories fibred in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$, its proof shows that it certainly applies to categories fibred in groupoids over $(\mathrm{LNSch}/\mathsf{S}_0)$. According to Theorems B.3.8, B.3.10, and B.3.12, to show that $\mathsf{M}_{\mathcal{H}}$ is representable by an algebraic stack locally of finite type over the base scheme $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, which is either a field or an excellent Dedekind domain as in the assumption of Artin's criterion, it suffices to verify the following conditions:

1. $\mathsf{M}_{\mathcal{H}}$ is a stack for the étale topology.
2. $\mathsf{M}_{\mathcal{H}}$ is locally of finite presentation.
3. Suppose ξ and η are two objects of $\mathsf{M}_{\mathcal{H}}(U)$ over some scheme $U \to \mathsf{S}_0$, then $\underline{\mathrm{Isom}}_U(\xi,\eta)$ is an algebraic space locally of finite type.
4. For k and ξ_0 as above, which defines a functor $\mathsf{Def}_{\xi_0} : \hat{\mathsf{C}} \to (\mathrm{Sets})$ as in Section 2.2.1, the functor Def_{ξ_0} is effectively prorepresentable.

Let us begin with condition 1. Consider tuples of the form $(A,\lambda,i,\alpha_{\mathcal{H}})$ parameterized by $\mathsf{M}_{\mathcal{H}}$. For any such tuple, we have the associated relatively ample invertible sheaf $\mathcal{L} := (\mathrm{Id}_A \times \lambda)^* \mathcal{P}_A$. The pair $(A, \mathcal{L})$ satisfy étale descent (or rather fpqc descent) by [58, VIII, 7.8]. The additional structures λ, i, and $\alpha_{\mathcal{H}}$ also satisfy étale descent, because they are defined by collection of morphisms, or étale-locally-defined orbits of morphisms. Hence $\mathsf{M}_{\mathcal{H}}$ is a stack for the étale topology.

Condition 2 is true because the objects of $\mathsf{M}_{\mathcal{H}}$ are defined by schemes and morphisms that are of finite presentation. (See Remark 1.3.1.2.)

To verify condition 3, note that the functor

$$\underline{\mathrm{Isom}}_U(\xi, \eta)$$

is representable by an algebraic space by the general theory of Hilbert schemes and by [9, Cor. 6.2]. Moreover, it is quasi-finite by Lemma 1.4.1.10. Therefore, it is proper (and hence finite) over U by the valuative criterion over discrete valuation rings, using the theory of Néron models. (Or we may use Proposition 3.3.1.7 below if we prefer.)

Remark 2.3.3.1. We have actually shown that the diagonal map $\Delta : \mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{M}_{\mathcal{H}}$ is finite. Hence $\mathsf{M}_{\mathcal{H}}$ is separated, as soon as it is representable by an algebraic stack.

Finally, condition 4 is already proved as Theorem 2.3.2.1. Hence, by Artin's criterion, we know that $\mathsf{M}_{\mathcal{H}}$ is representable by a separated algebraic stack locally of finite presentation of $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.

By Corollary 1.4.1.11, the objects of $\mathsf{M}_{\mathcal{H}}$ does not admit automorphisms when $\mathcal{H}$ is neat. Hence $\mathsf{M}_{\mathcal{H}}$ is also representable by an algebraic space when $\mathcal{H}$ is neat, (or whenever the existence of additional structures kills any possible nontrivial automorphism of objects.)

Let us show that $\mathsf{M}_{\mathcal{H}}$ is of finite type. By Definition A.6.1.5, an algebraic stack is of finite type if it is quasi-compact and locally of finite type. Hence (by Lemma A.6.1.4) it suffices to show that there is a map from a quasi-compact scheme to $\mathsf{M}_{\mathcal{H}}$. Let us describe briefly how such a map can be constructed.

Let S be any locally noetherian scheme, and let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be any object in $\mathsf{M}_{\mathcal{H}}(S)$. By definition of being a polarization (in 1.3.2.20), the invertible sheaf $\mathcal{L} := (\mathrm{Id}_A, \lambda)^* \mathcal{P}_A$ is relatively ample. Moreover, $\pi_*(\mathcal{L}^{\otimes 3})$ is locally free of finite rank over S by [97, Prop. 6.13]. Let $m := \mathrm{rank}_{\mathscr{O}_S}(\pi_*(\mathcal{L}^{\otimes 3})) - 1$. Whenever a local basis of sections of $\pi_*(\mathcal{L}^{\otimes 3})$ is chosen, the basis vectors define an isomorphism $r : \mathbb{P}_S(\pi_*(\mathcal{L}^{\otimes 3})) \xrightarrow{\sim} \mathbb{P}^m_S$ and defines an embedding of

A into $\mathbb{P}^m_S$, as $\mathcal{L}^{\otimes 3}$ is very ample on each fiber of A. Different choices give different embeddings r, which differ by an automorphism of $\mathbb{P}^m_S$ induced by the action of $\mathrm{PGL}(m+1)_S$. The image of the embeddings form a family of closed subschemes of $\mathbb{P}^m_S$, which are parameterized by the so-called Hilbert schemes. By applying Lemma 1.4.1.10 as above, the additional structures such as endomorphisms and level structures are parameterized by schemes quasi-compact over the above-mentioned Hilbert schemes. This shows that, if we consider the moduli problem $\tilde{\mathsf{M}}_{\mathcal{H}}$ parameterizing tuples of the form $(A, \lambda, i, \alpha_{\mathcal{H}}, r)$, then $\tilde{\mathsf{M}}_{\mathcal{H}}$ is representable by a scheme quasi-compact over S_0. On the other hand, we have a natural surjection $\tilde{\mathsf{M}}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}$ defined by forgetting the structure r. This shows that $\mathsf{M}_{\mathcal{H}}$ is also quasi-compact, as desired.

Remark 2.3.3.2. For a more comprehensive exposition of the use of Hilbert schemes, see for example [97, Ch. 7]. Certainly, we could have also proceeded with geometric invariant theory at this stage. But this is logically unnecessary (as we are going to work out the minimal compactifications anyway in Chapter 7), requires the main work in [97] to which we can do nothing but make a reference, and only allows us to omit the informative Section 2.3.2 and part of this Section 2.3.3. It is an interesting question whether there is any simple relation between the applicability of geometric invariant theory to the construction of moduli of abelian schemes, and the positivity of theta constants.

Finally, since $\mathsf{M}_{\mathcal{H}}$ is locally of finite presentation, the set of its points of finite type is dense. Since it is formally smooth at all of its points of finite type by Theorem 2.2.4.16, it is smooth everywhere.

This concludes the proof of Theorem 1.4.1.12.

2.3.4 Properties of Kodaira-Spencer Maps

To conclude this chapter, we list several properties of the Kodaira-Spencer maps for abelian schemes.

Definition 2.3.4.1. *Let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be a tuple over S parameterized by $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Then we define the sheaf $\underline{\mathrm{KS}}_{(A,\lambda)/S} = \underline{\mathrm{KS}}_{(A,\lambda,i,\alpha_{\mathcal{H}})/S}$ as the quotient*

$$(\underline{\mathrm{Lie}}^\vee_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S}) \Big/ \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ (i(b)^*(x)) \otimes y - x \otimes ((i(b)^\vee)^*(y)) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^\vee_{A/S}, \\ y,z \in \underline{\mathrm{Lie}}^\vee_{A^\vee/S}, \\ b \in \mathcal{O}}} .$$

Proposition 2.3.4.2. *Let* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ *be the base scheme over which* $\mathsf{M}_{\mathcal{H}}$ *is defined. Let* $S \to \mathsf{M}_{\mathcal{H}}$ *be any morphism over* S_0*, and let* $(A, \lambda, i, \alpha_{\mathcal{H}})$ *be the tuple over* S *associated to the morphism by the universal property of* $\mathsf{M}_{\mathcal{H}}$*. Suppose that* $\Omega^1_{S/\mathsf{S}_0}$ *is locally free over* $\mathscr{O}_S$*. Then the Kodaira-Spencer map* (*defined as in Definition 2.1.7.8*)

$$\mathrm{KS} = \mathrm{KS}_{A/S/\mathsf{S}_0} : \underline{\mathrm{Lie}}^{\vee}_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{A^\vee/S} \to \Omega^1_{S/\mathsf{S}_0}$$

satisfies

$$\mathrm{KS}(\lambda^*(y) \otimes z) = \mathrm{KS}(\lambda^*(z) \otimes y) \tag{2.3.4.3}$$

and

$$\mathrm{KS}((i(b)^*(x)) \otimes y) = \mathrm{KS}(x \otimes ((i(b)^\vee)^*(y))) \tag{2.3.4.4}$$

for any $x \in \underline{\mathrm{Lie}}^{\vee}_{A/S}$*,* $y, z \in \underline{\mathrm{Lie}}^{\vee}_{A^\vee/S}$*, and* $b \in \mathcal{O}$*, and hence induces a morphism*

$$\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda)/S} \to \Omega^1_{S/\mathsf{S}_0},$$

where $\underline{\mathrm{KS}}_{(A,\lambda)/S}$ *is defined as in Definition 2.3.4.1.*

Moreover, the morphism $S \to \mathsf{M}_{\mathcal{H}}$ *is* **étale** *if and only if it is flat and* KS *induces an* **isomorphism**

$$\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda)/S} \xrightarrow{\sim} \Omega^1_{S/\mathsf{S}_0}.$$

Proof. Let $\tilde{S}$ be first infinitesimal neighborhood of the image of S under the diagonal map $S \to S \underset{\mathsf{S}_0}{\times} S$ as in Section 2.1.7. Then the two relations (2.3.4.3) and (2.3.4.4) are satisfied because of Proposition 2.1.3.2, as any of the morphisms λ and $i(b)$ (for any $b \in \mathcal{O}$) can be lifted to the two pullbacks $\tilde{A}_1 := \mathrm{pr}_1^*(A)$ and $\tilde{A}_2 := \mathrm{pr}_2^*(A)$ of A under the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{S} \to S$.

Suppose the map $S \to \mathsf{M}_{\mathcal{H}}$ is étale. By restricting the map to local rings of points of finite type over the base S_0, we may assume that $(A, \lambda, i, \alpha_{\mathcal{H}})$ prorepresents the local deformation of some object associated to some $\xi_0 : \mathrm{Spec}(k) \to \mathsf{M}_{\mathcal{H}}$ studied in Section 2.2.1. Then the result follows from Corollaries 2.2.2.10 and 2.2.4.13. (Note that we do not need Corollaries 2.2.2.10 and 2.2.4.13 to know that $\underline{\mathrm{KS}}$ is a locally free sheaf. The local freeness of $\underline{\mathrm{KS}}$ follows from Proposition 1.2.2.4 and the assumption that (A, λ, i) satisfies the Lie algebra condition defined by the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$-module L_0 defined in Lemma 1.2.5.10.)

Conversely, suppose KS : $\underline{\mathrm{KS}}_{(A,\lambda)/S} \xrightarrow{\sim} \Omega^1_{S/\mathsf{S}_0}$ is an isomorphism. Note that above proof shows in particular that we have an isomorphism KS : $\underline{\mathrm{KS}}_{(A,\lambda)/\mathsf{M}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, where by abuse of notation we have also used (A,λ) to denote the universal abelian scheme and polarization over $\mathsf{M}_{\mathcal{H}}$. Let us denote the map $S \to \mathsf{M}_{\mathcal{H}}$ by f. Then the first morphism in the exact sequence

$$f^*\Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \to \Omega^1_{S/\mathsf{S}_0} \to \Omega^1_{S/\mathsf{M}_{\mathcal{H}}} \to 0$$

is an isomorphism, because the construction of $\underline{\mathrm{KS}}_{(A,\lambda)/S}$ commute with base changes, and because the association of Kodaira-Spencer maps is functorial. This shows that S is unramified over M_n, and hence étale because it is flat by assumption. □

Chapter 3

Structures of Semi-Abelian Schemes

In this chapter we summarize notions that are of fundamental importance in the study of degeneration of abelian varieties. The main objective is to understand the statement and the proof of theory of degeneration data, to be exhibited in Chapter 4. Our main references for these will be [33], [59], and in particular [96].

Technical results worth noting are Theorem 3.1.3.7 and Proposition 3.1.5.1 in Section 3.1; Propositions 3.3.1.7, 3.3.1.9, and 3.3.3.7 in Section 3.3; and Theorem 3.4.2.6 and 3.4.3.1, Lemma 3.4.3.3, and Proposition 3.4.4.1 in Section 3.4.

3.1 Groups of Multiplicative Type, Tori, and Their Torsors

3.1.1 Groups of Multiplicative Type

Let us summarize here several basic definitions and results about groups of multiplicative type. The main reference is [33, IX].

Definition 3.1.1.1 ([33, IX, 1.1]). *A* **group (scheme) of multiplicative type** *over a scheme S is a commutative group scheme over S that is fpqc locally isomorphic to a group scheme of the form $\underline{\mathrm{Hom}}(X, \mathbf{G}_{\mathrm{m}})$ for some abelian group X.*

A fundamental property of groups of multiplicative type is that they are *rigid* in the sense that they cannot be deformed. We describe this phenomenon as follows:

Theorem 3.1.1.2 (see [33, IX, 3.6 and 3.6 bis]). *Let S be a quasi-compact scheme, $\mathscr{I} \subset \mathscr{O}_S$ be a sheaf of ideals such that $\mathscr{I}^2 = 0$. Let $S_0 = \underline{\mathrm{Spec}}(\mathscr{O}_S/\mathscr{I})$ be the closed subscheme of S defined by $\mathscr{I}$. Let H_0 be a group of multiplicative type defined over S_0, let $G \to S$ a commutative group scheme smooth over S, and let $G_0 := G \underset{S}{\times} S_0$. Then:*

1. *The group H_0 can be uniquely lifted to a group H of multiplicative type over S. (Any two liftings are isomorphic by a unique morphism.)*
2. *Any homomorphism $u_0 : H_0 \to G_0$ can be uniquely lifted to a homomorphism $u : H \to G$. If moreover u_0 is a closed embedding, so is u.*

The definition of groups of multiplicative type can be weakened when we talk about group schemes of finite type over the base scheme S:

Theorem 3.1.1.3 (see [33, X, 4.5]). *Every group of multiplicative type that is* **of finite type** *over a base scheme S is étale locally isomorphic to a group scheme of the form $\underline{\mathrm{Hom}}(X, \mathbf{G}_{\mathrm{m}})$ for some abelian group X.*

Definition 3.1.1.4. *For a group H of multiplicative type of finite type, we denote by $\underline{X}(H) = \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$ the* **character group** *of H. It is an étale sheaf of finitely generated abelian groups. We say the group H is* **split** *if $\underline{X}(H)$ is a constant sheaf. We say the group H is* **isotrivial** *if there is a* **finite étale** *surjection $S' \to S$ such that $H \underset{S}{\times} S'$ is split.*

Definition 3.1.1.5. *A* **torus** *T over a scheme S is a group of multiplicative type of finite type such that $\underline{X}(T)$ is an étale sheaf of finitely generated* **free** *abelian groups. A torus is* **split** *(resp.* **isotrivial***) if it is split (resp. isotrivial) as a group of multiplicative type of finite type, i.e if $\underline{X}(T)$ is a constant sheaf (resp. if $\underline{X}(T)$ becomes constant after pullback to a finite étale surjection over S). The* **rank** *of a torus (which is defined locally on each connected component of S) is the rank of the étale sheaf $\underline{X}$ of finitely generated free abelian groups over S.*

Corollary 3.1.1.6. *The category of groups of multiplicative type of finite type (resp. tori) over a scheme S is anti-equivalent to the category of étale sheaves of finitely generated abelian groups (resp. finitely generated free abelian groups) over S, the equivalence being given by sending a group H to the étale sheaf $\underline{X}(H)$, and conversely by sending an étale sheaf $\underline{X}$ to the group scheme H defined by the étale group functor $\underline{\mathrm{Hom}}_S(\underline{X}, \mathbf{G}_{\mathrm{m},S})$.*

3.1.2 Torsors and Invertible Sheaves

Let us begin by reviewing the notion of torsors. Our main reference is [20, §6.4].

Definition 3.1.2.1. *Given a scheme $f : Z \to S$ over some base scheme S, a scheme $\mathcal{M}$ over Z, and a group scheme C over S acting on $\mathcal{M}$ by a morphism*

$$C_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} : (g, x) \mapsto gx,$$

where $C_Z = C \underset{S}{\times} Z$. Assume that $C_Z = C \underset{S}{\times} Z$ is (faithfully) flat and locally of finite presentation over Z. Then $\mathcal{M}$ is called a **C-torsor over** *Z (with respect to the fppf topology) if:*

1. *the structural morphism $\mathcal{M} \to Z$ is faithfully flat and locally of finite presentation, and*
2. *the morphism $C_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} \underset{Z}{\times} \mathcal{M}$ defined by $(g, x) \mapsto (gx, x)$ is an isomorphism.*

Viewing $C_Z \underset{Z}{\times} \mathcal{M}$ and $\mathcal{M} \underset{Z}{\times} \mathcal{M}$ as schemes over $\mathcal{M}$ with respect to the second projections, we see that the isomorphism in 2 is an isomorphism over $\mathcal{M}$. In other words, after applying the base change $\mathcal{M} \to Z$ to $\mathcal{M}$ and C_Z, they become isomorphic. The same is true for any base change $Y \to Z$ that factors through $\mathcal{M}$. As a result, if $C_Z \to Z$ satisfies any of the properties listed in [50, 2.7.1] and [52, 17.7.4], such as being smooth or of finite type, then $\mathcal{M} \to Z$ also does. (The references are applicable because we may assume that Z is affine and replace $\mathcal{M}$ by a quasi-compact subscheme when verifying the properties.)

If $\mathcal{M}(Z) \neq \emptyset$, the choice of any Z-valued point of $\mathcal{M}$ gives rise to an isomorphism over Z from C_Z to $\mathcal{M}$, and there is no essential difference between C_Z and the torsor $\mathcal{M}$ as a scheme. We say that the torsor $\mathcal{M}$ is *trivial* in this case. In general, if $\mathcal{M}(Z')$ is nonempty for any scheme $Z' \to Z$, then $\mathcal{M}$ is trivial after base change to $Z' \to Z$.

Proposition 3.1.2.2 (see [20, §2.2, Prop. 14, and §6.4])**.** *Suppose that C_Z is* **smooth**. *Then, under our assumptions on C_Z and $\mathcal{M}$ as above, $\mathcal{M}(Z')$ becomes nonempty after some surjective étale base change $Z' \to Z$. Therefore $\mathcal{M}$ is trivial after étale localization and corresponds to a class in $R^1 f_{*\text{ét}}(Z, C)$.*

Now suppose C is a split group of multiplicative type of finite type H over S. (Then the techniques of reduction to noetherian case by Theorem 1.3.1.3 applies.) By taking any étale covering $\{U_i\}$ of Z' above by schemes U_i over S, we can define alternatively a H-torsor $\mathcal{M}$ by giving an étale covering $\{U_i\}$ of Z over S, together with trivializations of H-torsors $\mathcal{M}|_{U_i} \cong (H_Z \underset{Z}{\times} U_i)$ and gluing isomorphisms $a_{ij} \in H(U_i \underset{Z}{\times} U_j)$ making the following diagram commutative:

$$\begin{tikzcd}
H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \arrow[r, "a_{ij} \times \mathrm{Id}", "\sim"'] \arrow[d] & H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \arrow[d] \\
(U_i \underset{Z}{\times} U_j) \arrow[r, "\mathrm{Id}"] & (U_i \underset{Z}{\times} U_j)
\end{tikzcd} .$$

As a special case, a $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$ on a base scheme Z is given by an étale covering $\{U_i\}$ of Z over S, together with trivializations of $\mathbf{G}_{\mathrm{m}}$-torsors $\mathcal{L}|_{U_i} \cong (\mathbf{G}_{\mathrm{m}} \times U_i)$ and gluing isomorphisms $a_{ij} \in \mathbf{G}_{\mathrm{m}}(U_i \underset{Z}{\times} U_j) = \mathscr{O}_Z^\times(U_i \underset{Z}{\times} U_j)$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \arrow[r, "a_{ij} \times \mathrm{Id}", "\sim"'] \arrow[d] & \mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \arrow[d] \\
(U_i \underset{Z}{\times} U_j) \arrow[r, "\mathrm{Id}"] & (U_i \underset{Z}{\times} U_j)
\end{tikzcd} .$$

In the absolute setting, we have the following generalization of *Hilbert Theorem 90*:

Theorem 3.1.2.3. *Let Z be a scheme. We have an isomorphism*

$$H^1_{\text{ét}}(Z, \mathbf{G}_{\mathrm{m}}) \xrightarrow{\sim} \mathrm{Pic}(Z),$$

where $\mathrm{Pic}(Z)$ *is the* (**absolute**) **Picard group**, *namely the group of isomorphism classes of invertible sheaves on Z.*

See for example [4, IX, 3.3], [39, 2.10], or [94, III, 4.9] for the proof.

Theorem 3.1.2.3 shows that $\mathbf{G}_{\mathrm{m}}$-torsors on Z are exactly invertible sheaves on Z. Since we are interested in the relative setting over S, we shall introduce the comparison between different topologies for the relative Picard functor.

Definition 3.1.2.4. *Let $f : Z \to S$ be a morphism of finite presentation. We say that $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_Z$* **holds universally** *if for any arbitrary morphism $S' \to S$, the canonical morphism*

$$\mathscr{O}_{S'} \to (f \underset{S}{\times} S')_* \mathscr{O}_{Z \underset{S}{\times} S'}$$

is an isomorphism.

Theorem 3.1.2.5. *Suppose a morphism $f : Z \to S$ of finite presentation and $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_Z$ holds universally. Then the natural maps*

$$\underline{\mathrm{Pic}}(Z/S) \hookrightarrow R^1 f_{*\mathrm{zar}} \mathbf{G}_{\mathrm{m}} \hookrightarrow R^1 f_{*\mathrm{\acute{e}t}} \mathbf{G}_{\mathrm{m}} \hookrightarrow R^1 f_{*\mathrm{fppf}} \mathbf{G}_{\mathrm{m}}$$

are all injections. Moveover, if f has a section, then all four maps are isomorphisms. If f has a section locally in the Zariski topology (resp. in the étale topology), then the latter three (resp. two) are isomorphisms.

See [73, Thm. 9.2.5] for the proof. (The assumption that schemes are locally noetherian in the beginning of [73] does not play any major role there.)

Remark 3.1.2.6. The assumption that $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_Z$ holds universally is true, for example, if Z is an abelian schemes over S. Note that it may fail even if Z is proper smooth over S but does not have geometrically connected fibers. (This issue will show up in our later construction of boundary charts.)

Assumption 3.1.2.7. *Let us assume from now that the scheme $f : Z \to S$ is of finite presentation, that $\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_Z$ holds universally, and that f admits a section $e_Z : S \to Z$.*

Under this assumption, $\underline{\mathrm{Pic}}(Z/S)$ is canonically isomorphic to $\underline{\mathrm{Pic}}_{e_Z}(Z/S)$ by rigidifications. We shall assume from now on that all our H-torsors $\mathcal{M}$ are rigidified. That is, each of them is equipped with a (necessarily unique) isomorphism $H \cong e_Z^*\mathcal{M}$.

As a result, since H is a split group of multiplicative type of finite type over S, which is naturally embedded as a subgroup of a split torus over S, we see that the gluing datum $\{\{U_i\}, \{a_{ij}\}\}$ as a class in $R^1 f_{*\text{ét}}(Z, H)$ can be identified with a class in $R^1 f_{*\text{zar}}(Z, H)$. That is, we may assume that there is a Zariski open covering $\{U'_i\}$ of Z over S trivializing the H-torsor $\mathcal{M}$ with gluing isomorphisms $\{a'_{ij}\}$ as above. Let us summarize the above as follows:

Corollary 3.1.2.8. *If H is a split group of multiplicative type of finite type over S, then any H-torsor over Z can be defined by a gluing datum $\{\{U_i\}, \{a_{ij}\}\}$ in the Zariski (or étale, or fppf) topology over S.*

Suppose that we have a H-torsor $\mathcal{M}$. Regarding the torsor as a relatively affine geometric object over Z, with projection $\pi : \mathcal{M} \to Z$, we can consider the push-forward $\pi_* \mathscr{O}_{\mathcal{M}}$ of the structural sheaf $\mathscr{O}_{\mathcal{M}}$ over $\mathscr{O}_Z$. Then $\pi_* \mathcal{M}$ is a sheaf of $\mathscr{O}_Z$-algebras over Z, and $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z}(\pi_* \mathscr{O}_{\mathcal{M}})$.

Convention 3.1.2.9. *By abuse of notations, we usually write simply $\mathscr{O}_{\mathcal{M}}$ for $\pi_* \mathscr{O}_{\mathcal{M}}$ in this and all similar cases, as long as π is relatively affine. We say in this case that we consider $\mathscr{O}_{\mathcal{M}}$ as an $\mathscr{O}_Z$-sheaf of algebras over Z.*

The pullback action of H_Z is given by isomorphisms

$$T_h^* : \mathscr{O}_{\mathcal{M}} \xrightarrow{\sim} \mathscr{O}_{\mathcal{M}}$$

for any $h \in H_Z$. Since H is commutative, we get a decomposition of $\mathscr{O}_{\mathcal{M}}$ into *weight spaces* (which are a priori quasi-coherent sheaves)

$$\mathscr{O}_{\mathcal{M}} = \bigoplus_{\chi \in \underline{X}(H) \cong \mathbb{Z}} \mathscr{O}_{\mathcal{M}, \chi}.$$

Using the fact that $\mathcal{M}$ is a H-torsor, which means it is isomorphic to H_Z after some étale surjective base change, we see that all the quasi-coherent sheaves $\mathscr{O}_{\mathcal{M},\chi}$ are actually invertible sheaves, since the property of being locally free of rank one is invariant under étale base change. Moreover, the canonical morphism $\mathscr{O}_{\mathcal{M},\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M},\chi'} \to \mathscr{O}_{\mathcal{M},\chi+\chi'}$ is an isomorphism, because it is an isomorphism after the étale surjective base change that trivializes $\mathcal{M}$.

If we have a Zariski open covering $\{U_i\}$ of Z over S such that we have $\mathcal{M}|_{U_i} \cong H_Z \underset{Z}{\times} U_i$ and gluing isomorphisms

$$a_{ij} \times \mathrm{Id} : H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \xrightarrow{\sim} H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j).$$

In terms of structural sheaves, we have isomorphisms $\mathscr{O}_{\mathcal{M}}|_{U_i} \cong \mathscr{O}_{H_Z}|_{U_i}$, and the multiplication

$$H_Z \xrightarrow{a_{ij}} H_Z$$

corresponds by definition of $\mathscr{O}_{\mathcal{M},\chi}$ to the multiplication

$$\mathscr{O}_{H_Z,\chi} \xleftarrow{\chi(a_{ij})} \mathscr{O}_{H_Z,\chi}\ ,$$

where the roles of H_Z's are reversed. In order to see how the trivializations of invertible sheaves $\mathscr{O}_{\mathcal{M}}|_{U_i} \cong \mathscr{O}_{H_Z}|_{U_i}$ are glued, we have to consider

$$\mathscr{O}_{H_Z,\chi} \xrightarrow{\chi(a_{ij})^{-1}} \mathscr{O}_{H_Z,\chi}\ .$$

That is, we glue compatibly

$$\begin{tikzcd}
\mathscr{O}_Z|_{U_i \underset{Z}{\times} U_j} \arrow[r, "\chi(a_{ij})^{-1}"] \arrow[d, dotted, leftrightarrow] & \mathscr{O}_Z|_{U_i \underset{Z}{\times} U_j} \arrow[d, dotted, leftrightarrow] \\
H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \arrow[r, "a_{ij}"] & H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j)
\end{tikzcd} ,$$

which means that $\mathscr{O}_{\mathcal{M},\chi}$ is obtained by gluing $\mathscr{O}_Z|_{U_i}$ using the gluing isomorphisms given by $\{\chi(a_{ij})^{-1}\}$, or by $(-\chi)(a_{ij})$ if we think of the character group additively. This datum $\{\{U_i\}, \{(-\chi)(a_{ij})\}\}$ is a class in $R^1 f_{*\mathrm{zar}}(Z, \mathbf{G}_{\mathrm{m}})$.

Given any invertible sheaf $\mathscr{N}$ on a base scheme Z, we obtain a $\mathbf{G}_{\mathrm{m}}$-torsor by setting $\mathcal{N} = \underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{N})$. If $\mathscr{N}$ is given by some gluing datum $\{\{U'_i\}, \{a'_{ij}\}\}$ as above, then we see that $\mathcal{N}$ is given by the same datum $\{\{U'_i\}, \{a'_{ij}\}\}$. (We can use either the Zariski, étale, or fppf topology in this construction.) Now consider the $\mathbf{G}_{\mathrm{m}}$-torsor given by the gluing datum $\{\{U_i\}, \{(-\chi)(a_{ij})\}\}$ for the invertible sheaf $\mathscr{O}_{\mathcal{M},\chi}$. We claim that this $\mathbf{G}_{\mathrm{m}}$-torsor can be obtained by a different process called *push-out*.

Let us introduce the push-out $\mathcal{M}_\chi$ of a H-torsor $\mathcal{M}$ by a character $\chi : H \to \mathbf{G}_{\mathrm{m}}$ as follows: It is defined as the quotient $\mathcal{M}_\chi$ of $\mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} \mathcal{M}$ under the equivalence relation

$$(x, tl) \sim (x\chi(t), l),$$

where x is a functorial point of $\mathbf{G}_{\mathrm{m},Z}$, where l is a functorial point of $\mathcal{M}$, where t is a functorial point of H_Z, and where tl denotes the image of l under the H-torsor action of t on $\mathcal{M}$. In other words, $\mathcal{M}_\chi$ is the quotient of $\mathcal{M}$ by

$\ker(\chi) \subset H_Z$ when χ is nontrivial, and is the trivial $\mathbf{G}_{\mathrm{m}}$-torsor $\mathbf{G}_{\mathrm{m},Z}$ on Z when χ is trivial. Alternatively, the push-out $\mathcal{M}_\chi$ is the largest quotient of $\mathcal{M}$ upon which H acts by χ.

Suppose the H-torsor $\mathcal{M}$ is given by local trivializations $\mathcal{M}|_{U_i} \cong (H_Z \underset{Z}{\times} U_i)$, where $\{U_i\}$ is an open covering in the Zariski topology, together with gluing isomorphisms

$$a_{ij} \times \mathrm{Id} : H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \xrightarrow{\sim} H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) .$$

Then the push-out $\mathbf{G}_{\mathrm{m}}$-torsor is given by the induced trivializations $\mathcal{M}_\chi|_{U_i} \cong \mathbf{G}_{\mathrm{m}} \times U_i$ and gluing isomorphisms

$$\chi(a_{ij}) \times \mathrm{Id} : \mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \xrightarrow{\sim} \mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) .$$

That is, $\mathcal{M}_\chi$ is given by the gluing datum $\{\{U_i\}, \{\chi(a_{ij})\}\}$, which corresponds to a class in $R^1 f_{*\mathrm{zar}}(Z, \mathbf{G}_{\mathrm{m}}) = \mathrm{Pic}(Z/S)$.

Note that $\mathcal{M}_\chi$ is naturally rigidified if $\mathcal{M}$ is. On the other hand, the torsor $\underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{M},\chi})$ has a natural rigidification as follows: From the rigidification $\xi : H \to e_Z^* \mathcal{M}$ of $\mathcal{M}$, we have by restriction to the χ-weight spaces an isomorphism $\xi_\chi^* : e_Z^* \mathscr{O}_{\mathcal{M},\chi} \cong \mathscr{O}_{H,\chi}$. The identity section e_H of H gives an augmentation homomorphism $e_H^* : \mathscr{O}_H = \underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{O}_{H,\chi} \to \mathscr{O}_S$, which restricts to a homomorphism $e_{H,\chi}^* : \mathscr{O}_{H,\chi} \to \mathscr{O}_S$ (which is an isomorphism because H is split) for each $\chi \in \underline{X}(H)$. Therefore, the torsor $\underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{M},\chi})$ has a distinguished element over e_Z whose composition with $e_{H,\chi}^* \xi_\chi^*$ is the identity map on $\mathscr{O}_S$. (Note that $e_Z^* \mathscr{O}_Z = \mathscr{O}_S$.) In other words, the torsor $\underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{M},\chi})$ can be rigidified.

By looking at the gluing datum, we get the following two propositions:

Proposition 3.1.2.10. *Under Assumption 3.1.2.7, let $\mathcal{M}$ be a rigidified H-torsor over some base scheme Z, where H is a split group of multiplicative type of finite type over S. Then the push-out operation defines a group homomorphism*

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S) : \chi \mapsto \mathcal{M}_\chi.$$

In particular, for each pair of $\chi, \chi' \in \underline{X}(H)$, there is a canonical isomorphism

$$\mathcal{M}_\chi \otimes \mathcal{M}_{\chi'} \cong \mathcal{M}_{\chi+\chi'} \tag{3.1.2.11}$$

respecting the rigidifications.

Proposition 3.1.2.12. *Under Assumption 3.1.2.7, let $\mathcal{M}$ be a rigidified H-torsor over some base scheme $Z \to S$, where H is a split group of multiplicative type of finite type over S. By considering the H_Z-action on the structural sheaf $\mathscr{O}_{\mathcal{M}}$, let $\mathscr{O}_{\mathcal{M},\chi}$ be the weight χ space under the action of H_Z, where $\chi \in \underline{X}(H)$ is a character. Let $\mathcal{M}_{-\chi}$ be the push-out of $\mathcal{M}$ by the character $(-\chi) : H \to \mathbf{G}_{\mathrm{m}}$. Then we have a (necessarily unique) isomorphism*

$$\mathcal{M}_{-\chi} \cong \underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{M},\chi})$$

respecting the rigidifications.

As a corollary, we have the following relations between $\mathbf{G}_{\mathrm{m}}$-torsors and invertible sheaves:

Corollary 3.1.2.13. *Under Assumption 3.1.2.7, let $\mathcal{L}$ be an $\mathbf{G}_{\mathrm{m}}$-torsor over some base scheme Z. Then we have a (necessarily unique) isomorphism*

$$\mathcal{L} = \mathcal{L}_1 \cong \underline{\mathrm{Isom}}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{L},-1})$$

respecting the rigidifications. Here the character $1 : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ is the identity map on $\mathbf{G}_{\mathrm{m}}$, and the character $-1 : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ is the inverse map.

3.1.3 Construction Using Sheaves of Algebras

Let us continue to assume Assumption 3.1.2.7 for $f : Z \to S$, so that isomorphisms between invertible sheaves or between torsors on Z are uniquely determined by rigidifications.

Proposition 3.1.3.1. *Under Assumption 3.1.2.7, let H be a split group of multiplicative type of finite type over S, with character group $\underline{X}(H)$, and suppose that we are given a group homomorphism*

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S) : \chi \mapsto \mathscr{M}_\chi.$$

In other words, suppose that we are given a family of rigidified invertible sheaves $\mathscr{M}_\chi$ on Z, indexed by $\chi \in \underline{X}(H)$, together with the unique isomorphisms $\Delta^*_{\chi,\chi'} : \mathscr{M}_\chi \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_{\chi'} \cong \mathscr{M}_{\chi+\chi'}$ inducing the canonical isomorphisms $\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_Z \cong \mathscr{O}_Z$ by the rigidifications. Set $\mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{M}_\chi$, which is equipped with a structure of sheaf of $\mathscr{O}_Z$-algebras $\Delta^* : \mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \to \mathscr{O}_{\mathcal{M}}$ defined by the isomorphisms $\Delta^*_{\chi,\chi'}$. Then $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z} \mathscr{O}_{\mathcal{M}}$ has a natural structure of a rigidified H-torsor such that $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_\chi$, where $\mathscr{O}_{\mathcal{M},\chi}$ is the weight χ space under H_Z-action.

Note that here we are abusing the notations by considering $\mathscr{O}_{\mathcal{M}}$ as an $\mathscr{O}_Z$-sheaf of algebras. More properly speaking we should use the push-forward in this case, or rather a completely different notation. However, it would be clear later that this will only increase the complexity of notations, which is unnecessary from my point of view.

Proof. Let us first define an action of H_Z on $\mathcal{M}$, namely a morphism $m : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M}$ satisfying the usual requirement for an action.

Let us write $\mathscr{O}_{H_Z} = \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{O}_{H_Z,\chi}$ as in the last section. Then $\mathscr{O}_{H_Z,\chi} \cong \mathscr{O}_Z$, because H_Z is a trivial H-torsor, and we can write

$$\mathscr{M}_\chi \overset{\text{can.}}{\cong} \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi \cong \mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi.$$

All these isomorphisms are uniquely determined if we require the invertible modules to be rigidified. In particular, we get a morphism

$$m^* : \mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi \to \bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \subset \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}},$$

which makes the following diagrams commutative:

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi \arrow[r, "m^*"] \arrow[dd, "m^*"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[d, "\mathrm{Id} \otimes m^*"] \\
 & \bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)) \arrow[d, "\text{can.}", "\wr"'] \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[r, "m^*_{H_Z} \otimes \mathrm{Id}"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} ((\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi \arrow[r, "m^*"] \arrow[dd, "\mathrm{Id}"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[d, hook] \\
 & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "e^*_{H_Z} \otimes \mathrm{Id}"] \\
\mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi \arrow[r, "\sim", "\text{can.}"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi
\end{tikzcd}$$

Note that the latter diagram is commutative because the map

$$e_{H_Z}^* : \mathscr{O}_{H_Z} = \bigoplus_{\chi \in \underline{X}(H)} \mathscr{O}_{H_Z,\chi} \to \mathscr{O}_Z$$

is defined locally by summation:

$$\sum_{\chi \in \underline{X}(H)} c_\chi X^\chi \mapsto \sum_{\chi \in \underline{X}(H)} c_\chi,$$

if we use $c_\chi X^\chi$ to denote symbolically an element in $\mathscr{O}_{H_Z,\chi}$ being sent to c_χ under $\mathscr{O}_{H_Z,\chi} \cong \mathscr{O}_Z$.

Translating the above diagrams back to diagrams of schemes, we get a morphism

$$m : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M},$$

which is an action because of the commutativity of the following diagrams:

$$\begin{tikzcd}
(H_Z \underset{Z}{\times} H_Z) \underset{Z}{\times} \mathcal{M} \arrow[r, "m_{H_Z} \times \mathrm{Id}"] \arrow[d, "\mathrm{can.}"', "\wr"] & H_Z \underset{Z}{\times} \mathcal{M} \arrow[dd, "m"] \\
H_Z \underset{Z}{\times} (H_Z \underset{Z}{\times} \mathcal{M}) \arrow[d, "\mathrm{Id} \times m"'] & \\
H_Z \underset{Z}{\times} \mathcal{M} \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{Id}"] \arrow[d, "(e_{H_Z}, \mathrm{Id})"'] & \mathcal{M} \arrow[d, "\mathrm{Id}"] \\
(H_Z \underset{Z}{\times} \mathcal{M}) \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

Now we check that $(m, \mathrm{pr}_2) : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is an isomorphism. Equivalently, we want the composition

$$\begin{tikzcd}
H_Z \underset{Z}{\times} \mathcal{M} \arrow[r, "\mathrm{Id} \times \Delta"] & H_Z \underset{Z}{\times} (\mathcal{M} \underset{Z}{\times} \mathcal{M}) \arrow[d, "\wr"', "\mathrm{can.}"] & \\
& (H_Z \underset{Z}{\times} \mathcal{M}) \underset{Z}{\times} \mathcal{M} \arrow[r, "m \times \mathrm{Id}"] & \mathcal{M} \underset{Z}{\times} \mathcal{M}
\end{tikzcd}$$

to be an isomorphism, where $\Delta : \mathcal{M} \to \mathcal{M} \underset{Z}{\times} \mathcal{M}$ is the diagonal map. In terms of sheaves of $\mathscr{O}_Z$-algebras, we want the composition

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[r, "m^* \times \mathrm{Id}"] & (\mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "\wr"', "\text{can.}"] & \\
& \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} (\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \arrow[r, "\mathrm{Id} \otimes \Delta^*"] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}
\end{tikzcd}$$

to be an isomorphism. This is true because this composition can be factored into a sequence of isomorphisms, in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "\wr"', "m^* \otimes \mathrm{Id}"] \arrow[dr, "m^* \otimes \mathrm{Id}"] & \\
\Big(\underset{\chi \in \underline{X}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{M}_\chi\Big) \arrow[r, hook] \arrow[d, "\wr"', "\text{can.}"] & (\mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "\wr"', "\text{can.}"] \\
\Big(\underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{O}_{H_Z,\chi}\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\underset{\chi' \in \underline{X}(H)}{\oplus} \mathscr{M}_\chi \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_{\chi'}\Big) \arrow[r, hook] \arrow[d, "\wr"', "\mathrm{Id} \otimes \Delta^*"] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} (\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \arrow[d, "\mathrm{Id} \otimes \Delta^*"] \\
\Big(\underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{O}_{H_Z,\chi}\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\underset{\chi'' \in \underline{X}(H)}{\oplus} \mathscr{M}_{\chi''}\Big) \arrow[r, equal] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}
\end{tikzcd}$$

The fact that $\mathscr{O}_{\mathcal{M}}$ is rigidified follows because we have compatible isomorphisms $e_Z^* \mathscr{M}_\chi \cong \mathscr{O}_S$, respecting the canonical isomorphism $\mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \cong \mathscr{O}_S$, coming from the rigidifications. These isomorphisms patch together and give an isomorphism

$$e_Z^* \mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{X}(H)}{\oplus} e_Z^* \mathscr{M}_\chi \cong \underset{\chi \in \underline{X}(H)}{\oplus} \mathscr{O}_S \cong \mathscr{O}_H,$$

which is the rigidification we want.

Finally, to verify the statement that $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_\chi$, observe that the multiplication by h where $h : Z \to H_Z = H \underset{S}{\times} Z$ is a Z-valued point of H, is given by

$$H_Z \xrightarrow[\sim]{\text{can.}} Z \underset{Z}{\times} H_Z \xrightarrow{h \times \mathrm{Id}} H_Z \underset{Z}{\times} H_Z \xrightarrow{m_{H_Z}} H_Z$$

in the case of multiplication on H_Z, and by

$$\mathcal{M} \xrightarrow[\sim]{\text{can.}} Z \underset{Z}{\times} \mathcal{M} \xrightarrow{h \times \mathrm{Id}} H_Z \underset{Z}{\times} \mathcal{M} \xrightarrow{m} \mathcal{M}$$

in the case of H_Z-action on $\mathcal{M}$. In terms of sheaves of algebras, the former case is given by

$$\begin{tikzcd}
\mathscr{O}_{H_Z} \arrow[r, "m^*_{H_Z}"] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z} \arrow[r, "h^* \otimes \mathrm{Id}"] & \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z} \cong \mathscr{O}_{H_Z} \quad , \\
\underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{O}_{H_Z,\chi} \arrow[u, equal] \arrow[r, "m^*_{H_Z}"] & \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \arrow[u, hook] \arrow[r, "h^* \otimes \mathrm{Id}"] & \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \arrow[u, "\text{can.}", "\wr"']
\end{tikzcd}$$

which suggests that

$$h^* : \underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{O}_{H_Z,\chi} \to \mathscr{O}_Z$$

is given by

$$\sum_{\chi\in\underline{X}(H)} c_\chi X^\chi \mapsto \sum_{\chi\in\underline{X}(H)} c_\chi \chi(h),$$

if we use $c_\chi X^\chi$ to denote symbolically an element in $\mathscr{O}_{H_Z,\chi}$ being sent to c_χ under $\mathscr{O}_{H_Z,\chi} \cong \mathscr{O}_Z$, and if we think of χ as a map

$$H(Z) \to \mathbf{G}_{\mathrm{m}}(Z) = \mathscr{O}_Z{}^\times \subset \mathscr{O}_Z.$$

The case for H_Z action on $\mathcal{M}$ in terms of sheaves of algebras is then given by

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} \arrow[r, "m^*"] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[r, "h^* \otimes \mathrm{Id}"] & \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \cong \mathscr{O}_{\mathcal{M}} \quad , \\
\underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{M}_\chi \arrow[u, equal] \arrow[r, "m^*_{H_Z}"] & \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[u, hook] \arrow[r, "h^* \otimes \mathrm{Id}"] & \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[u, "\text{can.}", "\wr"']
\end{tikzcd}$$

which by comparison implies that $\mathscr{M}_\chi$ is the weight χ space under H_Z-action in $\mathscr{O}_{\mathcal{M}}$, as desired. □

Remark 3.1.3.2. If we define the action by

$$m^* : \mathscr{O}_{\mathcal{M}} = \underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{M}_\chi \to \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_{H_Z,\gamma(\chi)} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \subset \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}},$$

where $\gamma \in \mathrm{GL}(\underline{X}(H))$, then we will have $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_{\gamma(\chi)}$. That is, we can twist the structure of a H-torsor by applying an action of $\mathrm{GL}(\underline{X}(H))$ to the way we index the sheaves $\mathscr{M}_\chi$ by characters.

Remark 3.1.3.3. Using the fact that every weight has to appear in $\mathscr{O}_{\mathcal{M}}$, because they do after an étale surjective base change that trivializes the torsor $\mathcal{M}$, we see that any possible structure of a H-torsor on $\mathcal{M}$ satisfies $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_{\gamma(\chi)}$ for some $\gamma \in \mathrm{GL}(\underline{X}(H))$. In particular, we can conclude that there is a unique structure of a H-torsor such that $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_\chi$, defined exactly as in the proposition.

Remark 3.1.3.4. In what follows, we should specify the action of H_Z by using the same indices as in the proof of the proposition. That is, if we write $\underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{M}_\chi$, then we assume that the action is given by

$$m^* : \mathscr{O}_{\mathcal{M}} = \underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{M}_\chi \to \underset{\chi\in\underline{X}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \subset \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}.$$

Corollary 3.1.3.5. *Under Assumption 3.1.2.7, let $\mathscr{L}$ be a rigidified invertible sheaf on Z. Identify $\underline{X}(\mathbf{G}_{\mathrm{m}}) = \mathbb{Z}$ naturally by sending* $(\mathrm{Id} : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}})$ *to* 1, *and let $\Delta^*_{i,j} : \mathscr{L}^{\otimes i} \underset{\mathscr{O}_Z}{\otimes} \mathscr{L}^{\otimes j} \cong \mathscr{L}^{\otimes i+j}$ be the canonical isomorphism, which respects the canonical isomorphism $\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_Z \cong \mathscr{O}_Z$ by the rigidification. Then $\mathcal{L}$ has a natural structure of a rigidified $\mathbf{G}_{\mathrm{m}}$-torsor such that $\mathscr{O}_{\mathcal{L},i} = \mathscr{L}^{\otimes -i}$. In particular, $\mathscr{O}_{\mathcal{L},-1} = \mathscr{L}$.*

To justify our choices above, assume that the scheme $f : Z \to S$ satisfies Assumption 3.1.2.7. Let $\{\{U_i\}, \{a_{ij}\}\}$ be a gluing datum of a rigidified invertible sheaf $\mathscr{L}$ in $\mathrm{Pic}_{e_Z}(Z/S) = R^1 f_{*\mathrm{zar}} \mathbf{G}_{\mathrm{m}}$. If we form

$$\mathcal{L} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z} \underset{i\in\mathbb{Z}}{\oplus} \mathscr{L}^{\otimes -i},$$

then locally

$$\mathcal{L}|_{U_i} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{U_i}} \underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{O}_{U_i}^{\otimes -i} \cong \mathbf{G}_{\mathrm{m},U_i},$$

and the gluing datum is given by the same $\{\{U_i\}, \{a_{ij}\}\}$ in $R^1 f_{*\mathrm{zar}} \mathbf{G}_{\mathrm{m}}$, as we have seen explicitly in the previous section. As a result, $\mathcal{L}$ corresponds to the same class as $\mathscr{L}$ in $R^1 f_{*\mathrm{zar}} \mathbf{G}_{\mathrm{m}}$.

The same argument as above shows that

$$\overline{\mathcal{L}} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z} \underset{i\in\mathbb{Z}_{\geq 0}}{\oplus} \mathscr{L}^{\otimes -i}$$

gives a line bundle over Z whose gluing datum defines the same class as $\mathscr{L}$ in $R^1 f_{*\mathrm{zar}}\mathbf{G}_{\mathrm{m}}$. In this case, $\mathcal{L}$ can be embedded as an open subscheme in $\overline{\mathcal{L}}$, which fits into our intuition that $\mathcal{L}$ is the open subscheme of $\overline{\mathcal{L}}$ with zero-section deleted.

Indeed, in our proof of Proposition 3.1.3.1, a part from the verification that $\mathcal{M}$ is a H-torsor, the definition of an action of H_Z on $\mathcal{M}$ have not used the full strength of our assumptions. In particular, let X' be any semi-subgroup of $\underline{X}(H)$, then we can define a sheaf of algebras by setting

$$\mathscr{O}_{\overline{\mathcal{M}}} = \underset{\chi \in X'}{\oplus} \mathscr{L}_\chi,$$

and define an action by

$$m^* : \mathscr{O}_{\overline{\mathcal{M}}} = \underset{\chi \in X'}{\oplus} \mathscr{L}_\chi \to \underset{\chi \in X'}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{L}_\chi).$$

It can be verified easily that this defines an action $m : H_Z \underset{Z}{\times} \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ on $\overline{\mathcal{M}} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z} \mathscr{O}_{\overline{\mathcal{M}}}$ such that $\mathscr{O}_{\overline{\mathcal{M}},\chi} = \mathscr{L}_\chi$. The descriptions of the sheaves of algebras make $\mathcal{M}$ naturally embedded as an open subscheme of $\overline{\mathcal{M}}$, in the sense that the embedding is actually H_Z-equivariant. These leads naturally to the theory of toroidal embeddings of torsors. (See [72] and [15], and see Section 6.1.)

Example 3.1.3.6. For a rigidified invertible sheaf $\mathscr{L}$ on Z, if we consider the semi-subgroup $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}$, then we get the line bundle $\mathcal{L} = \underline{\mathrm{Spec}}_{\mathscr{O}_Z} \underset{i \in \mathbb{Z}_{\geq 0}}{\oplus} \mathscr{L}^{\otimes -i}$, in which case $\mathscr{O}_{\mathcal{L},i} = \mathscr{L}^{\otimes -i}$.

We can combine Propositions 3.1.2.10 and 3.1.3.1 as the following:

Theorem 3.1.3.7. *Let H be a group of multiplicative type of finite type over S. Let $Z \to S$ be a scheme that satisfies Assumption 3.1.2.7. Then the category of rigidified H-torsors up to H-equivariant isomorphisms over Z is anti-equivalent to the category of maps*

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S),$$

between étale sheaves over S.

Remark 3.1.3.8. This equivalence is also functorial in terms of H. As a result, the theorem works for nonsplit H as well by étale descent.

3.1.4 Group Structures on Torsors

Let us now develop a theory of *relative Hopf algebras* that describes group schemes G over S that are relative affine over some (possibly non-affine) group schemes A over S. Let H be a split group of multiplicative type of finite type over S. Our precise goal in mind is to describe H-torsors $\mathcal{M}$ over an abelian scheme A that admit group structures.

Suppose that we are given group schemes $G \to S$ and $A \to S$, together with a *relative affine* group scheme homomorphism $G \to A$ over S. Then we have the following commutative diagrams:

(i) Multiplication m_G of G covering the multiplication m_A of A:

$$\begin{tikzcd}
G \underset{S}{\times} G \arrow[r, "m_G"] \arrow[d, "\text{str.}"'] & G \arrow[d, "\text{str.}"] \\
A \underset{S}{\times} A \arrow[r, "m_A"'] & A
\end{tikzcd}$$

(ii) Identity e_G of G covering the identity e_A of A:

$$\begin{tikzcd}
S \arrow[r, "e_G"] \arrow[d, equal] & G \arrow[d, "\text{str.}"] \\
S \arrow[r, "e_A"'] & A
\end{tikzcd}$$

(iii) Inverse $[-1]_G$ of G covering the inverse $[-1]_A$ of A:

$$\begin{tikzcd}
G \arrow[r, "{[-1]_G}"] \arrow[d, "\text{str.}"'] & G \arrow[d, "\text{str.}"] \\
A \arrow[r, "{[-1]_A}"'] & A
\end{tikzcd}$$

(iv) Diagonal Δ_G of G covering the diagonal Δ_A of A:

$$\begin{tikzcd}
G \arrow[r, "\Delta_G"] \arrow[d, "\text{str.}"'] & G \underset{S}{\times} G \arrow[d, "\text{str.}"] \\
A \arrow[r, "\Delta_A"'] & A \underset{S}{\times} A
\end{tikzcd}$$

(v) Structural morphism of G covering the structural morphism of A:

$$\begin{tikzcd}
G \arrow[r, "\text{str.}"] \arrow[d, "\text{str.}"'] & S \arrow[d, equal] \\
A \arrow[r, "\text{str.}"'] & S
\end{tikzcd}$$

Note that G is relative affine over A, and this is true after arbitrary base change. Therefore, by taking push-forward and by abuse of language, we may view the structural sheaf $\mathscr{O}_G$ of G as a sheaf of $\mathscr{O}_A$-modules. To save notation, let us denote by pr_1, pr_2, pr_3, pr_{12}, pr_{23}, etc the projections from products of copies of A. Therefore we can translate the above morphisms m_G, e_G, and $[-1]_G$ into homomorphisms of sheaves of algebras:

(i) Comultiplication

$$m^* : m_A^* \mathscr{O}_G \to \mathscr{O}_{G \underset{S}{\times} G} = \mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G$$

as a homomorphism of $\mathscr{O}_{A \underset{S}{\times} A}$-algebras.

(ii) Counit

$$e^* : e_A^* \mathscr{O}_G \to \mathscr{O}_S$$

as a homomorphism of $\mathscr{O}_S$-algebras.

(iii) Coinverse

$$[-1]^* : [-1]_A^* \mathscr{O}_G \to \mathscr{O}_G$$

as a homomorphism of $\mathscr{O}_A$-algebras.

(iv) Codiagonal

$$\Delta^* : \Delta_A^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) = \mathscr{O}_G \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_G \to \mathscr{O}_G$$

as a homomorphism of $\mathscr{O}_A$-algebras.

(v) Structural morphism

$$(\mathrm{str.}_A)^* \mathscr{O}_S = \mathscr{O}_A \to \mathscr{O}_G$$

as a homomorphism of $\mathscr{O}_A$-algebras.

The associativity of the comultiplication can be checked by the commutativity of the following diagram of $\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}$-algebras (with all the unspecified tensor products over $\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}$):

$$\begin{tikzcd}
(m_A \times \mathrm{Id})^* m_A^* \mathscr{O}_G \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "(m_A \times \mathrm{Id})^*(m^*)"'] & (\mathrm{Id} \times m_A)^* m_A^* \mathscr{O}_G \arrow[d, "(\mathrm{Id} \times m_A)^*(m^*)"] \\
(m_A \times \mathrm{Id})^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & (\mathrm{Id} \times m_A)^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\wr"', "\text{can.}"] \\
\mathrm{pr}_{12}^*(m_A^* \mathscr{O}_G) \otimes \mathrm{pr}_3^* \mathscr{O}_G \arrow[d, "\mathrm{pr}_{12}^*(m^*)"'] & \mathrm{pr}_1^* \mathscr{O}_G \otimes \mathrm{pr}_{23}^*(m_A^* \mathscr{O}_G) \arrow[d, "\mathrm{pr}_{23}^*(m^*)"] \\
(\mathrm{pr}_1^* \mathscr{O}_G \otimes \mathrm{pr}_2^* \mathscr{O}_G) \otimes \mathrm{pr}_3^* \mathscr{O}_G \arrow[r, "\sim", "\text{can.}"'] & \mathrm{pr}_1^* \mathscr{O}_G \otimes (\mathrm{pr}_2^* \mathscr{O}_G \otimes \mathrm{pr}_3^* \mathscr{O}_G)
\end{tikzcd}$$

The validity of the counit can be checked by the commutativity of the following diagram of $\mathscr{O}_S$-algebras:

$$\begin{tikzcd}
(\mathrm{Id} \times e_A)^* m_A^* \mathscr{O}_G \arrow[d, "(\mathrm{Id} \times e_A)^*(m^*)"'] \arrow[rdddd, bend left, "\text{can.}", "\wr"'] & \\
(\mathrm{Id} \times e_A)^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & \\
\mathscr{O}_G \underset{\mathscr{O}_S}{\otimes} e_A^* \mathscr{O}_G \arrow[d, "\mathrm{Id} \otimes e^*"'] & \\
\mathscr{O}_G \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_G
\end{tikzcd}$$

The validity of the coinverse can be checked by the commutativity of the following diagram of $\mathscr{O}_A$-algebras:

$$\begin{tikzcd}
\Delta_A^*(\mathrm{Id}\times[-1]_A)^*m_A^*\mathscr{O}_G \arrow[d, "\Delta_A^*(\mathrm{Id}\times[-1]_A)^*(m^*)"'] \arrow[rddd, bend left, "\wr"', "\text{can.}"] & \\
\Delta_A^*(\mathrm{Id}\times[-1]_A^*)^*(\mathrm{pr}_1^*\mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & \\
\Delta_A^*(\mathrm{pr}_1^*\mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*([-1]_A^*\mathscr{O}_G)) \arrow[d, "\Delta_A^*(\mathrm{Id}\otimes[-1]^*)"'] & \\
\mathscr{O}_G \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_G \arrow[r, "\Delta^*"] & \mathscr{O}_G
\end{tikzcd}$$

If the $\mathscr{O}_A$-algebra $\mathscr{O}_G$ satisfies all these conditions, then we get a group structure on G covering the one on A.

As an application, let us assume that $A \to S$ is a group scheme that satisfies Assumption 3.1.2.7, and that G is a H-torsor $\mathcal{M}$ over A, where H is a split group of multiplicative type of finite type over S.

Proposition 3.1.4.1. *Under the above hypothesis, suppose that we are given a group homomorphism*

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_e(A/S) : \chi \mapsto \mathscr{M}_\chi,$$

such that we have (*unique*) *isomorphisms*

$$m_\chi^* : m_A^*\mathscr{M}_\chi \xrightarrow{\sim} \mathrm{pr}_1^*\mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*\mathscr{M}_\chi$$

for each $\mathscr{M}_\chi$. Then the H-torsor $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\underset{\chi\in\underline{X}(H)}{\oplus}\mathscr{M}_\chi)$ defined in Proposition 3.1.3.1 has a group structure covering the one of A.

Proof. Recall that in the proof of Proposition 3.1.3.1, the structure of a sheaf of algebra

$$\Delta^* : \mathscr{O}_\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\mathcal{M} \to \mathscr{O}_\mathcal{M}$$

on $\mathcal{M} = \underline{\operatorname{Spec}}_{\mathscr{O}_A} \bigoplus\limits_{\chi \in \underline{X}(H)} \mathscr{M}_\chi$ is given by the unique isomorphisms

$$\Delta^*_{\chi,\chi'} : \mathscr{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{M}_{\chi'} \to \mathscr{M}_{\chi+\chi'}$$

respecting the rigidifications. Moreover, by assumption, there are unique isomorphisms

$$m^*_\chi : m^*_A \mathscr{M}_\chi \xrightarrow{\sim} \operatorname{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \operatorname{pr}_2^* \mathscr{M}_\chi,$$

which by pull-back using $(\mathrm{Id}, [-1]_A)^*$ induces

$$[-1]^*_\chi : [-1]^*_A \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{M}_\chi^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{M}_{-\chi}$$

respecting the rigidifications

$$e^*_A \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{O}_S.$$

Let $e^*_H : \mathscr{O}_H = \bigoplus\limits_{\chi \in \underline{X}(H)} \mathscr{O}_{H,\chi} \to \mathscr{O}_S$ be the counit for the group H, which induces a natural isomorphism $\mathscr{O}_{H,\chi} \xrightarrow{\sim} \mathscr{O}_S$ by restriction. Let us compose the rigidifications with the inverse isomorphisms $\mathscr{O}_S \cong \mathscr{O}_{H,\chi}$ and write

$$e^*_\chi : e^*_A \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{O}_{H,\chi}.$$

Let us define the comultiplication m^* by

$$\begin{tikzcd}
m^*_A \mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi \in \underline{X}(H)} m^*_A \mathscr{M}_\chi \arrow[r, "\bigoplus\limits_{\chi \in \underline{X}(H)} m^*_\chi"] \arrow[dr, bend right=15, "\text{def.}", "m^*"'] & \bigoplus\limits_{\chi \in \underline{X}(H)} (\operatorname{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \operatorname{pr}_2^* \mathscr{M}_\chi) \arrow[d, hook] \\
& \operatorname{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \operatorname{pr}_2^* \mathscr{O}_G
\end{tikzcd},$$

define the counit e^* by

$$\begin{tikzcd}
e^*_A \mathscr{O}_{\mathcal{M}} = \bigoplus\limits_{\chi \in \underline{X}(H)} e^*_A \mathscr{M}_\chi \arrow[r, "\bigoplus\limits_{\chi \in \underline{X}(H)} e^*_\chi"] \arrow[drr, bend right=15, "\text{def.}", "e^*"'] & \bigoplus\limits_{\chi \in \underline{X}(H)} \mathscr{O}_{H,\chi} \arrow[r, "\text{can.}", "\sim"'] & \mathscr{O}_H \arrow[d, "e^*_H"] \\
& & \mathscr{O}_S
\end{tikzcd},$$

and define the coinverse $[-1]^*$ by

$$\begin{tikzcd}
{[-1]_A^*\mathscr{O}_{\mathcal{M}} = \bigoplus_{\chi\in\underline{X}(H)} [-1]_A^*\mathscr{M}_\chi} \arrow[r, "{\bigoplus_{\chi\in\underline{X}(H)} [-1]^*_\chi}"] \arrow[dr, bend right=15, "\text{def.}", "{[-1]^*}"'] & \bigoplus_{\chi\in\underline{X}(H)} \mathscr{M}_{-\chi} \arrow[d, "\text{index reversing}"] \\
 & \mathscr{O}_G
\end{tikzcd}$$

Note that the index reversing above induces the coinverse $[-1]_H^*$ on $e_A^*\mathscr{O}_G \cong \mathscr{O}_H$.

The associativity of the comultiplication can be checked by the commutativity of the following diagram of $\mathscr{O}_{A\underset{S}{\times}A\underset{S}{\times}A}$-algebras (with all the unspecified tensor products over $\mathscr{O}_{A\underset{S}{\times}A\underset{S}{\times}A}$):

$$\begin{tikzcd}
(m_A\times\mathrm{Id})^*m_A^*(\bigoplus_{\chi\in\underline{X}(H)}\mathscr{M}_\chi) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "(m_A\times\mathrm{Id})^*(m^*)"'] & (\mathrm{Id}\times m_A)^*m_A^*(\bigoplus_{\chi\in\underline{X}(H)}\mathscr{M}_\chi) \arrow[d, "(\mathrm{Id}\times m_A)^*(m^*)"] \\
\bigoplus_{\chi\in\underline{X}(H)}(m_A\times\mathrm{Id})^*(\mathrm{pr}_1^*\mathscr{M}_\chi \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\mathscr{M}_\chi) \arrow[d, "\text{can.}"', "\wr"] & \bigoplus_{\chi\in\underline{X}(H)}(\mathrm{Id}\times m_A)^*(\mathrm{pr}_1^*\mathscr{M}_\chi \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\mathscr{M}_\chi) \arrow[d, "\wr"', "\text{can.}"] \\
\bigoplus_{\chi\in\underline{X}(H)}(\mathrm{pr}_{12}^*(m_A^*\mathscr{M}_\chi)\otimes\mathrm{pr}_3^*\mathscr{M}_\chi) \arrow[d, "\mathrm{pr}_{12}^*(m^*)"'] & \bigoplus_{\chi\in\underline{X}(H)}(\mathrm{pr}_1^*\mathscr{M}_\chi\otimes\mathrm{pr}_{23}^*(m_A^*\mathscr{M}_\chi)) \arrow[d, "\mathrm{pr}_{23}^*(m^*)"] \\
\bigoplus_{\chi\in\underline{X}(H)}((\mathrm{pr}_1^*\mathscr{M}_\chi\otimes\mathrm{pr}_2^*\mathscr{M}_\chi)\otimes\mathrm{pr}_3^*\mathscr{M}_\chi) \arrow[r, "\sim", "\text{can.}"'] & \bigoplus_{\chi\in\underline{X}(H)}(\mathrm{pr}_1^*\mathscr{M}_\chi\otimes(\mathrm{pr}_2^*\mathscr{M}_\chi\otimes\mathrm{pr}_3^*\mathscr{M}_\chi))
\end{tikzcd}$$

The validity of the counit can be checked by the commutativity of the following diagram of $\mathscr{O}_S$-algebras:

$$\begin{tikzcd}
(\mathrm{Id}\times e_A)^* m_A^* \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi \arrow[d, "(\mathrm{Id}\times e_A)^*(m^*)"'] \arrow[rddd, bend left, "\text{can.}", "\wr"'] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\mathrm{Id}\times e_A)^*(\mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi) \arrow[d, "\text{can.}"', "\wr"] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{M}_\chi \underset{\mathscr{O}_S}{\otimes} e_A^* \mathscr{M}_\chi) \arrow[d, "\mathrm{Id}\otimes e^*"'] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{M}_\chi \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S) \arrow[r, "\sim", "\text{can.}"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi
\end{tikzcd}$$

Note that the diagram is commutative because the last two compositions in the definition of e^* actually cancel the inverse of the isomorphism $\mathscr{O}_{H,\chi} \xrightarrow{\sim} \mathscr{O}_S$ we used in defining e_χ^*.

The validity of the coinverse can be checked by the commutativity of the following diagram of $\mathscr{O}_A$-algebras:

$$\begin{tikzcd}
\Delta_A^*(\mathrm{Id}\times[-1]_A)^* m_A^* (\bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi) \arrow[d, "\Delta_A^*(\mathrm{Id}\times[-1]_A)^*(m^*)"'] \arrow[rddd, bend left, "\text{can.}", "\wr"'] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\Delta_A^*(\mathrm{Id}\times[-1]_A^*)^*(\mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi)) \arrow[d, "\text{can.}"', "\wr"] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\Delta_A^*(\mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*([-1]_A^* \mathscr{M}_\chi))) \arrow[d, "\Delta_A^*(\mathrm{Id}\otimes[-1]^*)"'] & \\
\bigoplus\limits_{\chi\in\underline{X}(H)} (\mathscr{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{M}_\chi) \arrow[r, "\Delta^*"'] & \bigoplus\limits_{\chi\in\underline{X}(H)} \mathscr{M}_\chi
\end{tikzcd}$$

Since the $\mathscr{O}_A$-algebra $\mathscr{O}_{\mathcal{M}}$ satisfies all these conditions, we do get a group structure on $\mathcal{M}$ covering the one on A. □

Remark 3.1.4.2. The only non-routine part of the proof is the definitions of m_χ^*, $[-1]_\chi^*$, and e_χ^*, which rely on the fact that we have isomorphisms

$$m_\chi^* : m_A^* \mathscr{M}_\chi \xrightarrow{\sim} \mathrm{pr}_1^* \mathscr{M}_\chi \otimes \mathrm{pr}_2^* \mathscr{M}_\chi$$

for each $\mathscr{M}_\chi$.

Corollary 3.1.4.3. *Suppose that $A \to S$ is a group scheme that satisfies Assumption 3.1.2.7, and we have a rigidified $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$ together with an isomorphism*

$$m_A^* \mathcal{L} \xrightarrow{\sim} \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}$$

respecting the rigidifications. Then $\mathcal{L}$ has a group structure covering the one of A.

This follows directly from Proposition 3.1.4.1 and Corollary 3.1.3.5.

Corollary 3.1.4.4. *Suppose A is an abelian scheme over a base scheme S, H is a split group of multiplicative type of finite type over S, and we have a group homomorphism*

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_e^0(A/S) : \chi \mapsto \mathscr{M}_\chi.$$

Then the H-torsor $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\bigoplus_{\chi \in \underline{X}(H)} \mathscr{M}_\chi)$ defined as in Proposition 3.1.3.1 has a group structure covering the one of A.

This is true because $\mathscr{L} \in \underline{\mathrm{Pic}}_e^0(A/S)$ if and only if there exists a unique isomorphism $m_A^* \mathscr{L} \xrightarrow{\sim} \mathrm{pr}_1^* \mathscr{L} \otimes \mathrm{pr}_2^* \mathscr{L}$.

3.1.5 Group Extensions

Let H be a split group of multiplicative type of finite type over S as before. Let $\mathcal{M}$ be a torsor over an abelian scheme A over a base scheme S. (Note that in this case the morphism $A \to S$ satisfies Assumption 3.1.2.7.) Suppose that $\mathcal{M}$ admits a group structure over S such that the structural projection $\mathcal{M} \to A$ is a group scheme homomorphism over S. Let $e : S \to \mathcal{M}$ be the identity section of $\mathcal{M}$. Then the orbit of e under H gives an embedding of H into $\mathcal{M}$, whose image is isomorphic to the kernel of $\mathcal{M} \to A$. In particular, we get an exact sequence

$$0 \to H \to \mathcal{M} \to A \to 0,$$

which is an extension of the abelian scheme A by the group H over S.

Proposition 3.1.5.1. *The category of commutative group scheme extensions E of A by H up to isomorphisms of the form*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\wr"] & A \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r] & E' \arrow[r] & A \arrow[r] & 1
\end{tikzcd}$$

over S is anti-equivalent to the category of maps

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_e^0(A/S) = A^\vee,$$

between étale sheaves over S.

Proof. Let $\mathscr{M}_\chi$ be the rigidified invertible sheaf corresponding to the rigidified $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}_{-\chi}$ defined by push-out by $(-\chi) \in \underline{X}(H)$. This can be described as the largest quotient $\mathcal{M}_{-\chi}$ of $\mathcal{M}$ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, "-\chi"'] & \mathcal{M} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathcal{M}_{-\chi} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

We have already seen that this defines a homomorphism

$$\underline{X}(H) \to \underline{\mathrm{Pic}}_e(A/S) : \chi \mapsto \mathscr{M}_\chi,$$

and conversely that any such homomorphism defines a torsor by $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\bigoplus_{\chi \in \underline{X}(H)} \mathscr{M}_\chi)$. The question is whether $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}_e^0(A/S)$.

We claim that $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}_e^0(A/S)$ if $\mathcal{M}$ has a group structure covering the one of A. In this case, the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}_{-\chi}$ would have a group structure covering the one of A, which gives an isomorphism

$$\mathrm{pr}_1^* \mathcal{M}_{-\chi} \otimes \mathrm{pr}_2^* \mathcal{M}_{-\chi} \cong m_A^* \mathcal{M}_{-\chi}$$

over $A \underset{S}{\times} A$. This isomorphism corresponds to an isomorphism

$$\mathrm{pr}_1^* \mathscr{M}_\chi \otimes \mathrm{pr}_2^* \mathscr{M}_\chi \cong m_A^* \mathscr{M}_\chi$$

in $\underline{\mathrm{Pic}}_e(A \underset{S}{\times} A/S)$. In other words,

$$m_A^* \mathscr{M}_\chi \otimes \mathrm{pr}_1^* \mathscr{M}_\chi^{\otimes -1} \otimes \mathrm{pr}_2^* \mathscr{M}_\chi^{\otimes -1} \tag{3.1.5.2}$$

is trivial in $\underline{\mathrm{Pic}}_e(A \underset{S}{\times} A/S)$. By the theory of abelian varieties [99] and the rigidity lemma [97, Prop. 6.1], this is true if and only if $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}_e^0(A/S)$.

The converse is just Corollary 3.1.4.4. □

Remark 3.1.5.3. The only crucial properties we have used in the proof about $\underline{\mathrm{Pic}}_e^0$ of abelian schemes are that we have rigidifications and that $m_A^* \mathscr{M}_\chi \otimes \mathrm{pr}_1^* \mathscr{M}_\chi^{\otimes -1} \otimes \mathrm{pr}_2^* \mathscr{M}_\chi^{\otimes -1}$ is trivial in $\underline{\mathrm{Pic}}_e(A \underset{S}{\times} A/S)$ if and only if $\mathscr{M}$ lies in $\underline{\mathrm{Pic}}_e^0(A/S)$.

3.2 Biextensions and Cubical Structures

3.2.1 Biextensions

Definition 3.2.1.1 (Mumford)**.** *Let G, H, C be three* (*usual*) *commutative groups. A* **biextension** *of $G \times H$ by C is a set B on which C acts freely, together with a map*

$$B \xrightarrow{\pi} G \times H$$

making $G \times H$ into the quotient B/C, together with two **partial multiplication maps**

$$+_1 : B \underset{H}{\times} B \to B, \qquad +_2 : B \underset{G}{\times} B \to B,$$

where

$$B \underset{H}{\times} B := \{(x, y) : \pi(x) \text{ and } \pi(y) \text{ have same } H\text{-components}\}$$

and

$$B \underset{G}{\times} B := \{(x, y) : \pi(x) \text{ and } \pi(y) \text{ have same } G\text{-components}\}.$$

These are to subject to the requirements

1. *For any $h \in H$, $B_h^1 := \pi^{-1}(G \times \{h\})$ is a commutative group under $+_1$, π is a surjective homomorphism of B_h^1 onto G, and via the action of C on B_h^1, C is isomorphic to the kernel of π.*

2. *For any $g \in G$, $B_g^2 := \pi^{-1}(\{g\} \times H)$ is an abelian group under $+_2$, π is a surjective homomorphism of B_g^2 onto H, and via the action of C on B_g^2, C is isomorphic to the kernel of π.*

3. For any $x, y, u, v \in B$ such that

$$\pi(x) = (g_1, h_1), \quad \pi(y) = (g_2, h_1), \quad \pi(u) = (g_1, h_2), \quad \pi(v) = (g_2, h_2),$$

we have

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v).$$

Remark 3.2.1.2. The conventions we use for $+_1$ and $+_2$ is different from Mumford's in [98], and hence different from most of the existing ones in the literature.

Definition 3.2.1.3. *Let G, H, C, be three commutative group schemes over a base scheme S. A* **biextension** *of $G \underset{S}{\times} H$ by C over S is a representable functor B that associates to each scheme T over S a biextension of $G(T) \times H(T)$ by $C(T)$.*

Proposition 3.2.1.4. *For any abelian scheme $A \to S$, the Poincaré $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{P}_A$ on $A \underset{S}{\times} A^\vee$ (defined using Theorem 3.1.2.5) has a canonical structure of a biextension of $A \underset{S}{\times} A^\vee$ by $\mathbf{G}_{\mathrm{m}}$.*

This is essentially the *theorem of square* for (rigidified) invertible sheaves on the abelian scheme A over S.

3.2.2 Cubical Structures

Let G and C be group schemes over a base scheme S. Let $\mathcal{L}$ be any rigidified C-torsor on a group scheme G.

Then we may define functorially

$$\mathcal{D}_2(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^*\,\mathcal{L}^{\otimes -1} \otimes \mathrm{pr}_2^*\,\mathcal{L}^{\otimes -1},$$

which is a C-torsor on $G \underset{S}{\times} G$, and similarly

$$\mathcal{D}_3(\mathcal{L}) = m_{123}^*\mathcal{L} \otimes m_{12}^*\mathcal{L}^{\otimes -1} \otimes m_{23}^*\mathcal{L}^{\otimes -1} \otimes m_{13}^*\mathcal{L}^{\otimes -1} \otimes \mathrm{pr}_1^*\,\mathcal{L} \otimes \mathrm{pr}_2^*\,\mathcal{L} \otimes \mathrm{pr}_3^*\,\mathcal{L},$$

(where m_{123}, m_{12}, etc are the multiplication maps summing up the corresponding components,) which is a C-torsor on $G \underset{S}{\times} G \underset{S}{\times} G$. Here functoriality means the same operations define morphisms $\mathcal{D}_2(f)$ and $\mathcal{D}_3(f)$ for morphisms between C-torsors (and hence also for sections of $\mathcal{L}$ because they are by definition isomorphisms between $\mathcal{L}$ and the trivial C-torsor).

Lemma 3.2.2.1. *There is a canonical* **symmetry isomorphism** *of the invertible sheaf* $\mathcal{D}_2(\mathcal{L})$,

$$s^*\mathcal{D}_2(\mathcal{L}) \xrightarrow{\sim} \mathcal{D}_2(\mathcal{L}),$$

that covers the switching isomorphism $s : A \underset{S}{\times} A \xrightarrow{\sim} A \underset{S}{\times} A$ *of the two factors of* $A \underset{S}{\times} A$.

Remark 3.2.2.2. There are also canonical symmetry isomorphisms for $\mathcal{D}_3(\mathcal{L})$ covering the permutations of the factors of $A \underset{S}{\times} A \underset{S}{\times} A$, which we will not use explicitly. For more information, and also more formal properties of $\mathcal{D}_n$ in general, see [96, I].

Note that $\mathcal{D}_3(\mathcal{L})$ can be constructed from $\mathcal{D}_2(\mathcal{L})$ in two ways. We have thus two canonical isomorphisms:

$$\begin{tikzcd}
(m \underset{S}{\times} \mathrm{pr}_3)^*\mathcal{D}_2(\mathcal{L}) \otimes \mathrm{pr}_{13}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \otimes \mathrm{pr}_{23}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \arrow[dr, "\sim", "\xi_1"'] & \\
& \mathcal{D}_3(\mathcal{L}) \\
(\mathrm{pr}_1 \underset{S}{\times} m)^*\mathcal{D}_2(\mathcal{L}) \otimes \mathrm{pr}_{12}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \otimes \mathrm{pr}_{13}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \arrow[ur, "\xi_2", "\sim"'] &
\end{tikzcd} \tag{3.2.2.3}$$

Lemma 3.2.2.4. *Commutative group scheme structures on a* C*-torsor* $\mathcal{M}$ *over a group scheme* G *correspond bijectively to sections of* $\mathcal{D}_2(\mathcal{M})$ *over* $G \underset{S}{\times} G$.

Proof. This is a special case of [59, VII, 1.1.6 and 1.2]. □

Note that a section τ of $\mathcal{D}_3(\mathcal{L})$ defines two sections $\xi_1^{-1}(\tau)$ and $\xi_2^{-1}(\tau)$, which correspond respectively to two sections of

$$(m \underset{S}{\times} \mathrm{pr}_3)^*\mathcal{D}_2(\mathcal{L}) \otimes \mathrm{pr}_{13}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \otimes \mathrm{pr}_{23}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1}$$

and

$$(\mathrm{pr}_1 \underset{S}{\times} m)^*\mathcal{D}_2(\mathcal{L}) \otimes \mathrm{pr}_{12}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1} \otimes \mathrm{pr}_{13}^*\,\mathcal{D}_2(\mathcal{L})^{\otimes -1}.$$

This gives two *partial multiplication maps* on $\mathcal{D}_2(\mathcal{L})$.

Definition 3.2.2.5. *A* **cubical structure** *on a C-torsor $\mathcal{L}$ over a group scheme G over S is a section τ of the torsor $\mathcal{D}_3(\mathcal{L})$ on $G \underset{S}{\times} G \underset{S}{\times} G$ such that, via the isomorphisms in (3.2.2.3), the two partial multiplication maps on $\mathcal{D}_2(\mathcal{L})$ defined by the two sections $\xi_1^{-1}(\tau)$ and $\xi_2^{-1}(\tau)$ define a biextension structure of $G \underset{S}{\times} G$ by C.*

Definition 3.2.2.6. *A* **cubical torsor** *is a torsor together with a cubical structure.*

Definition 3.2.2.7. *A* **morphism** *of cubical torsors $(\mathcal{L}, \tau) \to (\mathcal{L}', \tau')$ is a morphism $f : \mathcal{L} \to \mathcal{L}'$ of C-torsors such that $\mathcal{D}_3(f) : \mathcal{D}_3(\mathcal{L}) \to \mathcal{D}_3(\mathcal{L}')$ verifies $\mathcal{D}_3(f)(\tau) = \tau'$.*

We denote by $\mathrm{CUB}_S(G, C)$ the category of cubical C-torsors on G.

We leave it to the reader to make explicit the notions of inverse images of a cubical torsor by a morphism $G' \to G$, of the image by a change of group structure $C \to C'$, of trivial cubical torsors, of tensor products (or sum) of C-torsors, of inverse cubical torsors, etc. With the operation of tensor products, $\mathrm{CUB}_S(G, C)$ is a *strictly commutative Picard category*, in the sense of [4, XVIII, 1.4.2].

Definition 3.2.2.8. *A* **trivialization** *of an object $(\mathcal{L}, \tau)$ in $\mathrm{CUB}_S(G, C)$ is an isomorphism from the trivial object to $(\mathcal{L}, \tau)$, namely a* **section** *σ of $\mathcal{L}$ over G, such that*

$$\mathcal{D}_3(\sigma) = \tau \tag{3.2.2.9}$$

Remark 3.2.2.10. The set of trivializations of an object $(\mathcal{L}, \tau)$ in $\mathrm{CUB}_S(G, C)$ is, if nonempty, a torsor under the group

$$\mathrm{Hom}^{(2)}(G, C),$$

the pointed maps of degree two from G to C, namely those maps $f : G \to C$ such that $\mathcal{D}_3(f) = 1$. In particular, there might be more than one way to trivialize a cubical torsor if $\mathrm{Hom}^{(2)}(G, C)$ is nontrivial.

Remark 3.2.2.11. When $C = \mathbf{G}_{\mathrm{m}}$, the group $\mathrm{Hom}^{(2)}(G, C)$ can be identified with $\mathrm{Hom}(G, C)$ under some mild assumptions on G and S, by Lemma 3.2.2.12 below.

Let us recall the following form of *Rosenlicht's Lemma*:

Lemma 3.2.2.12 ([59, VIII, 4.1])**.** *Let k be a field, Z and W be two k-schemes of finite type, separated, and geometrically connected, with points e_Z and e_W rational over k. Then every morphism $Z \times W \to \mathbf{G}_{\mathrm{m}}$ that is trivial over the two subschemes $Z \times e_W$ and $e_Z \times W$ is trivial.*

(See [26] for a modern proof.)

3.2.3 A Fundamental Example

The most important example of cubical structures are given by the so-called *theorem of cube* on abelian schemes, which can be generalized in the following form:

Proposition 3.2.3.1 (see [96, I, 2.6])**.** *Let G be a smooth commutative group scheme with geometrically connected fibers over a base scheme S. Suppose that one of the following conditions is satisfied:*

1. *G is an abelian scheme over S.*
2. (*cf. [21, 2.4]*) *S is normal and fibers of G over maximal points of S are multiple extensions of abelian varieties, tori* (*not necessarily split*)*, and the groups $\mathbf{G}_{\mathrm{a}}$.*

Then the forgetful functor

$$\mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S}) \to \mathrm{TORSRIG}_S(G, \mathbf{G}_{\mathrm{m},S})$$

(*where* $\mathrm{TORSRIG}_S(G, \mathbf{G}_{\mathrm{m},S})$ *is the category of $\mathbf{G}_{\mathrm{m}}$-torsors on G rigidified along identity section of A*) *is an equivalence of categories.*

Remark 3.2.3.2. When merely condition 2 is satisfied, two isomorphic objects in $\mathrm{TORSRIG}_S(G, \mathbf{G}_{\mathrm{m},S})$ might not be uniquely isomorphic, and two isomorphic objects in $\mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S})$ might not be uniquely isomorphic either. Therefore the intuitive statement that *theorem of cube* holds under condition 2 should not be misunderstood as the existence of a canonical cubical isomorphism for each rigidified torsor.

3.2.4 The Group $\mathcal{G}(\mathcal{L})$ for Abelian Schemes

Let S be a scheme, A an abelian scheme over S, $\mathcal{L}$ a $\mathbf{G}_{\mathrm{m}}$-torsor on A, rigidified at the identity section (hence cubical, by Proposition 3.2.3.1). Let $A^{\vee}$ be the dual abelian scheme of A over S. By Construction 1.3.2.10, we know that $\mathcal{L}$ defines canonically a symmetric morphism $\lambda_{\mathcal{L}} : A \to A^{\vee}$. Put

$$K(\mathcal{L}) = \ker \lambda_{\mathcal{L}}. \tag{3.2.4.1}$$

This is a closed subgroup scheme of A. The restriction of the morphism

$$\mathrm{Id}_A \times \lambda_{\mathcal{L}} : A \underset{S}{\times} A \to A \underset{S}{\times} A^{\vee}$$

to $A \underset{S}{\times} K(\mathcal{L})$ gives an isomorphism of biextensions

$$\mathcal{D}_2(\mathcal{L})|_{A \underset{S}{\times} K(\mathcal{L})} \xrightarrow{\sim} [(\mathrm{Id}_A \times \lambda_{\mathcal{L}})^* \mathcal{P}_A]|_{A \underset{S}{\times} K(\mathcal{L})} \xrightarrow{\sim} (\mathrm{Id}_A \times \lambda_{\mathcal{L}}|_{K(\mathcal{L})})^* (\mathcal{P}_A|_{A \underset{S}{\times} e_{A^{\vee}}}).$$

As $\mathcal{P}_A|_{A \underset{S}{\times} e_{A^{\vee}}}$ is the trivial biextension, we obtain a *canonical trivialization* of $\mathcal{D}_2(\mathcal{L})|_{A \underset{S}{\times} K(\mathcal{L})}$, from which we deduce a structure of commutative group scheme on $\mathcal{L}|_{K(\mathcal{L})}$, as a central extension

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{L}|_{K(\mathcal{L})} \to K(\mathcal{L}) \to 0$$

of $K(\mathcal{L})$ by $\mathbf{G}_{\mathrm{m}}$, as well as a *left action* (by switching the factor)

$$* : \mathcal{L}|_{K(\mathcal{L})} \underset{S}{\times} \mathcal{L} \to \mathcal{L}$$

of $\mathcal{L}|_{K(\mathcal{L})}$ on the torsor $\mathcal{L}$.

On the other hand, we have the familiar group scheme $\mathcal{G}(\mathcal{L})$ over S defined as follows: For any scheme S' over S, the group $\mathcal{G}(\mathcal{L})(S')$ consists of pairs (a, ξ), where $a \in K(\mathcal{L})(S')$, and where $\tilde{a}$ is an automorphism over S' of the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}_S$ making the diagram

$$\begin{tikzcd}
\mathcal{L}_{S'} \arrow[r, "\xi", "\sim"'] \arrow[d] & \mathcal{L}_{S'} \arrow[d] \\
A_{S'} \arrow[r, "\sim", "T_a"'] & A_{S'}
\end{tikzcd}$$

commute. (See for example [99, §23] or [95].)

Proposition 3.2.4.2 ([96, I, 4.4])**.** *With the above notations and hypotheses, the morphism*

$$\mathcal{L}|_{K(\mathcal{L})} \to \mathcal{G}(\mathcal{L}) \tag{3.2.4.3}$$

over S that associates the couple (a, ξ_u) to each point $u \in \mathcal{L}|_{K(\mathcal{L})}$ over $x \in K(\mathcal{L})$, where $\xi_u(v) = u * v$ for all $v \in \mathcal{L}$ (the symbol $*$ denotes the left action of $\mathcal{L}|_{K(\mathcal{L})}$ on $\mathcal{L}$ defined above), is an isomorphism of central extensions (*of $K(\mathcal{L})$ by $\mathbf{G}_{\mathrm{m}}$*)*. The inverse isomorphism is given by the association*

$$(a, \xi) \mapsto \xi(\varepsilon_{\mathcal{L}}),$$

where $\varepsilon_{\mathcal{L}} \in \mathcal{L}|_{e_A}(S)$ is the rigidification at the origin $e_A \in K(L)$, namely the identity section of the extension $\mathcal{L}|_{K(\mathcal{L})}$.

Let us include the proof for the convenience of the reader.

Proof of Proposition 3.2.4.2. As (3.2.4.3) is obviously a morphism of $\mathbf{G}_{\mathrm{m}}$-torsors over $K(\mathcal{L})$ (hence an isomorphism), it suffice to see that this is a morphism of groups, which is nothing but the fact that $*$ is an action of group. The last assertion is equally trivial. ☐

3.2.5 Descending Structures

If G, H, and C are three commutative group schemes over a base scheme S, we denote by $\mathrm{EXT}_S(G, C)$ (resp. $\mathrm{BIEXT}_S(G, H; C)$) in accordance with [59], the category of commutative extensions of G by C (resp. the biextensions of $G \times H$ by C).

Let us begin by including results in [59, VIII] concerning the descent of biextensions:

Proposition 3.2.5.1 (see [59, VIII, 3.4])**.** *Let P be a smooth group scheme of finite presentation and with geometrically connected fibers over a base scheme S, and let Q be a torus over S. Then the category of biextensions of $P \times Q$ by $\mathbf{G}_{\mathrm{m}}$ is equivalent to the punctual category, which means every biextension of $P \times Q$ by $\mathbf{G}_{\mathrm{m}}$ is trivial.*

Corollary 3.2.5.2 (see [59, VIII, 3.5])**.** *Let P be a smooth group scheme of finite presentation and with geometrically connected fibers over a base scheme S. Let T is a torus, and let*

$$0 \to T \to Q \to Q' \to 0$$

be an exact sequence of commutative group schemes over S. Then the inverse image functor

$$\mathrm{BIEXT}_S(P, Q'; \mathbf{G}_{\mathrm{m},S}) \to \mathrm{BIEXT}_S(P, Q; \mathbf{G}_{\mathrm{m},S})$$

is an equivalence of categories.

Now let us turn to cubical torsors. Let T be a *torus* over S, and let

$$0 \to T \xrightarrow{i} G \xrightarrow{\pi} H \to 0$$

be an exact sequence of *smooth* commutative group schemes *with connect fibers* over S. Moreover, let us take $C = \mathbf{G}_{\mathrm{m},S}$. Then we have the following results:

Proposition 3.2.5.3 ([96, I, 7.2.1])**.** *For all torus T over S, the category $\mathrm{CUB}_S(T, \mathbf{G}_{\mathrm{m},S})$ is equivalent to the category $\mathrm{EXT}_S(T, \mathbf{G}_{\mathrm{m},S})$ of commutative group extensions of T by $\mathbf{G}_{\mathrm{m},S}$.*

Proposition 3.2.5.4 ([96, I, 7.2.2])**.** *In the above setting, the category $\mathrm{CUB}_S(H, \mathbf{G}_{\mathrm{m},S})$ is equivalent to the category of couples $(\mathcal{L}, s)$, where $\mathcal{L} \in \mathrm{Ob}\,\mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S})$, and where s is a trivialization of $i^*\mathcal{L}$ in $\mathrm{CUB}_S(T, \mathbf{G}_{\mathrm{m},S})$ (namely, by Proposition 3.2.5.3, in $\mathrm{EXT}_S(T, \mathbf{G}_{\mathrm{m},S})$).*

Corollary 3.2.5.5 ([96, I, 7.2.3])**.** *In the above setting, let $\mathcal{L}$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on G. Then:*

1. *$\mathcal{L} \otimes [-1]_G^* \mathcal{L}$ comes canonically from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on H.*
2. *There exists an étale surjective map $S' \to S$ such that $\mathcal{L}_{S'} = \mathcal{L} \underset{S}{\times} S'$ comes from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on $H_{S'} = H \underset{S}{\times} S'$.*
3. *If all torus over S are isotrivial (defined as in Definition 3.1.1.5), then we may suppose in statement 2 that S' is finite étale over S.*

Remark 3.2.5.6 ([96, I, 7.2.3]). The assumption in statement 3 is satisfied in particular when S is locally noetherian and *normal* by [33, X, 5.16], or when S is the spectrum of a *complete* noetherian local ring by [33, X, 3.3].

Corollary 3.2.5.7 (cf. [96, I, 7.2.4])**.** *Suppose S is locally noetherian and normal, and let $S' \to S$ be any finite étale surjection that splits T. (Such an $S' \to S$ always exists by Remark 3.2.5.6 above.) Then $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'}, \mathbf{G}_{\mathrm{m},S'}) = 0$, and hence any cubical $\mathbf{G}_{\mathrm{m}}$-torsor on $G_{S'}$ comes from some $H_{S'}$ (by Proposition 3.2.5.4).*

Proof. It suffices to show that $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'},\mathbf{G}_{\mathrm{m},S'})=0$ under the assumptions. Note that the group $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'},\mathbf{G}_{\mathrm{m},S'})$ is canonically isomorphic to $H^1_{\mathrm{fppf}}(S',\underline{\mathbb{Z}}_{S'})$ because group schemes are sheaves in the fppf topology. By [57, Thm. 11.7], whose assumptions are trivially satisfied by $\underline{\mathbb{Z}}_{S'}$, we have a canonical isomorphism $H^1_{\text{ét}}(S',\underline{\mathbb{Z}}_{S'})\xrightarrow{\sim}H^1_{\mathrm{fppf}}(S',\underline{\mathbb{Z}}_{S'})$. By [4, IX, 3.6(ii)], we have $H^1_{\text{ét}}(S',\underline{\mathbb{Z}}_{S'})=0$ when S' is geometrically unibranch, or equivalently by [52, IV, 18.8.15] when all strict localizations of S' are irreducible. This is certainly satisfied when S' is locally noetherian and normal. Hence $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'},\mathbf{G}_{\mathrm{m},S'})=0$, as desired. □

3.3 Semi-Abelian Schemes

3.3.1 Generalities

Definition 3.3.1.1. *A* **semi-abelian scheme** *is a group scheme $\pi:G\to S$ that is separated, smooth, commutative, and such that each fiber G_s of G (for $s\in S$) is an extension of an abelian variety A_s by a torus T_s,*

$$0\to T_s\to G_s\to A_s\to 0.$$

Remark 3.3.1.2. The geometric fibers of G are hence automatically connected.

Remark 3.3.1.3. In the above definition, A_s and T_s are uniquely determined by G_s as follows: T_s is the largest smooth connected affine subgroup scheme of G_s, and A_s is the quotient of G_s by T_s. The torus T_s is called the *torus part* of G_s.

Remark 3.3.1.4. Smoothness is not a strong condition for families. A smooth family can contain empty fibers over the base scheme. However, since a smooth scheme is étale locally isomorphic to an open subscheme of an affine r-space $\mathbb{A}^r_S$ for some integer r, the existence of the *identity section* as a group scheme forces the relative dimensions of a semi-abelian scheme to be locally constant. This is much stronger than having no empty fibers.

Remark 3.3.1.5. As explained in Remark 3.3.1.3, any fiber G_s of G over a point $s\in S$ is uniquely an extension of some abelian scheme A_s by some torus T_s. Accordingly, the torsion points of G_s is an extension of group schemes of the form

$$0\to T[n]_s\to G[n]_s\to A[n]_s\to 0.$$

By taking any integer $n > 0$ that is invertible in $k(s)$, and by taking the ranks over any geometric point $\bar{s}$, we see that

$$\operatorname{rk}(G[n]_s) = \operatorname{rk}(T[n]_s)\operatorname{rk}(A[n]_s) = n^{\dim_s(T_s)} n^{2\dim_s(A_s)}.$$

This shows that we can calculate the rank of T_s by rank of the subscheme $G[n]_s$ of G_s over s.

Remark 3.3.1.6. By smoothness of the semi-abelian scheme $G \to S$, we know that torsion points of a fiber G_s over a point $s \in S$ extends to a local section near the point s. Note that this extension is unique by separateness of G. Let $\bar{s}$ be a geometric point over s. For each fixed integer $n \geq 1$ that is invertible in the residue field $k(s)$, we may find an étale neighborhood U of $\bar{s}$ in S such that all the n-torsion points of $G_{\bar{s}}$ extends to sections over U. In other words, $G[n]_s$ extends to a subscheme of $G[n]_U$. By uniqueness of the extension, this subscheme descends to an open neighborhood of s. By Remark 3.3.1.5, this shows that the function $s \mapsto \operatorname{rk}(\underline{X}(s))$ is upper semi-continuous on S. Note that the rank of the torus part may jump.

Let us quote some other useful results from [37, Ch. I, §2]:

Proposition 3.3.1.7. *Let S be a noetherian normal scheme, and G and H be two semi-abelian schemes over S. Suppose that over a dense open subscheme U of S there is a homomorphism $\phi_U : H_U \to G_U$. Then ϕ_U extends uniquely to a homomorphism $\phi : H \to G$ over S.*

This is originally proved in a more general setting as [110, IX, 1.4]. A direct proof for the special case of semi-abelian schemes was given in [37, Ch. I, Prop. 2.7].

Remark 3.3.1.8 ([37, Ch. I, Rem. 2.8]). It follows that a semi-abelian scheme $G \to S$ all whose fibers are abelian varieties is proper (namely an abelian scheme) over S: Use the valuative criterion to reduce to the case where S is the spectrum of a discrete valuation ring, and then use the fact that, by Proposition 3.3.1.7, G is isomorphic to the Néron model of its generic fiber.

A variant of the above proposition concerns the torus part:

Proposition 3.3.1.9. *Let S be a noetherian normal scheme, U be a dense open subscheme of S, and G be semi-abelian scheme over S. If a torus H_U over U is a closed subgroup of G_U, then the closure of H_U in G is a torus $H \to S$ contained in G.*

This is originally proved in a more general setting as [110, IX, 2.4]. A direct proof for the special case of semi-abelian schemes was given in [37, Ch. I, Prop. 2.9].

Remark 3.3.1.10 (p. 12 of [37]). If we drop the assumption that S is normal, then Propositions 3.3.1.7 and 3.3.1.9 both become *false*.

Now suppose $G \to S$ is a semi-abelian scheme, with an arbitrary base S that does not have to be normal.

Suppose S is the spectrum of a discrete valuation ring V with generic point η and special point s. Then the generic torus T_η extends to a subtorus of G, whose special fiber is contained in T_s. This inclusion defines a canonical surjection $X(s) \to X(\eta)$. As a result, there exists an étale constructible sheaf $\underline{X}$ on $S = \operatorname{Spec} V$, whose fiber in $\bar{s}$ (respectively $\bar{\eta}$) is $X(s)$ (respectively $X(\eta)$), such that the map above is the specialization map associated to the sheaf.

In general, we have the following:

Theorem 3.3.1.11 ([37, Ch. I, Thm. 2.10]). *Suppose $G \to S$ is a semi-abelian scheme. There exists a unique étale constructible sheaf $\underline{X} = \underline{X}(G)$ on S such that for any $s \in S$ the restriction of $\underline{X}$ to $\{s\}$ is equal to $X(s)$, and such that for any map $\operatorname{Spec}(V) \to S$ with V a discrete valuation ring, the pullback of $\underline{X}$ to $\operatorname{Spec}(V)$ is the constructible sheaf above. Furthermore, formation of $\underline{X}$ is functorial and commutes with any base change, and for any torus T over S with character group $\underline{Y}$, $\operatorname{Hom}_S(T, G)$ is isomorphic to $\operatorname{Hom}_S(\underline{X}, \underline{Y})$.*

Corollary 3.3.1.12 ([37, Ch. I, Cor. 2.11]). *Assume that the rank of $X(s)$ is locally constant on S. Then G is globally an extension of an abelian scheme by a torus. That is, there exists an exact sequence $0 \to T \to G \to A \to 0$ in which T is a torus and A is an abelian scheme. In particular, G is a torus if all its fibers are.*

For any semi-abelian scheme $G \to S$ we obtain a stratification of S by locally closed subset $S_r = \{s \in S : \operatorname{rk}(X(s)) = r\}$. The closure of S_r is contained in the union of the S_i for $i \geq r$, and over each S_r the semi-abelian scheme G is globally the extension of an abelian scheme by a torus.

3.3.2 Extending Structures

Let G be a semi-abelian scheme over a base scheme S. For any scheme S' over S, we denote by

$$\mathrm{Res}_{S,S'} : \mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S}) \to \mathrm{CUB}_{S'}(G_{S'}, \mathbf{G}_{\mathrm{m},S'}) \tag{3.3.2.1}$$

the functor of inverse image. We denote by U either an *open subscheme* of S that is *schematically dense* in S, or the *generic point* of S if S is irreducible. It is clear that the functor $\mathrm{Res}_{S,U}$ is already *faithful* in this case. Then natural questions are when $\mathrm{Res}_{S,U}$ is fully faithful, and when it is an equivalence of categories.

Proposition 3.3.2.2 (see [96, II, 3.2.1, 3.2.2, 3.2.3])**.** *Let G be a semi-abelian scheme over a* **normal** *base scheme S. Then:*

1. *The functor $\mathrm{Res}_{S,U}$ is* **fully faithful***.*
2. *If S is* **regular** *at points of $S - U$, then $\mathrm{Res}_{S,U}$ is an equivalence.*

Theorem 3.3.2.3 (cf. [96, II, 3.3])**.** *Let G be a semi-abelian scheme over a* **normal** *base scheme S. Let $\mathcal{L}_U$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on G_U, satisfying any one of the following properties:*

1. *The underlying torsor of $\mathcal{L}_U$ is* **of finite order** *in $\mathrm{Pic}(G_U)$.*
2. *The underlying torsor of $\mathcal{L}_U$ is* **symmetric***. Namely, there exists an isomorphism $\mathcal{L}_U \overset{\sim}{\to} [-1]^*_{G_U}\mathcal{L}_U$ of $\mathbf{G}_{\mathrm{m}}$-torsors on G_U.*

Then $\mathcal{L}_U$ is in the essential image of $\mathrm{Res}_{S,U}$.

Now let us state the important *semi-stable reduction theorem*:

Theorem 3.3.2.4 ([37, Ch. I, Thm. 2.6])**.** *Let V be a discrete valuation ring, let K be the fraction field of V, and let G_K be a semi-abelian variety over K. Then there exists a finite extension V' of V, with fraction field K', such that $G_{K'} = G_K \underset{K}{\otimes} K'$ extends to a semi-abelian scheme over $\mathrm{Spec}(V')$.*

Let us include the proof for the convenience of the reader.

Proof Theorem 3.3.2.4. When G_K is an abelian variety, the theorem is well-known and well-documented, in which case it means that the Néron model of $G_{K'}$ contains a semi-abelian open subgroup scheme over some finite extension K' of K. (See for example [59, X] or [5].)

When G_K is an extension of an abelian scheme A_K by some torus T_K, we may take some finite extension V' of V (and hence K' of K) so that $T_{K'}$ becomes a split torus and so that $A_{K'}$ extends to a semi-abelian scheme A over $\mathrm{Spec}(V')$. Suppose G' is any smooth group scheme over $\mathrm{Spec}(V')$ whose geometric fibers are connected. By Lemma 3.2.2.4, any group scheme extension $G''_{K'}$ of $G'_{K'}$ by $\mathbf{G}_{\mathrm{m},K'}$ defines a rigidified $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}_{K'}$ on $G'_{K'}$ together with a section of $\mathcal{D}_2(\mathcal{L}_{K'})$. In particular, $\mathcal{L}_{K'}$ admits a cubical structure. Hence, by Proposition 3.3.2.2, $\mathcal{L}_{K'}$ extends uniquely to a cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$ on G', with a unique section of $\mathcal{D}(\mathcal{L})$ extending the one of $\mathcal{D}_2(\mathcal{L}_{K'})$ by Theorem 3.3.2.6. By Lemma 3.2.2.4 again, this shows that $G''_{K'}$ extends to a group scheme extension G'' of G' by $\mathbf{G}_{\mathrm{m}}$ over $\mathrm{Spec}(V')$. Since $T_{K'}$ is a split torus, which is isomorphic to a product of $\mathbf{G}_{\mathrm{m}}$, the above argument of extending G'' over G' proves the theorem by induction on number of copies of $\mathbf{G}_{\mathrm{m},K'}$ in $T_{K'}$. □

Remark 3.3.2.5 (p. 9 of [37]). The proof of Theorem 3.3.2.4 shows that, after a finite extension of K, there exists an extension G of G_K such that the torus part of G_K extends to a closed subtorus of G.

To finish the section, let us include the following result concerning biextensions:

Theorem 3.3.2.6 (cf. [96, II, 3.6]). *Let S be a normal integral scheme with generic point η, and let G and H be two semi-abelian schemes over S. Then the natural functor*

$$\mathrm{BIEXT}_S(G,H;\mathbf{G}_{\mathrm{m},S}) \to \mathrm{BIEXT}_\eta(G_\eta,H_\eta;\mathbf{G}_{\mathrm{m},\eta})$$

is an equivalence of categories.

3.3.3 Raynaud Extensions

Let R be a noetherian domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S = \mathrm{Spec}(R)$, η the generic point of S, $R_i := R/I^{i+1}$, $S_i = \mathrm{Spec}(R/I^{i+1})$, where $i \geq 0$ is any integer. Let $G \to S$ be a semi-abelian scheme, and let $G_i := G \underset{S}{\times} S_i$. Let $S_{\mathrm{for}} := (S_i)_{i\geq 0}$ and $G_{\mathrm{for}} :=$

$(G_i)_{i\geq 0}$ be the formal schemes formed by the projective systems, which can be identified with the formal completions of respectively S and G along their closed fibers S_0 and G_0. Alternatively, we may set $S_{\mathrm{for}} := \mathrm{Spf}(R, I)$ and $G_{\mathrm{for}} := G \underset{S}{\times} S_{\mathrm{for}}$.

Assume that G_0 *is an* **extension** *of an abelian variety* A_0 *by an* **isotrivial** *torus* T_0 (*defined as in Definition 3.1.1.5*).

Remark 3.3.3.1. This may not be true for any semi-abelian scheme over S_0. In [37, Ch. II, §1, p. 33], they considered also those G such that G_0 is an extension of an abelian scheme A_0 by a torus T_0, without the isotriviality assumption on T_0. Since we cannot justify their claim that T_{for} (or rather $\underline{X}_{\mathrm{for}}$) is algebraizable without the isotriviality assumption, and since the assumption is also made by Grothendieck in [59, IX, 7.2.1, 7.2.2], we shall be content with our understanding of this more limited case (which nevertheless suffices for the construction of compactifications).

By Theorem 3.1.1.2, T_0 can be lifted uniquely to a multiplicative subgroup scheme T_i of G_i for every i. The quotient of G_i by T_i is an abelian scheme because it is smooth and trivially proper. Therefore we obtain (as in [59, IX, 7]) an exact sequence of formal group schemes

$$0 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \xrightarrow{\pi_{\mathrm{for}}} A_{\mathrm{for}} \to 0 \tag{3.3.3.2}$$

over S_{for}, which is a compatible system (for all $i \geq 0$) of exact sequences of group schemes

$$0 \to T_i \to G_i \xrightarrow{\pi_i} A_i \to 0 \tag{3.3.3.3}$$

over S_i, where A_i is an abelian scheme over S_i, and where T_i is a torus over S_i. Then we have the morphism $c_{\mathrm{for}} : \underline{X}_{\mathrm{for}} \to A^\vee_{\mathrm{for}}$, (or rather the projective system $(c_i : \underline{X}_i \to A^\vee_i)_{i\geq 0}$ of morphisms,) which corresponds to the sequence (3.3.3.2) (or rather the sequence (3.3.3.3)) under Proposition 3.1.5.1.

We have the following formal version of Corollary 3.2.5.5 (see [96, IV, 2.1] for the first two statements):

Corollary 3.3.3.4. *Let* $\mathcal{L}_{\mathrm{for}}$ *be a cubical* $\mathbf{G}_{\mathrm{m}}$*-torsor on* G_{for} (*namely a system* $(\mathcal{L}_i)_{i\geq 0}$ *of cubical* $\mathbf{G}_{\mathrm{m}}$*-torsors on* G_i*, together with the transition isomorphisms*). *Then:*

1. $\mathcal{L}_{\mathrm{for}} \otimes [-1]^*\mathcal{L}_{\mathrm{for}}$ *comes canonically from a cubical* $\mathbf{G}_{\mathrm{m}}$*-torsor on* A_{for}.

2. *There exists an étale surjection $S'_{\mathrm{for}} \twoheadrightarrow S_{\mathrm{for}}$ such that $(\mathcal{L}_{\mathrm{for}})_{S'_{\mathrm{for}}}$ comes from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on $(A_{\mathrm{for}})_{S'_{\mathrm{for}}}$. This morphism is algebraizable to a unique morphism $S' \to S$, but $S' \to S$ is not necessarily étale* (*due to potential issues of finiteness*).

3. *If all torus over S_0 are isotrivial* (*defined as in Definition 3.1.1.5*), *then we may suppose in statement 2 that the surjection $S' \to S$ is finite étale.*

Proof. Statement 1 follows by applying statement 1 of Corollary 3.2.5.5 to the base schemes S_i.

As for statement 2, by Propositions 3.2.5.4 and 3.2.5.3, it suffices to trivialize the restriction of $\mathcal{L}_{\mathrm{for}}$ to T_{for} as an *extension*. By [33, X, 3.2], it suffices to trivialize $\mathcal{L}_0$ over T_0, which can be achieved after base change to some étale algebra $R_0 \hookrightarrow R'_0$. Then the unique formally étale I-adic complete R-algebra R' (which we shall explain in Remark 3.3.3.5 below) defines the morphism $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ whose formal completion along S_0 gives the étale surjective morphism $S'_{\mathrm{for}} \to S_{\mathrm{for}}$.

Under the assumption on S_0 in statement 3, we may assume that R'_0 is finite over R_0 in the above argument, and accordingly that the unique R' above is finite over R (by Theorem 2.3.1.5, or rather by [47, III, 5.4.5]). That is, we may assume that the surjection $S' \to S$ is finite étale. □

Remark 3.3.3.5. The unique lifting R' can be realized concretely as follows: For any integer $i \geq 0$, let $R_i := R/I^{i+1}$. Start with the given étale R_0-algebra R'_0. For any integer $k \geq 0$, suppose R'_i is defined for any $0 \leq i \leq k$. Note that $R_{k+1}/I^{k+1} \cdot R_{k+1} \cong R_k$, and $(I^{k+1})^2 = 0$ in R_{k+1}. Hence, by Lemma 2.1.1.6 (or rather by [52, IV, 18.1.2]), there is a unique étale R_{k+1}-algebra R'_{k+1} such that $R'_{k+1}/I^k \cdot R'_{k+1} \cong R'_k$. Repeating this process, we obtain the definition of R'_k for any integer $k \geq 0$, which form a projective system compatible with their structural maps as an R-algebra. Then we define the R-algebra $R' := \varprojlim_k R'_k$, which is the unique formally étale over R by construction.

Remark 3.3.3.6. If R_0 is a separably closed field k, then the second assertion shows that the natural functor

$$\mathrm{CUB}_{S_{\mathrm{for}}}(A_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}}) \to \mathrm{CUB}_{S_{\mathrm{for}}}(G_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}})$$

is essentially surjective.

Proposition 3.3.3.7. *Let G be a semi-abelian scheme over S such that G_0 is an extension of an abelian scheme A_0 by an isotrivial torus T_0* (*defined as in Definition 3.1.1.5*)*. Suppose that there exists a cubical invertible sheaf on G that is* **relatively ample** *over S. Then the extension* (*3.3.3.2*) *is* **algebraizable** *by Theorem 2.3.1.5. Namely, there exist uniquely* (*over S*) *an abelian scheme A, a torus T, and an extension*

$$0 \to T \to G^{\natural} \to A \to 0 \tag{3.3.3.8}$$

whose associated extension of formal schemes is (*3.3.3.2*).

Definition 3.3.3.9. *The extension* (*3.3.3.8*) *is called [59, IX] the* **Raynaud extension** *attached to G.* (*See [111].*)

Remark 3.3.3.10. If S is *normal*, then G is quasi-projective over S by [110, XI, 1.13], a theorem of Grothendieck. That is, there exists an ample invertible sheaf $\mathcal{L}$ on G. Moreover, by Proposition 3.2.3.1, the invertible sheaf $\mathcal{L}$ admits a cubical structure as soon as it is rigidified. Therefore, the hypothesis of the existence of an ample cubical sheaf in Proposition 3.3.3.7 is automatic when S is normal.

The following proof is adapted from [96, IV, 2.2]. (See also [59, X, 7].)

Proof of Proposition 3.3.3.7. Let $\mathcal{L}$ be any cubical invertible sheaf on G that is *relatively ample* over S. Then, by statement 1 of Corollary 3.3.3.4, $\mathcal{L}_{\text{for}} \otimes [-1]^*\mathcal{L}_{\text{for}}$ comes canonically from a cubical $\mathbf{G}_{\text{m}}$-torsor $\mathcal{M}_{\text{for}}$ on A_{for}, which is *ample* by [110, XI, 1.11]. By Theorem 2.3.1.5 (or rather [47, III, 5.4.5]), the pair $(A_{\text{for}}, \mathcal{M}_{\text{for}})$ is algebraizable. In other words, there is an abelian scheme A over S, together with an ample invertible sheaf $\mathcal{M}$ over A, such that $(A_{\text{for}}, \mathcal{M}_{\text{for}}) \cong (A, \mathcal{M}) \underset{S}{\times} S_{\text{for}}$.

On the other hand, T_{for} is given by an étale sheaf $\underline{X}_{\text{for}}$ over S_{for}. By [33, X, 3.2], there is a torus T over S, with character group $\underline{X}$, such that $T_{\text{for}} \cong T \underset{S}{\times} S_{\text{for}}$ and $\underline{X}_{\text{for}} \cong \underline{X} \underset{S}{\times} S_{\text{for}}$. Moreover, we may treat the underlying scheme of $\underline{X}$ as a disjoint union of schemes that are *finite étale* over S. By Theorem 2.3.1.4 (or rather [47, III, 5.4.1]), the morphism $c_{\text{for}} : \underline{X}_{\text{for}} \to A^{\vee}_{\text{for}}$ is algebraizable by a unique morphism $c : \underline{X} \to A^{\vee}$. This gives an extension as in (3.3.3.8) whose formal completion is (3.3.3.2), as desired. □

Proposition 3.3.3.11. *With the same setting as in Proposition 3.3.3.7, the natural functor*

$$\mathrm{CUB}_{S_{\mathrm{for}}}(G^\natural, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}}) \to \mathrm{CUB}_{S_{\mathrm{for}}}(G_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}}) \qquad (3.3.3.12)$$

induced by the isomorphism $G^\natural_{\mathrm{for}} \simeq G_{\mathrm{for}}$ *is fully faithful, and the essential image of (3.3.3.12) contains all* $\mathcal{L}_{\mathrm{for}}$ *coming from a cubical* $\mathbf{G}_{\mathrm{m}}$*-torsor on* A_{for}*. In particular, if all torus over* S_0 *are isotrivial, then (3.3.3.12) is an equivalence of categories.*

The same proof as in [96, IV, 2.2] applies to our case. We do not include the proof here because it does not involve information that we will need later. Note that the arguments in [96, IV, 2.2] concerning essential surjectivity uses descent to A_{for}.

3.4 The Group $K(\mathcal{L})$ and Applications

3.4.1 Quasi-Finite Subgroups of a Semi-Abelian Scheme over a Henselian Base

Let S be the spectrum of a *Henselian* noetherian local ring R, s be its closed point, and k be its residue field. Suppose that we are given a semi-abelian scheme $G \to S$ over S (defined as in Definition 3.3.1.1). If X is a scheme over S that is *quasi-finite* and separated over S, then we denote by X^{f} (the *finite part* of X) its largest finite subscheme over S. We have thus a decomposition

$$X = X^{\mathrm{f}} \amalg X' \qquad (3.4.1.1)$$

(as in [59, IX, 2.2.3]), where X^{f} is finite over S and where the closed fiber X'_0 of X' is empty.

Now let us take X to be a closed subgroup scheme H of G, where H is quasi-finite over S. Then H^{f} is an open and closed subgroup scheme of H. If H_1 is a closed subgroup of H, then we verify immediately that

$$H_1^{\mathrm{f}} = H_1 \cap H^{\mathrm{f}}. \qquad (3.4.1.2)$$

Suppose moreover that H is *flat* over S. Then H^{f} is flat as well, since it is open in H. Let $T_0 \subset G_0$ be the maximal torus of the closed fiber G_0. The group $H \cap T_0$ is of multiplicative type. Hence, by [59, IX, 6.1], it extends to a unique finite subgroup of H, flat and of multiplicative type, which we denote by

$$H^{\mu} \subset H \qquad \text{(the \textit{torus part} of } H). \tag{3.4.1.3}$$

We have obviously $H^{\mu} \subset H^{\mathrm{f}}$. If $H_1 \subset H$ is closed and flat over S, then we have

$$H_1^{\mu} = H_1 \cap H^{\mu}. \tag{3.4.1.4}$$

Note that if $S' \to S$ is a *local* morphism of Henselian local schemes, then the formations of H^{μ} and H^{f} commute with the base change $S' \to S$. Finally, we put

$$H^{\mathrm{ab}} := H^{\mathrm{f}}/H^{\mu}. \tag{3.4.1.5}$$

This is a finite and flat group scheme over S, whose closed fiber H_0^{ab} can be identified with a subgroup of the *abelian part* G_0/T_0 of G_0. We call this the *abelian part* of H.

Now let U be a noetherian scheme over S, and let H_U be a *closed* subgroup scheme of $G_U = G \underset{S}{\times} U$ that is *quasi-finite and flat* over U. (In our applications, U will often be an open subscheme of S.) There then exists $m \geq 1$ such that $H_U \subset G_U[m]$. Since G is a semi-abelian scheme, the kernel $G[m]$ of multiplication by m is a closed subgroup of G, which is *flat* and *quasi-finite* over S. (See [59, IX, 2.2.1].) We may thus put

$$H_{U/S}^{\mathrm{f}} := H_U \cap (G[m]^{\mathrm{f}})_U \tag{3.4.1.6}$$

and

$$H_{U/S}^{\mu} := H_U \cap (G[m]^{\mu})_U. \tag{3.4.1.7}$$

(We denote here $G[m]^{\mathrm{f}} = (G[m])^{\mathrm{f}}$, etc.)

By applying (3.4.1.2) and (3.4.1.4) to the inclusions $G[m_1] \subset G[m_2]$ when $m_1 | m_2$, we see that the definitions (3.4.1.6) and (3.4.1.7) do not depend on the choice of m. The two subgroups of H_U thus defined are *finite* over U, because they are closed in $(G[m]^{\mathrm{f}})_U$. Moreover, $H_{U/S}^{\mathrm{f}}$ is *open* in H_U, which implies that $H_{U/S}^{\mathrm{f}}$ is *finite and flat* over U. The same is true for $H_{U/S}^{\mu}$:

Lemma 3.4.1.8 ([96, IV, 1.3.3])**.** *The group $H_{U/S}^{\mu}$ defined in (3.4.1.7) is flat over U.*

By Lemma 3.4.1.8, we can define the *abelian part*

$$H_{U/S}^{\mathrm{ab}} := H_{U/S}^{\mathrm{f}}/H_{U/S}^{\mu} \tag{3.4.1.9}$$

of H_U. This is a finite and flat group scheme over U.

When $U = S$, it is clear that the groups $H^{\mathrm{f}}_{U/S}$, $H^{\mu}_{U/S}$, and $H^{\mathrm{ab}}_{U/S}$ coincide with the groups H^{f}_{U}, $H^{\mu}_{U/S}$, and H^{ab}_{U} of (3.4.1.1), (3.4.1.3), and (3.4.1.5). Moreover, for a fixed scheme S, their formation commutes with any base change $U' \to U$ between schemes over S. More generally, if we have a commutative diagram

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & U \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

where S' is locally Henselian and where the morphism $S' \to S$ is *local*, then by putting $H_{U'} = H_U \underset{U}{\times} U'$, we have

$$H^{\mathrm{f}}_{U'/S'} = H^{\mathrm{f}}_{U'/S} = (H^{\mathrm{f}}_{U/S}) \underset{U}{\times} U', \tag{3.4.1.10}$$

and the same holds for the torus and abelian parts. (Note that the first equality results already from $(G_{S'}[m])^{\mathrm{f}} = G[m]^{\mathrm{f}} \underset{S}{\times} S'$.)

3.4.2 Statement of the Theorem on the Group $K(\mathcal{L})$

Now we retain the hypotheses and notations of Section 3.4.1, with moreover the assumption that R is *complete*. (That is, we also incorporate the assumptions in 3.3.3, so that Raynaud extensions of G could be defined.) Proceeding as explicitly in [59, IX, 7.3] for the quasi-finite subscheme $G[m]$, we obtain canonical isomorphisms

$$(G[m])^{\mu} \simeq T[m] \tag{3.4.2.1}$$

$$(G[m])^{\mathrm{f}} \simeq G^{\natural}[m] \tag{3.4.2.2}$$

$$(G[m])^{\mathrm{ab}} \simeq A[m] \tag{3.4.2.3}$$

of group schemes over S, characterized by the condition that they induce respectively, over the formal completions, the natural isomorphisms

$$(G[m])^{\mathrm{f}}_{\mathrm{for}} \simeq (G_{\mathrm{for}})[m] \simeq (G^{\natural}_{\mathrm{for}})[m] \simeq (G^{\natural}[m])_{\mathrm{for}}$$
$$(G[m])^{\mu}_{\mathrm{for}} \simeq (T[m])_{\mathrm{for}}$$
$$(G[m])^{\mathrm{ab}}_{\mathrm{for}} \simeq (A[m])_{\mathrm{for}}.$$

Now let U be a scheme over S, and let H_U, as in Section 3.4.1, be a *closed* subscheme of G_U, which is *flat and quasi-finite* over U. Then we have the finite and flat groups $H^{\mu}_{U/S}$, $H^{\mathrm{f}}_{U/S}$, and $H^{\mathrm{ab}}_{U/S}$ over U, contained respectively in $(G[m]^{\mu})_U$, $(G[m]^{\mathrm{f}})_U$, and $(G[m]^{\mathrm{ab}})_U$ as soon as $H_U \subset G[m]_U$. From the isomorphisms (3.4.2.1), (3.4.2.2), and (3.4.2.3), we have finite and flat groups over U, denoted following Grothendieck as $H^{\flat}_{U/S}$, $H^{\natural}_{U/S}$, and $H^{\sharp}_{U/S}$, which are defined by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{\mu}_{U/S} \arrow[r] \arrow[d, "\wr"'] & H^{\mathrm{f}}_{U/S} \arrow[r] \arrow[d, "\wr"'] & H^{\mathrm{ab}}_{U/S} \arrow[r] \arrow[d, "\wr"'] & 0 , \\
0 \arrow[r] & H^{\flat}_{U/S} \arrow[r] \arrow[d, hook] & H^{\natural}_{U/S} \arrow[r] \arrow[d, hook] & H^{\sharp}_{U/S} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (T[m])_U \arrow[r] \arrow[d, hook] & (G^{\natural}[m])_U \arrow[r] \arrow[d, hook] & (A[m])_U \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & T \arrow[r] & G^{\natural} \arrow[r] & A \arrow[r] & 0
\end{tikzcd} \tag{3.4.2.4}$$

where the rows are exact, and where the isomorphisms between the first two rows are induced by respectively (3.4.2.1), (3.4.2.2), and (3.4.2.3). On the other hand, it is clear that the subgroups $H^{\flat}_{U/S}$, etc, of T, $G^{\natural}$, and A are independent of the choice of m.

Remark 3.4.2.5. It is true that $H^{\mathrm{f}}_{U/S}$ and $H^{\natural}_{U/S}$ are canonically isomorphic. However, at least for our purpose, it will be more convenient to maintain the difference of notations.

We can now state the main theorem of [96, IV]:

Theorem 3.4.2.6 ([96, IV, 2.4])**.** *Let S be a* **normal** *locally noetherian integral scheme, with generic point η. Let $G \xrightarrow{f} S$ be a* **semi-abelian** *scheme over S (Definition 3.3.1.1) such that its generic fiber G_η is an abelian variety, and let $\mathcal{L}_\eta$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor on G_η. Suppose that:*

- *$\mathcal{L}_\eta$ is* **nondegenerate**. *In other words, the group $K(\mathcal{L}_\eta)$ is* **finite**. *(See (3.2.4.1).) (The group $K(\mathcal{L}_\eta)$ will be denoted K_η).*
- *$\mathcal{L}_\eta$ admits a cubical extension $\mathcal{L}_S$ over S.*

(*The first hypothesis is satisfied, for example, if* $\mathcal{L}_\eta$ *is* **ample**. *The second is satisfied, for example, if* S *is* **regular** (*by Proposition 3.3.2.2*), *or if* $\mathcal{L}_\eta$ *is* **symmetric** (*by Theorem 3.3.2.3*).)

Then:

1. $K_\eta := K(\mathcal{L}_\eta)$ *extends to a unique closed subgroup* K_S *of* G *that is* **flat** *and quasi-finite over* S. *In this case,* K_S *is necessarily the schematic closure of* K_η *in* G.

2. *There exists a unique alternating paring*

   $$\mathrm{e}_S^{\mathcal{L}_\eta} : K_S \underset{S}{\times} K_S \to \mathbf{G}_{\mathrm{m},S}$$

   extending the $\mathcal{L}_\eta$*-Weil paring* $\mathrm{e}^{\mathcal{L}_\eta}$. (*See [99, §23].*) *Moreover, if* K_S *is* **finite** *over* S, *then* $\mathrm{e}_S^{\mathcal{L}_\eta}$ *is a perfect duality.*

3. *Suppose that* $\mathcal{L}_\eta$ *admits a cubical extension* $\mathcal{L}_S$ *over* G. *Then the restriction of the biextension* $\mathcal{D}_2(\mathcal{L}_S)$ *to* $K_S \underset{S}{\times} G$ *is equipped with a trivialization extending the one of* $\mathcal{D}_2(\mathcal{L}_\eta)$ *defined in Section 3.2.4. As a result, by the same Section 3.2.4, the* $\mathbf{G}_{\mathrm{m}}$*-torsor* $\mathcal{G}(\mathcal{L}_S) = \mathcal{L}_S|_{K_S}$ *is equipped with a natural structure of central extension, denoted by*

   $$0 \to \mathbf{G}_{\mathrm{m},S} \to \mathcal{G}(\mathcal{L}_S) \to K_S \to 0,$$

   and with an action of this extension on $\mathcal{L}_S$. *Besides, the pairing* $\mathrm{e}_S^{\mathcal{L}_\eta}$ *of 2 coincides with the commutator pairing of the extension* $\mathcal{G}(\mathcal{L}_S)$.

4. (**Theorem of orthogonality**.) *Suppose that* S *is the spectrum of a* **Henselian local** *ring* R, *so that* (*as in Section 3.4.1*) K_S *is equipped with a filtration* $K_S^\mu \subset K_S^{\mathrm{f}} \subset K_S$. *Then* K_S^{f} *is the annihilator of* K_S^μ *for the pairing* $\mathrm{e}_S^{\mathcal{L}_\eta}$, *and the induced pairing on the quotient* K_S^{ab} *is a perfect duality. If moreover* K_S *is* **finite** *over* S, *then* $\mathrm{e}_S^{\mathcal{L}_\eta}$ *induces a perfect duality between* K_S^μ *and* K_S/K_S^{f}.

5. *Suppose that* $S = \mathrm{Spec}(R)$ *where* R *is* **complete local**, *and that* G *has the associated Raynaud extension*

   $$0 \to T \to G^\natural \to A \to 0$$

   (*defined as in Proposition 3.3.3.7*). *Then, by Proposition 3.3.3.11, there exists a unique cubical* $\mathbf{G}_{\mathrm{m}}$*-torsor* $\mathcal{L}^\natural$ *on* $G^\natural$ *whose formal completion* *is* $(\mathcal{L}_S)_{\mathrm{for}}$. *Suppose that* $\mathcal{L}^\natural$ *comes from a* (*cubical*) $\mathbf{G}_{\mathrm{m}}$*-torsor* $\mathcal{M}$ *on* A (*which is true in any case after a* **finite étale** *base change, by Corollary 3.2.5.5*). *Then the subgroup* $K_S^\sharp$ *of* (*3.4.2.4*) *coincides with* $K(\mathcal{M})$, *and the pairing induced by* $\mathrm{e}_S^{\mathcal{L}_\eta}$ *on* $K_S^\sharp \simeq K_S^{\mathrm{ab}}$ *by 4 coincides with* $\mathrm{e}^{\mathcal{M}}$.

3.4.3 Dual Semi-Abelian Schemes

Let S be a *normal* locally noetherian integral scheme, with generic point η. For any semi-abelian scheme $G \to S$ whose generic fiber G_η is an abelian scheme, it makes sense to talk about the dual abelian variety $(G_\eta)^\vee$ of G_η. Suppose $(G_\eta)^\vee$ extends to a semi-abelian scheme $G^\vee$, then it follows from Proposition 3.3.1.7 that any polarization $G_\eta \to (G_\eta)^\vee$ extends uniquely to a group scheme homomorphism $G \to G^\vee$. In particular, $G^\vee$ is unique up to unique isomorphism. Thus the main question is whether it exists at all.

The dual $(G_\eta)^\vee$ of G_η is usually constructed (in for example [99]) by forming a quotient $(G_\eta)^\vee = G_\eta/K(\mathcal{L}_\eta)$ for some ample line bundle $\mathcal{L}_\eta$ on G_η. Therefore the natural question is whether there is a suitable scheme $K_S(\mathcal{L}_\eta)$ that extends $K(\mathcal{L}_\eta)$ to a subscheme of G over S, so that we can form similarly a quotient $G/K_S(\mathcal{L}_\eta)$ that is representable as a scheme.

Therefore, as a corollary of Theorem 3.4.2.6, we can state the existence of the *dual semi-abelian scheme* as the following:

Theorem 3.4.3.1 (cf. [96, IV, 7.1]). *Let* S *be a* **normal** *locally noetherian integral scheme with generic point* η. *Let* G *be a semi-abelian scheme over* S *whose generic fiber* G_η *is an abelian variety. Then:*

1. *There exists a unique semi-abelian scheme over* S, *denoted by* $G^\vee$, *extending the dual variety* $(G_\eta)^\vee$ *of* G_η, *and a unique biextension* $\mathcal{P}$ *of* $G \underset{S}{\times} G^\vee$ *by* $\mathbf{G}_{\mathrm{m},S}$ *extending the Poincaré biextension* $\mathcal{P}_\eta$ *on* $G_\eta \underset{\eta}{\times} (G_\eta)^\vee$.

2. *Let* $\mathcal{L}$ *be a cubical* $\mathbf{G}_{\mathrm{m}}$*-torsor on* G *whose restriction* $\mathcal{L}_\eta$ *to* G_η *is nondegenerate. Let* $K_S(\mathcal{L}_\eta)$ *be the schematic closure of* $K(\mathcal{L}_\eta)$ *in* G *over* S. *Then the morphism* $\lambda_{\mathcal{L}_\eta} : G_\eta \to (G_\eta)^\vee$ *extends to a unique morphism* $\lambda_{\mathcal{L}_\eta,S} : G \to G^\vee$ *such that* $\ker(\lambda_{\mathcal{L}_\eta,S}) = K_S(\mathcal{L}_\eta)$. *Moreover, we have a canonical isomorphism of biextensions on* $G \underset{S}{\times} G$:

$$\mathcal{D}_2(\mathcal{L}) \overset{\sim}{\to} (\mathrm{Id}_G \times \lambda_{\mathcal{L}_\eta,S})^*\mathcal{P} \tag{3.4.3.2}$$

extending the usual isomorphism on $G_\eta \underset{\eta}{\times} G_\eta$.

We shall include the proof of [96, IV, 7.1] here, because the construction of the dual is important and instructive for our later purposes.

Proof of Theorem 3.4.3.1. Let us prove 1 first.

As remarked in the beginning of this section, the uniqueness of $G^\vee$ is a corollary of Proposition 3.3.1.7. The existence and the uniqueness of $\mathcal{P}$ follows from Theorem 3.3.2.6.

For the existence of $G^\vee$, we may suppose that S is local. Then by [110, IX, 1.13], there exists a $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$ over G, which is *relatively ample* over S. We may require that $\mathcal{L}$ is rigidified, so that a cubical structure of $\mathcal{L}$ exists by Proposition 3.2.3.1. Thus we have fulfilled the assumption in Theorem 3.4.2.6, so that $K(\mathcal{L}_\eta)$ extends to a flat and quasi-finite subscheme $K_S(\mathcal{L}_\eta)$ of G over S.

It remains to show that the quotient sheaf $G/K_S(\mathcal{L}_\eta)$ is a *scheme*, because $(G_\eta)^\vee$ is identified with $G_\eta/K(\mathcal{L}_\eta)$. More generally:

Lemma 3.4.3.3 ([96, IV, 7.1.2])**.** *Let G be a semi-abelian scheme over a locally noetherian base scheme S. Let K be a closed subgroup scheme of G, flat and quasi-finite over S. Suppose that one of the following two hypotheses is verified:*

1. *Locally for the étale topology over S, G is quasi-projective.*
2. *K is étale over S.*

Then the quotient sheaf G/K is a semi-abelian scheme over S.

Let us include the proof for the convenience of the reader.

Proof of Lemma 3.4.3.3. Let us first show that G/K is an *algebraic space* over S. This is clear in case 2. In case 1 the question is local over S for the étale topology. Hence we may suppose that G is quasi-projective over S and that K falls into

$$0 \to K^{\mathrm{f}} \to K \to E \to 0,$$

where K^{f} is finite and flat and E is étale over S. By [34, V, 4.1], and by the hypothesis of quasi-projectivity, the quotient $G' = G/K^{\mathrm{f}}$ is a *scheme*. Hence $G/K = G'/E$ is an algebraic space.

To see that G/K is a scheme, it suffices to remark that (locally over S) there exists $N \geq 1$ such that $K \subset G[N]$. Then there exists a *quasi-finite* morphism $\pi : G/K \to G$, and we can use [76, II, 6.16] (which says an algebraic space quasi-finite and separated over a scheme is a scheme). $\square$

Back to the proof of Theorem 3.4.3.1. Let us now prove 2. We define $\lambda_{\mathcal{L}_\eta,S}$ as the natural projection $G \to G/K_S(\mathcal{L}_\eta)$. The isomorphism (3.4.3.2) follows from Theorem 3.3.2.6. Finally, we have two structures of central extension over the restriction of the torsor $\mathcal{L}$ to $K_S(\mathcal{L}_\eta)$, which coincide at the generic point of S, which then coincide over the whole S by Proposition 3.3.2.2. $\square$

3.4.4 Dual Raynaud Extensions

Assume now that R be a noetherian normal domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S = \mathrm{Spec}(R)$, K the quotient field of R, $\eta = \mathrm{Spec}(K)$ the generic point of S, $R_i := R/I^{i+1}$, and $S_i = \mathrm{Spec}(R_i)$, for any integer $i \geq 0$. Let G be a semi-abelian scheme over S whose generic fiber G_η is an abelian scheme, and suppose that G_0 is an *extension* of an abelian scheme A_0 by an *isotrivial* torus T_0 (defined as in Definition 3.1.1.5). By Theorem 3.4.3.1 and its proof, there is a semi-abelian scheme $G^\vee$ whose generic fiber is the dual abelian variety of G_η, and the torus part of $G_0^\vee := G^\vee \underset{S}{\times} S_0$ is isotrivial because it is so for G. Hence we may consider the Raynaud extensions (defined as in Definition 3.3.3.9) associated to G and $G^\vee$, denoted respectively $0 \to T \to G^\natural \to A \to 0$ and $0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$ by abuse of notation. To justify this notation, we need:

Proposition 3.4.4.1. *The abelian part $A^\vee$ of $G^{\vee,\natural}$ is the dual abelian scheme of the abelian part A of G. In fact, the extension $\mathcal{P} = \mathcal{P}_G$ on $G \underset{S}{\times} G^\vee$ of the Poincaré invertible sheaf $\mathcal{P}_{G_\eta}$ on $G_\eta \underset{\eta}{\times} (G_\eta)^\vee$ constructed above (in Theorem 3.4.3.1) can be descended to the Poincaré invertible sheaf $\mathcal{P}_A$ on $A \underset{S}{\times} A^\vee$ after first passing to the formal completion.*

Proof. By [59, VIII, 3.5] (or Corollary 3.2.5.2), the formal completion $\mathcal{P}_{G,\mathrm{for}} = \{\mathcal{P}_G \underset{S}{\times} S_i\}_{i\geq 0}$ of $\mathcal{P}_G$ is a biextension on $G_{\mathrm{for}} \times (G^\vee)_{\mathrm{for}}$ that can be uniquely descended to a biextension $\mathcal{P}_{A,\mathrm{for}}$ on $A_{\mathrm{for}} \times (A^\vee)_{\mathrm{for}}$. Since A_{for} and $(A^\vee)_{\mathrm{for}}$ are both algebraic, so is $\mathcal{P}_{A,\mathrm{for}}$ by Corollary 2.3.1.3. Denote by $\mathcal{P}_A$ the algebraization of $\mathcal{P}_{A,\mathrm{for}}$. It remains to check that $\mathcal{P}_A$ actually defines a duality between A and $A^\vee$. It suffices to check this over η. Hence it suffices to check that the pairing $A[\ell^\infty] \times A^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m}}[\ell^\infty]$ on Barsotti-Tate groups induced by $\mathcal{P}_A$ is a perfect pairing over η, for every prime ℓ. The pairing $G^\natural[\ell^\infty] \times G^{\vee,\natural}[\ell^\infty] \to \mathbf{G}_{\mathrm{m}}[\ell^\infty]$ induced by $\mathcal{P}_{G,\mathrm{for}}$ factors through the previous pairing, because $\mathcal{P}_{G,\mathrm{for}}$

is the pull-back of $\mathcal{P}_{A,\mathrm{for}}$. On the other hand $G^\natural[\ell^\infty] \subset G[\ell^\infty]$, $G^{\vee,\natural}[\ell^\infty] \subset G^\vee[\ell^\infty]$ canonically, and the pairing $G[\ell^\infty] \times G^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m}}[\ell^\infty]$ induced by $\mathcal{P}_G$ extends the previous pairing. Since G and $G^\vee$ are abelian varieties over η, $G[\ell^\infty] \times G^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m}}[\ell^\infty]$ is a perfect pairing over η. If we can show that the annihilator of $G^\natural_\eta[\ell^\infty]$ in $G^\vee_\eta[\ell^\infty]$ is $T^\vee_\eta[\ell^\infty]$, the claim will follow. But this follows from similar statements for pairings defined by $\mathcal{D}_2(\mathcal{L})$. Since we may take $\mathcal{L}$ of the form $\mathcal{L}' \otimes [-1]^*\mathcal{L}'$ for some ample cubical invertible sheaf $\mathcal{L}'$ so that $\mathcal{L}_{\mathrm{for}}$ descends to an ample $\mathcal{M}_{\mathrm{for}}$ on A_{for}, the result follows. (This is the argument in [37, Ch. II, §2]. More details about various facts used in the above argument can be found in [59, IX].) ☐

By Proposition 3.1.5.1, the two Raynaud extensions $0 \to T \to G^\natural \to A \to 0$ and $0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$ are coded by two homomorphisms $c : \underline{X}(T) \to A^\vee$ and $c^\vee : \underline{X}(T^\vee) \to A$. We denote the two projections $G^\natural \to A$ and $G^{\vee,\natural} \to A^\vee$ by π and $\pi^\vee$ respectively, and let $\underline{X} = \underline{X}(T)$ and $\underline{Y} = \underline{X}(T^\vee)$. Pulling back the biextension $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$ by $c^\vee \times c$, we get a $\mathbf{G}_{\mathrm{m}}$-biextension of the étale group scheme $\underline{Y} \underset{S}{\times} \underline{X}$ by $\mathbf{G}_{\mathrm{m}}$.

By Proposition 3.3.1.7, any polarization $\lambda_\eta : G_\eta \to G^\vee_\eta$ extends uniquely to a homomorphism $\lambda_G : G \to G^\vee$. The functoriality of Raynaud extensions then gives us a homomorphism $\lambda^\natural : G^\natural \to G^{\vee,\natural}$, which induces a homomorphism $\lambda^\natural|_T : T \to T^\vee$ between the torus parts, and a polarization $\lambda_A : A \to A^\vee$ between the abelian parts. By Corollary 3.1.1.6, the homomorphism $\lambda^\natural|_T$ determines (and is determined by) a homomorphism $\phi : \underline{Y} \to \underline{X}$, and the two morphisms ϕ and λ_A satisfy the compatible relation $\lambda_A c^\vee = c\phi$.

Conversely, the two maps ϕ and λ_A satisfying $\lambda_A c^\vee = c\phi$ determines $\lambda^\natural$ uniquely, because every homomorphism from an abelian scheme to a torus is trivial, and more concretely because we can write down a map explicitly as follows: If we denote the rigidified invertible sheaf $(\mathrm{Id}, c(\chi))^*\mathcal{P}_A$ on A corresponding to $c(\chi) \in A^\vee$ as $\mathscr{M}_{c(\chi)} \in \underline{\mathrm{Pic}}^0_e(A/S)$, and denote the rigidified invertible sheaf $(c^\vee(\chi), \mathrm{Id})^*\mathcal{P}_A$ on $A^\vee$ corresponding to $c^\vee(\chi) \in (A^\vee)^\vee = A$ as $\mathscr{M}_{c^\vee(\chi)} \in \underline{\mathrm{Pic}}^0_e(A^\vee/S)$, then the map $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ can be described explicitly (by Propositions 3.1.2.10 and 3.1.5.1) as

$$\begin{aligned} G^\natural &= \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{X}(T)}{\oplus} \mathscr{M}_{c(\chi)}\Big) \\ &\to \lambda_A^*(G^{\vee,\natural}) = \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{X}(T^\vee)}{\oplus} \lambda_A^*\mathscr{M}_{c^\vee(\chi)}\Big) = \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{X}(T^\vee)}{\oplus} \mathscr{M}_{\lambda_A c^\vee(\chi)}\Big) \\ &= \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{X}(T^\vee)}{\oplus} \mathscr{M}_{c(\phi(\chi))}\Big). \end{aligned}$$

Let us record this observation as:

Lemma 3.4.4.2. *A map* $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ *is equivalent to a pair of maps* $\phi : \underline{Y} \to \underline{X}$ *and* $\lambda_A : A \to A^\vee$ *such that* $\lambda_A c^\vee = c\phi$*.*

Note that the two étale sheaves of finitely generated free abelian groups $\underline{X}$ and $\underline{Y}$ have the same rank, and ϕ is an injection with finite cokernel, because the map from T to its image in $T^\vee$ has finite kernel.

Remark 3.4.4.3. The degree of $\lambda^\natural$ is $[\underline{X} : \phi(\underline{Y})]$-times the degree of λ_A, and the degree of λ_G is $[\underline{X} : \phi(\underline{Y})]^2$-times the degree of λ_A. Indeed, we have the following commutative diagram for every integer n:

$$\begin{tikzcd}
0 \arrow[r] & G^\natural[n] \arrow[r] \arrow[d, "\lambda^\natural"'] & G[n] \arrow[r] \arrow[d, "\lambda_G"'] & \frac{1}{n}\underline{Y}/\underline{Y} \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & G^{\vee,\natural}[n] \arrow[r] & G^\vee[n] \arrow[r] & \frac{1}{n}\underline{X}/\underline{X} \arrow[r] & 0
\end{tikzcd}$$

Chapter 4

Theory of Degeneration for Polarized Abelian Schemes

In this chapter we reproduce the theory of degeneration data for abelian varieties, as elaborated in the original paper of [100] and in the first three chapters of [37]. Some substantial modifications have been introduced to make the statements compatible with our understanding of the proof. Although there is essentially nothing new in this chapter, we hope our rewriting of the subject will at least provide some corrections and clarifications even to experience readers.

The main objective in this chapter will be to state and proof Theorems 4.2.1.8, 4.4.18, and 4.6.3.30. Technical results worth noting are Propositions 4.3.2.11, 4.3.3.3, and 4.3.4.6 in Section 4.3; Proposition 4.5.1.14, Theorems 4.5.3.5 and Theorem 4.5.3.9, Proposition 4.5.3.10, and Theorem 4.5.4.12 in Section 4.5; and Propositions 4.6.1.5 and 4.6.2.9, and Theorem 4.6.3.13 in Section 4.6. Some of the differences among our work and the ones in [100] and [37], notably in the statements of Definitions 4.2.1.1 and 4.5.1.2, and Theorems 4.2.1.8 and 4.4.18, are explained in Remarks 4.1.1, 4.2.1.11 and 4.5.1.4.

4.1 The Setting

Let R be a noetherian normal domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S := \mathrm{Spec}(R)$, K the quotient field of R, $\eta := \mathrm{Spec}(K)$ the generic point of S, $R_i := R/I^{i+1}$, $S_i := \mathrm{Spec}(R_i)$, for any integer $i \geq 0$, and $S_{\mathrm{for}} = \mathrm{Spf}(R, I)$.

This setting will be assumed throughout the chapter, unless otherwise specified.

Remark 4.1.1. We will *not* need the following technical assumption made by Faltings and Chai in [37, Ch. II, §3, p. 36] (see Remark 4.1.1 below): For any étale R_0-algebra R'_0, its unique lifting R' to a formally étale I-adically complete R-algebra is normal. (See Remark 3.3.3.5 for the description of $R \to R'$.) Let us explain why we do not their assumption. The introduction of base extensions such as $R \to R'$ with $R_0 \to R'_0$ étale is necessary for splitting the various objects in the constructions. If the base extension $R \to R'$ is étale, then we may perform étale descent and obtain the result over R. The main reason for Faltings and Chai to make their assumption is that $R \to R'$ might fail to be locally of finite presentation in general, in which case it cannot be étale by definition. However, if we can arrange that the objects are split after base change by a *finite étale* morphism $R_0 \to R'_0$, then R' is finite over R (by [47, III, 5.4.5], or Theorem 2.3.1.5), and hence étale. This will be the case for us, because in our Definition 4.2.1.1 below, we require that G_0 is an extension of an abelian scheme by an *isotrivial* torus. Since we cannot justify the construction in [37] of the Raynaud extension in the case that the torus part of G_0 is not isotrivial (see Remark 3.3.3.1), we shall be content with a weaker theory (which nevertheless suffices for our construction of compactifications).

4.2 Ample Degeneration Data

The idea of introducing periods to degenerating abelian varieties is originally due to Tate in the case of elliptic curves, then due to Mumford [100] in the case that the degeneration is given by a split torus, and then due to Faltings and Chai [37, Ch. II] in the general case that the degeneration is a semi-abelian variety. In this section, we will introduce the idea of period maps, and state the main theorem of [37, Ch. II] that associates a period map to a degenerating abelian variety with a relatively ample invertible sheaf. The proof will be given in Section 4.3 using the Fourier expansions of theta functions.

4.2.1 Main Definitions and Main Theorem

First let us introduce the category of degenerating abelian schemes that we will study:

Definition 4.2.1.1. *Assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{ample}}$ *has objects of the form* $(G, \mathcal{L})$*, where:*

1. *G is a semi-abelian scheme over S such that G_η is an abelian variety, and such that $G_0 = G \underset{S}{\times} S_0$ is an* **extension**
$$0 \to T_0 \to G_0 \to A_0 \to 0.$$
of an abelian scheme A_0 by an **isotrivial** *torus T_0 (defined as in Definition 3.1.1.5) over S_0.*
2. *$\mathcal{L}$ is an invertible sheaf on G rigidified along the identity section (and hence endowed with a canonical cubical structure by Proposition 3.2.3.1) such that $\mathcal{L}_\eta$ is ample over G_η, and such that $\mathcal{L}_{\mathrm{for}}$ is in the essential image of (3.3.3.12).*

Let us now describe another category $\mathrm{DD}_{\mathrm{ample}}$, the category of *ample degeneration data.*

First, we need a semi-abelian scheme $G^\natural$ (defined as in Definition 3.3.1.1) that is globally an extension of an abelian scheme A by a torus T. What we have in mind is that $G^\natural$ should be the Raynaud extension (defined in Section 3.3.3) associated to G, so that $G^\natural_{\mathrm{for}} = G_{\mathrm{for}}$ and so that in particular $T_0 = T \underset{S}{\times} S_0$ and $A_0 = A \underset{S}{\times} S_0$ if T_0 and A_0 are as in Definition 4.2.1.1. By Proposition 3.1.5.1, we know that the structure of $G^\natural$ as a commutative group scheme extension of A by T is determined by a group scheme homomorphisms

$$c : \underline{X} = \underline{X}(T) \to A^\vee,$$

where $A^\vee$ is the dual abelian scheme that represents the functor $\underline{\mathrm{Pic}}_e(A/S)$ by definition. (See Theorem 1.3.2.6.)

Second, we need a notion of a period map $\iota : \underline{Y}_\eta \to G^\natural_\eta$. For reasons that will be seen later, we need $\underline{Y}$ to be an étale sheaf of free commutative groups of rank $r = \dim_S(T) = \mathrm{rank}_S(\underline{X})$. If we compose this ι with the natural projection $G^\natural_\eta \to A_\eta$, then we obtain a map

$$\underline{Y}_\eta \to A_\eta.$$

What we have in mind is that this should come from the map

$$c^\vee : \underline{Y} \to A$$

describing the Raynaud extension $G^{\vee,\natural}$ of $G^\natural$ defined by the dual $G^\vee$ of G, as described by Theorem 3.4.3.1 and Proposition 3.4.4.1.

Lemma 4.2.1.2. *With the setting as above, a group homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$ lying over $c^\vee : \underline{Y} \to A$ as above determines and is determined by a trivialization*

$$\tau^{-1} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_{A,\eta}$$

of the biextension $(c^\vee \times c)^*\mathcal{P}_{A,\eta}$ over the étale group scheme $(\underline{Y} \underset{S}{\times} \underline{X})_\eta$ over η.

The proof of Lemma 4.2.1.2 will be given in Section 4.2.2. For convenience we shall let

$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}^{\otimes -1}_{A,\eta}$$

be the inverse of τ^{-1}.

So far we have described data that are related to G. Next, we need data that are related to the cubical relatively ample invertible sheaf $\mathcal{L}$.

First, we need a cubical relatively ample invertible sheaf $\mathcal{L}^\natural$ over $G^\natural$. What we have in mind is that $\mathcal{L}^\natural_{\text{for}}$ and $\mathcal{L}_{\text{for}}$ should be canonically isomorphic over $G^\natural_{\text{for}} = G_{\text{for}}$, and there is a unique such $\mathcal{L}^\natural$ by Proposition 3.3.3.11 and by our assumption that $\mathcal{L}_{\text{for}}$ is in the essential image of (3.3.3.12). By Corollary 3.2.5.2, the $\mathbf{G}_{\text{m}}$-biextension $\mathcal{D}_2(\mathcal{L}^\natural)$ over $G^\natural \underset{S}{\times} G^\natural$ descends uniquely to a $\mathbf{G}_{\text{m}}$-biextension on $A \underset{S}{\times} A$, which (as a relatively ample invertible sheaf on $A \underset{S}{\times} A$) induces a polarization $\lambda_A : A \to A^\vee$ of A over S. In other words, this $\mathbf{G}_{\text{m}}$-biextension on $A \underset{S}{\times} A$ is isomorphism to $(\text{Id}_A \times \lambda_A)^*\mathcal{P}_A$. These statements are compatible with the corresponding statements over S_{for}. Moreover, we need a map $\phi : \underline{Y} \hookrightarrow \underline{X}$, which is dual to an isogeny from T to the torus part $T^\vee$ of $G^{\vee,\natural}$. We shall require that $\lambda_A c^\vee = c\phi$, which is equivalent to a map $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ by Lemma 3.4.4.2.

By Corollary 3.2.5.7, which is applicable because S is noetherian normal, there is a base change to a finite étale surjection over S over which the étale sheaf $\underline{X}$ is constant and the cubical invertible sheaf $i^*\mathcal{L}^\natural$ is trivial. (Here $i : T \to G^\natural$ is the canonical morphism.) In this case, any cubical trivialization

$s : i^*\mathcal{L}^\natural \xrightarrow{\sim} \mathscr{O}_T$ determines a relatively ample cubical invertible sheaf $\mathcal{M}$ on A and a cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$, both depending uniquely on the choice of s.

Second, we need an action $\underline{Y}_\eta$ on $\mathcal{L}^\natural_\eta$ over ι compatible with the T-action up to a character. Let us make this precise by the following:

Lemma 4.2.1.3. *With the setting as above, such an action determines and is determined by a cubical trivialization*

$$\psi : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*(\mathcal{L}^\natural_\eta)^{\otimes -1}$$

compatible with

$$\tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi) : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y},\eta} \xrightarrow{\sim} (c^\vee \times c\phi)^*\mathcal{P}^{\otimes -1}_{A,\eta},$$

in the sense that

$$\mathcal{D}_2(\psi) = \tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi).$$

The proof will be given in Section 4.2.3.

This compatibility makes sense because the biextension $\mathcal{D}_2(\mathcal{L}^\natural)$ uniquely descends to the biextension $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ of $A \underset{S}{\times} A$ over S. Note that this compatibility forces τ to be *symmetric* with respect to ϕ in the sense that $\tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi)$ is a trivialization of the biextension $(c^\vee \times c\phi)^*\mathcal{P}^{\otimes -1}_{A,\eta} = (c^\vee \times \lambda_A c^\vee)^*\mathcal{P}^{\otimes -1}_{A,\eta} = (c^\vee \times c^\vee)^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}^{\otimes -1}_{A,\eta}$ that is invariant under the symmetric isomorphism of $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$. Here $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ has a symmetric isomorphism because λ_A is symmetric by the definition of a polarization (in Definitions 1.3.2.15, 1.3.2.19, and 1.3.2.20), or equivalently because it is étale locally of the form $\mathcal{D}_2(\mathcal{M})$ for some invertible sheaf $\mathcal{M}$ on A. (See Lemma 3.2.2.1.)

The maps τ and ψ will be useful only if we have some suitable positivity conditions. By base change to a finite étale surjection over S if necessary, assume that both $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and that there is a cubical invertible sheaf $\mathcal{M}$ on A such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$. Then:

Definition 4.2.1.4. *The* **positivity condition** *for τ can be stated as follows: For every element y in Y, $\tau(y, \phi(y))$ extends to a section of the invertible sheaf $(c^\vee(y) \times c\phi(y))^*\mathcal{P}^{\otimes -1}_A$ on S, and for every nonzero element y in Y, $\tau(y, \phi(y))$ is congruent to zero modulo I in the sense that the section*

$\tau(y, \phi(y))$ induces a homomorphism $(c^\vee(y) \times c\phi(y))^* \mathcal{P}_A^{\otimes -1} \to \mathscr{O}_S$ whose image factors through $\underline{I}$, where $\underline{I}$ is the invertible subsheaf of $\mathscr{O}_S$ corresponding to the ideal $I \subset R$.

Definition 4.2.1.5. *The* **positivity condition** *for ψ can be stated as follows: Suppose we are given any positive integer n. Then for every element $y \in Y$, the section $\psi(y)$ extends to a section of the invertible sheaf $c^\vee(y)^* \mathcal{M}^{\otimes -1} = \mathcal{M}(c^\vee(y))^{\otimes -1}$ on S, and for all but finitely many y in Y, $\psi(y)$ is congruent to zero modulo I^n in the sense that $\psi(y)$ induces a homomorphism $\mathcal{M}(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is defined as above.*

Lemma 4.2.1.6. *The positivity condition for τ (in Definition 4.2.1.4) and the positivity condition for ψ (in Definition 4.2.1.5) are equivalent to each other.*

The proof will be given in Section 4.2.4.

Let us now state the definition of the category of degeneration data:

Definition 4.2.1.7. *Assumptions as in Section 4.1, the category $\mathrm{DD}_{\mathrm{ample}}$ has objects of the form $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$, with entries described as follows:*

1. *An abelian scheme A and a torus T over S, and an extension*

   $$0 \to T \to G^\natural \to A \to 0$$

   over S, which determines and is determined by a homomorphism

   $$c : \underline{X} \to A^\vee$$

   via the (negative of) push-out, where $\underline{X}$ is the character group of T.

2. *An étale sheaf of free commutative groups $\underline{Y}$ of rank $r = \dim_S(T)$.*

3. *A homomorphism $c^\vee : \underline{Y} \to A$, which determines an extension*

   $$0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$$

   over S, where $T^\vee$ is a torus with character group $\underline{Y}$.

4. *An injective homomorphism $\phi : \underline{Y} \to \underline{X}$ with finite cokernel.*

5. *A cubical ample invertible sheaf $\mathcal{L}^\natural$ on $G^\natural$ inducing a polarization $\lambda_A : A \to A^\vee$ of A over S such that $\lambda_A c^\vee = c\phi$, or equivalently a homomorphism from $G^\natural$ to $G^{\vee,\natural}$ inducing a polarization $\lambda_A : A \to A^\vee$ of A over S.*

6. *A symmetric trivialization*
$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$$
of the biextension $(c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$ over the étale group scheme $(\underline{X} \underset{S}{\times} \underline{Y})_\eta$, which determines a trivialization
$$\tau^{-1} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}$$
and hence a homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$ lying over $c^\vee$ by Lemma 4.2.1.2.

7. *A cubical trivialization*
$$\psi : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^* (\mathcal{L}^\natural_\eta)^{\otimes -1}$$
compatible with
$$\tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi) : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y},\eta} \xrightarrow{\sim} (c^\vee \times c\phi)^* \mathcal{P}_{A,\eta}^{\otimes -1},$$
in the sense that
$$\mathcal{D}_2(\psi) = \tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi),$$
which by Lemma 4.2.1.3 determines and is determined by an action of $\underline{Y}_\eta$ on $\mathcal{L}^\natural_\eta$ over ι compatible with ϕ.

The two trivializations τ and ψ are required to satisfy their respective positivity conditions, which are equivalent by Lemma 4.2.1.6.

Let us now state our slightly modification of the main theorem of [37, Ch. II] as follows:

Theorem 4.2.1.8 (cf. [37, Ch. II, Thm. 6.2])**.** *Assumptions as in Section 4.1, there is a functor*
$$\begin{aligned} \mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} &\to \mathrm{DD}_{\mathrm{ample}} \\ (G, \mathcal{L}) &\mapsto (A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi) \end{aligned}$$

called the association of **ample degeneration data***, which detects isomorphisms in the sense that it sends only isomorphisms to isomorphisms.* (*See Remark 4.2.1.12 below.*) *Moreover, the association of A, $\underline{X}$, $\underline{Y}$, c, and $c^\vee$ does not depend on the choice of $\mathcal{L}$, but only on its existence.* (*See Remark 4.2.1.9 below.*) *The association of ϕ, τ, and the map $\lambda_A : A \to A^\vee$ induced by $\mathcal{L}^\natural$* (*as explained above in this section*) *depends on the map $\lambda : G \to G^\vee$ induced by $\mathcal{L}$, but not on the particular choice of $\mathcal{L}$.* (*See Remark 4.2.1.10 below.*)

Apart from several basic justifications that will be given in Sections 4.2.2, 4.2.3, and 4.2.4, the remaining proof of Theorem 4.2.1.8 will be given in Section 4.3.

Remark 4.2.1.9. The association of A, $\underline{X}$, and c in Theorem 4.2.1.8 depends only on G, but not on $\mathcal{L}$, as they exists by the association of the Raynaud extension $G^\natural$ to G. The association of $\underline{Y}$ and $c^\vee$ in Theorem 4.2.1.8 depends also only on G, because it depends only on the dual $G^\vee$ of G defined in Theorem 3.4.3.1, and then on the association of $G^{\vee,\natural}$ to $G^\vee$ as described by Proposition 3.4.4.1.

Remark 4.2.1.10. The association of ι and equivalently τ in Theorem 4.2.1.8 depends *a priori* on the particular choice of $\mathcal{L}$, but we will show in Section 4.3.4 that it depends only on the map $\lambda : G \to G^\vee$ induced by $\mathcal{L}$. In fact, any two $\lambda_1, \lambda_2 : G \to G^\vee$ induce the same τ if $N_1\lambda_1 = N_2\lambda_2$ for any two integers $N_1, N_2 > 0$.

Remark 4.2.1.11. The claim in [37] that the independence should follow after proving the equivalences of categories in Theorem 4.4.18 using Mumford's constructions requires some further explanation. The argument they stated in the proof of [37, Ch. III, Thm. 7.1] is literally insufficient. We consulted Chai about this issue, and he kindly suggested a different approach that could repair the argument. We hope we could incorporate his argument in a future revision of this work. Nevertheless, since the abelian schemes in the moduli problems we consider are all polarized, we do not need this generality for the purpose of arithmetic compactifications.

Remark 4.2.1.12. The statement that the association of degeneration data sends only isomorphisms to isomorphisms has the following meaning: If we have any morphism from a pair $(G_1, \mathcal{L}_1)$ to another pair $(G_2, \mathcal{L}_2)$ over the same base scheme S, and if such a morphism induces an isomorphism on the associated degeneration data (in the obvious sense), then the morphism must be actually an isomorphism. This will be justified by Lemma 4.3.1.15. (The statement in [37, Ch. II, Thm. 6.2] is not grammatically clear about the a priori existence of a morphism from $(G_1, \mathcal{L}_1)$ to $(G_2, \mathcal{L}_2)$, which is indeed necessary.)

4.2.2 Equivalence Between ι and τ

Proof of Lemma 4.2.1.2. For simplicity, let us assume that the étale sheaves $\underline{X} = \underline{X}(T)$ and $\underline{Y} = \underline{X}(T^\vee)$ are constant with values respectively X and Y by base change to a finite étale surjection over S. Let us we interpret τ as a set of sections $\{\tau(y,\chi)\}_{y\in Y,\chi\in X}$ of $\mathcal{P}_A(c^\vee(y), c(\chi))_\eta^{\otimes -1}$ over η satisfying the bimultiplicative conditions described by the axioms of biextensions. Namely, for any $y_1, y_2 \in Y$ and $\chi \in X$, the section $\tau(y_1,\chi) \otimes \tau(y_2,\chi)$ is mapped to the section $\tau(y_1+y_2,\chi)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y_1), c(\chi))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y_2), c(\chi))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y_1+y_2), c(\chi))_\eta^{\otimes -1}$$

given by the first partial multiplication map of the biextension structure of $\mathcal{P}_{A,\eta}$, and, for any $y \in Y$ and $\chi_1, \chi_2 \in X$, the section $\tau(y,\chi_1) \otimes \tau(y,\chi_2)$ is mapped to the section $\tau(y,\chi_1+\chi_2)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y), c(\chi_1))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y), c(\chi_2))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y), c(\chi_1+\chi_2))_\eta^{\otimes -1}$$

given by the second partial multiplication map of the biextension structure of $\mathcal{P}_{A,\eta}$. We shall denote these bimultiplicative conditions symbolically by

$$\tau(y_1,\chi)\tau(y_2,\chi) = \tau(y_1+y_2,\chi)$$

and

$$\tau(y,\chi_1)\tau(y,\chi_2) = \tau(y,\chi_1+\chi_2),$$

if no confusion should arise. Then it follows from the compatibility between the two partial multiplication maps of the biextension structure of $\mathcal{P}_{A,\eta}$ that it makes sense to write

$$\tau(y_1,\chi_1)\tau(y_2,\chi_1)\tau(y_1,\chi_2)\tau(y_2,\chi_2) = \tau(y_1+y_2,\chi_1+\chi_2),$$

which gives the compatibility between two partial multiplication maps for the trivialization τ.

For each $y \in Y$ we have a collection of sections $\{\tau(y,\chi)\}_{\chi\in X}$ with $\tau(y,\chi) \in \mathcal{P}_A(c^\vee(y),c(\chi))_\eta^{\otimes -1}$. Let us denote the invertible sheaf $(\mathrm{Id},c(\chi))^*\mathcal{P}_A$ on A by $\mathscr{O}_\chi$, and denote the invertible sheaf $\mathcal{P}_A(c^\vee(y),c(\chi)) = (c^\vee(y)\times c(\chi))^*\mathcal{P}_A$ on S by $\mathscr{O}_\chi(c^\vee(y)) = (c^\vee(y))^*\mathscr{O}_\chi$. Considering the canonical isomorphism

$$\mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1} \cong \mathscr{O}_{S,\eta}$$

for every $\chi \in X$, we can interpret $\tau(y,\chi) \in \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ as "multiplication by $\tau(y,\chi)$":

$$\tau(y,\chi) : \mathscr{O}_X(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}.$$

Putting all the maps corresponding to all $\chi \in X$ together, we obtain a map of $\mathscr{O}_S$-sheaves

$$c^\vee(y)^*\mathscr{O}_{G,\eta} = c^\vee(y)^*(\underset{\chi\in X}{\oplus} \mathscr{O}_\chi)_\eta \xrightarrow{\sum \tau(y,\chi)} \mathscr{O}_{S,\eta}.$$

Here the notations make sense if we interpret $\mathscr{O}_{G^\natural}$ as an $\mathscr{O}_A$-sheaf (as in Section 3.1.4) using the fact that $G^\natural$ is relative affine over A, and if we interpret $\mathscr{O}_{S,\eta}$ as an $\mathscr{O}_S$-sheaf. For this to define a homomorphism of sheaves of algebras, we need to map $\tau(y,\chi_1)\otimes\tau(y,\chi_2)$ to $\tau(y,\chi_1+\chi_2)$ for each $y \in Y$ and $\chi_1,\chi_2 \in X$ under the algebra structure map given by

$$(c^\vee(y)^*\mathscr{O}_{\chi_1})_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} (c^\vee(y)^*\mathscr{O}_{\chi_2})_\eta \xrightarrow{\sim} (c^\vee(y)^*\mathscr{O}_{\chi_1+\chi_2})_\eta,$$

which is exactly the isomorphism

$$\mathcal{P}_A(c^\vee(y),c(\chi_1))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y),c(\chi_2))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y),c(\chi_1+\chi_2))_\eta^{\otimes -1}$$

given by the biextension structure of $\mathcal{P}_{A,\eta}$. Namely, we need the multiplicative condition

$$\tau(y,\chi_1)\tau(y,\chi_2) = \tau(y,\chi_1+\chi_2).$$

This is indeed the same as giving a point $\iota(y)$ in $G_\eta^\natural$ lying over $c^\vee(y)$ in A.

It still remains to show that τ defines a homomorphism $\iota : Y \to G_\eta$ lying over $c^\vee : Y \to A$, namely $\iota(y_1)+\iota(y_2) = \iota(y_1+y_2)$ for any $y_1,y_2 \in Y$ under the multiplication map of $G^\natural$. Recall that the multiplication map of $G^\natural$ is defined by

$$m^* : m_A^*\mathscr{O}_{G^\natural} = m_A^*(\underset{\chi\in X(T)}{\oplus}\mathscr{O}_\chi) = \underset{\chi\in X(T)}{\oplus} m_A^*\mathscr{O}_\chi$$

$$\xrightarrow{\sim} \underset{\chi\in X(T)}{\oplus}(\mathrm{pr}_1^*\,\mathscr{O}_\chi \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_\chi) \subset \mathrm{pr}_1^*\,\mathscr{O}_{G^\natural} \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{G^\natural} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{G^\natural\underset{S}{\times}G^\natural},$$

where each of the isomorphisms

$$m_A^*\mathscr{O}_\chi \xrightarrow{\sim} \mathrm{pr}_1^*\,\mathscr{O}_\chi \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_\chi$$

exists uniquely because $\mathscr{O}_\chi \in \underline{\mathrm{Pic}}_e^0(A/S)$ and because we require it to respect the rigidifications. Applying $(c^\vee(y_1)\times c^\vee(y_2))^*$ to this map, we get

$$(c^\vee(y_1)\times c^\vee(y_2))^*m^* :$$
$$(c^\vee(y_1)\times c^\vee(y_2))^*m_A^*\mathscr{O}_{G^\natural} \xrightarrow{\sim} \underset{\chi\in X}{\oplus}(c^\vee(y_1)\times c^\vee(y_2))^*(\mathrm{pr}_1^*\,\mathscr{O}_\chi \underset{\mathscr{O}_{A\underset{S}{\times}A}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_\chi),$$

which is essentially

$$c^\vee(y_1+y_2)^*\mathscr{O}_{G^\natural} \xrightarrow{\sim} \underset{\chi\in X}{\oplus}(c^\vee(y_1)^*\mathscr{O}_\chi \underset{\mathscr{O}_S}{\otimes} c^\vee(y_2)^*\mathscr{O}_\chi).$$

Therefore the compatibility $\iota(y_1)+\iota(y_2)=\iota(y_1+y_2)$ follows from the compatibility of the following diagram:

$$\begin{tikzcd}
\underset{\chi\in X}{\oplus} c^\vee(y_1+y_2)^*\mathscr{O}_{\chi,\eta} \arrow[rr, "\iota(y_1+y_2)^*"] \arrow[d, "(c^\vee(y_1)\times c^\vee(y_2))^*m^*"', "\wr"] & & \mathscr{O}_{S,\eta} \arrow[dd, "\text{can.}", "\wr"'] \\
\underset{\chi\in X}{\oplus}(c^\vee(y_1)^*\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_S}{\otimes} c^\vee(y_2)^*\mathscr{O}_{\chi,\eta}) \arrow[d, hook] & & \\
(\underset{\chi\in X}{\oplus} c^\vee(y_1)^*\mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_S}{\otimes} (\underset{\chi\in X}{\oplus} c^\vee(y_2)^*\mathscr{O}_{\chi,\eta}) \arrow[rr, "\iota(y_1)^*\otimes\iota(y_2)^*"'] & & \mathscr{O}_{S,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{S,\eta}
\end{tikzcd}$$

Since the morphism $\iota(y_1+y_2)^*$ is defined by the "multiplication by $\tau(y_1+y_2,\chi)$", while the morphism $\iota(y_1)^*\otimes\iota(y_2)^*$ is defined by the "multiplication by $\tau(y_1,\chi)\otimes\tau(y_2,\chi)$", the commutativity follows if $\tau(y_1,\chi)\otimes\tau(y_2,\chi)$ is mapped to $\tau(y_1+y_2,\chi)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y_1),c(\chi))_\eta^{\otimes -1} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A(c^\vee(y_2),c(\chi))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y_1+y_2),c(\chi))_\eta^{\otimes -1}$$

given by the biextension structure of $\mathcal{P}_{A,\eta}$. Namely, we need the multiplicative condition

$$\tau(y_1 + y_2, \chi) = \tau(y_1, \chi)\tau(y_2, \chi).$$

Now that we have settled the case where $\underline{X}$ and $\underline{Y}$ are constant, the general case follows by étale descent. More precisely, if the descent data of $\underline{X}$ of $\underline{Y}$ are given by isomorphisms $\gamma_X : X \xrightarrow{\sim} X$ and $\gamma_Y : Y \xrightarrow{\sim} Y$ between the underlying constant objects after base change to some finite étale surjection of S, then the isomorphisms γ_X and γ_Y, when substituted into objects in the explicit construction, also define the descent data for the morphisms ι and τ.

To summarize, a homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$ determines and is determined by a trivialization $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_{A,\eta}$ of biextensions, which is exactly the statement of Lemma 4.2.1.2. $\square$

4.2.3 Equivalence Between Action on $\mathcal{L}^\natural_\eta$ and ψ

The proof of Lemma 4.2.1.3 is similar to the proof of Lemma 4.2.1.2 in nature, but needs some extra preparations.

For simplicity, assume again that the étale sheaves $\underline{X} = \underline{X}(T)$ and $\underline{Y} = \underline{X}(T^\vee)$ are constant with values respectively X and Y by passing to a finite étale surjection over S. Assume further that we have made a choice of a cubical trivialization $s : i^*\mathcal{L} \cong \mathscr{O}_T$, so that by Proposition 3.2.5.4 we have a cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on A. These assumptions are justified by étale descent, as in Section 4.2.2.

Since $\mathcal{L}^\natural$ is an $\mathscr{O}_{G^\natural}$-module, and since $G^\natural$ is relative affine over A, we would like to think of $\mathcal{L}^\natural$ as an $\mathscr{O}_A$-sheaf of $\mathscr{O}_{G^\natural}$-modules, where $\mathscr{O}_{G^\natural}$ is an $\mathscr{O}_A$-sheaf of algebras (as in Section 3.1.4). This is the same as considering $\pi_*\mathcal{L}^\natural$ as a $\pi_*\mathscr{O}_{G^\natural}$-module, with the abuse of language of suppressing all the π_*'s in the notations. From now on, we shall adopt this abuse of language whenever possible.

By defining $\mathscr{O}_\chi$ as in Section 4.2.2, we can write

$$\mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$$

and

$$\mathcal{L}^\natural = \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} (\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi).$$

Let us denote $\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ by $\mathcal{M}_\chi$. Then the $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^\natural$, given by a map

$$\mathcal{L}^\natural \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{G^\natural} = (\underset{\chi\in X}{\oplus} \mathcal{M}_\chi) \underset{\mathscr{O}_A}{\otimes} (\underset{\chi\in X}{\oplus} \mathscr{O}_\chi) \to \mathcal{L}^\natural = \underset{\chi\in X}{\oplus} \mathcal{M}_\chi$$

of $\mathscr{O}_A$-modules, can be obtained by the canonical isomorphisms

$$\mathcal{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} = (\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \xrightarrow{\sim} \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi+\chi'} = \mathcal{M}_{\chi+\chi'}$$

determined by the unique isomorphism

$$\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \xrightarrow{\sim} \mathscr{O}_{\chi+\chi'}$$

respecting the rigidifications (and giving the $\mathscr{O}_A$-algebra structure of $\mathscr{O}_{G^\natural}$) for all $\chi, \chi' \in X$. To summarize, the $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^\natural$ is obtain by letting $\mathscr{O}_\chi$ act on $\mathcal{L}^\natural$ by *translation of weights by* χ.

If we take $\mathcal{M}' = \mathcal{M}_{\chi_0}$ for some $\chi_0 \in X$ and consider $\mathcal{L}^{\natural\prime} = \pi^*\mathcal{M}'$, then we get

$$\mathcal{L}^{\natural\prime} = \underset{\chi\in X}{\oplus} ((\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0}) \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) = \underset{\chi\in X}{\oplus} \mathcal{M}_{\chi+\chi_0}$$

The $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^{\natural\prime}$ is again given by *translation of weights by* χ:

$$\mathcal{M}_{\chi+\chi_0} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \xrightarrow{\sim} \mathcal{M}_{\chi+\chi'+\chi_0}.$$

The only difference is that every weight is shifted by χ_0.

As an $\mathscr{O}_A$-module, $\mathcal{L}^\natural$ can be canonically identified with $\mathcal{L}^{\natural\prime}$, by sending the subsheaf $\mathcal{M}_\chi$ of $\mathcal{L}^\natural$ identically to $\mathcal{M}_\chi$ of $\mathcal{L}^{\natural\prime}$ for all χ. Let us denote this map by Id_{χ_0}, because it is in fact the *identity map* on the *same* underlying $\mathscr{O}_{G^\natural}$-module. We claim that this isomorphism is also a cubical isomorphism. This is because the map Id_{χ_0} is given by putting together the canonical isomorphisms $\mathcal{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0} \xrightarrow{\sim} \mathcal{M}_{\chi+\chi_0}$, and so $\mathcal{D}_3(\mathrm{Id}_{\chi_0})$ is given by the canonical isomorphisms $\mathcal{D}_3(\mathcal{M}_\chi) \underset{\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}}{\otimes} \mathcal{D}_3(\mathscr{O}_{\chi_0}) \xrightarrow{\sim} \mathcal{D}_3(\mathcal{M}_{\chi+\chi_0})$. Now $\mathcal{D}_3(\mathscr{O}_{\chi_0})$, $\mathcal{D}_3(\mathcal{M}_\chi)$, and $\mathcal{D}_3(\mathcal{M}_{\chi+\chi_0})$ are all isomorphic to the trivial invertible sheaf by the usual theorem of cube (as in Proposition 3.2.3.1), with unique choices of isomorphisms respecting the rigidifications. Therefore, the isomorphism

$\mathcal{D}_3(\mathrm{Id}_{\chi_0})$ must agree with the canonical isomorphism between trivial invertible sheaves. On the other hand, the cubical structures on $\mathcal{L}$ and on $\mathcal{L}'$ differ by the pullback of the canonical cubical structure of $\mathscr{O}_{\chi_0}$, which is again built up by the same canonical isomorphisms in $\mathcal{D}_3(\mathrm{Id}_\chi)$. This justifies the claim.

By Rosenlicht's Lemma (given by Lemma 3.2.2.12) and by Remark 3.2.2.11, the set of cubical trivializations $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ is a torsor under $\mathrm{Hom}^{(1)}(T, \mathbf{G}_\mathrm{m})$, namely the character group X of T. By the above arguments, we see that this set as a X-module is equivalent with the choice of different $\mathscr{O}_{G^\natural}$-module structures on $\mathcal{L}^\natural$, on which a character $\chi \in X$ acts by *translation of weights by* χ.

Let us record the above observation as:

Lemma 4.2.3.1. *With assumptions as in Definition 4.2.1.7 and with the structural morphism $\pi : G^\natural \to A$ as above, all the different cubical invertible sheaves $\mathcal{M}$ and $\mathcal{M}'$ on A such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M} \cong \pi^*\mathcal{M}'$ (as cubical invertible sheaves) are related by $\mathcal{M}' \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ for some $\chi \in \underline{X} = \underline{X}(T)$.*

Proof of Lemma 4.2.1.3. To define an action of Y on $\mathcal{L}^\natural$ covering the multiplication by $\iota : Y \to G^\natural$, let us first understand the multiplication by $\iota(y)$ covering $c^\vee(y)$. It is given by the diagram:

$$\begin{tikzcd}
G^\natural_\eta \arrow[r, "\mathrm{can.}_{G^\natural}"] \arrow[d] & \eta \underset{\eta}{\times} G^\natural_\eta \arrow[r, "\iota(y)\times\mathrm{Id}"] \arrow[d] & G^\natural_\eta \underset{\eta}{\times} G^\natural_\eta \arrow[r, "m_{G^\natural}"] \arrow[d] & G^\natural_\eta \arrow[d] \\
A_\eta \arrow[r, "\mathrm{can.}_A"'] & \eta \underset{\eta}{\times} A_\eta \arrow[r, "c^\vee(y)\times\mathrm{Id}"'] & A_\eta \underset{\eta}{\times} A_\eta \arrow[r, "m_A"'] & A_\eta
\end{tikzcd}$$

By pulling back every sheaves of to the lower-left A_η, we get a diagram of

$\mathscr{O}_{A,\eta}$-algebras:

$$\begin{tikzcd}
T^*_{c^\vee(y)}\mathscr{O}_{G^\natural,\eta} \arrow[r, equal] \arrow[ddddr, bend right, "\sim", "T^*_{\iota(y)}"'] & (\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^* m_A^* \mathscr{O}_{G^\natural,\eta} \arrow[d, "(\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^*(m^*)"] \\
& (\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^*(\mathrm{pr}_1^*\,\mathscr{O}_{G^\natural,\eta} \underset{\mathscr{O}_{A_\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{G^\natural,\eta}) \arrow[d, equal] \\
& (\mathrm{can.}_A)^*(\mathrm{pr}_1^*\,c^\vee(y)^*\mathscr{O}_{G^\natural,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{G^\natural,\eta}) \arrow[d, "(\mathrm{can.}_A)^*(\iota(y)^*\otimes \mathrm{Id})"] \\
& (\mathrm{can.}_A)^*(\mathrm{pr}_1^*\,\mathscr{O}_{S,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{G^\natural,\eta}) \arrow[d, "(\mathrm{can.}_G)^*"] \\
& \mathscr{O}_{G^\natural,\eta}
\end{tikzcd}.$$

More precisely, this is defined by a sequence of isomorphisms:

$$\begin{tikzcd}
T^*_{c^\vee(y)}\mathscr{O}_{G^\natural,\eta} \arrow[r, equal] \arrow[ddddr, bend right, "\sim", "T^*_{\iota(y)}"'] & (\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^* m_A^* (\bigoplus\limits_{\chi\in X}\mathscr{O}_{X,\eta}) \arrow[d, "(\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^*(m^*)", "\wr"'] \\
& \bigoplus\limits_{\chi\in X}(\mathrm{can.}_A)^*(c^\vee(y)\times \mathrm{Id})^*(\mathrm{pr}_1^*\,\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{A_\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{\chi,\eta}) \arrow[d, equal] \\
& \bigoplus\limits_{\chi\in X}(\mathrm{can.}_A)^*(\mathrm{pr}_1^*\,c^\vee(y)^*\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{\chi,\eta}) \arrow[d, "(\mathrm{can.}_A)^*(\iota(y)^*\otimes \mathrm{Id})", "\wr"'] \\
& \bigoplus\limits_{\chi\in X}(\mathrm{can.}_A)^*(\mathrm{pr}_1^*\,\mathscr{O}_{S,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_{\chi,\eta}) \arrow[d, "(\mathrm{can.}_G)^*", "\wr"'] \\
& \bigoplus\limits_{\chi\in X}\mathscr{O}_{\chi,\eta} = \mathscr{O}_{G^\natural,\eta}
\end{tikzcd},$$

which relies essentially on the isomorphisms

$$\begin{aligned}
T^*_{c^\vee(y)}\mathscr{O}_{\chi,\eta} &\cong \mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(0)_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \\
&\text{(unique isomorphism in } \underline{\mathrm{Pic}}^0_e(A_\eta/eta)) \\
&\cong \mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \\
&\text{(rigidification of } \mathscr{O}_\chi(0)_\eta) \\
&\cong \mathscr{O}_{\chi,\eta} \\
&\text{(multiplication by } \tau(y,\chi) \in \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1})
\end{aligned} \tag{4.2.3.2}$$

for each $\chi \in X$.

If we want to define an action of Y on $\mathcal{L}^\natural_\eta$ (which commutes with the T-action up to a character), we need to write down an isomorphism between $T^*_{c^\vee(y)}\mathcal{L}^\natural_\eta$ and $\mathcal{L}^\natural_\eta$. Again the essential point is to understand the restriction of this isomorphism to the *weight* χ *space* $T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta}$. Let us first consider the invertible sheaf

$$T^*_{c^\vee(y)}\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes -1}.$$

Since $\mathcal{M}$ induces λ_A, we know that the above invertible sheaf is equivalent to the invertible sheaf in $\underline{\mathrm{Pic}}^0$ corresponding to the point by $\lambda_A c^\vee(y) = c\phi(y)$ in $A^\vee$, namely the invertible sheaf $\mathscr{O}_{\phi(y)}$. By adjusting the rigidifications properly, the equivalence becomes actually a uniquely determined isomorphism, which can be rewritten as

$$T^*_{c^\vee(y)}\mathcal{M} \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)) \cong \mathcal{M}_{\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)).$$

Then we have

$$T^*_{c^\vee(y)}\mathcal{M}_\chi \cong T^*_{c^\vee(y)}\mathcal{M} \underset{\mathscr{O}_A}{\otimes} T^*_{c^\vee(y)}\mathscr{O}_\chi \cong \mathcal{M}_{\chi+\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(c^\vee(y)).$$

Now, over the generic fiber, using the canonical isomorphisms

$$\mathcal{M}(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}(c^\vee(y))_\eta^{\otimes -1} \cong \mathscr{O}_{S,\eta}$$

for any $y \in Y$, we can interpret $\psi(y) \in \iota^*(y)^*(\mathcal{L}^\natural_\eta)^{\otimes -1} \cong \mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ as "multiplication by $\psi(y)$":

$$\psi(y) : \mathcal{M}(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}.$$

Similarly, for any $\chi \in X$, we have the "multiplication by $\tau(y, \chi)$":

$$\tau(y, \chi) : \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}.$$

Putting these together, we have (symbolically) the "multiplication by $\psi(y)\tau(y, \chi)$"

$$\psi(y)\tau(y, \chi) : \mathcal{M}_\chi(c^\vee(y))_\eta \cong \mathcal{M}(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta},$$

which enables us to define

$$\psi(y)\tau(y, \chi) : T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta} \cong \mathcal{M}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y))_\eta \to \mathcal{M}_{\chi+\phi(y),\eta}.$$

These maps patches together into an isomorphism

$$\tilde{T}_{\iota(y)} : T^*_{c^\vee(y)}\mathcal{L}^\natural_\eta \to \mathcal{L}^\natural_\eta,$$

with the shift of weights by $\phi(y)$ described above.

For this to define an action of Y on $\mathcal{L}^\natural_\eta$, we need to show the commutativity of the following diagram:

$$\begin{tikzcd}
T^*_{c^\vee(y_1+y_2)}\mathcal{L}^\natural_\eta \arrow[rr, "\tilde{T}_{\iota(y_1+y_2)}", "\sim"'] \arrow[dr, "\sim", "T^*_{c^\vee(y_1)}(\tilde{T}_{\iota(y_2)})"'] & & \mathcal{L}^\natural_\eta \\
& T^*_{c^\vee(y_1)}\mathcal{L}^\natural_\eta \arrow[ur, "\sim", "\tilde{T}_{\iota(y_1)}"'] &
\end{tikzcd}$$

In terms of weight spaces, this is

$$\begin{tikzcd}
T^*_{c^\vee(y_1+y_2)}\mathcal{M}_{\chi,\eta} \arrow[rr, "\psi(y_1+y_2)\tau(y_1+y_2,\chi)", "\sim"'] \arrow[dr, "\sim", "T^*_{c^\vee(y_1)}(\psi(y_2)\tau(y_2,\chi))"'] & & \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2),\eta} \\
& T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2),\eta} \arrow[ur, "\sim", "\psi(y_1)\tau(y_1,\chi+\phi(y_2))"'] &
\end{tikzcd} \quad (4.2.3.3)$$

Let us analyze the above diagram more closely. The first object

$$T^*_{c^\vee(y_1+y_2)}\mathcal{M}_\chi$$

is isomorphic to

$$\mathcal{M}_\chi(c^\vee(y_1) + c^\vee(y_2)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)}.$$

On the other hand, by pulling back the isomorphism

$$T_{c^\vee(y_2)}\mathcal{M}_\chi \xrightarrow{\sim} \mathcal{M}_{\chi+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2))$$

by $c^\vee(y_1)$, we get an isomorphism

$$\mathcal{M}_\chi(c^\vee(y_1)+c^\vee(y_2)) \xrightarrow{\sim} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)).$$

If we pull back the map

$$\psi(y_2)\tau(y_2,\chi) : T^*_{c^\vee(y_2)}\mathcal{M}_\chi \cong \mathcal{M}_{\chi+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \to \mathcal{M}_{\chi+\phi(y_2)}$$

by $T^*_{c^\vee(y_1)}$, then we get

$$\begin{aligned}
&T^*_{c^\vee(y_1)}(\psi(y_2)\tau(y_2,\chi)) : \\
&\quad T^*_{c^\vee(y_1)+c^\vee(y_2)}\mathcal{M}_\chi \cong (T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2)}) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \\
&\qquad\qquad\qquad \cong \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \\
&\quad \to T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2)} \cong \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)),
\end{aligned}$$

which is just the same "multiplication by $\psi(y_2)\tau(y_2,\chi)$" applied to a different invertible sheaf. Therefore, for the diagram (4.2.3.3) to be commutative, the essential point is to have (symbolically)

$$\psi(y_1+y_2)\tau(y_1+y_2,\chi) = \psi(y_1)\tau(y_1,\chi)\psi(y_2)\tau(y_2,\chi+\phi(y_1)),$$

which is simply the compatibility

$$\psi(y_1+y_2) = \psi(y_1)\psi(y_2)\tau(y_1,\phi(y_2)).$$

This is literally equivalent to

$$\psi(y_1+y_2)\psi(y_1)^{\otimes -1}\psi(y_2)^{\otimes -1} = \tau(y_1,\phi(y_2)),$$

or

$$\mathcal{D}_2(\psi)(y_1,y_2) = \tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi)(y_1,y_2),$$

which is the condition described in Lemma 4.2.1.3. □

Remark 4.2.3.4. By symmetric arguments we see that

$$\tau(y_1,\phi(y_2)) = \mathcal{D}_2(\psi)(y_1,y_2) = \mathcal{D}_2(\psi)(y_2,y_1) = \tau(y_2,\phi(y_1)).$$

Therefore $\tau \circ (\mathrm{Id}_{\underline{Y}} \times \phi)$ is a symmetric trivialization because $\mathcal{D}_2(\psi)$ is.

4.2.4 Equivalence Between The Positivity Condition for ψ and The Positivity Condition for τ

Definition 4.2.4.1. *Let R be a noetherian integral domain with fraction field K. Then we denote by $\mathrm{Inv}(R)$ the group of invertible R-submodules of K.*

Let v be a valuation of K. For an invertible R-submodule J of K, we define $v(J)$ to be the minimal value of v on nonzero elements of J.

Since R is noetherian and *normal*, we know that R is the intersection of the valuation rings of its discrete valuations defined by height one primes, namely $R = \bigcap\limits_{\substack{\mathfrak{p}\text{ prime}\\ \mathrm{ht}\,\mathfrak{p}=1}} R_{\mathfrak{p}}$. (See for example [91, Thm. 11.5].) Let us denote by Υ_1 the set of valuations of K defined by height one primes of R. Then we have:

Lemma 4.2.4.2. *A an invertible R-submodule J of K satisfy $J \subset R$ if and only if $v(J) \geq 0$ for all $v \in \Upsilon_1$.*

Lemma 4.2.4.3 ([122, pp. 95–96, proof of Thm. 35, §14, Ch. VI])**.** *Let R be a noetherian integral domain with fractional field K. Then, for any prime ideal $\mathfrak{p}$ of R, there is a discrete valuation $v : K^\times \to \mathbb{Z}$ of K such that $R \subset R_v$ and $\mathfrak{p} = R \cap \mathfrak{m}_v$, where R_v is the valuation ring of v, and where $\mathfrak{m}_v$ is the maximal ideal of R_v.*

Let us denote by Υ_I the set of discrete valuations v of K such that $I_v := I \underset{R}{\otimes} R_v \subsetneq R_v$. In other words, Υ_I is the set of discrete valuations of K that has center on $S_0 = \mathrm{Spec}(R_0)$. This is determined essentially only by $\mathrm{rad}(I)$, which is I because we assumed $\mathrm{rad}(I) = I$. Then we have:

Lemma 4.2.4.4. *An invertible R-submodule J of R satisfy $J \subset \mathrm{rad}(I) = I$ if and only if $v(J) > 0$ for all $v \in \Upsilon_I$.*

Proof of Lemma 4.2.1.6. The section $\psi(y)$ of $\mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ is an isomorphism $\mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$, which induces an isomorphism $\mathcal{M}(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. Since $\mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ has an integral structure given by $\mathcal{M}(c^\vee(y))^{\otimes -1}$, we can pullback this integral structure to an invertible R-submodule I_y in $K = \mathscr{O}_\eta$. In this case we can interpret $\psi(y)$ as an isomorphism $\psi(y) : \mathcal{M}(c^\vee(y)) \xrightarrow{\sim} \underline{I}_y$. If $I_y \subset I^n \subset R$ for some nonnegative integer n, we obtain a homomorphism $\mathcal{M}(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is the invertible subsheaf of $\mathscr{O}_S$ corresponding to the ideal $I \subset R$. We will write symbolically $\upsilon(\psi(y)) = \upsilon(I_y)$, as if $\psi(y)$ is an invertible R-submodule of K.

Similarly, the section $\tau(y,\chi)$ of $\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ is an isomorphism $\mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$, which induces an isomorphism $\mathscr{O}_\chi(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. Since $\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ has an integral structure given by $\mathscr{O}_\chi(c^\vee(y))^{\otimes -1}$, we can pullback this integral structure to an invertible R-submodule $I_{y,\chi}$ in $K = \mathscr{O}_\eta$. In this case we can interpret $\tau(y,\chi)$ as an isomorphism $\tau(y,\chi) : \mathscr{O}_\chi(c^\vee(y)) \xrightarrow{\sim} \underline{I}_{y,\chi}$. If $I_{y,\chi} \subset I^n \subset R$ for some nonnegative integer n, we obtain a homomorphism $\mathscr{O}_\chi(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is as above. The valuation $\upsilon(I_{y,\chi})$ is defined to be the minimal valuation of υ on nonzero elements in $I_{y,\chi}$. We will write symbolically $\upsilon(\tau(y,\chi)) = \upsilon(I_{y,\chi})$, as if $\tau(y,\chi)$ is an invertible R-submodule of K.

Note that we have symbolically, for any $y_1, y_2 \in Y$,

$$\psi(y_1 + y_2)\psi(y_1)^{-1}\psi(y_2)^{-1} = \tau(y_1, \phi(y_2)).$$

That is, $\psi(y)$ can be symbolically interpreted as a quadratic function in y, with associated bilinear pairing $\tau(y_1, \phi(y_2))$. In particular, for any $y_1, y_2 \in Y$, we have symbolically

$$\begin{aligned}
&(\psi(y_1 + y_2)\psi(y_1)^{-1})(\psi(y_1)\psi(y_1 - y_2)^{-1})^{-1} \\
&= \psi(y_1 + y_2)\psi(y_1 - y_2)\psi(y_1)^{-1}\psi(y_1)^{-1} \\
&= \psi(2y_1)\tau(y_1 + y_2, \phi(y_1 - y_2))^{-1}\psi(2y_1)^{-1}\tau(y_1, \phi(y_1)) \\
&= \tau(y_2, \phi(y_2)).
\end{aligned}$$

Now the implication from the positivity condition for ψ (defined in Definition 4.2.1.5) to the positivity condition for τ (defined in Definition 4.2.1.4) can be justified as follows:

Suppose $\upsilon \in \Upsilon_1$. The first half of the positivity condition for ψ implies that $\upsilon(\psi(y)) \leq 0$ for all but finitely many y in Y. In particular, if $y \neq Y$, and if we define a function $f_\upsilon : \mathbb{Z} \to \mathbb{Z}$ by $f_\upsilon(k) = \upsilon(\psi(ky))$, then the above calculation shows that

$$f_\upsilon(k+1) - 2f_\upsilon(k) + f_\upsilon(k-1) = \upsilon(\tau(y, \phi(y))).$$

If $\upsilon(\tau(y, \phi(y))) < 0$, then it is impossible that $f_\upsilon(k) \geq 0$ for all but finitely many k. As a result, we see that $\upsilon(\tau(y, \phi(y)) \geq 0$ for all $\upsilon \in \Upsilon_1$. By Lemma 4.2.4.2, we have $I_{y,\phi(y)} \subset R$ by noetherian normality of R.

Suppose $\upsilon \in \Upsilon_I$. The second half of the positivity condition for ψ implies that for any given value n_0, there can only be finitely many $y \in Y$ such that $\upsilon(\psi(y)) \leq n_0$. In particular, if $y \neq 0$, and if we again define a function $f_\upsilon : \mathbb{Z} \to \mathbb{Z}$ by $f_\upsilon(k) = \upsilon(\psi(ky))$, then for any given value n_0, there can only be finitely many integers k such that $f_\upsilon(k) \leq n_0$. The above calculation shows that if $\upsilon(\tau(y, \phi(y))) \leq 0$, then there will be infinitely many integers k such that $f_\upsilon(k) \leq f(0)$, which is a contradiction. As a result, we see that $\upsilon(\tau(y, \phi(y))) > 0$ for all $\upsilon \in \Upsilon_I$. By Lemma 4.2.4.4 and the known fact that $I_{y,\phi(y)} \subset R$, we have $I_{y,\phi(y)} \subset I$. This is the content of the positivity condition for τ.

Conversely, the positivity condition for τ shows that $\upsilon(\tau(y_1, \phi(y_2))$ defines a positive semi-definite form for all $\upsilon \in \Upsilon_1$, and defines a positive definite form for all $\upsilon \in \Upsilon_I$. Therefore, the associated quadratic form $\upsilon(\tau(y, \phi(y))$ is nonnegative for all $\upsilon \in \Upsilon_1$, and is positive for all $\upsilon \in \Upsilon_I$. This implies the positivity condition for ψ, using the fact that Y is finitely generated and the assumption that R is noetherian. $\square$

4.3 Fourier Expansions of Theta Functions

In this section we investigate the Fourier expansions of theta functions, namely the sections of $\Gamma(G, \mathcal{L})$, and use the result to prove Theorem 4.2.1.8.

4.3.1 Definition of ψ and τ

With the setting as in Section 4.1, suppose that the Raynaud extension of G is

$$0 \to T \xrightarrow{i} G^\natural \xrightarrow{\pi} A \to 0$$

over S, and suppose that the Raynaud extension of $G^\vee$ is

$$0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0,$$

so that $\underline{X} = \underline{X}(T)$ and $\underline{Y} = \underline{X}(T)$. Let us suppose that $\underline{X}$ and $\underline{Y}$ are constant, with values respectively X and Y, and suppose that a cubical trivialization $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ is chosen, so that we have a cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some ample invertible sheaf $\mathcal{M}$ on A. This gives us an isomorphism $\pi_*\mathcal{L}^\natural \cong \underset{\chi \in X}{\oplus} (\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathcal{M}) = \underset{\chi \in X}{\oplus} \mathcal{M}_\chi$, where $\pi_*\mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$ is the weight space decomposition under T-action, introduced in Section 4.2.3.

(Recall that in Section 4.2.2 we defined $\mathscr{O}_\chi := (\mathrm{Id}, c(\chi))^* \mathcal{P}_A$, the pullback of the Poincaré invertible sheaf $\mathcal{P}_A$ on $A \underset{S}{\times} A^\vee$ along the morphism $(\mathrm{Id}, c(\chi)) : A \to A \underset{S}{\times} A^\vee$, and that in Section 4.2.3 we defined $\mathcal{M}_\chi := \mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathcal{M}$.)

Let us we consider the formal completions

$$0 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \overset{\pi_{\mathrm{for}}}{\to} A_{\mathrm{for}} \to 0,$$

over S_{for} as a compatible system (for all $i \geq 0$) of exact sequences of group schemes

$$0 \to T_i \to G_i \overset{\pi_i}{\to} A_i \to 0$$

over S_i, where A_i is an abelian scheme over S_i, and where T_i is a torus over S_i. (See Section 3.3.3.)

Since

$$\Gamma(G_i, \mathscr{O}_{G_i}) = \Gamma(A_i, \pi_* \mathscr{O}_{G_i}) = \underset{\chi \in X}{\oplus} \Gamma(A_i, \mathscr{O}_\chi),$$

we may write

$$\Gamma(G_{\mathrm{for}}, \mathscr{O}_{G,\mathrm{for}}) = \varinjlim_i \Gamma(G_i, \mathscr{O}_{G_i}) = \underset{\chi \in X}{\hat{\oplus}} \Gamma(A_{\mathrm{for}}, \mathscr{O}_{\chi,\mathrm{for}}) = \underset{\chi \in X}{\hat{\oplus}} \Gamma(A, \mathscr{O}_\chi),$$

where $\underset{\chi \in X}{\hat{\oplus}}$ stands for *I-adic completion*, and where $\Gamma(A_{\mathrm{for}}, \mathscr{O}_{\chi,\mathrm{for}}) = \Gamma(A, \mathscr{O}_\chi)$ follows from Proposition 2.3.1.1. More concretely, if $f \in \Gamma(G_{\mathrm{for}}, \mathscr{O}_{G,\mathrm{for}})$, then we may form the *Fourier expansion of regular function*

$$f = \sum_{\chi \in X} \sigma_\chi(f),$$

where each $\sigma_\chi(f)$ lies in $\Gamma(A, \mathscr{O}_\chi)$, and where the sum is *$I$-adic convergent* in the sense that if we consider f as a limit of $(f \bmod I^{i+1}) \in \Gamma(G_i, \mathscr{O}_{G_i})$, then the corresponding sum $(\sum_{\chi \in X} \sigma_\chi(f) \bmod I^{i+1})$ has only finitely many nonzero terms for each i.

Similarly, we have

$$\Gamma(G_i, \mathcal{L}_i) = \Gamma(G_i, \pi_i^* \mathcal{M}_i) = \Gamma(A_i, \pi_{i,*} \pi_i^* \mathcal{M}_i) = \underset{\chi \in X}{\oplus} \Gamma(A_i, \mathcal{M}_{\chi,i}),$$

and we may write

$$\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \varinjlim_i \Gamma(G_i, \mathcal{L}_i) = \underset{\chi \in X}{\hat{\oplus}} \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi,\mathrm{for}}) = \underset{\chi \in X}{\hat{\oplus}} \Gamma(A, \mathcal{M}_\chi),$$

and we may write $s \in \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ in its *Fourier expansion of theta function*

$$s = \sum_{\chi \in X} \sigma_\chi(s),$$

where each $\sigma_\chi(s)$ lies in $\Gamma(A, \mathcal{M}_\chi)$, and where the sum is I-adic convergent in the sense that it has only finitely many nonzero terms modulo any power I^{i+1} of I as in the case of $\Gamma(G_{\mathrm{for}}, \mathscr{O}_{G,\mathrm{for}})$. Symbolically, we shall write $\sigma_\chi(s) \equiv 0 \pmod{I^{i+1}}$ *for all but finitely many* $\chi \in X$, *for each fixed* i.

If we consider the embedding

$$\Gamma(G, \mathcal{L}) \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}),$$

then we have an induced map

$$\sigma_\chi : \Gamma(G, \mathcal{L}) \to \Gamma(A, \mathcal{M}_\chi),$$

which extends naturally to

$$\sigma_\chi : \Gamma(G, \mathcal{L}) \underset{R}{\otimes} K \to \Gamma(A, \mathcal{M}_\chi) \underset{R}{\otimes} K.$$

Since $\Gamma(G_\eta, \mathcal{L}_\eta)$ and $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ are both finite dimensional because both $\mathcal{L}_\eta$ and $\mathcal{M}_{\chi,\eta}$ are ample, we know that

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \Gamma(G, \mathcal{L}) \underset{R}{\otimes} K$$

and

$$\Gamma(A_\eta, \mathcal{M}_{\chi,\eta}) = \Gamma(A, \mathcal{M}_\chi) \underset{R}{\otimes} K.$$

Therefore the map σ_χ above can be written as

$$\sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}).$$

This is now a map between K-vector spaces.

Note that (in the cases of either $\Gamma(G, \mathcal{L})$ or $\Gamma(G_\eta, \mathcal{L}_\eta)$) the maps σ_χ do depend on the choice of $\mathcal{M}$. We shall write $\sigma_\chi = \sigma_\chi^{\mathcal{M}}$ to signify this choice when necessary.

Remark 4.3.1.1. By Lemma 4.2.3.1, any different choice of $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ gives a different choices of $\mathcal{M}'$ such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$, which is necessarily of the form $\mathcal{M}' \cong \mathcal{M}_\chi = \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ for some $\chi \in \underline{X}$. This results in a shift of indices for the maps σ_χ above, and we will see later (in Remark 4.3.1.6) that this is harmless for defining the trivializations τ and ψ.

Since $\Gamma(G_\eta, \mathcal{L}_\eta)$ is finite dimensional, we expect that there might be some redundancy in the full collection $\{\sigma_\chi\}$ indexed by an infinite group. To make this precise, we will compare

$$\begin{aligned} T^*_{c^\vee(y)} \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) &\to \Gamma(A_\eta, T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta}) \\ &= \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}_\chi(c^\vee(y))_\eta \end{aligned}$$

(given by $T^*_{c^\vee(y)}\mathcal{M}_\chi \cong \mathcal{M}_{\chi+\phi(y)} \underset{R}{\otimes} \mathcal{M}_\chi(c^\vee(y)))$ with

$$\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}).$$

We claim that

$$\sigma_\chi \neq 0 \tag{4.3.1.2}$$

as a map (over the generic fiber η) for any $\chi \in X$, and we claim that for each pair of $y \in Y$ and $\chi \in X$ there exists a unique section $\psi(y,\chi)$ in $\mathcal{M}_\chi(c^\vee(y))^{\otimes -1}_\eta$ defining an isomorphism

$$\psi(y,\chi) : \mathcal{M}_\chi(c^\vee(y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$$

such that

$$\psi(y,\chi)\ T^*_{c^\vee(y)} \circ \sigma_\chi = \sigma_{\chi+\phi(y)}. \tag{4.3.1.3}$$

Using the canonical isomorphism

$$\mathcal{M}_\chi(c^\vee(y))^{\otimes -1}_\eta \cong \mathcal{M}(c^\vee(y))^{\otimes -1}_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))^{\otimes -1}_\eta,$$

we would like to write

$$\psi(y,\chi) = \psi(y)\tau(y,\chi)$$

for some sections $\psi(y) \in \mathcal{M}(c^\vee(y))^{\otimes -1}_\eta$ and $\tau(y,\chi) \in \mathscr{O}_\chi(c^\vee(y))^{\otimes -1}_\eta$, so that we can rewrite the above relation (4.3.1.3) as

$$\psi(y)\tau(y,\chi)\ T^*_{c^\vee(y)} \circ \sigma_\chi = \sigma_{\chi+\phi(y)}. \tag{4.3.1.4}$$

The choices of $\psi(y)$ and $\tau(y,\chi)$ are uniquely determined by the sections $\psi(y,\chi)$ if we impose the following assumptions:

(a) We shall require (symbolically)

$$\psi(0) = 1$$

in the sense that

$$\psi(0) : \mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathcal{M}(c^\vee(0))^{\otimes -1}_\eta = \mathcal{M}(0)^{\otimes -1}_\eta$$

coincides with the rigidification of $\mathcal{M}^{\otimes -1}_\eta$ along the identity section 0.

(b) We shall require (symbolically)

$$\tau(0,\chi) = 1$$

for any $\chi \in X$ in the sense that

$$\tau(0,\chi) : \mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathscr{O}_{\chi}(c^{\vee}(0))_{\eta}^{\otimes -1} = \mathscr{O}_{\chi}(0)_{\eta}^{\otimes -1}$$

coincides with the rigidification of $\mathscr{O}_{\chi,\eta}^{\otimes -1}$ along the identity section 0.

(c) We shall require (symbolically)

$$\tau(y,0) = 1 \tag{4.3.1.5}$$

for any $y \in Y$ in the sense that the inverse morphism of

$$\tau(y,0) : \mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathscr{O}_{0}(c^{\vee}(y))_{\eta}^{\otimes -1} \cong c^{\vee}(y)^{*}\mathscr{O}_{A,\eta}$$

coincides with the isomorphism giving the section $c^{\vee}(y) : \eta \to A_{\eta}$. Here $\mathscr{O}_{0}^{\otimes -1} \cong \mathscr{O}_{A}$ is the unique isomorphism given by the rigidification of $\mathcal{P}_{A}$, and the inverse of $\tau(y,0)$ is interpreted as a map $c^{\vee}(y)^{*}\mathscr{O}_{A,\eta} \xrightarrow{\sim} \mathscr{O}_{S,\eta}$.

Actually, (c) implies (a), and (a) implies (b). Therefore only (4.3.1.5) is essential.

Remark 4.3.1.6. If we replace $\mathcal{M}$ above by $\mathcal{M}' := \mathcal{M}_{\chi_0} = \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0}$ for some $\chi_0 \in \underline{X}$, then we are just shifting all the indices of σ_{χ} by χ_0. As a result, we obtain $\psi(y)\tau(y,\chi) = \psi'(y)\tau'(y,\chi - \chi_0)$ for any χ, and hence we have $\tau(y,\chi - \chi') = \tau'(y,\chi - \chi')$ for any two $\chi, \chi \in X$, which shows that $\tau = \tau'$. This shows that τ is independent of the $\mathcal{M}$ we choose. Moreover, the above relation shows that the two ψ and ψ' give the same cubical trivialization $\mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^{*}(\mathcal{L}_{\eta}^{\natural})^{\otimes -1}$. This shows that ψ is independent of the choice of $\mathcal{M}$ if we write it as a cubical trivialization of $\iota^{*}(\mathcal{L}_{\eta}^{\natural})^{\otimes -1}$ (rather than $(c^{\vee})^{*}\mathcal{M}_{\eta}^{\otimes -1}$). (This is called the *invariant formulation* in [37, Ch. II, §5].)

Remark 4.3.1.7. For any g in G, if we replace $\mathcal{L}$ by $\mathcal{L}' := T_{g}^{*}\mathcal{L}$, then the τ and ψ defined by $\mathcal{L}$ and by $\mathcal{L}'$ are the same. Indeed, we simply have to translate everything on $G_{\text{for}}^{\natural} = G_{\text{for}}$ by the formal section $g_{\text{for}} : S_{\text{for}} \to G_{\text{for}}^{\natural}$ defined by the I-completion of $g : S \to G^{\natural}$, and translate correspondingly everything on A_{for} by $\pi_{\text{for}}(g_{\text{for}}) : S_{\text{for}} \to A_{\text{for}}$. This algebraizes to a point $S \to A$, which we denote by $\pi(g)$ by abuse of notation. Then we have $(\mathcal{L}')^{\natural} \cong \pi^*\mathcal{M}'$ for $\mathcal{M}' = T^*_{\pi(g)}\mathcal{M}$. Thus we have translated the map

$$\sigma_\chi^{\mathcal{M}} : \Gamma(G, \mathcal{L}) \to \Gamma(A, \mathcal{M}_\chi)$$

to

$$\sigma_\chi^{\mathcal{M}'} : \Gamma(G, \mathcal{L}') \to \Gamma(A, \mathcal{M}'_\chi),$$

where $\mathcal{M}'_\chi := \mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$. Since the definitions of τ and ψ are given by comparing σ_χ with $\sigma_{\chi+\phi(y)}$, they remain the same if we replace $\mathcal{L}$ and $\mathcal{M}$ by respectively $\mathcal{L}'$ and $\mathcal{M}'$.

Proposition 4.3.1.8. *If we assume the relations (4.3.1.2), (4.3.1.4), and (4.3.1.5) above, then the following relations are natural consequences of the definitions:*

(*i*) *For any $y_1, y_2 \in Y$, we have (symbolically)*

$$\psi(y_1 + y_2) = \psi(y_1)\psi(y_2)\tau(y_1, \phi(y_2)) \tag{4.3.1.9}$$

under the $\mathbf{G}_{\mathrm{m}}$-torsor isomorphism

$$(c^\vee \times c^\vee)^*\mathcal{D}_2(\mathcal{M})_\eta^{\otimes -1} \cong (c^\vee \times c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1}.$$

By symmetry, we also have

$$\psi(y_1 + y_2) = \psi(y_2)\psi(y_1)\tau(y_2, \phi(y_1)). \tag{4.3.1.10}$$

(*ii*) *For any $y_1, y_2 \in Y$, we have (symbolically)*

$$\tau(y_1, \phi(y_2)) = \tau(y_2, \phi(y_1))$$

under the symmetry isomorphism of

$$(c^\vee \times c^\vee)^*\mathcal{D}_2(\mathcal{M})_\eta^{\otimes -1} \cong (c^\vee \times c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1}.$$

This is actually a formal consequence of (4.3.1.9) and (4.3.1.10).

(*iii*) *If we have (symbolically)*

$$\tau(y, \chi_1 + \chi_2) = \tau(y, \chi_1)\tau(y, \chi_2) \tag{4.3.1.11}$$

(under the biextension structure of $\mathcal{P}_{A,\eta}^{\otimes -1}$ *as in Section 4.2.2) for all* $\chi_1, \chi_2 \in X$ *and all* $y \in Y$*, then we have (symbolically)*

$$\tau(y_1 + y_2, \chi) = \tau(y_1, \chi)\tau(y_2, \chi) \tag{4.3.1.12}$$

(under the biextension structure of $\mathcal{P}_{A,\eta}^{\otimes -1}$ *as in Section 4.2.2) for* $\chi \in X$ *and all* $y_1, y_2 \in Y$*. (Note that (4.3.1.11) has to be proven independently later.)*

(*iv*) *For all but finitely many* $y \in Y$*, the section* $\psi(y)$ *extends to a section of* $\mathcal{M}(c^\vee(y))^{\otimes -1}$ *and is congruent to zero modulo* I*.*

This is a special case of the stronger statement: For any integer $n > 0$*, for all but finitely many* $y \in Y$*, the section* $\psi(y)$ *extends to a section of* $\mathcal{M}(c^\vee(y))^{\otimes -1}$ *and is congruent to zero modulo* I^n *(defined as in Definition 4.2.1.5).*

(*v*) *For all nonzero* $y \in Y$*, the section* $\tau(y, \phi(y))$ *extends to a section of* $(c^\vee(y) \times c\phi(y))^*\mathcal{P}_A^{\otimes -1}$ *and is congruent to zero modulo* I *(defined as in Definition 4.2.1.4).*

Proof. The claim (i) can be verified as follows: Consider the relations

$$\begin{aligned}
&\psi(y_1 + y_2)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 \\
&= \psi(y_1 + y_2)\tau(y_1 + y_2, 0)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 \\
&= \sigma_{\phi(y_1+y_2)}
\end{aligned}$$

and

$$\begin{aligned}
&\psi(y_1)\psi(y_2)\tau(y_1, \phi(y_2))\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 \\
&= \psi(y_1)\tau(y_1, \phi(y_2))\psi(y_2)\tau(y_2, 0)\, T^*_{c^\vee(y_1)} T^*_{c^\vee(y_2)} \circ \sigma_0 \\
&= \psi(y_1)\tau(y_1, \phi(y_2))\, T^*_{c^\vee(y_2)} \sigma_{\phi(y_2)} \\
&= \sigma_{\phi(y_1)+\phi(y_2)}.
\end{aligned}$$

The uniqueness of $\psi(y)\tau(y, \chi)$ then implies

$$\tau(y_1, \phi(y_2)) = \tau(y_2, \phi(y_1)),$$

which is (4.3.1.9).

The claim (ii) is just a formal consequence of (i).

The claim (iii) can be verified as follows: Consider the relations

$$\psi(y_1+y_2)\tau(y_1+y_2,\chi)\,T^*_{c^\vee(y_1)+c^\vee(y_2)}\circ\sigma_\chi=\sigma_{\chi+\phi(y_1+y_2)}$$

and

$$\begin{aligned}&\psi(y_1)\psi(y_2)\tau(y_1,\chi+\phi(y_2))\tau(y_2,\chi)\,T^*_{c^\vee(y_1)+c^\vee(y_2)}\circ\sigma_\chi\\&=\psi(y_1)\tau(y_1,\chi+\phi(y_2))\psi(y_2)\tau(y_2,\chi)\,T^*_{c^\vee(y_1)}T^*_{c^\vee(y_2)}\circ\sigma_\chi\\&=\psi(y_1)\tau(y_1,\chi+\phi(y_2))\,T^*_{c^\vee(y_2)}\sigma_{\chi+\phi(y_2)}\\&=\sigma_{\chi+\phi(y_1)+\phi(y_2)}.\end{aligned}$$

The uniqueness of $\psi(y)\tau(y,\chi)$ and (4.3.1.11) then implies

$$\begin{aligned}\psi(y_1+y_2)\tau(y_1+y_2,\chi)&=\psi(y_1)\psi(y_2)\tau(y_1,\chi+\phi(y))\tau(y_2,\chi)\\&=\psi(y_1)\psi(y_2)\tau(y_1,\chi)\tau(y_1,\phi(y_2))\tau(y_2,\chi).\end{aligned}$$

By cancelation using (4.3.1.9), we have

$$\tau(y_1+y_2,\chi)=\tau(y_1,\chi)\tau(y_2,\chi),$$

which is (4.3.1.12).

The claims (iv) and (v) can be verified as follows: It suffices to establish the positivity condition for τ, as the equivalence between the positivity conditions for τ and for ψ has already been established in Section 4.2.4.

By (4.3.1.2), there exists $s_0\in\Gamma(G_\eta,\mathcal{L}_\eta)$ such that $\sigma_0(s_0)\neq 0$. Since $\Gamma(G_\eta,\mathcal{L}_\eta)=\Gamma(G,\mathcal{L})\underset{R}{\otimes}K$, we may assume that $s_0\in\Gamma(G,\mathcal{L})$. Then we have $T^*_{c^\vee(y)}\circ\sigma_0(s_0)\neq 0$ for any $y\in Y$. On the other hand, we have $\sigma_{\phi(y)}(s_0)=\psi(y)\;T^*_{c^\vee(y)}\circ\sigma_0(s_0)\in\Gamma(A,\mathcal{M}_{\phi(y)})$ for all $y\in Y$. As a result, we have $\sigma_0(s_0)\in\Gamma(A,\mathcal{M}_0)\underset{R}{\otimes}I_y^{\otimes -1}$ for any $y\in Y$.

Let us fix a $y\neq 0$ in Y. Suppose there is a discrete valuation $\upsilon\in\Upsilon_1$ such that $\upsilon(\tau(y,\phi(y)))<0$. Then the calculation in Section 4.2.4 implies that we have $\lim_{k\to\infty}\upsilon(\psi(ky))\to-\infty$. In particular, we have $\sigma_0(s_0)\in\Gamma(A,\mathcal{M}_0)\underset{R}{\otimes}\mathfrak{m}_\upsilon^N$ for any $N>0$. Since $\Gamma(A,\mathcal{M}_0)$ is a finitely generated R-module, this is possible only when $\sigma_0(s_0)=0$, which is a contradiction. Thus we see that $\upsilon(\tau(y,\phi(y)))\geq 0$ for any $\upsilon\in\Upsilon_1$, which implies that $I_{y,\phi(y)}\subset R$ by noetherian normality of R. In other words, the section $\tau(y,\phi(y))$ of $\mathscr{O}_{\phi(y)}(c^\vee(y))_\eta^{\otimes -1}$ extends to a section of $\mathscr{O}_{\phi(y)}(c^\vee(y))^{\otimes -1}=$

$(c^\vee(y) \times c\phi(y))^* \mathcal{P}_A^{\otimes -1}$. This shows the first half of the positivity condition for τ.

For the second half, suppose that there is a $y \neq 0$ in Y such that $\tau(y, \phi(y))$ is not congruence to zero modulo I. Then $\upsilon(\tau(y, \phi(y))) = 0$ for some $\upsilon \in \Upsilon_I$, and hence $\upsilon(\psi(ky)) = k\upsilon(\psi(y))$ for all $k \in \mathbb{Z}$. Let $i > 0$ be an integer such that symbolically $\sigma_0(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. That is, the image of $\sigma_0(s_0)$ under the pullback $\Gamma(A, \mathcal{M}) \to \Gamma(A_{\upsilon,i}, \mathcal{M}_{\upsilon,i})$ is nonzero, where $A_{\upsilon,i}$ and $\mathcal{M}_{\upsilon,i}$ are respectively the pullbacks of A and $\mathcal{M}$ to $S_{\upsilon,i} = \mathrm{Spec}(R_\upsilon/\mathfrak{m}_\upsilon^{i+1})$. After composing with the pullback of the isomorphism $T^*_{c^\vee(ky)}$ to $S_{\upsilon,i}$, we have symbolically $T^*_{c^\vee(ky)} \circ \sigma_0(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. Since $\sigma_{\phi(ky)}(s_0) = \psi(ky)\, T^*_{c^\vee(ky)} \circ \sigma_0(s_0)$, we have $\upsilon(\sigma_{\phi(ky)}(s_0)) = \upsilon(\sigma_0(s_0)) + k\upsilon(\psi(y))$ for all $k \in \mathbb{Z}$. In particular, there exists infinitely many k such that $\sigma_{\phi(ky)}(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. This implies that there exists infinitely many k such that $\sigma_{\phi(ky)}(s_0) \not\equiv 0 \pmod{I^{i+1}}$, which contradicts the fact that $s_0 \equiv \sum\limits_{\chi \in X} \sigma_\chi(s_0) \pmod{I^{i+1}}$ is a finite sum for all i. This shows the full positivity condition for τ. □

Corollary 4.3.1.13. *Assuming the relations (4.3.1.2), (4.3.1.4), and (4.3.1.5) above, the association* $(G, \mathcal{L}) \mapsto (A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ *(described thus far) defines a functor* $\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$.

Proof. The point is that, if we assume relations (4.3.1.2), (4.3.1.4), and (4.3.1.5), the tuple $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ is associated in a functorial way and satisfies the required properties for an object in $\mathrm{DD}_{\mathrm{ample}}$. □

The use of the notation $\mathrm{F}_{\mathrm{ample}}$ will be justified after we prove Theorem 4.2.1.8.

Convention 4.3.1.14. *We shall say that* $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ *is the* **degeneration datum** *associated to* $(G, \mathcal{L})$.

We will see variants of this usage later when we study other kinds of additional structures.

Lemma 4.3.1.15 (see [37, Ch. II, p. 51])**.** *Suppose that we have a morphism* $f : G_1 \to G_2$ *between two pairs* $(G_1, \mathcal{L}_1)$ *and* $(G_2, \mathcal{L}_2)$ *in* $\mathrm{DEG}_{\mathrm{ample}}$ *in the sense that* $\mathcal{L}_1 \cong f^*\mathcal{L}_2$, *and suppose that* f *induces an isomorphism between the associated degeneration data in* $\mathrm{DD}_{\mathrm{ample}}$. *Then* f *is an isomorphism.*

Proof. Let us denote by $(A_i, \underline{X}_i, \underline{Y}_i, \phi_i, c_i, c_i^\vee, \mathcal{L}_i^\natural, \tau_i, \psi_i)$ the degeneration datum in $\mathrm{DD}_{\mathrm{ample}}$ associated to $(G_i, \mathcal{L}_i)$, for $i = 1, 2$. By assumption, the morphism $f : (G_1, \mathcal{L}_1) \to (G_2, \mathcal{L}_2)$ induces isomorphisms $f_A : A_1 \overset{\sim}{\to} A_2$, $f_X : \underline{X}_2 \overset{\sim}{\to} \underline{X}_1$, $f_Y : \underline{Y}_1 \overset{\sim}{\to} \underline{Y}_2$, $f^\natural : G_1^\natural \to G_2^\natural$, and $\mathcal{L}_1^\natural \overset{\sim}{\to} (f^\natural)^* \mathcal{L}_2^\natural$. Altogether, they induce a natural isomorphism $\Gamma(G^\natural_{2,\mathrm{for}}, (\mathcal{L}^\natural_{2,\mathrm{for}})^{\otimes n}) \overset{\sim}{\to} \Gamma(G^\natural_{1,\mathrm{for}}, (\mathcal{L}^\natural_{1,\mathrm{for}})^{\otimes n})$ for any integer n. The data of τ_1 and ψ_1 (resp. τ_2 and ψ_2) define an action of $\underline{Y}_1$ (resp. $\underline{Y}_2$) on $(\mathcal{L}^\natural_{1,\mathrm{for}})^{\otimes n}$ (resp. $(\mathcal{L}^\natural_{2,\mathrm{for}})^{\otimes n}$) for any integer $n \geq 0$, and the action characterizes the image of $\Gamma(G_{1,\eta}, (\mathcal{L}_{1,\eta})^{\otimes n}) \hookrightarrow \Gamma(G^\natural_{1,\mathrm{for}}, (\mathcal{L}^\natural_{1,\mathrm{for}})^{\otimes n}) \underset{R}{\otimes} K$ (resp. $\Gamma(G_{2,\eta}, (\mathcal{L}_{2,\eta})^{\otimes n}) \hookrightarrow \Gamma(G^\natural_{2,\mathrm{for}}, (\mathcal{L}^\natural_{2,\mathrm{for}})^{\otimes n}) \underset{R}{\otimes} K$). Therefore, the identification between (τ_1, ψ_1) and (τ_2, ψ_2) under the other isomorphisms induces an isomorphism from $\Gamma(G_{2,\eta}, \mathcal{L}_{2,\eta}^{\otimes n})$ to $\Gamma(G_{1,\eta}, \mathcal{L}_{1,\eta}^{\otimes n})$ for all $n \geq 0$. Since $\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ are ample, these isomorphisms give an isomorphism $G_{1,\eta} \overset{\sim}{\to} G_{2,\eta}$, which has to agree with the restriction of $f : G_1 \to G_2$ to the fiber over η. By Proposition 3.3.1.7, f must be an isomorphism, as desired. □

Now that we have seen the rather formal consequences of the definitions, the proof of Theorem 4.2.1.8 will be completed by verifying the relations (4.3.1.2), (4.3.1.4), and (4.3.1.5), and Proposition 4.3.4.6 below, using the theory of theta representations.

4.3.2 Relations Between Theta Representations

In this section we use the uniqueness of irreducible theta representations (which are algebro-geometric analogues of Heisenberg group representations) to prove (4.3.1.4) from (4.3.1.2).

Recall (from Theorem 3.4.2.6) that the group $K(\mathcal{L}_\eta)$ extends to a quasi-finite flat subgroup scheme $K(\mathcal{L})$ in G over S. The group scheme $K(\mathcal{L})$ has the *finite part* $K(\mathcal{L})^{\mathrm{f}}$, which is the largest finite subscheme of $K(\mathcal{L})$, and the *torus part* $K(\mathcal{L})^{\mu}$, which is isomorphic to $K(\mathcal{L})^\flat = K(\mathcal{L})^\natural \cap T$, a subgroup of T. We have the following exact sequences:

$$0 \to K(\mathcal{L})^{\mathrm{f}}_\eta \to K(\mathcal{L})_\eta \to K(\mathcal{L})_\eta / K(\mathcal{L})^{\mathrm{f}}_\eta \to 0$$

and

$$0 \to K(\mathcal{L})^{\mu}_\eta \to K(\mathcal{L})^{\mathrm{f}}_\eta \to K(\mathcal{M}_\eta) \to 0, \tag{4.3.2.1}$$

where:

1. $K(\mathcal{L})^{\mu}_\eta$ is isotropic under the pairing

$$\mathrm{e}^{\mathcal{L}_\eta} : K(\mathcal{L}_\eta) \underset{\eta}{\times} K(\mathcal{L}_\eta) \to \mathbf{G}_{\mathrm{m},\eta}$$

induced (as a commutator pairing) by the central extension structure

$$0 \to \mathbf{G}_{\mathrm{m},\eta} \to \mathcal{G}(\mathcal{L}_\eta) \to K(\mathcal{L}_\eta) \to 0. \tag{4.3.2.2}$$

2. $K(\mathcal{L})^{\mathrm{f}}_\eta$ is the annihilator of $K(\mathcal{L})^\mu_\eta$ under the pairing $\mathrm{e}^{\mathcal{L}_\eta}$.
3. $K(\mathcal{L})_\eta / K(\mathcal{L})^{\mathrm{f}}_\eta$ can be identified with the Cartier dual of $K(\mathcal{L})^\mu_\eta$ using the pairing $\mathrm{e}^{\mathcal{L}_\eta}$.

Over the whole base scheme S, the pairing $\mathrm{e}^{\mathcal{L}_\eta}$ extends to a pairing

$$\mathrm{e}^{\mathcal{L}_\eta}_S : K(\mathcal{L}) \underset{S}{\times} K(\mathcal{L}) \to \mathbf{G}_{\mathrm{m},S},$$

which is induced (as a commutator pairing) by the central extension structure

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0. \tag{4.3.2.3}$$

We still have the exact sequence

$$0 \to K(\mathcal{L})^\mu \to K(\mathcal{L})^{\mathrm{f}} \to K(\mathcal{M}) \to 0,$$

where $K(\mathcal{L})^\mu$ is still isotropic and where $K(\mathcal{L})^{\mathrm{f}}$ is still the annihilator of $K(\mathcal{L})^\mu$ under $\mathrm{e}^{\mathcal{L}_\eta}_S$.

Now $K(\mathcal{L})^\mu$ being isotropic in $K(\mathcal{L})$ under $\mathrm{e}^{\mathcal{L}}$ implies that the extension (4.3.2.3) splits over $K(\mathcal{L})^\mu$, namely there exists a splitting

$$K(\mathcal{L})^\mu \hookrightarrow \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^\mu}. \tag{4.3.2.4}$$

Among all possible splittings as above, there is a natural choice coming from the cubical isomorphism $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ which can be explained as follows: Recall (from (3.4.2.4)) that we have isomorphisms $K(\mathcal{L})^\natural \cong K(\mathcal{L})^{\mathrm{f}}$ and $K(\mathcal{L})^\flat \cong K(\mathcal{L})^\mu$ between finite flat group schemes over S, where $K(\mathcal{L})^\natural \subset G^\natural$, where $K(\mathcal{L})^\flat \subset T \subset G^\natural$, where $K(\mathcal{L})^{\mathrm{f}} \subset G$, and where $K(\mathcal{L})^\mu \subset G$. Using the canonical isomorphisms $\mathcal{L}^\natural_{\mathrm{for}} \cong \mathcal{L}_{\mathrm{for}}$ and $K(\mathcal{L})^\natural_{\mathrm{for}} \cong K(\mathcal{L})^{\mathrm{f}}_{\mathrm{for}}$ over the formal fiber, we have a canonical isomorphism $\mathcal{L}^\natural|_{K(\mathcal{L})^\natural_{\mathrm{for}}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}_{\mathrm{for}}}$, which by Theorem 2.3.1.2 (using finiteness of $K(\mathcal{L})^\natural \cong K(\mathcal{L})^{\mathrm{f}}$) algebraizes uniquely to a canonical isomorphism $\mathcal{L}^\natural|_{K(\mathcal{L})^\natural} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}}$. In particular, it induces a canonical isomorphism $\mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathcal{L}|_{K(\mathcal{L})^\mu}$ by restriction. Now using the fact that $K(\mathcal{L})^\mu \subset T$, and that we have a cubical isomorphism $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$, which can be interpreted alternatively as an isomorphism $s : \mathcal{L}^\natural|_T \cong \mathbf{G}_{\mathrm{m}}|_T$ of cubical $\mathbf{G}_{\mathrm{m}}$-torsors over T, we have an isomorphism $s|_{K(\mathcal{L})^\flat} : \mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathbf{G}_{\mathrm{m}}|_{K(\mathcal{L})^\flat}$ by restriction. Now the trivial $\mathbf{G}_{\mathrm{m}}$-torsor over $K(\mathcal{L})^\flat$ has a natural structure of a central extension of $K(\mathcal{L})^\flat$ by $\mathbf{G}_{\mathrm{m}}$, together with a natural choice of splitting given by the identity section of $\mathbf{G}_{\mathrm{m}}$. This gives us a natural choice of a splitting of $\mathcal{L}^\natural|_{K(\mathcal{L})^\flat}$ over $K(\mathcal{L})^\flat$, and hence gives us a natural choice of a splitting of $\mathcal{G}(\mathcal{L})^\mu = \mathcal{L}|_{K(\mathcal{L})^\mu}$ over $K(\mathcal{L})^\mu$ via the canonical isomorphism $\mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathcal{L}|_{K(\mathcal{L})^\mu}$ above. Let us fix this choice of the splitting from now on. We say that this is the choice *compatible with the cubical trivialization* $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$.

Note that the set of choices of the splitting (4.3.2.4) form a torsor under the group of group homomorphisms $\mathrm{Hom}_S(K(\mathcal{L})^\mu, \mathbf{G}_{\mathrm{m}})$, namely the character group of $K(\mathcal{L})^\mu$. Since $K(\mathcal{L})^\mu \cong K(\mathcal{L})^\flat = K(\mathcal{L})^\natural \cap T$ is the kernel of $T \twoheadrightarrow T^\vee$ induce by $G^\natural \to G^{\vee,\natural}$, wee see that $K(\mathcal{L})^\mu$ is canonically dual to $X/\phi(Y)$, where $\phi : Y \to X$ is by definition dual to $T \twoheadrightarrow T^\vee$ above. Therefore, we can identify the character group of $K(\mathcal{L})^\mu$ canonically with $X/\phi(Y)$.

Now that $\Gamma(G, \mathcal{L})$ is a representation of $\mathcal{G}(\mathcal{L})$, we have an action of $K(\mathcal{L})^\mu$ on $\Gamma(G, \mathcal{L})$ via the above chosen splitting (4.3.2.4). Since $K(\mathcal{L})^\mu$ is commutative, the representation $\Gamma(G, \mathcal{L})$ can be decomposed according to the character group of $K(\mathcal{L})^\mu$ discussed above, which is isomorphic to $X/\phi(Y)$. Hence we can write

$$\Gamma(G, \mathcal{L}) = \bigoplus_{\bar{\chi} \in X/\phi(Y)} \Gamma(G, \mathcal{L})_{\bar{\chi}},$$

where $\Gamma(G, \mathcal{L})_{\bar{\chi}}$ is the *weight-$\bar{\chi}$ subspace* of $\Gamma(G, \mathcal{L})$ under $K(\mathcal{L})^\mu$-action. Note that this depends on the choice of the splitting (4.3.2.4).

The notion of weight spaces under $K(\mathcal{L})^\mu$ and the weight spaces under T-action are compatible in the sense that we have

$$\Gamma(G, \mathcal{L})_{\bar{\chi}} \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})_{\bar{\chi}} = \hat{\bigoplus_{\mu \in \chi + \phi(Y)}} \Gamma(A, \mathcal{M}_\mu)$$

under

$$\Gamma(G, \mathcal{L}) \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \hat{\bigoplus_{\mu \in X}} \Gamma(A, \mathcal{M}_\mu),$$

because they are both determined by the cubical trivialization $s : i^*\mathcal{L}_{\mathrm{for}} \cong \mathscr{O}_{T_{\mathrm{for}}}$, which gives a T_{for}-action on the sections of $\mathcal{L}_{\mathrm{for}}$. Note that if we consider the induced formal action of $K(\mathcal{L})^\flat \subset T$ on $\mathcal{L}_{\mathrm{for}}$, the weights of T are given by elements in $\chi \in X$. However, since $K(\mathcal{L})^\flat$ is the kernel of $T \twoheadrightarrow T^\vee$ and

$\phi : Y = \underline{X}(T^\vee) \to X = \underline{X}(T)$ is given by composing a character of $T^\vee$ by $T \twoheadrightarrow T^\vee$, we see that $\phi(y)$ is trivial on $K(\mathcal{L})^\flat$ for any $y \in Y$. In particular, the value of χ on $K(\mathcal{L})^\flat$ depends only on the class $\bar{\chi}$ of χ in $X/\phi(Y)$.

Let us now consider the exact sequence

$$0 \to \mathcal{G}(\mathcal{L})^\mu \to \mathcal{G}(\mathcal{L})^{\mathrm{f}} \to K(\mathcal{M}) \to 0, \tag{4.3.2.5}$$

where

$$\mathcal{G}(\mathcal{L})^\mu = \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^\mu} = \mathcal{L}|_{K(\mathcal{L})^\mu}$$

and

$$\mathcal{G}(\mathcal{L})^{\mathrm{f}} = \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^{\mathrm{f}}} = \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}}$$

are restrictions of $\mathcal{G}(\mathcal{L}) = \mathcal{L}|_{K(\mathcal{L})}$ to the subgroups $K(\mathcal{L})^\mu$ and $K(\mathcal{L})^{\mathrm{f}}$. This exact sequence (4.3.2.5) is compatible with the previous (4.3.2.1) in the sense that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\mathcal{L})^\mu \arrow[r] \arrow[d] & \mathcal{G}(\mathcal{L})^{\mathrm{f}} \arrow[r] \arrow[d] & K(\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 , \\
0 \arrow[r] & K(\mathcal{L})^\mu \arrow[r] & K(\mathcal{L})^{\mathrm{f}} \arrow[r] & K(\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

where the first two vertical maps (from the left-hand side) are the canonical morphisms.

The existence of a splitting of the extension

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}(\mathcal{L})^\mu \to K(\mathcal{L})^\mu \to 0,$$

namely the fact that $K(\mathcal{L})^\mu$ is isotropic under the pairing $\mathrm{e}^{\mathcal{L}}_S$, implies that $\mathcal{G}(\mathcal{L})^\mu$ is commutative. Moreover, the choice (4.3.2.4) of this splitting above gives us a (commutative) action of $K(\mathcal{L})^\mu$ on $\Gamma(G, \mathcal{L})$, as we have seen above. Let

$$h_{\bar{\chi}} : \mathcal{G}(\mathcal{L})^\mu \to \mathbf{G}_{\mathrm{m}}$$

be the group scheme homomorphism that is identity on $\mathbf{G}_{\mathrm{m}}$ and is $\bar{\chi}$ on $K(\mathcal{L})^\mu$ via the above chosen splitting, so that $h_{\bar{\chi}}$ reflects the character of the (commutative) action of $\mathcal{G}(\mathcal{L})^\mu$ on $\Gamma(G, \mathcal{L})$. If we push-out $\mathcal{G}(\mathcal{L})^\mu$ by $h_{\bar{\chi}}$ in

$$0 \to \mathcal{G}(\mathcal{L})^\mu \to \mathcal{G}(\mathcal{L})^{\mathrm{f}} \to K(\mathcal{M}) \to 0,$$

then we get a push-out group scheme $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\mathcal{L})^{\mu} \arrow[r] \arrow[d, "h_{\bar{\chi}}"'] & \mathcal{G}(\mathcal{L})^{\mathrm{f}} \arrow[r] \arrow[d, dotted] & K(\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 , \\
0 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \arrow[r] & K(\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

Lemma 4.3.2.6. *The push-out* $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ *is naturally isomorphic to*

$$0 \to \mathbf{G}_{\mathrm{m}} \to \mathcal{G}(\mathcal{M}_{\chi}) \to K(\mathcal{M}_{\chi}) \to 0.$$

Proof. Given the chosen cubical trivialization $s : i^*\mathcal{L}^{\natural} \cong \mathscr{O}_T$, or $s : \mathcal{L}^{\natural}|_T \cong \mathbf{G}_{\mathrm{m}}|_T$, we obtain a splitting of $\mathcal{L}^{\natural}|_T$ over T given by the identity section of $\mathbf{G}_{\mathrm{m}}|_T$, and all the other possible cubical trivializations can be identified with maps from T to $\mathbf{G}_{\mathrm{m}}$, namely the character group of T. Since the cubical isomorphism $\mathcal{L}^{\natural} \cong \pi^*\mathcal{M}$ restricts to s over T, the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}$ on A can be identified with the weight-0 subsheaf of $\pi_*\mathcal{L}$ under T-action (defined by the splitting from T to $\mathbf{G}_{\mathrm{m}}|_T \cong \mathcal{L}^{\natural}|_T$), or the invariant subsheaf of $\mathcal{L}$ under T-action described above. If the splitting is modified by adding a character $-\chi$ of T to the splitting of $\mathbf{G}_{\mathrm{m}}|_T$ over T, which we denote by $s_{\chi} : i^*\mathcal{L}^{\natural} \cong \mathscr{O}_T$, then T acts by $-\chi$ on $\mathcal{M}$ as a subsheaf of $\mathcal{L}^{\natural}$, which implies that the invariant subspace should be given by $\mathcal{M}_{\chi} = \mathscr{O}_{\chi} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}$, together with a different isomorphism $\mathcal{L}^{\natural} \cong \pi^*\mathcal{M}_{\chi}$ that restricts to the modified trivialization $s_{\chi} : i^*\mathcal{L}^{\natural} \cong \mathscr{O}_T$.

If we restrict s_{χ} to $s_{\chi}|_{K(\mathcal{L})^{\flat}} : \mathcal{L}^{\natural}|_{K(\mathcal{L})^{\flat}} \cong \mathbf{G}_{\mathrm{m}}|_{K(\mathcal{L})^{\flat}}$, then we obtain a splitting of $\mathcal{G}(\mathcal{L})^{\mu}$ over $K(\mathcal{L})^{\mu}$, compatible with the cubical trivialization s_{χ} (instead of s). Note that this slitting maps $K(\mathcal{L})^{\mu}_{\eta}$ isomorphically onto the kernel of $h_{\bar{\chi}}$ over the generic fiber. The group $K(\mathcal{L})^{\flat}$ is the kernel of the map $\bar{\lambda} : G^{\natural} \to \lambda_A^* G^{\vee,\natural} = A \underset{\lambda_A, A^{\vee}}{\times} G^{\vee,\natural}$ induced by $\lambda^{\natural} : G^{\natural} \to G^{\vee,\natural}$, which fits into the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, two heads, "\text{dual to } \phi"'] & G^{\natural} \arrow[r] \arrow[d, dotted, "\bar{\lambda}"] \arrow[dd, dotted, bend right=60, "\lambda^{\natural}"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^{\vee} \arrow[r] \arrow[d, equal] & \lambda_A^* G^{\vee,\natural} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\lambda_A"] & 0 \\
0 \arrow[r] & T^{\vee} \arrow[r] & G^{\vee,\natural} \arrow[r] & A^{\vee} \arrow[r] & 0
\end{tikzcd}$$

Any cubical trivialization $s_\chi|_{K(\mathcal{L})^\flat} : \mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathbf{G}_{\mathrm{m}}|_{K(\mathcal{L})^\flat}$ as above corresponds (by similar reasonings of descent theory behind Proposition 3.2.5.4) to a cubical torsor $\mathcal{L}^\natural_{\bar{\chi}}$ on $\lambda_A^* G^{\vee,\natural}$, together with a cubical isomorphism $\mathcal{L}^\natural \cong \bar{\lambda}^* \mathcal{L}^\natural_{\bar{\chi}}$ that restricts to the cubical trivialization $s_\chi|_{K(\mathcal{L})^\flat} : \mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathbf{G}_{\mathrm{m}}|_{K(\mathcal{L})^\flat}$ above. We can think of $\mathcal{L}^\natural_{\bar{\chi}}$ as the subsheaf of $\bar{\lambda}_* \mathcal{L}^\natural$ on which $K(\mathcal{L})^\flat$ acts by $\bar{\chi}$.

Let $\bar{\pi} : \lambda_A^* G^{\vee,\natural} \to A$ be the structural map. The compatibility between the splitting of $\mathcal{G}(\mathcal{L})^\mu$ over $K(\mathcal{L})^\mu$ with s_χ gives us cubical isomorphisms $\mathcal{L}^\natural \cong \bar{\lambda}^* \mathcal{L}^\natural_{\bar{\chi}}$, $\mathcal{L}^\natural_{\bar{\chi}} \cong \bar{\pi}^* \mathcal{M}_\chi$, and $\mathcal{L}^\natural \cong \pi^* \mathcal{M}_\chi$ such that the cubical isomorphism $\mathcal{L}^\natural \cong \bar{\lambda}^* \bar{\pi}^* \mathcal{M}_\chi$ induced by the first two cubical isomorphisms agrees with the third one.

The underlying schemes of $\mathbf{G}_{\mathrm{m}}$-torsors can be fit into a commutative diagram

$$\begin{tikzcd}
\mathcal{L}^\natural \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{L}^\natural_{\bar{\chi}} \arrow[dl] \arrow[dd] \\
& \mathcal{M}_\chi \arrow[dd] & \\
G^\natural \arrow[rr] \arrow[dr] & & \lambda_A^* G^{\vee,\natural} \arrow[dl] \\
& A &
\end{tikzcd}$$

respecting the rigidifications along the identities, which by restriction gives a commutative diagram

$$\begin{tikzcd}
\mathcal{L}^\natural|_{K(\mathcal{L})^\natural} \arrow[rr] \arrow[dr] & & \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat} \arrow[dl] \\
& \mathcal{M}_\chi|_{K(\mathcal{M}_\chi)} &
\end{tikzcd}$$

of group schemes. The rigidifications give compatible group scheme homomorphisms from $\mathbf{G}_{\mathrm{m}}$ to these group schemes, which in particular give a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] \arrow[d, equal] & \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat} \arrow[r] \arrow[d] & K(\mathcal{L})^\natural/K(\mathcal{L})^\flat \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathbf{G}_{\mathrm{m}} \arrow[r] & \mathcal{M}_\chi|_{K(\mathcal{M}_\chi)} \arrow[r] & K(\mathcal{M}_\chi) \arrow[r] & 0
\end{tikzcd}$$

in which the middle vertical arrow is forced to be an isomorphism of group schemes. Note that $K(\mathcal{M}_\chi) = K(\mathcal{M})$ does not depend on χ. The group scheme $\mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat}$ can be identified with the quotient of $\mathcal{L}^\natural|_{K(\mathcal{L})^\natural} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}} = \mathcal{G}(\mathcal{L})^{\mathrm{f}}$ by $\ker(h_{\bar{\chi}})$, which is just $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$. Moreover, we have $\mathcal{M}_\chi|_{K(\mathcal{M}_\chi)} = \mathcal{G}(\mathcal{M}_\chi)$ by definition. Therefore we have proved the claim that $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ is naturally isomorphic to $\mathcal{G}(\mathcal{M}_\chi)$. ☐

Since $\mathcal{G}(\mathcal{L})^{\mathrm{f}}$ is the annihilator of $\mathcal{G}(\mathcal{L})^{\mu}$ under the commutator pairing of $\mathcal{G}(\mathcal{L})$, or equivalently since $K(\mathcal{L})^{\mathrm{f}}$ is the annihilator of $K(\mathcal{L})^{\mu}$ under the pairing $\mathrm{e}^{\mathcal{L}}$ induced by the commutator pairing of $\mathcal{G}(\mathcal{L})$, we may interpret $\Gamma(G, \mathcal{L})_{\bar{\chi}}$ as a $\mathcal{G}(\mathcal{L})^{\mathrm{f}}$-invariant submodule of $\Gamma(G, \mathcal{L})$. Therefore the push-out group scheme $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ acts naturally on $\Gamma(G, \mathcal{L})_{\bar{\chi}}$, because $\ker(h_{\bar{\chi}})$ has no action, and because push-out by $h_{\bar{\chi}}$ just means forming the quotient by $\ker(h_{\bar{\chi}})$.

Lemma 4.3.2.7. *Under the identification $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$ given by Lemma 4.3.2.6, the map*

$$\sigma_\chi : \Gamma(G, \mathcal{L})_{\bar{\chi}} \to \Gamma(A, \mathcal{M}_\chi)$$

is $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$-equivariant.

Proof. By definition, the map σ_χ above factors through

$$\begin{aligned}\Gamma(G, \mathcal{L})_{\bar{\chi}} \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})_{\bar{\chi}} = \Gamma(\lambda_A^* G^{\vee,\natural}_{\mathrm{for}}, \mathcal{L}^\natural_{\bar{\chi},\mathrm{for}}) = \hat{\bigoplus_{\mu \in \phi(Y)}} \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi+\mu,\mathrm{for}}) \\ \to \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi,\mathrm{for}}) = \Gamma(A, \mathcal{M}_\chi),\end{aligned} \tag{4.3.2.8}$$

where the first inclusion is invariant under $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$-action.

Since $\mathcal{D}_2(\mathcal{L}^\natural_{\bar{\chi}})$ (together with its canonical trivialization descended from

$\mathcal{D}_2(\mathcal{L}^\natural)$) descends down to $\mathcal{D}_2(\mathcal{M}_\chi)$, we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu}) \underset{S}{\times} \mathcal{L}^\natural_{\bar{\chi}} \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{L}^\natural_{\bar{\chi}} \arrow[dd] \arrow[dr] & . \\
& (\mathcal{M}_\chi|_{K(\mathcal{M}_\chi)}) \underset{S}{\times} \mathcal{M}_\chi \arrow[rr] \arrow[dd] & & \mathcal{M}_\chi \arrow[dd] \\
(K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu) \underset{S}{\times} \lambda_A^* G^{\vee,\natural} \arrow[rr] \arrow[dr] & & \lambda_A^* G^{\vee,\natural} \arrow[dr] & \\
& K(\mathcal{M}_\chi) \underset{S}{\times} A \arrow[rr] & & A
\end{tikzcd}$$

By identifying $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu}$ and $\mathcal{G}(\mathcal{M}_\chi) \cong \mathcal{M}_\chi|_{K(\mathcal{M}_\chi)}$ as in the proof of Lemma 4.3.2.6, we may interpret the two rectangles in the diagram (4.3.2) as describing the actions of $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ and $\mathcal{G}(\mathcal{M}_\chi)$ on respectively $\Gamma(G, \mathcal{L})_{\bar{\chi}}$ and $\Gamma(A, \mathcal{M}_\chi)$. Then the $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$-equivariance of (4.3.2.8) follows from the commutativity of the diagram (4.3.2), as desired. □

Let us now pullback everything to the generic point η. Then we obtain an equivariant morphism

$$\sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \to \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}) \tag{4.3.2.9}$$

between two representations of the *same* group.

Lemma 4.3.2.10. *Both of the representations in (4.3.2.9) are irreducible. As a result, the map σ_χ in (4.3.2.9) is an intertwining operator between two irreducible representations, which is either zero or unique up to a nonzero multiple in K.*

Proof. Note that $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ is an irreducible representation of $\mathcal{G}(\mathcal{M}_{\chi,\eta})$. Therefore it suffices to show that the ranks, or rather dimensions, of the modules at both sides of (4.3.2.9) are the same.

By the Riemann-Roch theorem [99, §16], we know that

$$\dim \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}) = \deg(\lambda_A)^{1/2}$$

for any $\chi \in X$. By Remark 3.4.4.3,

$$\dim \Gamma(G_\eta, \mathcal{L}_\eta) = \deg(\lambda_G)^{1/2} = [X : \phi(Y)] \deg(\lambda_A)^{1/2}.$$

Since

$$\dim \Gamma(G_\eta, \mathcal{L}_\eta) = \sum_{\bar{\chi} \in X/\phi(Y)} \dim \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \geq \sum_{\bar{\chi} \in X/\phi(Y)} \dim \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}),$$

(where in the expression $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ we may take any representative χ for each $\bar{\chi} \in X/\phi(Y)$,) and since both sides of the inequality are now equal to $[X : \phi(Y)] \deg(\lambda_A)^{1/2}$, we must have

$$\dim \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} = \dim \Gamma(A_\eta, \mathcal{L}_{\chi,\eta})$$

for any $\chi \in X$, as desired. □

Proposition 4.3.2.11. *If (4.3.1.2) (i.e. $\sigma_\chi \neq 0$) is true for any arbitrary $\chi \in X$, then (4.3.1.4) is true.*

Proof. By (4.3.2.8) above, $\sigma_\chi = 0$ on $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}'}$ if $\bar{\chi}' \neq \bar{\chi}$ in $X/\phi(Y)$. By assumption, $\sigma_\chi \neq 0$ for any arbitrary $\chi \in X$ on the whole $\Gamma(G_\eta, \mathcal{L}_\eta)$. Therefore, we must have a surjection from $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}}$ to $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$. By Lemma 4.3.2.10, we see that σ_χ is a nonzero intertwining operator between two irreducible representations, which is unique up to a nonzero multiple in K.

Now let us compare the two maps

$$T^*_{c^\vee(y)} \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$$

and

$$\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}).$$

When restricted to the *weight-$\bar{\chi}$ subspace* $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} = \Gamma(G_\eta, \mathcal{L}_\eta)_{\overline{\chi+\phi(y)}}$, they become respectively

$$T^*_{c^\vee(y)} \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$$

and

$$\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta)_{\overline{\chi+\phi(y)}} \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}),$$

both of which are nonzero equivariant homomorphisms between irreducible representations of $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} = \mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\overline{\chi+\phi(y)}} \cong \mathcal{G}(\mathcal{M}_{\chi,\eta})$. Therefore they must be proportional if we can identity the two spaces $\Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$ and $\Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta})$ as K-vector spaces. Up to a nonzero multiple in K, we can choose the identification so that the two maps are *identical.* This is equivalent to saying that there exists a section $\psi(y)\tau(y,\chi) \in \mathcal{M}(c^\vee(y))^{\otimes -1}_\eta$ satisfying

$$\psi(y)\tau(y,\chi)\ T^*_{c^\vee(y)} \circ \sigma_\chi = \sigma_{\chi+\phi(y)},$$

which is just the desired (4.3.1.4). □

The proof for the assumption that (4.3.1.2) (i.e. $\sigma_\chi \neq 0$) is true for any $\chi \in X$ will be given in the next section using the so-called *addition formula for theta functions*.

4.3.3 Addition Formula

In this section, we introduce the addition formula of theta functions, and prove both (4.3.1.2) and (4.3.1.11). This will conclude the proof of Theorem 4.2.1.8.

Since the points in (4.3.1.2) and (4.3.1.11) are about inequalities and equalities, it suffices to base change from R to a complete discrete valuation ring using an adic injection, and show the inequalities and equalities over the complete discrete valuation ring. Note that complete discrete valuation rings are nice because they are *Nagata*. (See [90, 31.A]. See also the proof of Proposition 4.5.2.18 and Remark 4.5.2.19 below.) In this case, the normalizations of R in finite algebraic extensions of $K = \mathrm{Frac}(R)$ are again discrete valuation rings. Hence, for the purpose of proving (4.3.1.2) and (4.3.1.11), we may and will assume that R is a complete discrete valuation ring, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We will also perform further base changes $R \to R'$ to finite flat extensions of complete discrete valuation rings whenever necessary.

Remark 4.3.3.1. The assumption that $S_0 = \mathrm{Spec}(k)$ is the spectrum of a field is convenient for the following purpose. Later we will have to replace $\mathcal{L}$ by some other cubical invertible sheaf, and for $(G, \mathcal{L})$ to qualify as an object of $\mathrm{DEG}_{\mathrm{ample}}$ (defined as in Definition 4.2.1.1), we have to verify that $\mathcal{L}_{\mathrm{for}}$ lies in the essential image of (3.3.3.12). Let us claim that this is automatic. By Corollary 3.2.5.7, there is a finite étale extension of S_0 over which $\mathcal{L}_0 \cong \pi_0^*\mathcal{M}_0$

for some $\mathcal{M}_0$ on A_0. As explained in the proof of Corollary 3.3.3.4, we have accordingly a finite formally étale extension of S_{for} over which $\mathcal{L}_{\mathrm{for}} \cong \pi^*_{\mathrm{for}}\mathcal{M}_{\mathrm{for}}$ for some $\mathcal{M}_{\mathrm{for}}$ on A_{for}. By Theorems 2.3.1.4 and 2.3.1.2, there is a finite étale extension of S over which $\mathcal{M}_{\mathrm{for}}$ algebraizes to some $\mathcal{M}$ on A. Hence $\mathcal{L}^\natural := \pi^*\mathcal{M}$ satisfies $\mathcal{L}^\natural_{\mathrm{for}} \cong \mathcal{L}_{\mathrm{for}}$ and descends to S, which shows that $\mathcal{L}_{\mathrm{for}}$ lies in the essential image of (3.3.3.12), as desired.

Let us consider the isogeny $\Phi : G \underset{S}{\times} G \to G \underset{S}{\times} G$ defined by $(x, y) \mapsto (x + y, x - y)$ for functorial points x and y of G. Suppose $\mathcal{L}$ is a symmetric cubical invertible sheaf. That is, $[-1]^*\mathcal{L} \cong \mathcal{L}$ either as a $\mathbf{G}_{\mathrm{m}}$-torsor or cubical $\mathbf{G}_{\mathrm{m}}$-torsor (which are equivalent by Proposition 3.2.3.1). Then we have a canonical isomorphism

$$\Phi^*(\mathrm{pr}_1^*\,\mathcal{L} \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*\,\mathcal{L}) \xrightarrow{\sim} \mathrm{pr}_1^*\,\mathcal{L}^{\otimes 2} \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*\,\mathcal{L}^{\otimes 2},$$

where pr_1 and pr_2 are the two projections. More generally, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent cubical invertible sheaves, then we have a canonical isomorphism

$$\Phi^*(\mathrm{pr}_1^*\,\mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*\,\mathcal{L}_2) \xrightarrow{\sim} \mathrm{pr}_1^*(\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2) \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*(\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}_2).$$

Here the condition that $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent means $\mathscr{N} := \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2^{\otimes -1}$ as a $\mathbf{G}_{\mathrm{m}}$-torsor has the structure of a commutative group scheme. In other words, it means $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent over the generic fiber G_η in the usual sense of algebraic equivalence for invertible sheaves over an abelian variety. For the proof, note that by Proposition 3.3.2.2 it suffices to verify the statements over the generic fibers. If $\mathcal{L}_1 = \mathcal{L}_2$, then it is just the theorem of cube pulled back from $G_\eta \underset{\eta}{\times} G_\eta \underset{\eta}{\times} G_\eta$ to $G_\eta \underset{\eta}{\times} G_\eta$ via $(x, y) \mapsto (x, y, -y)$. So one is reduced to the case that $\mathcal{L}_1$ is trivial and that $\mathcal{L}_2$ is algebraically equivalent to the trivial invertible sheaf, which is again immediate.

Lemma 4.3.3.2 (see [37, Ch. II, Lem. 4.3])**.** *Let $f : H' \to H$ be an isogeny of semi-abelian schemes over R such that both H_η and H'_η are abelian schemes, and such that both the torus parts of $H_0 = H \underset{R}{\otimes} k$ and $H'_0 = H' \underset{R}{\otimes} k$ are split tori. Let $H^\natural$ (resp. $H'^\natural$) be the Raynaud extension of H, with torus part T*

(resp. T') and abelian part A (resp. A'). The isogeny f induces an isogeny between Raynaud extensions,

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r, "i'"] \arrow[d, "f_T"'] & H'^{\natural} \arrow[r, "\pi'"] \arrow[d, "f^{\natural}"'] & A' \arrow[r] \arrow[d, "f_A"'] & 0 , \\
0 \arrow[r] & T \arrow[r, "i"'] & H^{\natural} \arrow[r] & A \arrow[r, "\pi"'] & 0
\end{tikzcd}$$

where the morphism f_T can be described by duality by a morphism $f_T^* : X(T) \to X(T')$.

Let $\mathcal{F}$ be a cubical invertible sheaf on H, and let $\mathcal{F}' := f^*\mathcal{F}$. Choose a trivialization $s : i^*\mathcal{F}^{\natural} \cong \mathscr{O}_T$, which a cubical invertible sheaf $\mathcal{N}$ on A such that $\mathcal{F}^{\natural} \cong \pi^*\mathcal{N}$. Then the pullback $s' := f_T^*(s)$ is a trivialization $s' : (i')^*(\mathcal{F}')^{\natural} \cong \mathscr{O}_{T'}$, which determines a cubical invertible sheaf $\mathcal{N}'$ on A' such that $(\mathcal{F}')^{\natural} \cong (\pi')^*(\mathcal{N}')$. Let the invertible sheaves $\mathcal{N}_\chi := \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ (resp. $\mathcal{N}'_{\chi'} := \mathcal{N}' \underset{\mathscr{O}_{A'}}{\otimes} \mathscr{O}_{\chi'}$) be determined as usual for each $\chi \in \underline{X}(T)$ (resp. $\chi' \in \underline{X}(T')$) as the weight χ-subspace (resp. weight χ'-subspace) of $\pi_*\mathcal{F}^{\natural}$ (resp. $(\pi')_*(\mathcal{F}')^{\natural}$). Then they satisfy the natural compatibility $f_A^*\mathcal{N}_\chi \cong \mathcal{N}'_{f_T^*(\chi)}$. In particular, $\mathcal{N}' \cong f_A^*\mathcal{N}$.

For each section s of $\Gamma(H, \mathcal{F})$, we have a decomposition

$$s = \sum_{\chi \in X(T)} \sigma_\chi^{\mathcal{N}}(s),$$

where each $\sigma_\chi^{\mathcal{N}}(s)$ is an element in $\Gamma(A, \mathcal{N}_\chi)$. Similarly, for each section s' of $\Gamma(H', f^*\mathcal{F})$, we have a decomposition

$$s' = \sum_{\chi' \in X(T')} \sigma_{\chi'}^{\mathcal{N}'}(s'),$$

where each $\sigma_{\chi'}^{\mathcal{N}'}(s')$ is an element in $\Gamma(A', \mathcal{N}'_{\chi'})$. Then, with the compatible choices above, we have

$$f_A^*(\sigma_\chi^{\mathcal{N}}(s)) = \sigma_{f_T^*(\chi)}^{\mathcal{N}'}(f^*s),$$

where

$$f_A^* : \Gamma(A, \mathcal{N}_\chi) \to \Gamma(A', \mathcal{N}'_{f_T*(\chi)})$$

denotes the canonical morphism.

The proof of this lemma is straightforward.

Applying this lemma to the isogeny $\Phi : G \underset{S}{\times} G \to G \underset{S}{\times} G$ defined above, we have:

Proposition 4.3.3.3 (addition formula; see [37, Ch. II, p. 40])**.** *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be algebraically equivalent cubical invertible sheaves on G such that $\mathcal{L}_1^\natural \cong \pi^* \mathcal{M}_1$ and such that $\mathcal{L}_2^\natural \cong \pi^* \mathcal{M}_2$ for some cubical invertible sheaves on A, where $\pi : G^\natural \to A$ is the structural morphisms. Let $\mathcal{F} := \mathrm{pr}_1^* \mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2$ and $\mathcal{F}' := \Phi^* \mathcal{F}$. Let $\mathcal{N} := \mathrm{pr}_1^* \mathcal{M}_1 \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathcal{M}_2$ and let $\mathcal{N}' := \Phi_A^* \mathcal{N}$, where Φ_A is the isogeny induced by Φ on the abelian part, given similarly by $(x, y) \mapsto (x + y, x - y)$ for any functorial points x and y of A. Then we have $\mathcal{F} \cong (\pi \times \pi)^* \mathcal{N}$, $\mathcal{F}' \cong (\pi \times \pi)^* \mathcal{N}'$, and*

$$\Phi_A^* (\sigma_{(\chi,\mu)}^{\mathcal{N}} (\mathrm{pr}_1^* s_1 \otimes \mathrm{pr}_2^* s_2)) = \sigma_{(\chi+\mu,\chi-\mu)}^{\mathcal{N}'} (\Phi^* (\mathrm{pr}_1^* s_1 \otimes \mathrm{pr}_2^* s_2))$$

for any $s_1, s_2 \in \Gamma(G, \mathcal{L})$ and for any $\chi, \mu \in X$.

Since $\sigma_{(\chi,\mu)}^{\mathcal{N}} = \sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}$ and $\sigma_{(\chi+\mu,\chi-\mu)}^{\mathcal{N}'} = \sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2}$, we may rewrite the addition formula as

$$\Phi_A^* \circ (\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}) = (\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2}) \circ \Phi^*. \tag{4.3.3.4}$$

Here the domain and ranges of the maps can be described in the following diagram:

$$\begin{tikzcd}
\Gamma(G, \mathcal{L}_1) \underset{R}{\otimes} \Gamma(G, \mathcal{L}_2) \arrow[d, "\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}"'] \arrow[dr, bend left, "\Phi^*"] & \\
\Gamma(A, \mathcal{M}_{1,\chi}) \underset{R}{\otimes} \Gamma(A, \mathcal{M}_{2,\mu}) \arrow[dr, bend right, "\Phi_A^*"'] & \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2) \underset{R}{\otimes} \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^* \mathcal{L}_2) \arrow[d, "\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2}"] \\
& \Gamma(A, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2)_{\chi+\mu}) \underset{R}{\otimes} \Gamma(A, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}_2)_{\chi-\mu})
\end{tikzcd}$$

Now let us assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, and specialize to the generic fiber. By the arguments in Section 4.3.2, it suffices to show that (4.3.2.9) is nonzero, which is equivalent to being injective. Then the picture becomes:

$$\begin{tikzcd}
\Gamma(G_\eta, \mathcal{L}_{1,\eta})_{\bar{\chi}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{2,\eta})_{\bar{\mu}} \arrow[d, "\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}"'] \arrow[rdd, bend left, "\mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}} \circ \Phi^*"] & \\
\Gamma(A_\eta, \mathcal{M}_{1,\chi,\eta}) \underset{K}{\otimes} \Gamma(A_\eta, \mathcal{M}_{2,\mu,\eta}) \arrow[rdd, bend right, "\Phi_A^*"'] & \\
& \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{L}_{2,\eta})_{\overline{\chi+\mu}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_{2,\eta})_{\overline{\chi-\mu}} \arrow[d, "\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2}"] \\
& \Gamma(A_\eta, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2)_{\chi+\mu,\eta}) \underset{K}{\otimes} \Gamma(A_\eta, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_2)_{\chi-\mu,\eta})
\end{tikzcd}$$

Here the $\bar{\chi}$ and $\bar{\mu}$ on the left-hand side are classes in $X/\phi(Y)$, while the $\overline{\chi+\mu}$ and $\overline{\chi-\mu}$ on the right-hand side are classes in $X/2\phi(Y)$. We have to use 2ϕ instead of ϕ for the right-hand side, because $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent, and so that the polarization defined by $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2$ and $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}_2$ are twice of the one defined by $\mathcal{L}_1$ and $\mathcal{L}_2$. The map $\mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}}$ above is the projection to the $(\overline{\chi+\mu}, \overline{\chi-\mu})$-weight space.

Remark 4.3.3.5. The left-hand side of (4.3.3.4), when restricted to $\Gamma(G_\eta, \mathcal{L}_{1,\eta})_{\bar{\chi}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{2,\eta})_{\bar{\mu}}$, is nonzero if and only if $\sigma_\chi^{\mathcal{M}_1} \neq 0$ and $\sigma_\mu^{\mathcal{M}_2} \neq 0$, in which case both of them are injective by the representation theory of theta groups. For the same reason, the right-hand side of (4.3.3.4), when restricted to $\Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{L}_{2,\eta})_{\overline{\chi+\mu}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_{2,\eta})_{\overline{\chi-\mu}}$, is nonzero if and only if $\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, $\sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, and $\mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}} \circ \Phi^* \neq 0$.

Lemma 4.3.3.6. *Given any ample cubical invertible sheaf $\mathcal{L}_0$ on G, there exists a finite flat base extension $R \to R'$ of complete discrete valuation rings such that $\mathcal{L}_0$ is algebraically equivalent to a* **symmetric** *ample cubical invertible sheaf $\mathcal{L}$ over R'.*

Proof. It suffices to show that after replacing $K = \mathrm{Frac}(R)$ by a finite algebraic extension $K \to K'$ the generic fiber $\mathcal{L}_{0,\eta}$ of $\mathcal{L}_0$ is algebraically equivalent to some symmetric ample invertible sheaf $\mathcal{L}_\eta$. Then R' is simply the normalization of R in K', and $\mathcal{L}$ is the unique cubical invertible sheaf extending

$\mathcal{L}_\eta$ (by Proposition 3.2.3.1 and Theorem 3.3.2.3). Let $\lambda_{0,\eta}$ be the polarization on G_η induced by $\mathcal{L}_{0,\eta}$. By replacing K by a finite algebraic extension, we may assume that $\lambda_{0,\eta} = 2\lambda_{1,\eta}$ for some polarization $\lambda_{1,\eta}$ on G_η. Then $\mathcal{L}_\eta := (\mathrm{Id}_{G_\eta}, \lambda_{1,\eta})^*\mathcal{P}_{G_\eta}$ is a *symmetric* ample invertible sheaf on G_η inducing the same polarization $\lambda_{0,\eta}$. As a result, $\mathcal{L}_\eta$ and $\mathcal{L}_{0,\eta}$ are algebraically equivalent to each other, as desired. $\square$

Assume for the time being that the ample cubical invertible sheaf $\mathcal{L}$ is *symmetric*. Moreover, after base change to a finite étale extension if necessary, let us assume that $\mathcal{L} \cong \pi^*\mathcal{M}$ for some $\mathcal{M}$. (See Remark 4.3.3.1.)

Because $\Gamma(G, \mathcal{L})_{\bar{\chi}} \neq 0$ for all $\bar{\chi} \in X/\phi(Y)$, we know that for all $\bar{\chi} \in X/\phi(Y)$, there exists χ in $\bar{\chi}$ such that $\sigma_\chi^{\mathcal{M}} \neq 0$. In particular, $\sigma_{\phi(y_0)}^{\mathcal{M}} \neq 0$ for some $y_0 \in Y$. Then as in Remark 4.3.3.5, (4.3.3.4) tells us that $\mathrm{pr}_{\bar{0},\bar{0}} \circ \Phi^* \neq 0$, $\sigma_{\phi(2y_0)}^{\mathcal{M}^{\otimes 2}} \neq 0$ and $\sigma_0^{\mathcal{M}^{\otimes 2}} \neq 0$.

Lemma 4.3.3.7. *The subset* $Z := \{\chi \in \phi(Y) : \sigma_\chi^{\mathcal{M}} \neq 0\}$ *is a subgroup of* $\phi(Y)$.

Proof. We have seen that $0 \in Z$. If χ, $\mu \in Z$, then (4.3.3.4) gives $\sigma_{2\chi}^{\mathcal{M}^{\otimes 2}} \neq 0$ and $\sigma_{2\mu}^{\mathcal{M}^{\otimes 2}} \neq 0$. Applying (4.3.3.4) again, we get $\sigma_{\chi+\mu}^{\mathcal{M}} \neq 0$ and $\sigma_{\chi-\mu}^{\mathcal{L}} \neq 0$. Hence $\chi + \mu \in Z$ and $\chi - \mu \in Z$, as desired. $\square$

Lemma 4.3.3.8. *We have* $\sigma_\chi^{\mathcal{M}} \neq 0$ *for all* $\chi \in \phi(Y)$.

Proof. Suppose this is not true. By Lemma 4.3.3.7, with the subgroup Z defined there, there is a proper subgroup W of $\phi(Y)$ containing Z and of finite index in $\phi(Y)$. We must show that this leads to a contradiction.

Note that such a W corresponds to a nontrivial finite flat subgroup H of T containing $K(\mathcal{L})^\mu$. The trivialization of $i^*\mathcal{L}_{\mathrm{for}}$ induces a section of $\mathcal{G}(\mathcal{L})^\mu \to K(\mathcal{L})^\mu$, and hence an action of $K(\mathcal{L})^\mu$ on $\mathcal{L}$. Therefore $\mathcal{L}$ descends to a cubical ample invertible sheaf $\underline{\mathcal{L}}$ on the quotient semi-abelian scheme $\underline{G} := G/K(\mathcal{L})^\mu$. The existence of W means that any $K(\mathcal{L})^\mu$-invariant section $s \in \Gamma(G, \mathcal{L})$ has a Fourier expansion $s = \sum\limits_{\chi \in W} \sigma_\chi^{\mathcal{M}}(s)$. Passing to $\underline{G}$, we are reduced to the case that $\phi(Y) = X$. Translating $\mathcal{L}$ by elements of $G(R)$, we conclude that for any $\mathcal{L}'$ algebraically equivalent to $\mathcal{L}$ and any section s' of $\mathcal{L}'_\eta$, the Fourier expansion of s' involves only the entries for $\chi \in W$, provided that we have chosen a suitable trivialization of $i^*\mathcal{L}$.

Now we quote the following result of Mumford:

Proposition 4.3.3.9 (Mumford; recorded as [37, Ch. I, Prop. 5.3])**.** *Suppose B is an abelian variety over an algebraically closed field k, and suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two ample line bundles on B. Then $\Gamma(B, \mathcal{L}_1 \underset{\mathscr{O}_B}{\otimes} \mathcal{L}_2)$ is spanned by the image of*

$$\Gamma(B, \mathcal{L}_1 \underset{\mathscr{O}_B}{\otimes} \mathcal{N}) \underset{k}{\otimes} \Gamma(B, \mathcal{L}_2 \underset{\mathscr{O}_B}{\otimes} \mathcal{N}^{\otimes -1}) \to \Gamma(B, \mathcal{L}_1 \underset{\mathscr{O}_B}{\otimes} \mathcal{L}_2),$$

for $\mathcal{N}$ running over a Zariski dense subset of $B^\vee(k)$.

Proof. The space spanned by the images is stable under $\mathcal{G}(\mathcal{L}_1 \underset{\mathscr{O}_B}{\otimes} \mathcal{L}_2)$. ☐

Back to the proof of Lemma 4.3.3.8. Note that Proposition 4.3.3.9 implies that every element of $\Gamma(G, \mathcal{L}^{\otimes n})$ has a Fourier expansion involving only those $\chi \in W$. Since this statement can be verified after suitable base change $R \hookrightarrow R'$, we may assume that $\Gamma(G, \mathcal{L}_\eta^{\otimes n})$ is generated by elements of the form $T_{x_1}^*(s) \otimes \ldots \otimes T_{x_n}^*(s)$ for $n \geq 1$, where $x_1, \ldots, x_n \in G(R)$ satisfy $x_1 + \ldots + x_n = 0$. Then the implication is clear.

As a result, we see that elements of $\Gamma(G, \mathcal{L}^{\otimes n})$ factor through G/H. But this contradicts the fact that $\mathcal{L}$ is ample on G. Hence $Z = \phi(Y)$, as desired. ☐

Now we are ready to prove (4.3.1.2) and (4.3.1.11).

Proof of (4.3.1.2). Let $\mathcal{L}_1$ be any given ample cubical invertible sheaf on G such that $\mathcal{L}_1^\natural \cong \pi^* \mathcal{M}_1$ for some cubical invertible sheaf $\mathcal{M}_1$ on A. Then we know from Lemmas 4.3.3.6 and 4.3.3.8 (and also Remark 4.3.3.1) that, after performing a finite flat base extension if necessary, there exists an ample cubical invertible sheaf $\mathcal{L}_2$ on G that is algebraically equivalent to $\mathcal{L}_1$, such that $\mathcal{L}_2^\natural \cong \pi^* \mathcal{M}_2$ for some cubical invertible sheaf $\mathcal{M}_2$ on A, with the additional property that $\sigma_\mu^{\mathcal{M}_2} \neq 0$ for all $\mu \in \phi(Y)$. Given any $\chi \in X$, there exists ξ such that $\xi \equiv \chi \pmod{\phi(Y)}$ and such that $\sigma_\xi^{\mathcal{M}_1} \neq 0$, for the same reason as above. Then (4.3.3.4) implies $\mathrm{pr}_{\overline{\xi+\mu},\overline{\xi-\mu}} \circ \Phi^* \neq 0$, $\sigma_{\xi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, and $\sigma_{\xi-\mu}^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2} \neq 0$, for all $\mu \in \phi(Y)$. Writing $\chi = \xi + \mu$, then we have in particular $\mathrm{pr}_{\bar{\chi},\bar{\chi}} \circ \Phi^* \neq 0$, $\sigma_\chi^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, and $\sigma_\chi^{\mathcal{M}_1 \otimes [-1]^* \mathcal{M}_2} \neq 0$. By (4.3.3.4) again, we have $\sigma_\chi^{\mathcal{M}_1} \neq 0$, as desired. ☐

Proof of (4.3.1.11). For any $\chi, \mu \in X$ and $y \in Y$, we have by definition

$$\sigma_{\chi+\phi(y)}^{\mathcal{M}}(\,\cdot\,) = \psi(y)\tau(y,\chi)\; T_{c^\vee(y)}^* \circ \sigma_\chi^{\mathcal{M}}(\,\cdot\,)$$

and

$$\sigma^{\mathcal{M}}_{\chi+\mu+\phi(y)}(\,\cdot\,) = \psi(y)\tau(y,\chi+\mu)\; T^*_{c^\vee(y)} \circ \sigma^{\mathcal{M}}_{\chi+\mu}(\,\cdot\,).$$

Then (4.3.3.4) gives

$$\Phi^*_A \circ (\sigma^{\mathcal{M}}_{\chi+\phi(y)} \otimes \sigma^{\mathcal{M}}_{\mu+\phi(y)}) = (\sigma^{\mathcal{M}\otimes\mathcal{M}}_{\chi+\mu+2\phi(y)} \otimes \sigma^{\mathcal{M}\otimes[-1]^*\mathcal{M}}_{\chi-\mu}) \circ \Phi^*$$

and

$$\Phi^*_A \circ (\sigma^{\mathcal{M}}_{\chi} \otimes \sigma^{\mathcal{M}}_{\mu}) = (\sigma^{\mathcal{M}\otimes\mathcal{M}}_{\chi+\mu} \otimes \sigma^{\mathcal{M}\otimes[-1]^*\mathcal{M}}_{\chi-\mu}) \circ \Phi^*.$$

Let us also denote by $\tau^{\mathcal{L}}$ and $\tau^{\mathcal{L}^{\otimes 2}}$, and $\psi^{\mathcal{L}}$ and $\psi^{\mathcal{L}^{\otimes 2}}$, the different trivializations associated to respectively $\mathcal{L}$ and $\mathcal{L}^{\otimes 2}$. (They do not depend on the choices of respectively $\mathcal{M}$ and $\mathcal{M}^{\otimes 2}$, because different choices only shift the indices of $\sigma^{\mathcal{M}}_{\chi}$ and $\sigma^{\mathcal{M}^{\otimes 2}}_{\chi}$, as explained in Remark 4.3.1.6.) Since translation by $(\iota(y),\iota(y)) \in G \underset{S}{\times} G$ corresponds to translation by $(2\iota(y),0)$ under Φ, we can conclude that

$$\psi^{\mathcal{L}}(y)^2\tau^{\mathcal{L}}(y,\chi)\tau^{\mathcal{L}}(y,\mu) = \psi^{\mathcal{L}^{\otimes 2}}(2y)\tau^{\mathcal{L}^{\otimes 2}}(2y,\chi+\mu).$$

Similarly, by substituting $\chi+\mu$ for χ and 0 for μ, we have

$$\psi^{\mathcal{L}}(y)^2\tau^{\mathcal{L}}(y,\chi+\mu) = \psi^{\mathcal{L}^{\otimes 2}}(2y)\tau^{\mathcal{L}^{\otimes 2}}(2y,\chi+\mu).$$

Comparing these two equations, we obtain the multiplicativity

$$\tau^{\mathcal{L}}(y,\chi+\mu) = \tau^{\mathcal{L}}(y,\chi)\tau^{\mathcal{L}}(y,\mu)$$

of τ in the second variable, as desired. □

4.3.4 Dependence of τ on the Choice of $\mathcal{L}$

In this section our goal is to show that τ does not depend on $\mathcal{L}$, but only on the map $\lambda : G \to G^\vee$ induced by $\mathcal{L}$. More precisely, suppose we have two pairs $(G,\mathcal{L}_1)$ and $(G,\mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}$, with both $\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ ample on G_η, such that the induced polarizations satisfy $N_1\lambda_{1,\eta} = N_2\lambda_{2,\eta}$ for some integers $N_1, N_2 > 0$. Suppose the associated degeneration data in $\mathrm{DD}_{\mathrm{ample}}$ (using the constructions so far) are respectively $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}^\natural_1,\tau_1,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}^\natural_2,\tau_2,\psi_2)$. Then our goal is to show that $\tau_1 = \tau_2$.

The starting point is the following:

Lemma 4.3.4.1 (cf. [37, Ch. II, Lem. 6.1])**.** *Let* $(G, \mathcal{L}_1)$ *and* $(G, \mathcal{L}_2)$ *be two pairs as in Theorem 4.2.1.8, and let* $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau_1, \psi_1)$ *and* $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2)$ *be respectively the associated degeneration data. Let* $\mathcal{L} := \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2$*. Let* $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ *be the degeneration datum associated to* $(G, \mathcal{L})$*. Suppose that we know already either* $\tau_1 = \tau_2$*,* $\tau = \tau_1$*, or* $\tau = \tau_2$*. Then* $\phi = \phi_1 + \phi_2$*,* $\tau = \tau_1 = \tau_2$*, and* $\psi = \psi_1\psi_2$*.*

Proof. Let us assume that $\mathcal{L}_1^\natural \cong \pi^*\mathcal{M}_1$ and $\mathcal{L}_2^\natural \cong \pi^*\mathcal{M}_2$ for some invertible cubical sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ on A, where $\pi : G^\natural \to A$ is the structural morphism. Then we also have $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for $\mathcal{M} := \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2$. Suppose we have nonzero sections $s_1 \in \Gamma(G, \mathcal{L}_1)$ and $s_2 \in \Gamma(G, \mathcal{L}_2)$, and let $s := s_1 \otimes s_2 \in \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2)$. Then $\sigma_\chi^{\mathcal{M}}(s) = \sum\limits_{\chi_1+\chi_2=\chi} \sigma_{\chi_1}^{\mathcal{M}_1}(s_1) \otimes \sigma_{\chi_2}^{\mathcal{M}_2}(s_2)$. Note that we have necessarily $\phi = \phi_1 + \phi_2$ in this case. If we replace χ by $\chi + \phi(y)$, if we replace χ_1 by $\chi_1 + \phi_1(y)$ and χ_2 by $\chi_2 + \phi_2(y)$ in the sum, and if we compare the maps over the generic fiber over η using (4.3.1.4), then we see that we are multiplying the summand corresponding to $\chi_1 + \chi_2 = \chi$ by

$$\psi_1(y)\tau_1(y, \chi_1)\psi_2(y)\tau_2(y, \chi_2) = \psi(y)\tau(y, \chi).$$

Note that the same is true if we have any χ_1' and χ_2' such that we have $\sigma_{\chi_1'}^{\mathcal{M}_1}(s_1) \neq 0$, such that $\sigma_{\chi_2'}^{\mathcal{M}_2}(s_2) \neq 0$, and such that $\sigma_{\chi'}^{\mathcal{M}}(s) \neq 0$ for $\chi_1' + \chi_2' = \chi'$. If $\tau_1 = \tau_2$ (resp. $\tau = \tau_1$, resp. $\tau = \tau_2$), then we have $\tau_1(y, \chi') = \tau_2(y, \chi') = \tau(y, \chi')$ (resp. $\tau_2(y, \chi_2') = \tau(y, \chi_2')$, resp. $\tau_1(y, \chi_1') = \tau(y, \chi_1')$). In particular, we see that $\tau_1(y, \cdot\,) = \tau_2(y, \cdot\,) = \tau(y, \cdot\,)$ over the subset of X consisting of those differences $\chi - \chi'$ between χ and χ' such that $\sigma_\chi^{\mathcal{M}}(s) \neq 0$ and $\sigma_{\chi'}^{\mathcal{M}}(s) \neq 0$ (resp. those χ_2 and χ_2' such that $\sigma_{\chi_2}^{\mathcal{M}_2}(s) \neq 0$ and $\sigma_{\chi_2'}^{\mathcal{M}_2}(s) \neq 0$, resp. those χ_1 and χ_1' such that $\sigma_{\chi_1}^{\mathcal{M}_1}(s) \neq 0$ and $\sigma_{\chi_1'}^{\mathcal{M}_1}(s) \neq 0$). The question is whether this subset of X is the whole of X.

Note that, as mentioned in Remark 4.3.1.7, for $i = 1, 2$, the two trivializations τ_i and ψ_i remain unchanged if we replace $\mathcal{L}_i$ and $\mathcal{M}_i$ by respectively $\mathcal{L}_i' := T_{g_i}^*\mathcal{L}_i$ and $\mathcal{M}_i' := T_{\pi(g_i)}^*\mathcal{M}_i$, where $\pi(g_i)$ is defined by algebraizing $\pi_{\mathrm{for}}(g_{i,for})$ by abuse of notations as in Remark 4.3.1.7. Let us consider the set of sections (g_1, g_2) of $G \underset{S}{\times} G$ such that $\lambda_1(g_1) + \lambda_2(g_2) = 0$. Then we have $\mathcal{L}_1' \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2' = T_{g_1}^*\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} T_{g_2}^*\mathcal{L}_2 \cong \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2 \cong \mathcal{L}$ for any such sections (g_1, g_2). Note that such pairs (g_1, g_2) is Zariski-dense in the identity component of the kernel of $\lambda_1 + \lambda_2 : G \underset{S}{\times} G \to G^\vee$. On the hand, since $\Gamma(G_\eta, \mathcal{L}_\eta)$ is spanned by the images of $\Gamma(G_\eta, T^*_{g_1}\mathcal{L}_{1,\eta}) \underset{K}{\otimes} \Gamma(G_\eta, T^*_{g_2}\mathcal{L}_{2,\eta})$ for such sections in the identity component of the kernel by Proposition 4.3.3.9, we see that there is always some (g_1, g_2) such that $\sigma^{\mathcal{M}}_\chi(s) \neq 0$ (resp. $\sigma^{\mathcal{M}_2}_{\chi_2}(s) \neq 0$, resp. $\sigma^{\mathcal{M}_1}_{\chi_1}(s) \neq 0$) for each particular χ (resp. χ_2, resp. χ_1) in X. This shows that $\tau(y, -) = \tau_1(y, -) = \tau_2(y, -)$ over all of X, and proves the lemma. □

Corollary 4.3.4.2. *Let $(G, \mathcal{L})$ be a pair as in Theorem 4.2.1.8, with associated degeneration datum $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$. Then the degeneration datum associated to $(G, \mathcal{L}^{\otimes n})$ is simply $(A, \underline{X}, \underline{Y}, n\phi, c, c^\vee, (\mathcal{L}^\natural)^{\otimes n}, \tau, n\psi)$. In particular, they have the same τ.*

Lemma 4.3.4.3. *Let $(G, \mathcal{L}_1)$ and $(G, \mathcal{L}_2)$ be two pairs as in Theorem 4.2.1.8, such that $\mathcal{L}_1 = f^*\mathcal{L}_2$ for some $f : G \to G$ whose restriction $f_\eta : G_\eta \to G_\eta$ to η is an isogeny. Let $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}^\natural_1, \tau_1, \psi_1)$ and $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}^\natural_2, \tau_2, \psi_2)$ be respectively the associated degeneration data. Let $f_Y : \underline{Y} \to \underline{Y}$ and $f^\natural : G^\natural \to G^\natural$ be the maps induced by $f : G \to G$, and let $\iota_1 : \underline{Y}_\eta \to G^\natural_\eta$ and $\iota_2 : \underline{Y}_\eta \to G^\natural_\eta$ be the maps corresponding to respectively τ_1 and τ_2. Then ι_1 and ι_2 are related by $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$.*

In particular, if $f = [-1] : G \to G$, then $\iota_1 = \iota_2$ as maps from $\underline{Y}_\eta$ to $G^\natural_\eta$, or equivalently $\tau_1 = \tau_2$ as trivializations of $(c^\vee \times c)^*\mathcal{P}^{\otimes -1}_{A,\eta}$.

Proof. By étale descent if necessary, we may assume that $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and that $\mathcal{L}^\natural_2 \cong \pi^*\mathcal{M}_2$ for some cubical invertible sheaf $\mathcal{M}_2$ on A, where $\pi : G^\natural \to A$ is the structural morphism. Let $\mathcal{M}_1 := f^*_A\mathcal{M}_2$, so that naturally $\mathcal{L}^\natural_1 \cong \pi^*\mathcal{M}_1$. Let us denote by $f_T : T \twoheadrightarrow T$ and $f_A : A \twoheadrightarrow A$ the isogenies induced by $f^\natural$, and let $f_{Y^\vee} : T^\vee \to T^\vee$ be the isogeny induced by $f^{\vee,\natural}$. Let $f_X : X \hookrightarrow X$ and $f_Y : Y \hookrightarrow Y$ be the maps corresponding respectively to f_T and $f_{T^\vee}$. Consider the morphism

$$f^* : \Gamma(G, \mathcal{L}_2) \to \Gamma(G, f^*\mathcal{L}_2) \cong \Gamma(G, \mathcal{L}_1)$$

given by pulling back the sections by f, and consider the corresponding weight spaces

$$f^*_A : \Gamma(A, \mathcal{M}_{2,\chi}) \to \Gamma(A, f^*_A(\mathcal{M}_{2,\chi})) \cong \Gamma(A, \mathcal{M}_{1,f_X(\chi)})$$

for any $\chi \in X$, where the last isomorphism follows from $f_{A^\vee}c = cf_X$. Note that we have $\phi_1 = f_X\phi_2 f_Y$, coming from the relation $\lambda_1 = f^\vee\lambda_2 f$, where λ_1

and λ_2 are defined respectively by $\mathcal{L}_1$ and $\mathcal{L}_2$. The comparison between $\sigma_\chi^{\mathcal{M}_2}$ and $\sigma_{\chi+\phi_2(f_Y(y))}^{\mathcal{M}_2}$ over η gives the relation

$$\psi_2(y)\tau_2(f_Y(y),\chi)\; T^*_{c^\vee(f_Y(y))} \circ \sigma_\chi^{\mathcal{M}_2} = \sigma_{\chi+\phi_2(f_Y(y))}^{\mathcal{M}_2},$$

while the comparison between $\sigma_{f_X(\chi)}^{\mathcal{M}_1}$ and $\sigma_{f_X(\chi)+\phi_1(y)}^{\mathcal{M}_1}$ gives the relation

$$\psi_1(y)\tau_1(y,f_X(\chi))\; T^*_{c^\vee(y)} \circ \sigma_{f_X(\chi)}^{\mathcal{M}_1} = \sigma_{f_X(\chi)+\phi_2(y)}^{\mathcal{M}_1}.$$

Since $f_X(\phi_2(f_Y(y))) = \phi_1(y)$, the two relations match when we replace y by $f_Y(y)$ in the first one, and replace χ by $f_X(\chi)$ in the second one. This shows that we have the natural functorial relation

$$\tau_1(y,f_X(\chi)) = f_A^*\tau_2(f_Y(y),\chi).$$

This is exactly what it means by $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$. □

Lemma 4.3.4.4. *Let $(G,\mathcal{L})$ be a pair as in Theorem 4.2.1.8. Then $(\mathrm{Id}_G,\lambda)^*\mathcal{P}_G \cong \mathcal{L} \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}$, where $\mathcal{P}_G$ is the unique $\mathbf{G}_{\mathrm{m}}$-biextension of $G \underset{S}{\times} G$ (given by Theorem 3.4.3.1) extending the Poincaré $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{P}_\eta$ of $G_\eta \underset{\eta}{\times} G_\eta$, and where $\lambda : G \to G^\vee$ is the unique morphism extending the polarization $G_\eta \to G_\eta^\vee$ induced by $\mathcal{L}_\eta$.*

Proof. The statement is standard on the fiber over η. The uniqueness of the extension of cubical structures forces $(\mathrm{Id}_G,\lambda)^*\mathcal{P}_G$ to agree with $\mathcal{L} \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}$ over S. □

Corollary 4.3.4.5. *Let $(G,\mathcal{L})$ be a pair as in Theorem 4.2.1.8. Then, in the degeneration datum $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)$ associated to $(G,\mathcal{L})$, the trivialization τ depends only on $(\mathrm{Id}_G,\lambda)^*\mathcal{P}_G = \mathcal{L} \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}$, and hence only on the λ induced by $\mathcal{L}$.*

Proof. Combine the above three lemmas. □

Proposition 4.3.4.6. *Let $(G,\mathcal{L}_1)$ and $(G,\mathcal{L}_2)$ be two pairs as in Theorem 4.2.1.8, and let $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau_1,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau_2,\psi_2)$ be respectively the associated degeneration data. Then $\tau_1 = \tau_2$ if the there are positive integers N_1 and N_2 such that $N_1\lambda_1 = N_2\lambda_2$, where $\lambda_1,\lambda_2 : G \to G$ are the maps associated to respectively $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Proof. This is just a combination of Corollaries 4.3.4.2 and 4.3.4.5. □

The proof of Theorem 4.2.1.8 is now complete.

4.4 Equivalences of Categories

Our goal in this section is to define, following [37, Ch. III], the categories $\mathrm{DEG}_{\mathrm{pol}}$, $\mathrm{DEG}_{\mathrm{IS}}$, DEG, $\mathrm{DD}_{\mathrm{pol}}$, $\mathrm{DD}_{\mathrm{IS}}$, and DD, and state the main Theorem 4.4.18 that sets up the following equivalence of categories using the so-called Mumford's construction:

$$\mathrm{DEG}_{\mathrm{ample}} \approx \mathrm{DD}_{\mathrm{ample}},$$
$$\mathrm{DEG}_{\mathrm{pol}} \approx \mathrm{DD}_{\mathrm{pol}},$$

including in particular quasi-inverses to the functor $\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$ in Theorem 4.2.1.8 defined by Fourier expansions of theta functions (in Section 4.3) and to the functor $\mathrm{F}_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}} \to \mathrm{DD}_{\mathrm{pol}}$ to be defined below (in Definition 4.4.9, following Corollary 4.3.4.5). We have chosen to omit the equivalences

$$\mathrm{DEG}_{\mathrm{IS}} \approx \mathrm{DD}_{\mathrm{IS}}$$
$$\mathrm{DEG} \approx \mathrm{DD}$$

claimed by [37, Ch. III, Cor. 7.2], as we cannot justify their main theorem [37, Ch. III, Thm. 7.1] that is necessary for showing even the existence of a well-defined functor

$$\mathrm{F} : \mathrm{DEG} \to \mathrm{DD}$$

coming from

$$\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$$

by forgetting the invertible sheaf $\mathcal{L}$ in each pair $(G, \mathcal{L})$ defining an object in DEG over S. (See Remark 4.2.1.10 and Section 4.3.4.) Nevertheless, we will still construct functors in the reversed direction for each of the categories $\mathrm{DD}_{\mathrm{ample}}$, $\mathrm{DD}_{\mathrm{pol}}$, $\mathrm{DD}_{\mathrm{IS}}$, and DD in Section 4.5.4, as elaborated in [37, Ch. III, §6 and §7].

First, let us give the definitions of the categories DEG, $\mathrm{DEG}_{\mathrm{pol}}$, and $\mathrm{DEG}_{\mathrm{IS}}$ as follows:

Definition 4.4.1. *Assumptions as in Section 4.1, the category* DEG *has objects G with same conditions as in Definition 4.2.1.1.*

Note that any G in DEG has an invertible sheaf $\mathcal{L}$ with $\mathcal{L}_\eta$ ample over G_η because the base scheme S is normal. (See [110, XI, 1.13]; cf. also Remark 3.3.3.10.)

Definition 4.4.2. *Assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{pol}}$ *has objects of the form* (G, λ_η)*, where* G *is an object in* DEG*, and where* $\lambda_\eta : G_\eta \to G_\eta^\vee$ *is a polarization of* G_η*.*

Remark 4.4.3. Since the base scheme is noetherian normal as assumed in Section 4.1, the polarization $\lambda_\eta : G_\eta \to G_\eta^\vee$ extends to a morphism $\lambda : G \to G^\vee$. Therefore it is unambiguous to write objects of $\mathrm{DEG}_{\mathrm{pol}}$ as (G, λ).

Definition 4.4.4. *Assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{IS}}$ *has objects of the form* $(G, \mathcal{F})$*, where* G *is an object in* DEG*, and where* $\mathcal{F}$ *is an invertible sheaf on* G *rigidified along the identity section* (*and hence endowed with a unique cubical structure by Proposition 3.2.3.1*)*.*

Next, let us give the definitions of the categories DD, $\mathrm{DD}_{\mathrm{pol}}$, and $\mathrm{DD}_{\mathrm{IS}}$ as follows:

Remark 4.4.5. The polarization $\lambda_A : A \to A^\vee$ is induced from $\mathcal{L}^\natural$ in this setting as described in Theorem 4.2.1.8.

Definition 4.4.6. *Assumptions as in Section 4.1, the category* $\mathrm{DD}_{\mathrm{pol}}$ *has objects of the form* $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$*, with entries described* (*together with the positivity condition for* τ) *as in Definition 4.2.1.7.*

Remark 4.4.7. In Definition 4.2.1.7 the symmetry property of τ could be omitted in the statements because it is rather a byproduct of the existence of ψ and the compatibility between τ, ϕ, and ψ. But now we need to emphasize the symmetric property because τ is appearing alone.

Remark 4.4.8. Any object $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ of $\mathrm{DD}_{\mathrm{pol}}$ can be canonically extended to an object of $\mathrm{DD}_{\mathrm{ample}}$ as follows: Let $\mathcal{M} := (\mathrm{Id}, \lambda_A)^* \mathcal{P}_A$. Let $\mathcal{L}^\natural := \pi^* \mathcal{M}$, where $\pi : G^\natural \to A$ is the structural morphism. Then $\mathcal{L}^\natural$ admits a canonical $\underline{Y}$-action over η given by

$$\begin{aligned}\psi := (\mathrm{Id}_Y, \phi)^* \tau :\ \mathbf{1}_{\underline{Y},\eta} &\xrightarrow{\sim} (c^\vee, c\phi)^* \mathcal{P}_{A,\eta}^{\otimes -1} \cong (c^\vee, \lambda_A c^\vee)^* \mathcal{P}_{A,\eta}^{\otimes -1} \\ &\cong (c^\vee)^* \mathcal{M}_\eta^{\otimes -1} \cong \iota^* (\mathcal{L}_\eta^\natural)^{\otimes -1}.\end{aligned}$$

Then $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ defines an object of $\mathrm{DD}_{\mathrm{ample}}$. However this object induces *twice* the original polarization.

Definition 4.4.9. *Following Corollary 4.3.4.5, the functor* $\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$ *induces a functor from* $\mathrm{DEG}_{\mathrm{pol}}$ *to* $\mathrm{DD}_{\mathrm{pol}}$*, which we denote by the association*

$$\begin{aligned}\mathrm{F}_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}} &\to \mathrm{DD}_{\mathrm{pol}} \\ (G, \lambda) &\mapsto (A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau),\end{aligned}$$

where A, λ_A, $\underline{X}$, $\underline{Y}$, ϕ, c, and $c^\vee$ are defined canonically by (G,λ), and where τ is defined by any $(G,\mathcal{L})$ in $\mathrm{DEG}_{\mathrm{ample}}$ *such that $\mathcal{L}$ induces a multiple of $\lambda : G \to G^\vee$.*

Remark 4.4.10. For the definition of τ, we can always take $\mathcal{L}$ to be $(\mathrm{Id},\lambda)^*\mathcal{P}$, where $\mathcal{P}$ is as in Theorem 3.4.3.1 (cf. Remark 4.4.8).

Definition 4.4.11. *Assumptions as in Section 4.1, the category* DD *has objects of the form $(G^\natural, \iota : \underline{Y}_\eta \to G^\natural_\eta)$, or equivalently of the form $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$, with entries described as in Definition 4.2.1.7, which can be extended to objects of* $\mathrm{DD}_{\mathrm{pol}}$*, or equivalently, to* $\mathrm{DD}_{\mathrm{ample}}$.

Remark 4.4.12. The positivity condition of τ cannot be stated for a tuple $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$ without having some compatible choices of λ_A and ϕ. Indeed, it does not make sense to consider the tuples in DD without considering its extendability to an object in $\mathrm{DD}_{\mathrm{pol}}$ or $\mathrm{DD}_{\mathrm{ample}}$.

Remark 4.4.13. Similarly, the symmetry condition of τ cannot be stated for a tuple $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$ without having some compatible choice of λ_A and ϕ.

Definition 4.4.14. *Assumptions as in Section 4.1, the category* $\mathrm{DD}_{\mathrm{IS}}$ *has objects of the form $(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)$, with entries A, $\underline{X}$, $\underline{Y}$, c, $c^\vee$, and τ described as in Definition 4.2.1.7, and with the remaining entries explained as follows:*

1. *$\mathcal{F}^\natural$ is a cubical invertible sheaf on $G^\natural$ defined by $c : \underline{X} \to A^\vee$, defining a $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{D}_2(\mathcal{F}^\natural)$ of $G^\natural \underset{S}{\times} G^\natural$, which (by Corollary 3.2.5.2) descends uniquely to a $\mathbf{G}_{\mathrm{m}}$-biextension of $A \underset{S}{\times} A$ that (as an invertible sheaf on $A \underset{S}{\times} A$) induces a homomorphism $\lambda_A : A \to A^\vee$ of abelian schemes over S (by the universal property of $A^\vee$, as in Construction 1.3.2.10).*
2. *$f_Y : \underline{Y} \to \underline{X}$ is a homomorphism such that $f_A c^\vee = c f_Y$. In other words, f_A and f_Y induces a homomorphism $f^\natural : G^\natural \to G^{\vee,\natural}$.*
3. *An action of $\underline{Y}_\eta$ on $\mathcal{F}^\natural_\eta$ over ι compatible with f_Y. This action determines and is determined by a trivialization*

$$\zeta : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*\mathcal{F}^{\otimes -1}_\eta$$

compatible with

$$\tau \circ (\mathrm{Id}_{\underline{Y}} \times f_Y) : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y}, \eta} \xrightarrow{\sim} (c^\vee \times c f_Y)^* \mathcal{P}_{A,\eta}^{\otimes -1},$$

in the sense that

$$\mathcal{D}_2(\zeta) = \tau \circ (\mathrm{Id}_{\underline{Y}} \times f_Y).$$

This compatibility makes sense because the biextension $\mathcal{D}_2(\mathcal{F}^\natural)$ *on* $G^\natural \underset{S}{\times} G^\natural$ *uniquely descends to a biextension of* $A \underset{S}{\times} A$*, which by 1 is just* $(\mathrm{Id}_A \times f_A)^* \mathcal{P}_A$*.*

Each 9-tuple is required to satisfy two further conditions: First, $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ *has to define an object in* DD*. Second,* ζ *has to satisfy the following* **finiteness condition***: After passing to a finite étale surjection over* S *that makes* $\underline{Y}$ *constant, there is an integer* $n \geq 1$ *such that* $I_y \cdot I_{y,\phi(y)}^{\otimes n} \subset R$ *(where* I_y *is the ideal defined by* ζ*) for any* y *in* Y*.*

Note that $\mathrm{DD}_{\mathrm{IS}}$ admits a structure of a (strictly commutative) Picard category over DD, defined by the following tensor operation:

Definition 4.4.15. *The tensor product of two tuples*

$$(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$$

and

$$(A, \underline{X}, \underline{Y}, f'_Y, c, c^\vee, \mathcal{F}^{\natural\prime}, \tau, \zeta')$$

in $\mathrm{DD}_{\mathrm{IS}}$*, denoted as*

$$(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta) \otimes (A, \underline{X}, \underline{Y}, f'_Y, c, c^\vee, \mathcal{F}^{\natural\prime}, \tau, \zeta'),$$

is defined to be the tuple

$$(A, \underline{X}, \underline{Y}, f_Y + f'_Y, c, c^\vee, \mathcal{F}^\natural \underset{\mathscr{O}_{G^\natural}}{\otimes} \mathcal{F}^{\natural\prime}, \tau, \zeta + \zeta').$$

The tensor inverse of $(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$*, denoted as*

$$(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)^{\otimes -1},$$

is defined to be

$$(A, \underline{X}, \underline{Y}, -f_Y, c, c^\vee, (\mathcal{F}^\natural)^{\otimes -1}, \tau, \zeta^{-1}).$$

These correspond to the natural tensor operations $(G, \mathcal{F}) \otimes (G, \mathcal{F}') = (G, \mathcal{F} \underset{\mathscr{O}_G}{\otimes} \mathcal{F}')$ and $(G, \mathcal{F})^{\otimes -1} = (G, \mathcal{F}^{\otimes -1})$ of $\mathrm{DEG}_{\mathrm{IS}}$ over DEG (also as a (strictly commutative) Picard category).

Since $\mathrm{DD}_{\mathrm{ample}}$ is naturally a full subcategory of $\mathrm{DD}_{\mathrm{IS}}$, we can talk about tensor produces of objects in $\mathrm{DD}_{\mathrm{ample}}$, which are naturally objects in $\mathrm{DD}_{\mathrm{IS}}$.

Lemma 4.4.16. *For any object $(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$, there exist tuples $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau, \psi_1)$ and $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau, \psi_2)$ in $\mathrm{DD}_{\mathrm{ample}}$ such that*

$$\begin{aligned}(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)& \\ = (A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau, \psi_1) &\otimes (A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau, \psi_2)^{\otimes -1}.\end{aligned} \tag{4.4.17}$$

In other words, we have $f_Y = \phi_1 - \phi_2$, $\mathcal{F}^\natural = \mathcal{L}_1^\natural \underset{\mathscr{O}_{G^\natural}}{\otimes} (\mathcal{L}_2^\natural)^{\otimes -1}$, and $\zeta = \psi_1 \psi_2^{-1}$.

Proof. By Definition 4.4.14 of $\mathrm{DD}_{\mathrm{IS}}$, the tuple $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ has to define an object in DD. By Definition 4.4.11 of DD, this means there is an object $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ in $\mathrm{DD}_{\mathrm{ample}}$ that extends $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$.

Let us translate the statements preceding Definition 4.2.1.7 almost verbatim to this context: Let $i : T \to G^\natural$ and $\pi : G^\natural \to A$ denote the canonical morphisms. By the normality assumption on S and by Corollary 3.2.5.7, there is a base change to a finite étale surjection over S over which the étale sheaf $\underline{X}$ is constant and the cubical invertible sheaves $i^*\mathcal{F}^\natural$ (resp. $i^*\mathcal{L}^\natural$) is trivial. In this case, any cubical trivialization $s_{\mathcal{F}^\natural} : i^*\mathcal{F}^\natural \xrightarrow{\sim} \mathscr{O}_T$ (resp. $s : i^*\mathcal{L}^\natural \xrightarrow{\sim} \mathscr{O}_T$) determines a cubical invertible sheaf $\mathcal{N}$ (resp. a relatively ample cubical invertible sheaf $\mathcal{M}$) on A and a cubical isomorphism $\mathcal{F}^\natural \cong \pi^*\mathcal{N}$ (resp. $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$), both depending uniquely on the choice of $s_{\mathcal{F}^\natural}$ (resp. s).

Now take a sufficiently large integer $N > 0$ such that $\mathcal{M}_1 := \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes N}$ and $\mathcal{M}_2 := \mathcal{M}^{\otimes N}$ are both relatively ample. Let $\mathcal{L}_1^\natural := \pi^*\mathcal{M}_1$ and $\mathcal{L}_2^\natural := \pi^*\mathcal{M}_2$. Note that $\mathcal{N} \cong \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2^{\otimes -1}$, and hence $\mathcal{F}^\natural \cong \mathcal{L}_1^\natural \underset{\mathscr{O}_A}{\otimes} \mathcal{L}_2^\natural$ by construction. Let $\phi_2 = N\phi$, and let $\psi_2 = \psi^N$. Note that $\phi_2 : Y \hookrightarrow X$ is injective because ϕ is, and ψ_2 satisfies the positivity and compatibility conditions because ψ does. Set $\phi_1 = f_Y + \phi_2$. By increasing N if necessary, we may assume that $\phi_1 : Y \hookrightarrow X$ is still injective. Similarly, set $\psi_1 = \zeta \cdot \psi_2$. By the finiteness condition for ζ as stated in Definition 4.4.14, we may assume that ψ_1 satisfies the positivity and compatibility conditions as ψ_2 does if N

is sufficiently large. As a result, by étale descent if necessary, we obtain two valid tuples

$$(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau, \psi_1)$$

and

$$(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau, \psi_2)$$

in $\mathrm{DD}_{\mathrm{ample}}$ over the original base scheme that realizes the relation (4.4.17), as desired. □

Theorem 4.4.18. *Assumptions as in Section 4.1, there are functors*

$$\begin{aligned}
&\mathrm{M}_{\mathrm{ample}} : \mathrm{DD}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}} : (A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi) \mapsto (G, \mathcal{L})\\
&\mathrm{M}_{\mathrm{pol}} : \mathrm{DD}_{\mathrm{pol}} \to \mathrm{DEG}_{\mathrm{pol}} : (A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \mapsto (G, \lambda_\eta)\\
&\mathrm{M}_{\mathrm{IS}} : \mathrm{DD}_{\mathrm{IS}} \to \mathrm{DEG}_{\mathrm{IS}} : (A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta) \mapsto (G, \mathcal{F})\\
&\mathrm{M} : \mathrm{DD} \to \mathrm{DEG} : (G^\natural, \iota : \underline{Y}_\eta \to G^\natural_\eta) \text{ or } (A, \underline{X}, \underline{Y}, c, c^\vee, \tau) \mapsto G
\end{aligned}$$

given rise from generalizations of Mumford's construction in [100], such that $\mathrm{M}_{\mathrm{ample}}$ *and* $\mathrm{M}_{\mathrm{pol}}$ *are equivalences of categories. The quasi-inverse of* $\mathrm{M}_{\mathrm{ample}}$ *and* $\mathrm{M}_{\mathrm{pol}}$ *are given respectively by* $\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$ *and* $\mathrm{F}_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}} \to \mathrm{DD}_{\mathrm{pol}}$ *defined above in Theorem 4.2.1.8 and Definition 4.4.9.*

The proof will be given in Section 4.5, in particular Section 4.5.4, following Faltings and Chai's generalization of Mumford's constructions.

4.5 Mumford's Construction

Strictly speaking, Mumford's construction is only explicitly given as a map from a certain subcollection of the *split objects* in $\mathrm{Ob}(\mathrm{DD}_{\mathrm{ample}})$ to $\mathrm{Ob}(\mathrm{DEG}_{\mathrm{ample}})$. (The precise description of such split objects will be described in Section 4.5.1.) In [37, Ch. III], Faltings and Chai show that this is sufficient for defining a functor from $\mathrm{DD}_{\mathrm{IS}}$ to $\mathrm{DEG}_{\mathrm{IS}}$ that induces quasi-inverses for $\mathrm{F}_{\mathrm{ample}}$ and $\mathrm{F}_{\mathrm{pol}}$. We will explain their arguments in Section 4.5.4.

4.5.1 Relatively Complete Models

Definition 4.5.1.1. *Assumptions as in Section 4.1, the category* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ *has objects consisting of 9-tuples* $(A, \mathcal{M}, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, \psi)$*, together with* *the positivity conditions for ψ and τ, as described in Definition 4.2.1.7, with $\underline{X}$ and $\underline{Y}$ constant sheaves with values respectively X and Y, and with $\mathcal{M}$ a relatively ample invertible sheaf on A. There is a natural map* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$ *defined by sending the tuple* $(A, \mathcal{M}, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, \psi)$ *to* $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural := \pi^*\mathcal{M}, \tau, \psi)$, *where* $\pi : G^\natural \to A$ *is the structural morphism.*

Then $\psi : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*(\mathcal{L}^\natural_\eta)^{\otimes -1}$ can be viewed as a trivialization of the cubical invertible sheaf $(c^\vee)^*\mathcal{M}_\eta^{\otimes -1}$ over the constant group Y_η, which extends to a section of $(c^\vee)^*\mathcal{M}^{\otimes -1}$ over Y_S and is congruent to zero modulo I (cf. Definition 4.2.1.5). Note that $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ is not embedded as a subcategory of $\mathrm{DD}_{\mathrm{ample}}$ because there is no unique choice of $\mathcal{M}$ satisfying $\mathcal{L}^\natural = \pi^*\mathcal{M}$. Indeed, the possible choices form a torsor under X (by Lemma 3.2.2.12, Remark 3.2.2.11, and Proposition 3.2.5.4).

Our goal is to start with an object $(A, \mathcal{M}, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ and construct an object $(G, \mathcal{L})$ in $\mathrm{DEG}_{\mathrm{ample}}$, which gives the tuple $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \phi, \mathcal{L}^\natural = \pi^*\mathcal{M}, \tau, \psi)$ via Theorem 4.2.1.8. One of Mumford's insight is to axiomatize the desirable conditions of partial compactifications of $(G^\natural, \mathcal{L}^\natural)$ on which Y acts, and define such partial compactifications as the so-called *relatively complete models.*

Definition 4.5.1.2. *A* **relatively complete model** *of an object* $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ *in* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ *as described above consists of the following data:*

(1) An integral scheme $P^\natural$ locally of finite type over A, (which is then equivalently locally of finite type over S,) containing $G^\natural$ as an open dense subscheme. We denote again by $\pi : P^\natural \to A$ the structural morphism.

(2) An invertible sheaf on $P^\natural$ extending the invertible sheaf $\pi^*\mathcal{M}$ on $G^\natural \subset P^\natural$, which we denote again by $\mathcal{L}^\natural$ by abuse of notation.

(3) An action of $G^\natural$ on $P^\natural$ extending the translation action of $G^\natural$ on itself. We shall denote this action by $T_g : P^\natural \to P^\natural$ for any functorial point g of $G^\natural$.

(4) An action of Y on $(P^\natural, \mathcal{L}^\natural)$ extending the action of Y on $(G^\natural_\eta, \pi^*\mathcal{M}_\eta)$. Note that since $G^\natural$ is embedded in $P^\natural$ as an open dense subscheme, which contains in particular the identity section of $G^\natural$, we obtain an extension

of $\iota : Y_\eta \to G^\natural_\eta$ *to sections of* $P^\natural$ *over* S*. We shall denote this action by* $S_y : P^\natural \to P^\natural$ *and* $\tilde{S}_y : S_y^* \mathcal{L}^\natural \to \mathcal{L}^\natural$ *for any* $y \in Y$*.*

(5) *An action of* $G^\natural$ *on the invertible sheaf* $\mathcal{N} = \mathcal{L}^\natural \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathcal{M}^{\otimes -1}$ *on* $P^\natural$ *extending the translation action of* $G^\natural$ *on its structural sheaf, where* $\pi : P^\natural \to A$ *is the structural morphism in* (*1*)*. We shall denote this action by* $\tilde{T}_g : T_g^* \mathcal{N} \to \mathcal{N}$ *for any functorial point* g *of* $G^\natural$*.*

Moreover, the above data are required to satisfy the following conditions:

(i) *There exists an open* $G^\natural$*-invariant subscheme* $U \subset P^\natural$ *of finite type over* S *such that* $P^\natural = \underset{y \in Y}{\cup} S_y(U)$*.*

(ii) $\mathcal{L}^\natural$ *is ample on* $P^\natural$ *in the sense that the complement of the zero sets of global sections of* $(\mathcal{L}^\natural)^{\otimes n}$*, for all* $n \geq 1$*, define a basis of the* (*Zariski*) *topology of* $P^\natural$*.* (*Note that in [48, II, 4.5] Grothendieck only defines ampleness on quasi-compact schemes. As Mumford remarked in [100, §2], this seems to be the best property among his equivalent defining properties for our purposes.*)

(iii) (**Completeness condition:**) *Let* $\Upsilon(G^\natural)$ *be the set of all valuations* υ *of the rational function field* $K(G^\natural)$ *of* $G^\natural$ *such that* $\upsilon(R) \geq 0$*, namely the underlying set of the* **Zariski's Riemann surface** *of* $K(G^\natural)/R$*. For any* υ *in* $\Upsilon(G^\natural)$*, denote by* x_υ *the center of* υ *on* A (*which exists because* A *is proper over* S)*, which can be interpreted as an* S*-valued point* $S \to A$*. Recall that for any* $y \in Y$ *and* $\chi \in X$*, we have the invertible* R*-submodule* $I_{y,\chi}$ *of* K *defined in Section 4.2.4. Then the completeness condition is: For any* $\upsilon \in \Upsilon(G^\natural)$ *such that* $\upsilon(I) > 0$*,* υ *has a center on* $P^\natural$ (*which then necessarily lie on* $P^\natural_0 := P^\natural \underset{S}{\times} S_0$) *if for any* $y \in Y$*, there exists an integer* $n > 0$ *such that* $\upsilon(I^{\otimes n}_{y,\phi(y)} \cdot x_\upsilon^*(\mathscr{O}_{\phi(y)})) \geq 0$*.*

Remark 4.5.1.3. In the completeness condition (iii), we need to allow valuations of rank greater than one. (See [122, Ch. VI] for a classical treatment on this topic.)

Remark 4.5.1.4. Our completeness condition is much weaker than Mumford's [100, Def. 2.1(ii)] and Faltings-Chai's [37, Ch III, Def 3.1(3)], as we only require one direction of the implications in the special case that $\upsilon(I) > 0$. We choose to weaken the condition because the original condition is only used in

Mumford's proof of [100, Prop. 3.3] to show that the connected components of $P_0^\natural$ are proper, and because it is exactly this weaker implication that is needed. (The proof of the similar statement in Faltings-Chai is omitted. This is why we refer to Mumford's original paper.) On the other hand, we could not fully justify the proofs of [100, Thm. 2.5] or [37, Prop. 3.3] using their stronger conditions. (Both of them are literally incomplete and requires some further justifications.) We would like to point out that the completeness condition is not used in the more recent works such as [1] in some slightly different context. In [1], they do not have to introduce the general notion of relatively complete models, because their goal is to obtain canonical constructions in the category of schemes with actions of semi-abelian schemes, but not in the category of group schemes. On the other hand, we do need to allow noncanonical choices, if we want to have group structures, and if we want to allow some constructions such as the one in the proof of Lemma 4.5.4.6 below. Since we do not really know if the original proofs of Mumford and Faltings-Chai could be completed within their own settings, we should be content with this weakening of the completeness condition.

We will prove the existence of relatively complete models by writing down explicitly a $\underline{\mathrm{Proj}}$ of an explicit sheaf of graded $\mathscr{O}_A$-algebras, under a stronger assumption to be specified later. We follow very closely the constructions in [100] and [37, Ch. III, §3]. The essential difference, as explained above in Remark 4.5.1.4, is that our completeness condition (iii) in Definition 4.5.1.2 is weaker than theirs.

Construction 4.5.1.5. Let $\pi : G^\natural \to A$ denote the canonical morphism, and let us write $\mathscr{O}_{G^\natural}$ instead of $\pi_*\mathscr{O}_{G^\natural}$ by abuse of notation as before. Let us define two sheaves of graded $\mathscr{O}_A$-algebras

$$\mathcal{S}_1 := \underset{n\geq 0}{\oplus} \mathscr{O}_{G^\natural}\theta^n = \sum_{n\geq 0}(\underset{\chi\in X}{\oplus}\mathscr{O}_\chi)\theta^n$$

and

$$\mathcal{S}_2 := \underset{n\geq 0}{\oplus}(\underset{\chi\in X}{\oplus}(\mathcal{M}^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_\chi))\theta^n,$$

where θ is an indeterminate of degree 1 giving the gradings.

Note that $G^\natural$ acts on $\mathscr{O}_{G^\natural}$ (as a sheaf of $\mathscr{O}_A$-algebras) naturally by translation, or by the map between sheaves of $\mathscr{O}_{A\underset{S}{\times}A}$-algebras

$$m^* : m_A^*\mathscr{O}_{G^\natural} \to \mathrm{pr}_1^*\,\mathscr{O}_{G^\natural}\underset{\mathscr{O}_{G^\natural\underset{S}{\times}G^\natural}}{\otimes}\mathrm{pr}_2^*\,\mathscr{O}_{G^\natural},$$

whose restriction to the χ-weight space factors through the isomorphism

$$m_\chi^* : m_A^* \mathscr{O}_\chi \xrightarrow{\sim} \mathrm{pr}_1^* \mathscr{O}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_\chi$$

given by the theorem of square on $A \underset{S}{\times} A$. For each $g \in G^\natural$, these can be written as maps of $\mathscr{O}_A$-sheaves

$$\tilde{T}_g : T_{\pi(g)}^* \mathscr{O}_\chi \to \mathscr{O}_\chi,$$

which induces an action of $G^\natural$ on $\mathcal{S}_1$ given by

$$\tilde{T}_g : T_{\pi(g)}^* \mathscr{O}_\chi \xrightarrow{\sim} \mathscr{O}_\chi,$$

covering the translation by $\pi(g)$ on A. In particular, if $t \in T$, then $\pi(t) = e_A$, and we have

$$\tilde{T}_t : \mathscr{O}_\chi \xrightarrow{\sim} \mathscr{O}_\chi$$

given exactly by multiplication by $\chi(t)$.

Recall that (in Section 4.3.1) the Y-action on $(\pi^* \mathcal{M})_\eta = \underset{\chi \in X}{\oplus} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta})$ is given by

$$\begin{aligned} \psi(y)\tau(y,\chi) : T_{c^\vee(y)}^* (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) & \\ \cong \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y)) & \xrightarrow{\sim} \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}. \end{aligned}$$

This action extends to an action on the whole of $\mathcal{S}_2$ by

$$\tilde{S}_y = \psi(y)^n \tau(y,\chi) : \ T_{c^\vee(y)}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+n\phi(y)},$$

which covers the translation by $c^\vee(y)$ on A. Note that this agrees with $\tilde{T}_{\iota(y)}$ when $n = 0$.

Following [100] and [37], we define a *star* $\bigstar$ in X to be a finite subset of X generating X (as an abelian group) such that $0 \in \bigstar$ and $-\bigstar = \bigstar$.

Let $\bigstar$ be such a star, we define two subsheaves of graded $\mathscr{O}_A$-algebras of respectively $\mathcal{S}_{1,\eta}$ and $\mathcal{S}_{2,\eta}$ by

$$\mathcal{R}_{1,\bigstar} = \mathscr{O}_A[(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)})\theta]_{y \in Y, \alpha \in \bigstar}$$

and

$$\begin{aligned}\mathcal{R}_{2,\bigstar} &= \mathscr{O}_A[(I_y \cdot I_{y,\alpha} \cdot \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\alpha+\phi(y)})\theta]_{y\in Y,\alpha\in\bigstar} \\ &= \mathscr{O}_A[\tilde{S}_y(T^*_{c^\vee(y)}(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\alpha))\theta]_{y\in Y,\alpha\in\bigstar}.\end{aligned}$$

Note that here we have used the isomorphisms $\psi(y) : \mathcal{M}(c^\vee(y)) \overset{\sim}{\to} \underline{I}_y$ and $\tau(y,\alpha) : \mathscr{O}_\chi(c^\vee(y)) \overset{\sim}{\to} \underline{I}_{y,\alpha}$ coming from the very definitions of $\underline{I}_y$ and $\underline{I}_{y,\alpha}$ given in Section 4.2.4.

Then $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar}) = \underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{2,\bigstar})$ by the usual construction of $\underline{\mathrm{Proj}}$. We shall denote this scheme by $P^\natural = P^\natural_{(\phi,\psi),\bigstar}$, with structural morphism $\pi : P^\natural \to A$. Regarding $P^\natural_{(\phi,\psi),\bigstar}$ as $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar})$ endows it with an invertible sheaf $\mathcal{N}$ that is relatively ample over A (in the appropriate sense for morphisms only locally of finite type as in Definition 4.5.1.2), because sections of powers of $\mathcal{N}$ must give a basis of $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar})$ by the very construction of $\underline{\mathrm{Proj}}$. On the other hand, regarding it as $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{2,\bigstar})$ endows it with an invertible sheaf $\mathcal{L}^\natural = \mathcal{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^*\mathcal{M}$, which is ample over S (again in the sense of Definition 4.5.1.2) because $\mathcal{M}$ is ample.

The pair $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ admits a natural $G^\natural$-action from $\tilde{T}$ on $\mathcal{S}_{1,\eta}$, because $T^*_{\pi(g)}\mathcal{R}_{1,\bigstar}$, for any functorial point g of $G^\natural$, is generated by $T^*_{\pi(g)}(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)})\theta$, and we have

$$\tilde{T}_g : T^*_{\pi(g)}(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)}) \overset{\sim}{\to} I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)}$$

because $\underline{I}_y$ and $\underline{I}_{y,\alpha}$ are defined by invertible R-submodules in K, which are not affected under translation by $\pi(g)$ on $A \to S$. We shall denote this action by $T_g : P^\natural \overset{\sim}{\to} P^\natural$, for any functorial point g of $G^\natural$.

Moreover, the pair $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ admits a natural Y-action from $\tilde{S}$ on $\mathcal{S}_{2,\eta}$, because $T^*_{c^\vee(z)}\mathcal{R}_{2,\bigstar}$, for $z \in Y$, is generated by $T^*_{c^\vee(z)}\tilde{S}_y(T^*_{c^\vee(y)}(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\alpha))\theta$, and $\tilde{S}_z T^*_{c^\vee(z)}\tilde{S}_y T^*_{c^\vee(y)} = \tilde{S}_{z+y}T^*_{c^\vee(z+y)}$ by the definition of $\tilde{S}$. We shall denote this action by $S_z : P^\natural \overset{\sim}{\to} P^\natural$ and $\tilde{S}_z : S_z^*\mathcal{L}^\natural \overset{\sim}{\to} \mathcal{L}^\natural$.

To make $P^\natural_{(\phi,\psi),\bigstar}$ a relatively complete model, we need one additional condition:

Condition 4.5.1.6. *We have $I_y \cdot I_{y,\alpha} \subset R$ for all $y \in Y$ and all $\alpha \in \bigstar$.*

This condition is not necessarily satisfied for any given split object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$. To make it possible, we need to modify the entries related to polarizations according to the following:

Lemma 4.5.1.7 (cf. [37, Ch. III, Prop. 3.2])**.** *Let R be a noetherian normal domain with fraction field K. Let $a, a' : Y \to \mathrm{Inv}(R)$ be multiplicatively quadratic maps, $b : Y \times X \to \mathrm{Inv}(R)$ be a bimultiplicative map, and $\phi, \phi' : Y \to X$ be homomorphisms between free abelian groups of the same finite rank such that ϕ is injective and such that*

$$a(y_1 + y_2) = a(y_1) \cdot a(y_2) \cdot b(y_1, \phi(y_2))$$

and

$$a'(y_1 + y_2) = a'(y_1) \cdot a'(y_2) \cdot b(y_1, \phi'(y_2))$$

for any $y_1, y_2 \in Y$. Suppose moreover that $a(y) \subset R$ and $a'(y) \subset R$ for all but finitely many y's in Y. Then for every α in X, there exists an integer $n_0 > 0$ such that

$$a(y) \cdot b(y, 2n\phi(y)) \cdot b(y, \alpha) \subset R,$$

and

$$a'(y) \cdot b(y, 2n\phi(y)) \cdot b(y, \alpha) \subset R$$

for all $y \in Y$, for every $n \geq n_0$.

Proof. Since X and Y are finitely generated, there are finitely many height one prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ such that for any height one prime ideal $\mathfrak{p}$, $\mathfrak{p} \neq \mathfrak{p}_i$ for any $1 \leq i \leq k$, we have $a(y) \cdot R_\mathfrak{p} = R_\mathfrak{p}$, $a'(y) \cdot R_\mathfrak{p} = R_\mathfrak{p}$, and $b(y, \chi) \cdot R_\mathfrak{p} = R_\mathfrak{p}$ for all $y \in Y$ and all $\chi \in X$. Since $R = \cap R_\mathfrak{p}$ for $\mathfrak{p}$ running through all height one prime ideals in R, we are reduced to the case where R is a discrete valuation ring by taking the maximum of the n_0 obtained from each $R_{\mathfrak{p}_i}$, for $1 \leq i \leq k$. Let υ be the associated discrete valuation. Then $\upsilon \circ a$ is a real-valued quadratic function, whose associated bilinear pairing $\upsilon \circ b$ is positive semi-definite and symmetric on $Y \times \phi(Y) \subset Y \times X$.

Note that $\upsilon(b(\,\cdot\,, \phi(\,\cdot\,)))$ defines a positive semi-definite symmetric bilinear pairing on $(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R})$, which restricts to $\upsilon(b(\,\cdot\,, \,\cdot\,))$ on $Y \times X$ when we realize both Y and X as lattices inside $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Suppose $y \in Y$ is not in $\mathrm{Rad}(\upsilon(b(\,\cdot\,, \phi(\,\cdot\,))))$, the radical (namely the annihilator of the whole space) of the positive semi-definite symmetric bilinear pairing $\upsilon(b(\,\cdot\,, \phi(\,\cdot\,)))$ on Y. Then there is some $z \in Y$ such that

$\upsilon(b(y,\phi(z))) \neq 0$. If $\upsilon(b(y,\phi(y))) = 0$, then $\upsilon(b(ny+z,\phi(ny+z))) = 2n\upsilon(b(y,\phi(z))) + \upsilon(b(z,\phi(z))) < 0$ for some $n \in \mathbb{Z}$, which contradicts the positive semi-definiteness of $\upsilon(b(\,\cdot\,,\phi(\,\cdot\,)))$. As a result, $\upsilon(b(y,\phi(y))) > 0$ if and only if $y \notin \mathrm{Rad}(\upsilon(b(\,\cdot\,,\phi(\,\cdot\,))))$.

We claim that it suffices to show that there is an integer $n_1 > 0$ such that

$$\upsilon(b(y,2n\phi(y)) \cdot b(y,\alpha)) \geq 0$$

for any $y \in Y$ and any $n \geq n_1$. This is because there are only finitely many y such that $\upsilon(a(y)) < 0$ or $\upsilon(a'(y)) < 0$. If $\upsilon(a(y)) < 0$, then by the same argument as in Section 4.2.4, the fact that $a(ky) \subset R$ for all but finitely many $k \in \mathbb{Z}$ forces $\upsilon(b(y,\phi(y))) > 0$. Similarly, if $\upsilon(a'(y)) < 0$, then $\upsilon(b(y,\phi'(y))) > 0$ shows that $y \notin \mathrm{Rad}(\upsilon(b(\,\cdot\,,\phi(\,\cdot\,))))$, and hence $\upsilon(b(y,\phi(y))) > 0$. In either case, there is an integer n_2 such that $\upsilon(a(y)b(y,n_2\phi(y))) \geq 0$ and $\upsilon(a'(y))b(y,n_2\phi(y)) \geq 0$ for all $y \in Y$, and it suffices to take $n_0 = n_1 + n_2$. This proves claim.

Now let us assume that $\upsilon(b(\,\cdot\,,\phi(\,\cdot\,)))$ is positive definite by replacing $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ by its quotient by $\mathrm{Rad}(\upsilon(b(\,\cdot\,,\phi(\,\cdot\,))))$. The images of Y and X under this quotient are again lattices, because $\mathrm{Rad}(\upsilon(b(\,\cdot\,,\phi(\,\cdot\,))))$ is rationally defined. Then $\upsilon(b(\,\cdot\,,\phi(\,\cdot\,)))$ defines a norm $\|\cdot\|_\upsilon$ defined by $\|y\|_\upsilon := \upsilon(b(y,\phi(y)))^{1/2}$ on the real vector space $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and we have the Cauchy-Schwarz inequality

$$|\upsilon(b(y,\alpha))| \leq \|y\|_\upsilon \|\alpha\|_\upsilon.$$

As a result, we have

$$\upsilon(b(y,2n\phi(y))b(y,z)) \geq 2n\|y\|_\upsilon^2 - \|y\|_\upsilon\|\alpha\|_\upsilon \geq (2n\|y\|_\upsilon - \|\alpha\|_\upsilon)\|y\|_\upsilon.$$

Now it suffices to show that there is a single n_1 so that $\|y\|_\upsilon \geq \frac{1}{2n_1}\|\alpha\|_\upsilon$ for any $y \neq 0$ in Y. Note that Y is a lattice in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, which is discrete with respect to the topology defined by $\|\cdot\|_\upsilon$. Therefore there is an open ball $\{y \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \|y\|_\upsilon < r\}$ that contains no element of Y but 0. Hence any integer $n_1 > \frac{1}{2r}\|\alpha\|_\upsilon$ will do. $\square$

Corollary 4.5.1.8. *Given an object* $(A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi)$ *in* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$, *and given an invertible sheaf* $\mathcal{M}'$ *on* A *that is either ample or trivial. Suppose that we have* ϕ' *and* ψ' *such that* $(A,X,Y,\phi',c,c^\vee,(\mathcal{L}^\natural)' = \pi^*\mathcal{M}',\tau,\psi')$ *defines an object in* $\mathrm{DD}_{\mathrm{IS}}$. *Then there is an integer* $n_0 > 0$ *such that* *both the tuples* $(A, \mathcal{M}^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, (2n+1)\phi, c, c^\vee, \tau, \psi^{n+1}[-1]^*\psi^n)$ *and* $(A, \mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, \phi' + 2n\phi, c, c^\vee, \tau, \psi'\psi^n[-1]^*\psi^n)$ *satisfy Condition 4.5.1.6 if* $n \geq n_0$.

Proof. Let I_y and $I_{y,\alpha}$ be defined by the tuple $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ as usual. Let I'_y be either trivial, in the case that $\mathcal{M}'$ is trivial, or otherwise defined by the tuple $(A, \mathcal{M}', X, Y, \phi', c, c^\vee, \tau, \psi')$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ as usual. (Note that we can define I_y for any object in $\mathrm{DD}_{\mathrm{IS}}$.)

Let us consider the maps $a : Y \to \mathrm{Inv}(R) : y \mapsto I_y$, $a' : Y \to \mathrm{Inv}(R) : y' \mapsto I'_y$, and $b : Y \times X \to \mathrm{Inv}(R) : (y, \chi) \mapsto I_{y,\chi}$, which together with ϕ and ϕ' satisfy the requirement of Lemma 4.5.1.7. Then there is an integer $n_0 \geq 2$ such that $I_y \cdot I_{y,2n\phi(y)} \cdot I_{y,\alpha} \subset R$ and $I'_y \cdot I_{y,2n\phi(y)} \cdot I_{y,\alpha} \subset R$ for any $n \geq n_0$ and any of the finitely many $\alpha \in \bigstar$. Note that Condition 4.5.1.6 for the tuple $(A, \mathcal{M}^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, (2n+1)\phi, c, c^\vee, \tau, \psi^{n+1}[-1]^*\psi^n)$ is given by $I_{(n+1)y} \cdot I_{-ny} \cdot I_{y,\alpha} = I_y \cdot I_{y,n(n+1)\phi(y)} \cdot I_{y,\alpha} \subset R$, and Condition 4.5.1.6 for the tuple $(A, \mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, \phi' + 2n\phi, c, c^\vee, \tau, \psi'\psi^n[-1]^*\psi^n)$ is given $I'_y \cdot I_{ny} \cdot I_{-ny} \cdot I_{y,\alpha} = I'_y \cdot I_{y,n^2\phi(y)} \cdot I_{y,\alpha} \subset R$, which are both satisfied because $n(n+1) \geq 2n$ (resp. $n^2 \geq 2n$) and $I_{y,\phi(y)} \subset R$ anyway. ☐

In particular, we see that we can replace any tuple $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ by $(A, \mathcal{M}^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, (2n+1)\phi, c, c^\vee, \tau, \psi^{n+1}[-1]^*\psi^n)$ or $(A, \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^* \mathcal{M}^{\otimes n}, X, Y, 2n\phi, c, c^\vee, \tau, \psi^n[-1]^*\psi^n)$, so that any of the latter two satisfies Condition 4.5.1.6 with the same underlying object in DD.

By the very definition of $\underline{\mathrm{Proj}}$, $P^\natural_{(\phi,\psi),\bigstar}$ is covered by relatively affine open subschemes

$$U_{y,\alpha} := \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[I_y^{-1} \cdot I_{y,\alpha}^{-1} \cdot I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)}]_{\beta \in \bigstar, z \in Y}),$$

with α running over elements in $\bigstar$ and y running over elements of Y, which are all integral schemes over S. Moreover, it is clear from the construction that S_z maps $U_{y,\alpha}$ to $U_{z+y,\alpha}$. Therefore, there are only finitely many Y-orbits in the collection $\{U_{y,\alpha}\}_{y \in Y, \alpha \in \bigstar}$, with representatives given by $U_{0,\alpha}$ for α running through elements in $\bigstar$.

Note that

$$U_{0,0} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta+\phi(z)}]_{\beta \in \bigstar, z \in Y}) = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\mathscr{O}_\beta]) = G^\natural$$

because $I_z \cdot I_{z,\beta} \subset R$ for any $z \in Y$, $\alpha \in \bigstar$, as in Condition 4.5.1.6. Hence $P^\natural$ contains $G^\natural$ as a dense open subscheme.

We would like to show that all the affine open subschemes $U_{y,\alpha}$ of $P^\natural$ are of finite type over $S = \operatorname{Spec}(R)$. For this purpose, we need the following basic lemma:

Lemma 4.5.1.9 (cf. [100, Lem. 1.3])**.** *Suppose that for every $y \in Y$, $y \neq 0$, we are given an integer $n_y > 0$. Then there exists a finite set of elements $y_1, \ldots, y_k \in Y$, $y_i \neq 0$, and a finite set $Q \subset Y$, such that for all $z \in Y - Q$,*

$$I_{y_i,\phi(z)} = I_{z,\phi(y_i)} \subset I_{y_i,\phi(y_i)}^{\otimes n_{y_i}} \tag{4.5.1.10}$$

for some y_i, $1 \leq i \leq k$.

Proof. Since Y is finitely generated, there are finitely many height one prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ such that for any height one prime ideal $\mathfrak{p} \neq \mathfrak{p}_i$ for any $1 \leq i \leq k$, we have $I_{y,\phi(z)} \cdot R_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $y, z \in Y$. Since $R = \cap R_{\mathfrak{p}}$ for $\mathfrak{p}$ running through all height one prime ideals in R, we only need to consider those discrete valuations $\upsilon_j \in \Upsilon_1$ (defined as in Definition 4.2.4.2) associated respectively to $\mathfrak{p}_j$, and verify (4.5.1.10) by evaluating υ_j.

Note that $B_j(y, z) := \upsilon_j(I_{y,\phi(z)})$ defines a positive semi-definite symmetric pairing on $Y \times Y$, which extends by linearity to a positive semi-definite symmetric pairing on $(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. For all $y \in Y$, $y \neq 0$, let

$$C_y := \left\{ \begin{matrix} z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} : B_j(y, z) > n_y B_j(y, y) \text{ for all } j \\ \text{such that } y \notin \operatorname{Rad}(B_j), \text{ the radical of } B_j \end{matrix} \right\}$$

Note that C_y is a nonempty convex open subset of $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and we have $r \cdot C_y \subset C_y$ if $r \in \mathbb{R}_{\geq 1}$.

Let us first claim that

$$\bigcup_{N=1}^{\infty} \bigcup_{\substack{y \in Y \\ y \neq 0}} \tfrac{1}{N} C_y = Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} - \{0\}.$$

The radicals $\operatorname{Rad}(B_j)$ are all spanned by elements in Y. In particular, they are rationally defined subspaces. Therefore, for any $z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we can find $y \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ such that:

1. If $z \in \operatorname{Rad}(B_j)$, then $y \in \operatorname{Rad}(B_j)$ as well.

2. If $z \notin \mathrm{Rad}(B_j)$, then $B_j(y,z) > 0$.

As a result, for $z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we can approximate z by an element in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ in the intersection of those $\mathrm{Rad}(B_j)$ containing z. On the other hand, for those $\mathrm{Rad}(B_j)$ not containing z, we have $B_j(z,z) > 0$ (as in the proof of Lemma 4.5.1.7), and hence $B_j(y,z) > 0$ when y is close to z. Therefore $N \cdot z \in C_y$ if N is sufficiently large, and hence $z \in \frac{1}{N} C_y$. This proves the claim.

By compactness of the unit sphere in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ (defined using any Euclidean norm for the finite-dimensional $\mathbb{R}$-vector space), there exist finitely many $y_i \in Y$, $y_i \neq 0$, and integers $N_i > 0$, such that

$$\bigcup_{i=1}^{k} \tfrac{1}{N_i} C_{y_i} \supset (\text{unit sphere}).$$

Then we also have

$$\bigcup_{i=1}^{k} C_{y_i} \supset \bigcup_{\substack{r \in \mathbb{R} \\ r \geq N := \max\limits_{1 \leq i \leq k} (N_i)}} r \cdot (\text{unit sphere}).$$

Let Q be the set of elements of Y that lie inside the ball of radius $N = \max\limits_{1 \leq i \leq k} (N_i)$, which is finite because Y is discrete. Then for any $z \in Y - Q$, there is a y_i such that $B_j(z, y_i) \geq n_j B_j(y_i, y_i)$, as desired. $\square$

Proposition 4.5.1.11 (cf. [100, Prop. 2.4])**.** *For any $\alpha \in \bigstar$, the affine open subscheme $U_{0,\alpha}$ of $P^\natural$ is of finite type over S.*

Proof. Note that $U_{0,\alpha}$ is the $\underline{\mathrm{Spec}}_{\mathscr{O}_A}$ for the sheaf of $\mathscr{O}_A$-algebras

$$\mathscr{O}_A[I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}]_{\beta \in \bigstar, z \in Y}$$

generated by the $\mathscr{O}_A$-sheaves modules of finite type

$$\mathscr{M}_{z,\beta} := I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}.$$

We would like to show that it suffices to take finitely many $\mathscr{M}_{z,\beta}$'s as generators.

Note that for $y \in Y$, we have the following relation:

$$\begin{aligned}
\mathscr{M}_{z,\beta} &= I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)} \\
&= I_{y+(z-y)} \cdot I_{y+(z-y),\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(y)+\phi(z-y)} \\
&= I_y \cdot I_{z-y} \cdot I_{y,\phi(z-y)} \cdot I_{y,\alpha} \cdot I_{y,\beta-\alpha} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\phi(y)} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)} \\
&= I_{y,\phi(z-y)} \cdot I_{y,\beta-\alpha} \cdot (I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\phi(y)}) \cdot (I_{z-y} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)}) \\
&= I_{y,\phi(z-y)+\beta-\alpha} \cdot \mathscr{M}_{y,\alpha} \cdot \mathscr{M}_{z-y,\beta}.
\end{aligned}$$

For any integer n, let us write

$$\begin{aligned} I_{y,\phi(z-y)+\beta-\alpha} &= I_{y,\phi(z)} \cdot I_{y,\phi(y)}^{\otimes -1} \cdot I_{y,\beta-\alpha} \\ &= (I_{y,\phi(z)} \cdot I_{y,\phi(y)}^{\otimes -n}) \cdot (I_{y,\phi(y)}^{\otimes n-2} I_{y,\beta-\alpha}) \cdot I_{y,\phi(y)}. \end{aligned}$$

By Lemma 4.5.1.7, there is an integer $n_0 > 0$ such that $I_{y,\phi(y)}^{\otimes n_0 - 2} \cdot I_{y,\beta-\alpha} \subset R$ for all y and for all of the finitely many $\beta \in \bigstar$. By Lemma 4.5.1.9, there is a finite subset $\{y_1, \ldots, y_k\}$ of nonzero elements in Y, and a finite subset $Q \subset Y$, such that for all $z \in Y - Q$, there is some y_i such that $I_{y_i,\phi(z)} \cdot I_{y_i,\phi(y_i)}^{\otimes -n_0} \subset R$. Note that $I_{y_i,\phi(y_i)} \subset I$ by the positivity condition of τ, because $y_i \neq 0$. As a result, for all $z \in Y - Q$, there is some y_i such that

$$\mathscr{M}_{z,\beta} \subset I \cdot \mathscr{M}_{y_i,\alpha} \cdot \mathscr{M}_{z-y_i,\beta}.$$

We may repeat this process until $z - y_i \in Q$. Let us claim that this process always stops in a finite number of steps. Then we can claim that we only need the finitely many generators $\mathscr{M}_{y_i,\alpha}$ and $\mathscr{M}_{q,\beta}$ for $q \in Q$ instead of all of $\mathscr{M}_{z,\beta}$. This will prove the proposition.

Note that by Condition 4.5.1.6,

$$\mathscr{M}_{z,\beta} = I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)} \subset \mathscr{O}_{\beta-\alpha+\phi(z)}$$

for any $z \in Y$ and any $\beta \in \bigstar$. If we have

$$\begin{aligned} \mathscr{M}_{z,\beta} &\subset I \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{z-y_{i_1},\beta} \\ &\subset I^2 \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{y_{i_2},\alpha} \cdot \mathscr{M}_{z-y_{i_1}-y_{i_2},\beta} \\ &\subset \ldots \\ &\subset I^m \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{y_{i_2},\alpha} \cdot \ldots \cdot \mathscr{M}_{y_{i_m},\alpha} \cdot \mathscr{M}_{z-\sum\limits_{1 \leq i \leq m} y_{i_m},\beta}, \end{aligned}$$

then

$$\begin{aligned} \mathscr{M}_{z,\beta} &\subset I^m \cdot \mathscr{O}_{\phi(y_{i_1})} \cdot \mathscr{O}_{\phi(y_{i_2})} \cdot \ldots \cdot \mathscr{O}_{\phi(y_{i_m})} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)-\sum\limits_{1 \leq i \leq m} \phi(y_{i_m})} \\ &\subset I^m \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}. \end{aligned}$$

If this happens for all $m > 0$, then we have

$$\mathscr{M}_{z,\beta} \subset \bigcap_{m=1}^{\infty} (I^m \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}) = 0,$$

because $\mathscr{O}_{\beta-\alpha+\phi(z)}$ is a sheaf of $\mathscr{O}_A$-modules of finite type, and R is noetherian. This is impossible. Therefore the process must stop in a finite number of steps, and the claim is proved. □

Corollary 4.5.1.12. *For any $y \in Y$ and $\alpha \in \bigstar$, the affine open subscheme $U_{y,\alpha}$ of $P^\natural$ is of finite type over S.*

Proof. Simply because $U_{y,\alpha} = S_y(U_{0,\alpha})$. $\square$

Corollary 4.5.1.13. *If we take U to be the finite union of those $U_{0,\alpha}$ with α running over elements in $\bigstar$, then U is of finite type over S, and we have $P^\natural = \bigcup\limits_{y\in Y} S_y(U)$. In particular, $P^\natural$ is locally of finite type.*

Proposition 4.5.1.14 (cf. [100, Thm. 2.5] and [37, Ch. III, Prop. 3.3])**.** *Assume that Condition 4.5.1.6 is satisfied. Then $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ is a relatively complete model.*

Proof. So far we have constructed $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ with all the data (1) – (5) in Definition 4.5.1.2, and we have verified (i) and (ii) in the definition as well. It only remains to verify (iii), namely the *completeness condition.*

Suppose $\upsilon \in \Upsilon(G^\natural)$, namely υ is a valuation of the rational function field $K(G^\natural)$ of $G^\natural$ such that $\upsilon(R) \geq 0$. We need to show that υ has a center on $P^\natural_{(\phi,\psi),\bigstar}$ if and only if for any $\chi \in X$, there exists $y \in Y$ such that $\upsilon(I_{y,\chi} \cdot x_\upsilon^*(\mathscr{O}_\chi)) \geq 0$. (Recall that x_υ is the center of υ on A, which exists uniquely by properness of A over S.)

Since $P^\natural_{(\phi,\psi),\bigstar}$ is the union of $U_{y,\alpha}$, each of which is relative affine over A, the valuation υ has a center on $P^\natural_{(\phi,\psi),\bigstar}$ if and only if it has one on $x_\upsilon^*(U_{y,\alpha})$ for some $y \in Y$ and some $\alpha \in \bigstar$. By construction,

$$x_\upsilon^*(U_{y,\alpha}) = \operatorname{Spec}(R[I_y^{-1} \cdot I_{y,\alpha}^{-1} \cdot I_z \cdot I_{z,\beta} \cdot x_\upsilon^*(\mathscr{O}_{\beta-\alpha+\phi(z-y)})]_{\beta\in\bigstar,z\in Y}).$$

Having a center on $x_\upsilon^*(U_{y,\alpha})$ means the values of υ are positive on all the generators over R of the affine ring. (Here for convenience we are confusing the $\mathscr{O}_S$-sheaves with R-modules.) In other words, there is a $y \in Y$ and $\alpha \in \bigstar$ such that $\upsilon(I_z \cdot I_{z,\beta} \cdot x_\upsilon^*(\mathscr{O}_{\beta+\phi(z)})) \geq \upsilon(I_y \cdot I_{y,\alpha} \cdot x_\upsilon^*(\mathscr{O}_{\alpha+\phi(y)}))$ for all $z \in Y$ and $\beta \in \bigstar$. That is, the minimum of $\{\upsilon(I_y \cdot I_{y,\alpha} \cdot x_\upsilon^*(\mathscr{O}_{\alpha+\phi(y)})\}_{y\in Y,\alpha\in\bigstar}$ exists.

As remarked in the proof of [37, Ch. III, Prop. 3.3], we have to modify Mumford's original argument in the last few steps of the proof of [100, Thm. 2.5] to show this. Since the correction proposed by [37, Ch. III, Prop. 3.3] is not completely correct either, we would like to propose a modification under the weaker assumption that $\upsilon(I) > 0$ in our weaker version of the completeness condition. (See Remark 4.5.1.4.)

To save notations, let us denote by $\mathscr{N}_{z,\alpha} := I_z \cdot I_{z,\alpha} \cdot \mathscr{O}_{\alpha+\phi(z)}$, for any $z \in Y$ and $\alpha \in \bigstar$, and denote by $\upsilon(x_v^* \mathscr{N}_{z,\alpha}) := \upsilon(I_z \cdot I_{z,\alpha} \cdot x_v^*(\mathscr{O}_{\alpha+\phi(z)}))$. Assume that $\upsilon(I) > 0$, and assume that for any $y \in Y$, there exists an integer $n_y > 0$ such that $\upsilon(I^{\otimes n_y}_{y,\phi(y)} \cdot x_v^*(\mathscr{O}_{\phi(y)}) \geq 0$. Our goal is to show that $\min\limits_{z \in Y} \upsilon(x_v^* \mathscr{N}_{z,\alpha})$ exist for each of the finitely many $\alpha \in \bigstar$. Let us fix a choice of α.

Consider the following relation for any $y \in Y$ (cf. the proof of Proposition 4.5.1.11):

$$\begin{aligned}
\mathscr{N}_{z,\alpha} &= I_z \cdot I_{z,\alpha} \cdot \mathscr{O}_{\alpha+\phi(z)} \\
&= I_{y+(z-y)} \cdot I_{y+(z-y),\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)+\phi(z-y)} \\
&= I_y \cdot I_{z-y} \cdot I_{y,\phi(z-y)} \cdot I_{y,\alpha} \cdot I_{z-y,\alpha} \cdot \mathscr{O}_{\phi(y)} \cdot \mathscr{O}_{\alpha+\phi(z-y)} \\
&= I_{y,\phi(z-y)} \cdot I_{y,\alpha} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \cdot (I_{z-y} \cdot I_{z-y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(z-y)}) \\
&= I_{y,\phi(z-y)+\alpha} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \cdot \mathscr{N}_{z-y,\alpha}
\end{aligned}$$

For any integer n, let us write:

$$\begin{aligned}
&I_{y,\phi(z-y)+\alpha} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \\
&= I_{y,\phi(y)} \cdot (I_{y,\phi(z)} \cdot I^{\otimes -2-n-n_y}_{y,\phi(y)}) \cdot (I^{\otimes n}_{y,\phi(y)} \cdot I_{y,\alpha}) \cdot (I_y \cdot I^{\otimes n_y}_{y,\phi(y)} \cdot \mathscr{O}_{\phi(y)}),
\end{aligned}$$

where $n_y > 0$ is an integer determined by the assumption so that $\upsilon(I^{\otimes n_y}_{y,\phi(y)} \cdot x_v^*(\mathscr{O}_{\phi(y)}) \geq 0$. By Lemma 4.5.1.7, there is an integer $n_0 > 0$ such that $I^{\otimes n_0}_{y,\phi(y)} \cdot I_{y,\alpha} \subset R$ for all y. By Lemma 4.5.1.9, there is a finite subset $\{y_1, \ldots, y_k\}$ of nonzero elements in Y, and a finite subset $Q \subset Y$, such that for all $z \in Y - Q$, there is some y_i such that $I_{y_i,\phi(z)} \cdot I^{\otimes -2-n_0-n_y}_{y_i,\phi(y_i)} \subset R$. Note that $I_{y_i,\phi(y_i)} \subset I$ by the positivity condition of τ, because $y_i \neq 0$, and note that $\upsilon(I) > 0$ by assumption. As a result, for all $z \in Y - Q$, there is some y_i such that

$$\mathscr{N}_{z,\beta} \subset I \cdot (I_{y_i} \cdot I^{\otimes n_{y_i}}_{y_i,\phi(y_i)} \cdot \mathscr{O}_{\phi(y_i)}) \cdot \mathscr{N}_{z-y_i,\alpha},$$

and so, by taking the value of υ at x_v,

$$\begin{aligned}
\upsilon(x_v^* \mathscr{N}_{z,\alpha}) &\geq \upsilon(I) + \upsilon(I_{y_i} \cdot I^{\otimes n_{y_i}}_{y_i,\phi(y_i)} \cdot x_v^*(\mathscr{O}_{\phi(y_i)})) + \upsilon(x_v^* \mathscr{N}_{z-y_i,\beta}) \\
&> \upsilon(x_v^* \mathscr{N}_{z-y_i,\alpha}).
\end{aligned}$$

Note that here we have used $I_y \subset R$ and $\upsilon(R) \geq 0$. This shows that the minimum of $\upsilon(x_v^* \mathscr{N}_{z,\beta})$ occurs in the finite set Q, and proves the existence of the center of υ on $P^\natural$ (or rather $P^\natural_0 := P^\natural \underset{S}{\times} S_0$). $\square$

Remark 4.5.1.15. The construction of the relatively complete model $P^\natural$ using $P^\natural_{(\phi,\psi),\bigstar}$ is functorial with the choice of $\mathcal{M}$ and the specification of the star $\bigstar$ in X. If we replace $\mathcal{M}$ on A by a different descended form of $\mathcal{L}^\natural$, with all the other data unchanged, then the different between the two choices of $\mathcal{M}$ is given by an element in X (by Lemma 3.2.2.12, Remark 3.2.2.11, and Proposition 3.2.5.4). If this element lies in $\phi(Y)$, then the new relatively complete model $P^\natural$ can be obtained from the old one by replacing the inclusion $G^\natural \subset P^\natural$ by its transforms under S_y. By construction, the Mumford quotients $G \subset P$ are unchanged under transforms under S_y. Therefore the essential choice lies in the group $X/\phi(Y)$. It is not surprising that this group also shows up in Section 4.3.2 when we analyze the structure of $\Gamma(G, \mathcal{L})$!

4.5.2 Construction of The Quotient

Suppose now that we are given a relatively complete model $(P^\natural, \mathcal{L}^\natural)$ of an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$, (no matter it satisfies Condition 4.5.1.6 or not.) The goal of this section is to construct some kind of "quotient" of $(G^\natural, \mathcal{L}^\natural)$ by Y in an appropriate sense.

Remark 4.5.2.1. To avoid unnecessary misunderstandings, let us emphasize that *almost nothing in Sections 4.5.2 and 4.5.3 is due to us.* The arguments in [100] are sometimes even quoted verbatim without being mentioned. We could have avoided this by giving references to [100] and sketches of necessary modifications as in [37]. Since one of the main points of writing this chapter is to understand the first three chapters of [37], we have decided to adopt the contrary approach. We hope the reader could understand that *almost every single step is originally due to Mumford.* We hope that Mumford's work is so well-known that there should be no confusion!

Lemma 4.5.2.2 (cf. [100, Lem. 1.4])**.** *For any $y \in Y$, and any $\chi \in X$, there exists an integer $n > 0$ such that $I^{\otimes n}_{y,\phi(y)} \cdot I_{y,\chi} \subset R$.*

Proof. As in the proof of Lemma 4.5.1.9, by noetherian normality of R, it suffices to show that there is a common integer $n > 0$ such that $I^{\otimes n}_{y,\phi(y)} \cdot I_{y,\chi} \cdot R_\mathfrak{p} \subset R_\mathfrak{p}$ for any height one prime ideal $\mathfrak{p}$. Since Y and X are finitely generated, there are finitely many height one prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ such that for any height one prime ideal $\mathfrak{p} \neq \mathfrak{p}_i$ we have $I_{y,\chi} \cdot R_\mathfrak{p} = R_\mathfrak{p}$. Hence it suffice to find integers $n > 0$ for the finitely many ideals $\mathfrak{p}_i$, or equivalently to find one integer $n_i > 0$ for each $\mathfrak{p}_i$.

Let $v_i \in \Upsilon_1$ (defined as in Lemma 4.2.4.2) be the valuation associated to $\mathfrak{p}_i$. Then $B_i(z,\mu) := v_i(I_{z,\mu})$ defines a pairing on $Y \times X$, which extends to a positive semi-definite symmetric pairing on $(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \cong (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (X \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ by $\phi : Y \hookrightarrow X$. We need to show that there is an integer n_i such that $n_i B_i(y, \phi(y)) + B_i(y,\chi) \geq 0$. Suppose $B_i(y,\phi(y)) = 0$, then y must lie in $\mathrm{Rad}(B_i)$ as in the proof of Lemma 4.5.1.7. Hence $B_i(y,\chi) = 0$ as well and there is nothing to prove. Otherwise $B_i(y,\phi(y)) > 0$, and hence there exist an integer $n_i > 0$ such that $n_i B_i(y,\phi(y)) + B_i(y,\chi) \geq 0$, as desired. ☐

Proposition 4.5.2.3 (cf. [100, Prop. 3.1] and [37, Ch. III, Prop. 4.1])**.** *For any $y \in Y$, we have $\mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \subset \pi_* \mathscr{O}_{P^\natural} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$. That is, $\mathscr{O}_{\phi(y)}$ defines regular functions on $P^\natural$ over* $\mathrm{Spec}(R[I_{y,\phi(y)}^{\otimes -1}])$*.*

Proof. First note that $\iota(y)$ extends to an element of $G^\natural(R[I_{y,\phi(y)}^{\otimes -1}])$ for any y. It suffices to know that the sections $\tau(y,\chi)$ can be defined over $R[I_{y,\phi(y)}^{\otimes -1}]$ for any y and any $\chi \in X$, which follows from Lemma 4.5.2.2 above. The translation action $T_{\iota(y)} : P^\natural \xrightarrow{\sim} P^\natural$ and the action $S_y : P^\natural \xrightarrow{\sim} P^\natural$ has to agree whenever they are both defined, and hence it makes sense to compare the $G^\natural$-action

$$\tilde{T}_{\iota(y)} : T_{\iota(y)}^* \mathscr{N} \xrightarrow{\sim} \mathscr{N}$$

on $\mathscr{N}$ with the isomorphism

$$\tilde{S}_y : S_y^* \mathscr{N} \xrightarrow{\sim} \mathscr{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$$

deduced from the Y-action $\tilde{S}_y : S_y^* \mathscr{L}^\natural \xrightarrow{\sim} \mathscr{L}^\natural$ on $\mathscr{L}^\natural$ over $R[I_{y,\phi(y)}^{\otimes -1}]$. This gives an isomorphism $\mathscr{N} \xrightarrow{\sim} \mathscr{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ of invertible sheaves over $R[I_{y,\phi(y)}^{\otimes -1}]$, or equivalently an isomorphism

$$\zeta : \mathscr{O}_{P^\natural} \xrightarrow{\sim} \mathscr{O}_{P^\natural} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$$

over $R[I_{y,\phi(y)}^{\otimes -1}]$, both extending the isomorphism $\mathscr{O}_{G^\natural} \xrightarrow{\sim} \mathscr{O}_{G^\natural} \underset{\mathscr{O}_{G^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ by shifting the weight spaces in the decomposition $\pi_* \mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$ under T-action.

The T-action on $P^\natural$ also gives a decomposition

$$\pi_* \mathscr{O}_{P^\natural} = \underset{\chi \in X}{\oplus} (\pi_* \mathscr{O}_{P^\natural})_\chi.$$

Since $G^\natural \subset P^\natural$, we have a canonical inclusion $(\pi_* \mathscr{O}_{P^\natural})_\chi \subset \mathscr{O}_\chi$. In particular, we have $(\pi_* \mathscr{O}_{P^\natural})_0 = \mathscr{O}_A$ because $\mathscr{O}_A$ is already contained in $\pi_* \mathscr{O}_{P^\natural}$ via the structural map $\pi : P^\natural \to A$. The isomorphism $\zeta : \mathscr{O}_{P^\natural} \xrightarrow{\sim} \mathscr{O}_{P^\natural} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ over $R[I_{y,\phi(y)}^{\otimes -1}]$ induces a collection of isomorphisms $(\pi_* \mathscr{O}_{P^\natural})_{\chi + \phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \xrightarrow{\sim} (\pi_* \mathscr{O}_{P^\natural})_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$. In particular, by taking $\chi = 0$, we obtain an isomorphism $(\pi_* \mathscr{O}_{P^\natural})_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \xrightarrow{\sim} \mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$, which shows that the above-mentioned canonical inclusion $(\pi_* \mathscr{O}_{P^\natural})_{\phi(y)} \subset \mathscr{O}_{\phi(y)}$ becomes identities $(\pi_* \mathscr{O}_{P^\natural})_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] = \mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$, as desired. □

Corollary 4.5.2.4 (cf. [100, Cor. 3.2] and [37, Ch. III, Cor. 4.2])**.** *As subschemes of $P^\natural$, we have $G^\natural_\eta = P^\natural_\eta$.*

Proof. Over η, $\mathscr{O}_{\phi(y)}$ defines regular functions on $P^\natural_\eta$ for all $y \in Y$. Since there is some integer $N \geq 1$ such that $NX \subset \phi(Y)$, we see that $\mathscr{O}_\chi$ defines regular functions on the normalization of $P^\natural$ over η for every $\chi \in X$. Since $G^\natural = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\underset{\chi \in X}{\oplus} \mathscr{O}_\chi)$ is a normal subscheme of $P^\natural$, this forces the normalization of $P^\natural$ to agree with $G^\natural$ over η. Then certainly $P^\natural_\eta = G^\natural_\eta$. □

The following technical lemma and its proof is quoted verbatim from [100, Lem. 3.4]:

Lemma 4.5.2.5. *Let $f : Z \to Z'$ be a morphism locally of finite type, with Z an irreducible scheme but Z' arbitrary. If f satisfies the valuative criterion for properness for all valuations, then f is proper.*

Proof. The usual valuative criterion (such as for example [48, II, 7.3]) would hold if we know that f was of finite type. It suffices to prove that f is quasi-compact. As this is a topological statement, we may replace Z by Z_{red}, and Z' by Z'_{red}. By looking locally on the base, we may assume Z' is affine, say $\mathrm{Spec}(A)$. Finally, we may assume that f is dominating. Then A is a subring of the rational function field $K(Z)$ of Z. Let $\mathscr{Z}$ denote the Zariski's Riemann surface of $K(Z)/A$. (See [122, p. 110] or [91, p. 73].) Set-theoretically $\mathscr{Z}$ is simply the set of valuations υ on $K(Z)$ such that $\upsilon(A) \geq 0$. By the valuative criterion, every υ has a uniquely determined center on Z, and hence there is a natural map $\pi : \mathscr{X} \to Z$ taking υ to its center. Since $\mathscr{X}$ is a quasi-compact topological space (by [91, Thm. 10.5]), and since π is continuous and surjective, we see that Z is quasi-compact, as desired. ☐

Proposition 4.5.2.6 (cf. [100, Prop. 3.3])**.** *Every irreducible component of* $P_0^\natural := P^\natural \underset{S}{\times} S_0$ *is proper over* $A_0 := A \underset{S}{\times} S_0$*, or equivalently, over* S_0*.*

Proof. By Lemma 4.5.2.5 above, it suffices to show that if Z is any component of $P_0^\natural$, and if υ is any valuation of its rational function field $K(Z)$ such that $\upsilon(R_0) \geq 0$, then υ has a center on Z. To show this, let υ_1 be a valuation of $K(G^\natural)$ such that $\upsilon(R) \geq 0$ and such that the center of υ_1 is Z, and let υ_2 be the composite of the valuations υ and υ_1. Note that $\upsilon_2(I) > 0$ as Z is a scheme over $\operatorname{Spec}(R_0)$. Let x_{υ_1} and x_{υ_2} be respectively the centers of υ_1 and υ_2 on A, which exist by properness of A. By Proposition 4.5.2.3, we know that for any $y \in Y$, there is an integer $n > 0$ such that $\upsilon_1(I_{y,\phi(y)}^{\otimes n} \cdot x_{\upsilon_1}^*(\mathscr{O}_{\phi(y)})) \geq 0$. By positivity condition of τ, we know that $I_{y,\phi(y)} \subset I$ if $y \neq 0$. Since υ_1 has center on Z, which is a scheme over $\operatorname{Spec}(R_0)$, we may increase n and assume that $\upsilon_1(I_{y,\phi(y)}^{\otimes n} \cdot x_{\upsilon_1}^*(\mathscr{O}_{\phi(y)})) > 0$ when $y \neq 0$ and hence $\upsilon_2(I_{y,\phi(y)}^{\otimes n} \cdot x_{\upsilon_2}^*(\mathscr{O}_{\phi(y)})) > 0$. Otherwise $y = 0$, and we have $\upsilon_2(I_{y,\phi(y)}^{\otimes n} \cdot x_{\upsilon_2}^*(\mathscr{O}_{\phi(y)})) = \upsilon_2(x_{\upsilon_2}^*(\mathscr{O}_A)) > 0$ by definition of x_{υ_2}. In either case, it follows that υ_2 has a center on $P^\natural$ by the completeness condition (iii) in Definition 4.5.1.2. This implies that υ has a center on Z and concludes the proof. ☐

Corollary 4.5.2.7 (cf. [100, Cor. 3.5])**.** *Let* $U_0 := U \underset{S}{\times} S_0$*, where* U *is as in condition* (*1*) *of Definition 4.5.1.2. Then the closure* $\overline{U_0}$ *of* U_0 *in* $P_0^\natural$ *is proper over* S_0*.*

Proof. This is true simply because U is of finite type over S, and hence U_0 has only finitely many irreducible components. ☐

Proposition 4.5.2.8 (cf. [100, Prop. 3.6] and [37, Ch. III, Prop. 4.5])**.** *There is a finite subset* $Q \subset Y$ *such that* $S_y(\overline{U_0}) \cap S_z(\overline{U_0}) = \emptyset$ *if* $y - z \neq Q$*.*

Proof. First let us claim that the actions of Y and T on $(P^\natural, \mathcal{L}^\natural)$ satisfy the following commutation relation: For any functorial point t of T and any $y \in Y$, the underlying actions on P satisfies $S_y \circ T_t = T_t \circ S$, and the two isomorphisms

$$\tilde{T}_t \circ T^*(\tilde{S}_y) : T_t^* \circ S_y^*(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) \xrightarrow{\sim} \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi+\phi(y)}$$

and

$$\tilde{S}_y \circ S^*(\tilde{T}_t) : S_y^* \circ T_t^*(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) \xrightarrow{\sim} \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi+\phi(y)}$$

with same source and target differ by the relation

$$\tilde{T}_t \circ T^*(\tilde{S}_y) = \phi(y)(t)\tilde{S}_y \circ S^*(\tilde{T}_t).$$

To show this relation, it suffices to verify them over $(G_\eta^\natural, \mathcal{L}_\eta^\natural)$, which is then automatic by the assumption on ι.

Let us consider the action of $T_0 := T \underset{S}{\times} S_0$ on P_0. Let F be the fixed-point subscheme of P_0. Since T_0 is a connected solvable linear algebraic group, Borel's fixed point theorem (see for example [118, Thm. 6.2.6]) shows that T_0-fixed points exists on every irreducible components of P_0, as each of them is proper by Proposition 4.5.2.6. In other words, F intersects every irreducible component of P_0. Since U_0 is of finite type, $\overline{U_0} \cap F$ has only finitely many connected components $C_1, \ldots, C_n$. Since T_0 leaves every point of C_i invariant, it acts via a character χ_i on each $(C_i, \mathcal{L}^\natural|_{C_i})$. By the commutation relation above, T_0 acts on the restriction of $\mathcal{L}^\natural$ to the connected components of $S_y(C_1), \ldots, S_y(C_n)$ of $S_y(\overline{U_0})$ via the characters $\chi_1 + \phi(y), \ldots, \chi_n + \phi(y)$. Then the proposition follows if we take simply $Q := \phi^{-1}(\{\chi_i - \chi_j : 1 \leq i, j \leq n\})$. □

Corollary 4.5.2.9 (cf. [100, Cor. 3.7])**.** *The group Y acts freely on $P_0^\natural$.*

Proof. This follows from Proposition 4.5.2.8 because $P_0^\natural = \underset{y \in Y}{\cup} S_y(\overline{U_0})$. □

Proposition 4.5.2.10 (cf. [100, Prop. 3.8])**.** *The special fiber $P_0^\natural$ is connected.*

Proof. Our first goal is to reduce the question to a simpler question over a base scheme that is the spectrum of a discrete valuation ring, over which (we will see in the proof below that) $P^\natural$ enjoys a much stronger completeness condition than condition (iii) of Definition 4.5.1.2.

Since R is complete in the I-adic topology, and since R has no idempotents, we know that R_0 has no idempotents either. In other words, S_0 is connected. Therefore $G_0^\natural$ is a connected open subset of $P_0^\natural$, and it determines a canonical connected component of $P_0^\natural$. Suppose there is point x that does not lie on this connected component of $P_0^\natural$. Let υ be a discrete rank one valuation of $K(P^\natural) = K(G^\natural)$ (equality by Corollary 4.5.2.4) such that $\upsilon(R) \geq 0$

and has center x. Let $R_v := \{k \in K : v(k) \geq 0\}$. Let $S_v := \operatorname{Spec}(R_v)$ and $P^\natural_v := P^\natural \underset{S}{\times} S_v$. Note that R_v is by definition a discrete rank one valuation ring with quotient field K, so S and S_v have the same generic point, and $P^\natural$ and $P^\natural_v$ has the same generic fiber. Let $(P^\natural_v)'$ be the closure of the generic fiber of $P^\natural_v$ in $P^\natural_v$. By definition, $(P^\natural_v)'$ is an integral scheme with the same rational function field as $P^\natural$, namely $K(G^\natural)$, and it is locally of finite type over the valuation ring R_v. Note that S_v has only two points: its generic point and its closed point. Let $(P^\natural_v)'_0$ be the fiber of $(P^\natural_v)'$ over the closed point of S_v. Then there is a natural morphism

$$(P^\natural_v)'_0 \to P^\natural_0.$$

Let us claim that both x and $G^\natural_0$ lies in the image, which will then imply that $(P^\natural_v)'_0$ is disconnected as well. For x, let $\tilde{R}_v \subset K(G^\natural)$ be the valuation ring of v. Since we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(\tilde{R}_v) \arrow[r] \arrow[d] & P^\natural \arrow[d] \\
S_v \arrow[r] & S
\end{tikzcd}$$

by definition of v, we have a morphism $\operatorname{Spec}(\tilde{R}_v) \to P^\natural_v$ taking the generic point of $\operatorname{Spec}(\tilde{R}_v)$ to the generic fiber of $P^\natural_v$. This shows that the morphism factors through $(P^\natural_v)'$. The image x' of the closed point in $(P^\natural_v)'_0$ lies over x and justifies the claim for x. For $G^\natural$, note that $G^\natural \subset P^\natural$ and $G^\natural$ is smooth over S. So $G^\natural \underset{S}{\times} S_v \subset P^\natural_v$ and it is smooth over S_v. This forces $G^\natural \underset{S}{\times} S_v \subset (P^\natural_v)'$ as every point in closed fiber must come from specialization of a point in the generic fiber. This justifies the claim for $G^\natural$ as well.

Let us claim that this $(P^\natural_v)'$ satisfies a stronger completeness condition than the one stated in condition (iii) of Definition 4.5.1.2:

Lemma 4.5.2.11 (cf. [100, Lem. 3.9])**.** *Let π be a generator of the maximal ideal of R_v, so that $(\pi) = I \cdot R_v$ in R_v. Let v' be a valuation of $K(G^\natural)$ such that $v'(R_v) \geq 0$ and $v'(\pi) > 0$. Then v' has a center on $(P^\natural_v)'$ if and only if, for all $\chi \in X$, there is some integer $n > 0$ such that $-nv'(\pi) \geq v'(x^*_{v'}(\mathscr{O}_\chi)) \geq nv'(\pi)$.*

Here $x_{v'}$ is the center of v' on $A_v := A \underset{S}{\times} S_v$, which exists because A_v is proper over S_v.

Proof of Proposition 4.5.2.11. By the original completeness condition (iii) in Definition 4.5.1.2 for $P^\natural$, we know that υ' with $\upsilon'(\pi) > 0$ has a center on $P^\natural_\upsilon = P^\natural \underset{S}{\times} S_\upsilon$ if for any $y \in Y$, there exists an integer $n > 0$ such that $\upsilon'(I^{\otimes n}_{y,\phi(y)} \cdot x^*_{\upsilon'}(\mathscr{O}_{\phi(y)})) \geq 0$. Since $I_{y,\phi(y)} \cdot R_\upsilon \subset I \cdot R_\upsilon = (\pi)$ when $y \neq 0$, and since $\phi(Y)$ has finite index in X, the original completeness condition can be restated as: For any $\chi \in X$, there exists some integer $n > 0$ such that $-n\upsilon'(\pi) \geq \upsilon'(x^*_{\upsilon'}(\mathscr{O}_\chi)) \geq n\upsilon'(\pi)$ as in the statement of the lemma. Since this center necessarily lies in the closure of the generic fiber of $P^\natural_{\upsilon'}$ in $P^\natural_\upsilon$, we see that the new condition holds for $(P^\natural_\upsilon)'$.

For the converse, note that $x^*_{\upsilon'}(\mathscr{O}_\chi)$ can be identified as an R_υ-submodule of K, and hence has a generator. Note also that $I_{y,\phi(y)} \subset I$ and $I \cdot R_\upsilon = (\pi)$ in R_υ. Hence the statement of Proposition 4.5.2.3 that $\mathscr{O}_\chi$ defines regular functions on $P^\natural$ over $R[I^{\otimes -1}_{y,\phi(y)}]$ implies that there is an integer $n > 0$ such that both $\upsilon'(\pi^n x^*_{\upsilon'}(\mathscr{O}_{\phi(y)})) \geq 0$ and $\upsilon'(\pi^n x^*_{\upsilon'}(\mathscr{O}_{-\phi(y)})) \geq 0$. Since $\phi(Y)$ has finite index in X, this shows that it is also true that, for all $\chi \in X$, there is some integer $n > 0$ such that $-n\upsilon'(\pi) \geq \upsilon'(x^*_{\upsilon'}(\mathscr{O}_\chi)) \geq n\upsilon'(\pi)$. ☐

Remark 4.5.2.12. The argument for the proof of the converse might not work for general υ as there might be infinitely many generators for $x^*_{\upsilon'}(\mathscr{O}_{\phi(y)})$. But it does work if υ is rank one. (See Remark 4.5.1.4 for some discussion on related issues.)

Back to the proof of Proposition 4.5.2.10. For simplicity, let us replace R by R_υ, S by S_υ, $P^\natural$ by its integral subscheme $(P^\natural_\upsilon)'$, and $G^\natural$ by $G^\natural \underset{S}{\times} S_\upsilon$ from now on. We shall also replace T by $T \underset{S}{\times} S_\upsilon$ and A by $A \underset{S}{\times} S_\upsilon$, so that $G^\natural$ is still the extension of A by T, and so that $P^\natural$ is still a scheme over A. Let us take π as in Lemma 4.5.2.11 to be any element of R such that (π) defines the maximal ideal of R. Then the old ideal I in the old R is replaced by the new maximal ideal (π) in the new R. The essential two properties we need to know about this new $P^\natural$ are as follows:

1. The generic fibers of $P^\natural$ and $G^\natural$ are the same.
2. (Restatement of Lemma 4.5.2.11:) For any valuation υ of $K(G^\natural)$ such that $\upsilon(R) \geq 0$ on R and $\upsilon(\pi) > 0$, υ has a center on $P^\natural$ if and only if for all $\chi \in X$, there is some integer $n > 0$ such that $-n\upsilon(\pi) \geq \upsilon(x^*_\upsilon(\mathscr{O}_\chi)) \geq n\upsilon(\pi)$. Here x_υ is the center of υ on A, which exists

> because A is proper over $S = \operatorname{Spec}(R)$. (Do not confuse the v here with the v in the previous paragraphs.)

Then we claim that this new $P^\natural$ over S cannot have a disconnected special fiber. This will contradict the choice of x and this new $P^\natural$, and conclude the proof.

Let $\chi_1, \ldots, \chi_r$ be a basis of X. For any n, define

$$P^{(n)} = \underline{\operatorname{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\pi^n \mathscr{O}_{\chi_1}, \pi^n \mathscr{O}_{-\chi_1}, \ldots, \pi^n \mathscr{O}_{\chi_r}, \pi^n \mathscr{O}_{-\chi_r}]).$$

Locally over A, over which each of the $\mathscr{O}_{\chi_i}$, for $i = 1, \ldots, r$, is principal and generated by some element u_i, we have an isomorphism

$$\begin{aligned} P^{(n)} &\cong \underline{\operatorname{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\pi^n u_1, \pi^n u_1^{-1}, \ldots, \pi^n u_r, \pi^n u_r^{-1}]) \\ &\cong \underline{\operatorname{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[u_1, v_1, \ldots, u_r, v_r]/(u_1 v_1 - \pi^{2n}, \ldots, u_r v_r - \pi^{2n})), \end{aligned}$$

where $v_r := u_r^{-1}$. This shows that $P^{(n)}$ is a relative complete intersection in $\mathbb{A}^{2r}_A$ over A, and is smooth over A outside a subset of codimension two. In particular, it is a normal scheme. Note that by construction $P^{(n)}$ is isomorphic to $G^\natural$ when π is invertible. Therefore $P^{(n)}$ and $P^\natural$ has the same generic fiber isomorphic to the one of $G^\natural$, and hence all of them has the same rational function field $K(G^\natural)$. Let $Z^{(n)} \subset P^{(n)} \underset{A}{\times} P^\natural$ be the join of this birational correspondence. By the restatement of Lemma 4.5.2.11 above, all valuations of $K(G^\natural)$ with a center on $P^{(n)}$ also have a center on P. Since $Z^{(n)}$ is locally of finite type over $P^{(n)}$, by Lemma 4.5.2.5, we see that $Z^{(n)} \to P^{(n)}$ is proper because it satisfies the valuative criterion for properness. By Zariski's connected theorem (see [47, III, 4.3.1]), this shows that all fibers of $Z^{(n)} \to P^{(n)}$ are connected, as this is the case over the generic fiber. Note that the closed fiber of $P^{(n)}$ is, locally over $A_0 := A \underset{R}{\otimes} R/(\pi)$, isomorphic to $\underline{\operatorname{Spec}}_{\mathscr{O}_{A_0}}(\mathscr{O}_{A_0}[u_1, v_1, \ldots, u_r, v_r]/(u_1 v_1, \ldots, u_r v_r))$, which is clearly connected. Therefore the closed fiber of $Z^{(n)}$ is connected.

If we set

$$W_n = \overline{\operatorname{pr}_2(Z_0^{(n)})} \subset P^\natural,$$

then W_n is connected too. By the above restatement of Lemma 4.5.2.11, every valuation v with a center x on $P_0^\natural$ has a center on *some* $P^{(n)}$. Therefore x can be lifted to a point of $Z^{(n)}$ for some n, or equivalently $x \in W_n$. This shows that $P_0^\natural = \underset{n}{\cup} W_n$. Note that, from the explicit expressions of the sheaves of algebras defining the schemes, there is a map from $P^{(m)}$ to $P^{(n)}$ when $n|m$, which is an isomorphism over the generic fiber. In particular, W_n and W_m have nontrivial intersection when $n|m$. This shows that $P_0^\natural = \underset{n}{\cup} W_n$ is connected, as desired. ☐

Proposition 4.5.2.13 (cf. [100, Thm. 3.10] and [37, Ch. III, Prop./Def. 4.8])**.** *For every integer $i \geq 1$, let $P_i^\natural := P^\natural \underset{S}{\times} S_i$. There exists a scheme P_i projective over $S_i = \operatorname{Spec}(R/I^{i+1})$ and an étale surjective morphism $\pi_i : P_i^\natural \to P_i$ such that P_i is the quotient of $P_i^\natural$ as fpqc-sheaf and such that the ample invertible sheaf $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ on $P_i^\natural$ descends to an ample invertible sheaf $\mathcal{L}_i$ on P_i. The schemes P_i fit together as i varies and form a formal scheme P_{for}, and the ample invertible sheaves $\mathcal{L}_i$ also fit together and form a formal ample invertible sheaf $\mathcal{L}_{\mathrm{for}}$ on P_{for}. Hence the pair $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ algebraizes uniquely to a pair $(P, \mathcal{L})$, where P is a projective scheme over S, and where $\mathcal{L}$ is a relatively ample invertible sheaf on P.*

Proof. By Proposition 4.5.2.8, there exists an integer $k \geq 1$ such that, under the action of the subgroup $kY \subset Y$, no two points of any open set

$$S_y(U) \underset{S}{\times} S_i$$

are identified. Thus we can form a quotient

$$\pi_i' : P^\natural \underset{S}{\times} S_i = \underset{y\in Y}{\cup} S_y(U) \underset{S}{\times} S_i \to P_i'$$

by simply gluing the open sets $S_y(U) \underset{S}{\times} S_i$ on their overlaps. Since Y acts on $\mathcal{L}^\natural \underset{R}{\otimes} R_i$, the invertible sheaf $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ descends to an invertible sheaf $\mathcal{L}_i'$ on P_i'.

Choose representatives $y_1, \ldots, y_t$ in Y for the cosets of kY in Y. The restriction of π_i' gives a surjection

$$\overset{t}{\underset{j=1}{\cup}} S_{y_j}(U) \underset{S}{\times} S_i \to P_i',$$

and hence also a surjection

$$\overset{t}{\underset{j=1}{\cup}} S_{y_j}(\overline{U_0}) \underset{S}{\times} S_i \to P_i'. \tag{4.5.2.14}$$

Since the scheme on the left-hand side of (4.5.2.14) is a finite union of schemes *proper* over S_i by Proposition 4.5.2.6, we see that P_i' is also proper over S_i.

The invertible sheaf $\mathcal{L}'_i$ on P'_i pulls back to the restriction of $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ on the left-hand side of (4.5.2.14), which is ample there. Then $\mathcal{L}'_i$ on P'_i is also ample by *Nakai's criterion.* (See [74].)

Without going into technical details, let us explain Nakai's criterion as follows: The criterion states that a Cartier divisor on a complete algebraic scheme is ample if and only if it is *arithmetically positive.* When the algebraic scheme in question is a nonsingular variety, this condition simply means its n-th power has strictly positive intersection number with all the n-dimensional subvarieties with $n > 0$. In general, one defines an analogue condition using polynomials defined by the Euler-Poincaré characteristics of the corresponding invertible sheaves. In any case, these conditions can be checked on each irreducible component. Since the two sides of (4.5.2.14) are locally glued from isomorphic irreducible components, the condition are trivially the same for the two sides. Hence the ampleness of $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ is equivalent to the ampleness of $\mathcal{L}'_i$, as desired.

Finally, the *finite group* Y/kY acts freely (by Corollary 4.5.2.9) on the *projective scheme* P'_i, and on the ample sheaf $\mathcal{L}'_i$. Hence a quotient $P_i = P'_i/(Y/kY)$ exists in the category of projective schemes, in the sense that $\mathcal{L}'_i$ also descends to an ample invertible sheaf $\mathcal{L}_i$ on P_i.

These projective schemes $(P_i, \mathcal{L}_i)$ obviously fit together and form a projective formal scheme $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ over $S_{\mathrm{for}} = \mathrm{Spf}(R, I)$. Hence, by Theorem 2.3.1.5, we know that $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ algebraizes to some projective scheme $(P, \mathcal{L})$, where $\mathcal{L}$ is relatively ample over $S = \mathrm{Spec}(R)$. □

Construction 4.5.2.15 (cf. [100, §3, p. 297] and [37, Ch. III, Def. 4.9]). We will pick out an open subscheme G of P, which can be viewed as a quotient of $G^\natural$ by the action of $\iota(Y)$. Moreover, $(G, \mathcal{L})$ with $\mathcal{L}$ standing for $\mathcal{L}|_G$ can be viewed as a quotient of $\mathcal{L}$ by the action of Y defined by ψ. Equivalently, we shall define the complement C of G in P. We shall proceed in the following steps:

1. $G^{\natural,*} := \bigcup\limits_{y \in Y} S_y(G^\natural) \subset P^\natural$ is an open subscheme of $P^\natural$, which form an open subscheme G_i of P_i under the quotient by Y over S_i. The formation of quotient is compatible between different n's as in the case of P_i, and we obtain an open formal subscheme $G_{\mathrm{for}} := \varinjlim G_i \subset P_{\mathrm{for}}$. G_{for} is canonically isomorphic to $G^\natural_{\mathrm{for}}$ since G_i is canonically isomorphic to $G^\natural_i = G^\natural \underset{S}{\times} S_i$.

2. $C^\natural := (P^\natural - G^{\natural,*})_{\mathrm{red}}$ is a closed reduced subscheme of $P^\natural$, which form a reduced closed subscheme C_i of P_i under the quotient by Y over S_i, whose underlying topological space is $P_i - G_i$. The formation of quotient is again compatible between different i's. Hence we obtain a closed formal subscheme $C_{\mathrm{for}} := \varinjlim C_i \subset P_{\mathrm{for}}$ whose underlying topological space is $P_{\mathrm{for}} - G_{\mathrm{for}}$.

3. C_{for} algebraizes to a reduced closed subscheme C of P. Define G to be the open subscheme $P - C$ of C. Then the formal completion of G is canonically isomorphic to G_{for}, and hence canonically isomorphic to $G^\natural_{\mathrm{for}}$.

We will prove in Section 4.5.3 that G is a group scheme whose group structure is compatible with the one of $G^\natural$ via the canonical isomorphism between G_{for} and $G^\natural_{\mathrm{for}}$. We shall regard this G as the quotient of $G^\natural$ by $\iota(Y)$.

Let us record some properties of G. Our first claim is that G is smooth over S. This follows from a general criterion provided by Mumford:

Proposition 4.5.2.16 (cf. [100, Prop. 4.1]). *Let $f_j : P_j \to S$, $j = 1, 2$ be two schemes over S, such that f_2 is proper. Suppose that there is an étale surjective morphism*

$$\pi : P_{1,\mathrm{for}} \to P_{2,\mathrm{for}}$$

between the I-adic completions of P_1 and P_2, and suppose that there are two closed subschemes $C_1 \subset P_1$ and $C_2 \subset P_2$ such that:

1. *$P_1 - C_1$ is smooth over S of relative dimension r.*

2. *If $C_{j,\mathrm{for}}$, $j = 1, 2$ are the formal completions of respective C_j, $j = 1, 2$. Then we have an inclusion of subschemes $C_1 \subset \pi^{-1}(C_2)$,* (*not just subsets.*)

Then we can conclude that $P_2 - C_2$ is also smooth over S of relative dimension r.

Let us include (almost verbatim) the proof here for the convenience of the reader.

Proof. First we need to check that $P_2 - C_2$ is flat over S. Let $\mathscr{M} \subset \mathscr{N}$ be two $\mathscr{O}_S$-modules. Consider the two kernels

$$0 \to \mathscr{K}_1 \to \mathscr{M} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_1} \to \mathscr{N} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_1}$$

and

$$0 \to \mathscr{K}_2 \to \mathscr{M} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_2} \to \mathscr{N} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_2}.$$

Since $P_1 - C_1$ is flat over S, we have $\mathrm{Supp}(\mathscr{K}_1) \subset C_1$, and hence, for all $x \in P_1$, we have $(\mathscr{K}_1)_x \cdot (\mathscr{I}_{C_1})_x^{\otimes n} = (0)$ for some integer $n > 0$. Taking I-adic completions, we get two exact sequences

$$0 \to \mathscr{K}_{1,\mathrm{for}} \to \mathscr{M} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{1,\mathrm{for}}} \to \mathscr{N} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{1,\mathrm{for}}}$$

and

$$0 \to \mathscr{K}_{2,\mathrm{for}} \to \mathscr{M} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{2,\mathrm{for}}} \to \mathscr{N} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{2,\mathrm{for}}},$$

and it follows that $\mathscr{K}_{1,\mathrm{for}} \cong \pi^*\mathscr{K}_{2,\mathrm{for}}$ because π is flat. Since $(\mathscr{K}_{1,\mathrm{for}})_x \cdot (\mathscr{I}_{C_{1,\mathrm{for}}})_x^{\otimes n} = (0)$ for all $x \in P_{1,\mathrm{for}}$, and since $C_1 \subset \pi^{-1}(C_2)$ as subschemes, it follows that $(\mathscr{K}_{2,\mathrm{for}})_{\pi(x)} \cdot (\mathscr{I}_{C_{2,\mathrm{for}}})_{\pi(x)}^{\otimes n} = (0)$. This implies that, in an open neighborhood of $f_2^{-1}(S_0)$, $\mathscr{K}_2$ is killed by $\mathscr{I}_{C_2}$, and hence $\mathrm{Supp}(\mathscr{K}_2) \subset C_2$ in that neighborhood. Since P_2 is proper over S, all closed points of P_2 lie over S_0, and hence $\mathrm{Supp}(\mathscr{K}_2) \subset C_2$ everywhere.

To show that $P_2 - C_2$ is actually smooth over S, it suffices to show that, in addition to being flat, it is differentially smooth. (See [52, IV, 16.10.1].) Namely, $\Omega^1_{P_2/S}$ is locally free of rank r outside C_2, and $\mathrm{Sym}^i(\Omega^1_{P_2/S}) \to \mathscr{P}^i_{P_2/S}$ is an isomorphism outside C_2 for each $i \geq 0$. Since we know these are true for P_1 over S, we deduce in particular that the following two statements are true at all points x of $f_1^{-1}(S_0)$:

1. For all $g \in (\mathscr{I}_{C_1})_x$, $(\Omega^1_{P_1/S})_x \underset{\mathscr{O}_{P_1,x}}{\otimes} \mathscr{O}_{P_1,x}[1/g]$ is locally free of rank r over $\mathscr{O}_S[1/g]$.
2. For each $i \geq 0$, the kernel and cokernel of $\mathrm{Sym}^i(\Omega^1_{P_1/S})_x \to \mathscr{P}^i_{P_1/S,x}$ are killed by the powers of $(\mathscr{I}_{C_1})_x$.

These two facts imply the corresponding facts for the formal scheme $P_{1,\mathrm{for}}$. Since π is étale, $\pi^*(\Omega^1_{P_2/S}) \cong \Omega^1_{P_1/S}$ and $\pi^*(\mathscr{P}^i_{P_2/S}) \cong \mathscr{P}^i_{P_1/S}$. Hence the assumption that $C_1 \subset \pi^{-1}(C_2)$ as subschemes implies the corresponding facts for $P_{2,\mathrm{for}}$. Finally, these imply the two statements for P_2 over S at the points $x \in f_2^{-1}(S_0)$. Since P_2 is proper over S, they hold everywhere on P_2. This shows that $P_2 - C_2$ is differentially smooth over S as well. □

Proposition 4.5.2.17 (cf. [100, Cor. 4.2])**.** *The scheme G is smooth over S.*

Proof. In the context of Proposition 4.5.2.16, simply take $P_1 = P^\natural$, $P_2 = P$, $C_1 = C^\natural$, $C_2 = C$, and use the fact that $G^\natural$ (and hence $G^{\natural,*} = \underset{y \in Y}{\cup} S_y(G^\natural)$) is smooth over S. ☐

Proposition 4.5.2.18 (cf. [100, Prop. 4.2])**.** *The scheme P is irreducible.*

Proof. By base change to complete discrete valuation rings, we may assume that S is *Nagata*. (See [90, 31.A, and 31.C, Cor. 2].) Then we may replace $P^\natural$ by its normalization, and the excellency of S implies that $P^\natural_{\text{for}}$ is also normal. As a result, we know that P_{for} and P are also normal. By Proposition 4.5.2.10, $P^\natural_0$ is connected, and hence $P_0 = P^\natural_0/Y$ is connected. But P is proper over S, and hence P is connected too. This shows that P is irreducible because it is normal. ☐

Remark 4.5.2.19. As pointed out by [37, Ch. III, §0], this is the only place in the original paper of Mumford [100, Thm. 4.3] where the excellency of the base scheme S is used. Moreover, as in the proof of Proposition 4.5.2.18 above, only the property of being Nagata is involved. (Excellent rings are automatically Nagata, by [90, 33.H, Thm. 78].) Hence we can remove the excellency assumption by base change to complete discrete valuation rings.

Proposition 4.5.2.20 (cf. [37, Ch. III, Prop. 4.12] and [100, Cor. 4.9])**.** *As subschemes of P, we have $G_\eta = P_\eta$. In particular, G_η is proper over η.*

Proof. Since $G^\natural_\eta = P^\natural_\eta$, there is a nonzero element r of R annihilating $\mathscr{O}_{C^\natural}$. Then $\mathscr{O}_C$ is also annihilated by r. In particular, we have $G_\eta = P_\eta$ as well. This shows that G_η is proper over η because P is. ☐

4.5.3 Functoriality

So far we have constructed the relatively complete model $(P^\natural, \mathcal{L}^\natural)$ for an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\text{split}}_{\text{ample}}$, and constructed a "quotient" P of $P^\natural$ by Y containing an open smooth subscheme G as a "quotient" of $G^\natural$ by $\iota(Y)$. In this section, we would like to establish the group structure of G as a consequence of the functoriality of Mumford's construction.

Definition 4.5.3.1 (cf. [100, Def. 4.4] and [37, Ch. III, Def. 5.1])**.** *Let $G_1^\natural$ be a semi-abelian subscheme of $G^\natural$ such that $G_1^\natural$ is the extension of an abelian subscheme A_1 of A by a* (*necessarily split*) *subtorus T_1 of T. Let Y_1 be the subgroup $\iota^{-1}(G_{1,\eta}^\natural)$ of Y. Then we say that $G_1^\natural$ is* **integrable** *if* $\mathrm{rk}_{\mathbb{Z}}(Y_1) = \dim(T_1)$.

Construction 4.5.3.2. Let us start with an inclusion $j^\natural : G_1^\natural \hookrightarrow G^\natural$ without assuming that $G_1^\natural$ is integrable. Let us denote the inclusion $Y_1 \hookrightarrow Y$ by j_Y, the map $\iota|_{Y_1} : Y_1 \to G_{1,\eta}^\natural$ by ι_1. Let us denote by $c_1 : X_1 \to A_1^\vee$ the map giving the extension structure of $G_1^\natural$, and by $\pi_1 : G_1^\natural \to A_1$ the canonical morphism. The inclusion $j^\natural$ induces the inclusions $j_T : T_1 \hookrightarrow T$ and $j_A : A_1 \hookrightarrow A$, and hence the surjections $j_X : X \twoheadrightarrow X_1 := X(T_1)$ and $j_A^\vee : A^\vee \twoheadrightarrow A_1^\vee$, which are compatible under $c_1 j_X = j_A^\vee c$. Let $c_1^\vee : Y_1 \to A_1$ be the unique map extending $\pi_1\iota_1 : Y_1 \to A_{1,\eta}$ by the properness of A_1. Then we have another compatibility $j_A c_1^\vee = c^\vee j_Y$ coming from the definitions. Let $\phi_1 := j_X \phi j_Y : Y_1 \to X_1$, and let $\lambda_{A_1} := j_A^\vee \lambda_A j_A : A_1 \to A_1^\vee$. Note that λ_{A_1} is a polarization, because étale locally λ_A is induced by some ample invertible sheaf $\mathcal{M}$, and λ_{A_1} is simply the polarization induced by the restriction $\mathcal{M}|_{A_1}$ of $\mathcal{M}$ to A_1. Then we have the compatibility $c_1\phi_1 = \lambda_{A_1} c_1^\vee$, because all the outer circles are commutative in the following diagram:

$$\begin{tikzcd}
Y \arrow[rrr, "c^\vee"] \arrow[ddd, "\phi"'] & & & A \arrow[ddd, "\lambda_A"] \\
& Y_1 \arrow[ul, hook', "j_Y"] \arrow[r, "c_1^\vee"] \arrow[d, "\phi_1"'] & A_1 \arrow[ur, hook, "j_A"'] \arrow[d, "\lambda_{A_1}"] & \\
& X_1 \arrow[r, "c_1"'] & A_1^\vee & \\
X \arrow[ur, two heads, "j_X"] \arrow[rrr, "c"'] & & & A^\vee \arrow[ul, two heads, "j_A^\vee"']
\end{tikzcd}$$

The map $\iota_1 : Y_1 \to G_{1,\eta}^\natural$ corresponds to a trivialization of biextensions $\tau_1 : \mathbf{1}_{Y_1 \times X_1,\eta} \xrightarrow{\sim} (c_1^\vee \times c_1)^* \mathcal{P}_{A_1,\eta}^{\otimes -1}$. For each $y_1 \in Y_1$ and $\chi_1 \in X_1$, where $\chi_1 = j_X(\chi)$ for some element $\chi \in X$, we have actually $(c_1^\vee(y_1), c_1(\chi_1))^*\mathcal{P}_{A_1,\eta} \cong (c_1^\vee(y_1), c_1 j_X(\chi))^*\mathcal{P}_{A_1,\eta} \cong (c_1^\vee(y_1), j_A^\vee c(\chi))^*\mathcal{P}_{A_1,\eta} \cong (c^\vee j_Y(y_1), c(\chi))^*\mathcal{P}_{A,\eta}$. The map $\tau_1(y_1,\chi_1) : (c_1^\vee(y_1), c_1(\chi_1))^*\mathcal{P}_{A_1,\eta} \to \mathscr{O}_\eta$ is by definition the map $\tau(j_Y(y_1),\chi) : (c^\vee j_Y(y_1), c(\chi))^*\mathcal{P}_{A,\eta} \to \mathscr{O}_\eta$. Since the positivity of τ_1 is measured by the image of $(c_1^\vee(y_1), c_1(\phi_1(y_1)))^*\mathcal{P}_A$ under

$\tau_1(y_1, \phi_1(y_1))$, which is the same as the image of $(c^\vee j_Y(y_1), c(\phi j_Y(y_1)))^*\mathcal{P}_A$ under $\tau(j_Y(y_1), \phi j_Y(y_1))$, the positivity of τ implies the positivity of τ_1. Indeed, $\tau_1|_{\mathbf{1}_{Y_1 \times \phi_1(Y_1),\eta}} : \mathbf{1}_{Y_1 \times \phi_1(Y_1),\eta} \overset{\sim}{\to} (c_1^\vee \times c_1)^*\mathcal{P}_{A_1,\eta}^{\otimes -1}$ is simply the restriction of $\tau|_{Y \times \phi(Y),\eta} : \mathbf{1}_{Y \times \phi(Y),\eta} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ to $Y_1 \times \phi_1(Y_1)$. In particular, ϕ_1 must be injective, and $\mathrm{rk}_{\mathbb{Z}}(Y_1) \leq \dim_S(T_1)$. Moreover, the fact that τ_1 is just a restriction τ shows that $\psi_1 := \psi|_{Y_1} : \mathbf{1}_{Y_1,\eta} \overset{\sim}{\to} \iota_1^*(\mathcal{L}^\natural|_{G_1^\natural})_\eta^{\otimes -1}$ is actually a cubical trivialization compatible with τ_1.

If the equality $\mathrm{rk}_{\mathbb{Z}}(Y_1) = \dim_S(T_1)$ holds, namely if $G_1^\natural$ is integrable, then we have everything we need for defining an object in $\mathrm{DD}_{\mathrm{ample}}$ or $\mathrm{DD}_{\mathrm{pol}}$. We record this fact as:

Lemma 4.5.3.3. *Given a tuple* $(A, \lambda_A, X, Y, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{pol}}$ *(resp.* $(A, X, Y, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ *in* $\mathrm{DD}_{\mathrm{ample}}$*, resp.* $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ *in* $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}$*), and* $G_1^\natural$ *an integral semi-abelian subscheme of* $G^\natural$*. Then the tuple* $(A_1, \lambda_{A_1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1)$ *(resp.* $(A_1, X_1, Y_1, \phi_1, c_1, c_1^\vee, \mathcal{L}^\natural|_{G_1^\natural}, \tau_1, \psi_1)$*, resp.* $(A_1, \mathcal{M}_{A_1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1, \psi_1)$*) given as in Construction 4.5.3.2 defines an object in* $\mathrm{DD}_{\mathrm{pol}}$ *(resp.* $\mathrm{DD}_{\mathrm{ample}}$*, resp.* $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}$*).*

Suppose now that $(P^\natural, \mathcal{L}^\natural)$ is a relatively complete model of an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}$, and $G_1^\natural \hookrightarrow G^\natural$ an integrable semi-abelian subscheme of $G^\natural$. We shall construct a closed subscheme W of P such that the intersection of W with G is an open subscheme G_1 of W whose formal completion $G_{1,\mathrm{for}}$ along I is canonically isomorphic to $G_{1,\mathrm{for}}^\natural$. And we shall show that G_1 is the quotient of $G_1^\natural$ by $\iota_1(Y_1)$ in the sense of Mumford's construction.

We shall proceed in the following steps:

1. Let $W_1^\natural$ be the scheme-theoretic closure of $G_1^\natural$ in $P^\natural$. Then $W_1^\natural$ is invariant under the restriction of the action of Y to Y_1 as $G_1^\natural$ is. Moreover, $W_1^\natural$ is reduced because $G_1^\natural$ is.

2. Let $W_{1,\mathrm{for}}^\natural$ be the I-adic completion of $W_1^\natural$, which is also Y_1-invariant. Let us define a formal subscheme $W_{\mathrm{for}}^\natural$ by

$$W_{\mathrm{for}}^\natural := \bigcup_{y \in Y/Y_1} S_y(W_{1,\mathrm{for}}^\natural)$$

as a subscheme of $P_{\mathrm{for}}^\natural$. This is not well-defined unless we can show that the right-hand-side is a locally finite union.

3. Let $W_{\mathrm{for}} := W_{\mathrm{for}}^{\natural}/Y$, which is a closed subscheme of P_{for}.

4. W_{for} algebraizes to a closed subscheme W of P. Then we define G_1 by $G_1 := W \cap G$.

The only nontrivial step in this construction is Step 2 where we need to show the *local finiteness* of the union. This requires the integrability condition of $G_1^{\natural}$:

Proposition 4.5.3.4 (cf. [100, Prop. 4.5] and [37, Ch. III, Prop. 5.4])**.** *Let $G_1^{\natural}$ be an integrable semi-abelian subscheme of $G^{\natural}$, and let $W_1^{\natural}$ be the scheme-theoretic closure of $G_1^{\natural}$ in $P^{\natural}$. Then there is a finite subset $Q \subset Y$ such that*

$$(W_1^{\natural} \underset{S}{\times} S_0) \cap S_y(U_0) \neq \emptyset$$

for all $y \notin Q + Y_1$.

Proof. Let X' be the kernel of the surjection $X \twoheadrightarrow X_1$, which is the subgroup of characters of T that are identically trivial on T_1. Since $\tau_1(z, \chi) = 1$ for any $\chi \in X'$ and any $z \in Y_1$, we have $\phi^{-1}(X') \cap Y_1 = \{0\}$. Therefore we see that $Y_1 + \phi^{-1}(X')$ has finite index in Y. Let $Y' := \{y \in Y : ny \in Y_1 \text{ for some integer } n \geq 1\}$. Then $Y = Y' \oplus \phi^{-1}(X')$, and the R-submodules $I_{z,\chi}$ of K for $z \in Y$ and $\chi \in X'$ depends only on the value of z modulo Y'.

For any nonzero $y \in Y$, choose an integer $n_y \geq 1$ such that $I_{y,\phi(y)}^{\otimes n_y} \cdot \mathscr{O}_{\phi(y)}$ is contained in $(\pi|_U)_* \mathscr{O}_U$ and is congruent to zero modulo I. This is possible by Proposition 4.5.2.3, because U is of finite type. By Lemma 4.5.1.9, there is a finite subset $\{y_1, \ldots, y_k\}$ of $\phi^{-1}(X')$, and a finite subset Q' of Y, such that, for any $z \in Y$ such that $z \notin Q' + Y'$, there exists some y_j, $1 \leq j \leq k$, such that $I_{z,\phi(y_j)} \subset (I_{y_j,\phi(y_j)}^{\otimes n_{y_j}})$. Consider $\mathscr{O}_\chi|_{S_z(U)}$ as the direct image of a sheaf of rational functions on $S_z(U)$. Then it corresponds to the sheaf $I_{z,\chi} \cdot \mathscr{O}_\chi|_U$ on U under the translation isomorphism $S_z : U \xrightarrow{\sim} S_z(U)$. If $z \notin Q' + Y'$, then the sheaf of rational functions $I_{z,\phi(y_j)} \cdot \mathscr{O}_{\phi(y_j)}|_U$ on U consists of regular functions that are congruent to zero modulo I. Hence the sheaf of regular functions $\mathscr{O}_{\phi(y_j)}|_{S_z(U)}$ on $S_z(U)$ consists of regular functions congruent to zero modulo I. This proves Proposition 4.5.3.4 if we take Q to be a set of representatives of $(Q' + Y')/Y_1$. □

Now we have arrived at the main result of this section, namely the functoriality of Mumford's construction:

Theorem 4.5.3.5 (cf. [100, Thm. 4.6] and [37, Ch. III, Thm. 5.5])**.** *Let*

$$(A_1, \mathcal{M}_1, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1, \psi_1)$$

and

$$(A_2, \mathcal{M}_2, X_2, Y_2, \phi_2, c_2, c_2^\vee, \tau_2, \psi_2)$$

be two objects of $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}$ *with relatively complete models respectively* $(P_1^\natural, \mathcal{L}_1^\natural)$ *and* $(P_2^\natural, \mathcal{L}_2^\natural)$*. Let* G_1 *and* G_2 *be respectively the two schemes constructed as in Section 4.5.2. Suppose we have a homomorphism* $h : Y_1 \to Y_2$ *and an* S*-homomorphism* $f^\natural : G_1^\natural \to G_2^\natural$ *such that* $\iota_2 \circ h = f \circ \iota_1$*. Then there is a unique* S*-homomorphism* $f : G_1 \to G_2$ *such that the* I*-adic completion* $f_{\mathrm{for}} : G_{1,\mathrm{for}}^\natural \to G_{2,\mathrm{for}}^\natural$ *is identical to* $f_{\mathrm{for}}^\natural : G_{1,\mathrm{for}} \to G_{2,\mathrm{for}}$*.*

Proof. Consider the datum

$$(A_1 \underset{S}{\times} A_2, \mathrm{pr}_1^* \mathcal{M}_1 \underset{\mathscr{O}_{A_1 \underset{S}{\times} A_2}}{\otimes} \mathrm{pr}_2^* \mathcal{M}_2, X_1 \times X_2, Y_1 \times Y_2,$$
$$\phi_1 \times \phi_2, c_1 \times c_2, c_1^\vee \times c_2^\vee, \tau_1 \times \tau_2, \psi_1 \times \psi_2)$$

defining the semi-abelian scheme $G_1^\natural \underset{S}{\times} G_2^\natural$, and let us define

$$H^\natural := \mathrm{image}((\mathrm{Id}_{G^\natural}, f^\natural) : G_1^\natural \to G_1^\natural \underset{S}{\times} G_2^\natural)$$

to be the graph of $f^\natural$. This defines a sub-semi-abelian scheme of $G_1^\natural \underset{S}{\times} G_2^\natural$, which is integral because $\iota_2 \circ h = f \circ \iota_1$. Let us take any relatively complete models $P_1^\natural$ and $P_2^\natural$ for respectively $G_1^\natural$ and $G_2^\natural$ and the data coming with them. As Proposition 4.5.3.4 and the steps remarked before it shows, this $H^\natural$ induces a closed subscheme of $G_1 \underset{S}{\times} G_2$ as follows:

1. $W_1^\natural :=$ scheme-theoretic closure of $H^\natural$ in $P_1^\natural \underset{S}{\times} P_2^\natural$.

2. $W_{\mathrm{for}}^\natural := \underset{y \in Y_1 \times Y_2}{\cup} S_y(W_{1,\mathrm{for}}^\natural)$.

3. $W_{\mathrm{for}} := W_{\mathrm{for}}^\natural / (Y_1 \times Y_2) \subset P_{1,\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}}$.

4. $H := W \cap (G_1 \underset{S}{\times} G_2)$, where W is the algebraization of W_{for}.

We claim that H defines the graph of a morphism from G_1 to G_2.

First, we need to show that the projection $\mathrm{pr}_1 : W \to P_1$ is smooth of relative dimension zero outside $C_1 := (P_1 - G_1)_{\mathrm{red}}$. This follows by essentially the same argument used in the proof of Proposition 4.5.2.16, as $\mathrm{pr}_1 : W_1^\natural \to P_1^\natural$ is smooth of relative dimension zero outside $C_1^\natural := (P_1^\natural - G_1^\natural)_{\mathrm{red}}$. Locally at every point, $W_{\mathrm{for}}^\natural$ is the formal completion of a finite union $S_{y_1}(W_1^\natural) \cup \ldots \cup S_{y_k}(W_1^\natural)$ for some $y_j \in Y_1 \times Y_2$, and since this is also smooth of relative dimension zero outside $C_1^\natural$, the same is true for $\mathrm{pr}_1 : W_{\mathrm{for}}^\natural \to P_{1,\mathrm{for}}^\natural$. Here by *smooth outside* $C_1^\natural$, we *do not* mean just smooth at points of $P_{1,\mathrm{for}}^\natural - C_{1,\mathrm{for}}^\natural$. Instead, we mean smoothness in the sense of the two statements in the proof of Proposition 4.5.2.16, namely smoothness after localizing by the ideal $\mathscr{I}_{C_1^\natural}$. This property descends to smoothness for $\mathrm{pr}_1 : W_{\mathrm{for}} \to P_{1,\mathrm{for}}$, and hence for $\mathrm{pr}_1 : W \to P_1$ as well.

Second, we need to prove that $W \cap (P_1 \underset{S}{\times} C_2) \subset C_1 \underset{S}{\times} C_2$. This follows by descending a stronger ideal-theoretic property on the schemes before quotient: For every finite set of y_j's, we have

$$[S_{y_1}(W_1^\natural) \cup \ldots \cup S_{y_k}(W_1^\natural)] \cap (P_1^\natural \underset{S}{\times} C_2^\natural) \subset C_1^\natural \underset{S}{\times} C_2^\natural.$$

Hence, on $P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}}^\natural$, we have

$$\mathscr{I}_{W_{\mathrm{for}}^\natural} + \mathscr{I}_{P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} C_{2,\mathrm{for}}^\natural} \supset \mathscr{I}^{\otimes N}_{C_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} C_{2,\mathrm{for}}^\natural}$$

for some integer $N > 0$. This property descends and algebraizes, which shows that $W \cap (P_1 \underset{S}{\times} C_2) \subset C_1 \underset{S}{\times} C_2$. Since $\mathrm{pr}_1 : W \to P_1$ is a proper morphism, this proves that the restriction $\mathrm{pr}_1 : H \to G_1$ is also proper.

Combining these two results, it follows that $\mathrm{pr}_1 : H \to G_1$ is finite and étale. Note that, just as $G_{1,\mathrm{for}} = G_{1,\mathrm{for}}^\natural$ and $G_{2,\mathrm{for}} = G_{2,\mathrm{for}}^\natural$, the formal completion H_{for} of H is the graph of the formal morphism $f^\natural : G_{1,\mathrm{for}}^\natural \to G_{2,\mathrm{for}}^\natural$ defined by $H^\natural$. Therefore $\mathrm{pr}_1 : H \to G_1$ has degree one over S_0. Since G_1 is irreducible, this shows that it has degree one everywhere. This proves that there is an S-morphism $f : G_1 \to G_2$ extending the formal morphisms defined by $f^\natural$. Finally, since G_1 is irreducible, such an f is clearly determined by its restriction to $G_{1,\mathrm{for}}$. $\square$

Corollary 4.5.3.6 (cf. [100, Cor. 4.7] and [37, Ch. III, Cor. 5.6]). *The scheme G depends* (*up to isomorphism*) *only on* $(A, X, Y, c, c^\vee, \tau)$ *as an object of* DD, *and is independent of the choice of ϕ, ψ, $\mathcal{M}$, and the relatively complete model $(P^\natural, \mathcal{L}^\natural)$.*

Proof. Let

$$(A, \mathcal{M}_1, X, Y, \phi_1, c, c^\vee, \tau, \psi_2)$$

and

$$(A, \mathcal{M}_2, X, Y, \phi_2, c, c^\vee, \tau, \psi_2)$$

be any two tuples in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ that extend the same tuple $(A, X, Y, c, c^\vee, \tau)$ in DD. Let $(G_1, \mathcal{L}_1)$ and $(G_2, \mathcal{L}_2)$ be the constructed respectively for the two tuples above for some choices of relatively complete models. Then by applying Theorem 4.5.3.5 to the identities $\mathrm{Id}_{G^\natural} : G^\natural \xrightarrow{\sim} G^\natural$ and $\mathrm{Id}_Y : Y \xrightarrow{\sim} Y$, we see that there is a canonical isomorphism $(G_1, \mathcal{L}_1) \xrightarrow{\sim} (G_2, \mathcal{L}_2)$ that induces the identity $\mathrm{Id}_{G^\natural_{\mathrm{for}}} : G^\natural_{\mathrm{for}} \xrightarrow{\sim} G^\natural_{\mathrm{for}}$, as desired. ☐

Corollary 4.5.3.7 (cf. [100, Cor. 4.8] and [37, Ch. III, Cor. 5.7])**.** *G is a group scheme over S.*

Proof. Consider the tuples

$$(A \underset{S}{\times} A, \mathrm{pr}_1^* \mathcal{M} \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathcal{M}, X \times X, Y \times Y, \phi \times \phi, c \times c, c^\vee \times c^\vee, \tau \times \tau, \psi \times \psi)$$

and

$$(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$$

in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ so that the second tuple defines the object $(G, \mathcal{L})$. By applying Theorem 4.5.3.5 to the multiplication maps $m_{G^\natural} : G^\natural \underset{S}{\times} G^\natural \to G^\natural$ and $m_Y : Y \times Y \to Y$, we obtain a morphism $m_G : G \underset{S}{\times} G \to G$. By applying Theorem 4.5.3.5 to the inverse isomorphisms $[-1]_{G^\natural} : G^\natural \xrightarrow{\sim} G^\natural$ and $[-1]_Y : Y \xrightarrow{\sim} Y$, we obtain an isomorphism $[-1]_G : G \xrightarrow{\sim} G$. The compatibility relations for m_G and $[-1]_G$ to define a group structure on G are satisfied, because they are satisfied by $(m_{G^\natural}, m_Y)$ and $([-1]_{G^\natural}, [-1]_Y)$ and because of the uniqueness statement in Theorem in Theorem 4.5.3.5. ☐

Corollary 4.5.3.8 (cf. [100, Cor. 4.9] and [37, Ch. III, Cor. 5.8])**.** *The scheme G_η is an abelian variety.*

Proof. We have now seen that G_η is a group scheme, and that it is proper and smooth by Propositions 4.5.2.20 and 4.5.2.17. Since P is irreducible by Proposition 4.5.2.18, we see that $G_\eta = P_\eta$ is also irreducible. Hence G_η is an abelian variety. $\square$

To prove that G has connected fibers and is indeed a semi-abelian scheme over S, it suffices to have a description of torsion points of G. Let $G^{\natural,*} = \underset{y\in Y}{\cup} S_y(G^\natural) \subset P^\natural$ as before. For any $y \in Y$ and any positive integer n, let $\sigma_y : S \to G^{\natural,*}$ be the unique S-section of $G^{\natural,*}$ such that $\sigma_y(\eta) = \iota(y)$. This is nothing but the translation of the identity section $e_{G^\natural} : S \to G^\natural$ under the action S_y on $P^\natural$. Define ${}^{(n)}Z_y^\natural$ by the fiber product

$$\begin{tikzcd}
{}^{(n)}Z_y^\natural \arrow[r] \arrow[d] & G^{\natural,*} \arrow[d, "{[n]}"] \\
S \arrow[r, "\sigma_y"'] & G^{\natural,*}
\end{tikzcd},$$

where $[n] : G^{\natural,*} \to G^{\natural,*}$ is the multiplication by n. For any $z \in Y$, it is easy to see that S_z induces an isomorphism from ${}^{(n)}Z_y^\natural$ to ${}^{(n)}Z_{y+nz}^\natural$ for all y. Therefore the disjoint union $\coprod\limits_{y\in Y/nY} {}^{(n)}Z_y^\natural$ is well-defined and has the structure of a commutative group scheme over S in a canonical way.

Theorem 4.5.3.9 (cf. [100, Thm. 4.10]). *The group scheme $G[n]$ is canonically isomorphic to the S-group scheme $\coprod\limits_{y\in Y/nY} {}^{(n)}Z_y^\natural$ constructed above.*

Proof. Let $\overline{{}^{(n)}Z_y^\natural}$ denote the closure of ${}^{(n)}Z_y^\natural$ in $P^\natural$. By the valuation property of the relatively complete model $P^\natural$, it follows that all valuations of $\overline{{}^{(n)}Z_y^\natural}$ have centers on $\overline{{}^{(n)}Z_y^\natural}$. By Lemma 4.5.2.5, this implies that $\overline{{}^{(n)}Z_y^\natural}$ is proper over S. Let $(\overline{{}^{(n)}Z_y^\natural})_{\text{for}}$ be its I-adic completion. We have seen in the proof of Proposition 4.5.3.4 and Theorem 4.5.3.5 above that if ${}^{(n)}W_1^\natural$ is the closure in $P^\natural \underset{S}{\times} P^\natural$ of the graph of $x \mapsto nx$, then ${}^{(n)}W^\natural := \underset{y\in Y}{\cup} S_{(0,y)}({}^{(n)}W_{1,\text{for}}^\natural) \subset P_{\text{for}}^\natural \underset{S_{\text{for}}}{\times} P_{\text{for}}^\natural$ is a locally finite union. Since $\overline{{}^{(n)}Z_y^\natural} \underset{S}{\times} \sigma_y(S) \subset {}^{(n)}W_1^\natural$, and hence $\overline{{}^{(n)}Z^{\natural}_{y}} \underset{S}{\times} \sigma_0(S) \subset S_{(0,-y)}({}^{(n)}W^{\natural}_{1})$, it follows from Proposition 4.5.3.4 that $\underset{y\in Y}{\cup}(\overline{{}^{(n)}Z^{\natural}_{y}})_{\mathrm{for}} \subset P^{\natural}_{\mathrm{for}}$ is a locally finite union, and that

$$\underset{y\in Y}{\cup}(\overline{{}^{(n)}Z^{\natural}_{y}})_{\mathrm{for}} = {}^{(n)}W^{\natural}_{\mathrm{for}} \cap (P^{\natural}_{\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} \sigma_0(S_{\mathrm{for}})).$$

Taking quotients by Y and $Y \times Y$ respectively, we obtain a formal closed subscheme of $P^{\natural}_{\mathrm{for}}$,

$$(\overline{{}^{(n)}Z})_{\mathrm{for}} := \left[\underset{y\in Y}{\cup}(\overline{{}^{(n)}Z^{\natural}_{y}})_{\mathrm{for}}\right]/Y = {}^{(n)}W_{\mathrm{for}} \cap (P_{\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} \sigma_0(S_{\mathrm{for}})),$$

where ${}^{(n)}W_{\mathrm{for}} = {}^{(n)}W^{\natural}_{\mathrm{for}}/(Y \times Y)$. It follows that $(\overline{{}^{(n)}Z})_{\mathrm{for}}$ algebraizes to a subscheme $\overline{{}^{(n)}Z} \subset P$ such that

$$\overline{{}^{(n)}Z} = {}^{(n)}W \cap (P \underset{S}{\times} \sigma_0(S)),$$

where ${}^{(n)}W$ is the algebraization of ${}^{(n)}W_{\mathrm{for}}$. Hence ${}^{(n)}Z := \overline{{}^{(n)}Z} \cap G$ satisfies

$${}^{(n)}Z = {}^{(n)}H \cap (G \underset{S}{\times} \sigma_0(S)),$$

where ${}^{(n)}H \subset G \underset{S}{\times} G$ is the graph of the morphism $x \mapsto nx$. Thus we see that ${}^{(n)}Z$ is the kernel $G[n]$ of the multiplication by n in G.

Now for every *finite* subset $Y_0 \subset Y$, we have a formal morphism

$$\overline{p}_{\mathrm{for}} : \underset{y\in Y_0}{\cup}(\overline{{}^{(n)}Z^{\natural}_{y}})_{\mathrm{for}} \to (\overline{{}^{(n)}Z})_{\mathrm{for}}.$$

Since these formal schemes are the completions of the schemes $\underset{y\in Y_0}{\cup} \overline{{}^{(n)}Z^{\natural}_{y}}$ and $\overline{{}^{(n)}Z}$, which are *proper* over S, this formal morphism $\overline{p}_{\mathrm{for}}$ algebraizes uniquely to a morphism

$$\overline{p} : \underset{y\in Y_0}{\cup} \overline{{}^{(n)}Z^{\natural}_{y}} \to \overline{{}^{(n)}Z^{\natural}}$$

of schemes (by Theorem 2.3.1.4). Since $C^{\natural} = (P^{\natural} - G^{\natural})_{\mathrm{red}}$ is the inverse image of $C = (P - G)_{\mathrm{red}}$ under the étale map $P^{\natural} \to P$, we see that $\underset{y\in Y_0}{\cup}(\overline{{}^{(n)}Z^{\natural}_{y}})_{\mathrm{for}} \cap$

$C^\natural_{\rm for}$ is the inverse image of $(\overline{{}^{(n)}Z})_{\rm for} \cap C_{\rm for}$. Hence $\underset{y\in Y_0}{\cup} \overline{{}^{(n)}Z^\natural_y} \cap C^\natural$ is the inverse image of $\overline{{}^{(n)}Z} \cap C$. Therefore the above map restricts to a proper morphism

$$p : \underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y \to {}^{(n)}Z.$$

Next, note that

$$\overline{p}_{\rm for} : \underset{y\in Y}{\cup} (\overline{{}^{(n)}Z^\natural_y})_{\rm for} \to (\overline{{}^{(n)}Z})_{\rm for}$$

is étale and surjective. It follows that, for each fixed $y_0 \in Y$, there is a finite subset $Y_0 \subset Y$ such that $\overline{p}_{\rm for} : \underset{y\in Y_0}{\cup} (\overline{{}^{(n)}Z^\natural_y})_{\rm for} \to (\overline{{}^{(n)}Z})_{\rm for}$ is étale at all points of $(\overline{{}^{(n)}Z^\natural_{y_0}})_{\rm for}$, and is surjective. Therefore, its algebraization $\overline{p}$ has the same properties. On the other hand, when we intersect with $G^\natural$, the union $\underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y$ is disjoint. Hence it follows that the $p : \underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y \to {}^{(n)}Z$ is étale for every Y_0, and is surjective for Y_0 large enough.

Now it is clear that the diagram

$$\begin{tikzcd}
{}^{(n)}Z^\natural_y \arrow[rr, "S_z"] \arrow[dr, "p"'] & & {}^{(n)}Z^\natural_{y+nz} \arrow[dl, "p"] \\
& {}^{(n)}Z &
\end{tikzcd}$$

commutes. Therefore the map $p : \underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y \to {}^{(n)}Z$ is already surjective when Y_0 is a set of coset representatives of Y/nY. By identifying $\underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y$ with $\coprod_{y\in Y/nY} {}^{(n)}Z^\natural_y$, we see that $\underset{y\in Y_0}{\cup} {}^{(n)}Z^\natural_y$ has a natural group scheme structure, and it is clear that p has degree one over the points $\sigma_0(S_0) \subset G_0$. However, it is easy to see that over S_0, ${}^{(n)}Z^\natural_y$ is empty unless $y \in nY$, ${}^{(n)}Z^\natural_0$ is the kernel of $[n]$ in the part of the torus $G^\natural_0$ over S_0, and p is just the restriction to the kernel of $[n]$ of the canonical isomorphism between $G^\natural_0$ and G_0 over S_0. Thus we see that p has degree one, or equivalently, $p : \coprod_{y\in Y/nY} {}^{(n)}Z^\natural_y \overset{\sim}{\to} {}^{(n)}Z = G[n]$ is an isomorphism. □

Proposition 4.5.3.10 (cf. [37, Ch. III, Prop. 5.10]). *For any $s \in S$ and any fixed y and n, the following statements are equivalent:*

1. *$({}^{(n)}Z^{\natural}_y)_s$ is nonempty.*
2. *There exists $z \in Y$ such that $y - nz \in G^{\natural} \underset{S}{\times} \mathrm{Spec}(\mathscr{O}_{S,s})$.*
3. *There exists $z \in Y$ such that $I_{y-nz,\phi(y-nz)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$.*
4. *$y \in Y_s + nY$, where*
$$\begin{aligned} Y_s &:= \{z \in Y : I_{z,\phi(z)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}\} \\ &= \{z \in Y : z \in G^{\natural}(\mathrm{Spec}(\mathscr{O}_{S,s}))\}. \end{aligned}$$

Proof. The first implies the second because, by the fiber product definition of ${}^{(n)}Z^{\natural}_y$, it is nonempty if there is a subscheme T of $G^{\natural}$ such that its translation $S_z(T)$ as a subscheme of $S_z(G^{\natural})$ is mapped to $\sigma_y(s) = S_y(e_{G^{\natural}})$ under the multiplication $[n] : G^{\natural,*} \to G^{\natural,*}$. This shows that $S_{nz}([n](T)) = S_y(e_{G^{\natural}})$, and therefore $S_{y-nz}(e_{G^{\natural}})$ defines a subscheme of $G^{\natural}_s$. That is, $y - nz$ extends to a section over s, and hence over $\mathrm{Spec}(\mathscr{O}_{S,s})$ by smoothness of $G^{\natural}$. Conversely, if $y - nz$ extends to a section over s, and if $T = G^{\natural}[n]_s$ is the subscheme of n-torsion points in $G^{\natural}_s$, then $S_z(T)$ is mapped to $S_y(e_{G^{\natural}})$ under the multiplication $[n] : G^{\natural,*} \to G^{\natural,*}$, which shows that $({}^{(n)}Z^{\natural}_y)_s$ is nonempty. In fact, this is exactly the image of ${}^{(n)}Z^{\natural}_y$ in $S_z(G^{\natural})$, as we have seen from the above arguments.

To show that the second implies the third, note that if $w = y - nz$, then $\iota(w)$ extends to a section of $G^{\natural} \underset{S}{\times} \mathrm{Spec}(\mathscr{O}_{S,s})$ if and only if $I_{w,\chi} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$ for any $\chi \in X$, as these $I_{w,\chi}$ are defined by $\tau(w, \chi)$, or equivalently by $\iota(w)$. In particular, $I_{w,\phi(w)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$ if $\iota(w)$ does extend. Conversely, as always we may reduce to the case that $S = \mathrm{Spec}(R_v)$ for some discrete valuation ring R_v and s is the closed point of S. Consider the positive semi-definite pairing $B(\,\cdot\,,\,\cdot\,) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (X \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ defined by $(y, \chi) \mapsto \upsilon(I_{y,\chi})$. Then, as in the proof of Lemma 4.5.1.7, we see that $w \in \mathrm{Rad}(B)$ if and only if $B(w, \phi(w)) = 0$. In other words, $\iota(w)$ extends to a section of $G^{\natural} \underset{S}{\times} \mathrm{Spec}(\mathscr{O}_{S,s})$ if and only if simply $I_{w,\phi(w)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$.

The last statement is just a restatement of a combination of the second and the third statements. □

Corollary 4.5.3.11 (cf. [100, Cor. 4.11] and [37, Ch. III, Cor. 5.11])**.** *For a positive integer n, and for any $s \in S$, there is a natural exact sequence*

$$0 \to G^{\natural}[n]_s \to G[n]_s \to \tfrac{1}{n}Y_s/Y_s \to 0.$$

Taking limits, we obtain an exact sequence

$$0 \to (G_s^\natural)_{\text{tors}} \to (G_s)_{\text{tors}} \to Y_s \underset{\mathbb{Z}}{\otimes} (\mathbb{Q}/\mathbb{Z}) \to 0.$$

Proof. As we have seen in the proofs of Theorem 4.5.3.9 and Proposition 4.5.3.10, $G[n]_s$ is isomorphic to the union of the translates of $G^\natural[n]_s$ under a coset representative of Y_s/nY_s. Hence the result follows. □

Remark 4.5.3.12. This shows in particular that $G[n]_s$ always has a subgroup defined over s whose rank is at least as large as the cardinality of Y_s/nY_s.

Corollary 4.5.3.13 (cf. [100, Cor. 4.12] and [37, Ch. III, Cor. 5.12])**.** *The geometric fibers of G over S are all connected with trivial unipotent radical. That is, G is a semi-abelian scheme over S.*

Proof. This follows from the general fact that if a commutative algebraic group H over an algebraically closed field has the property that the p-primary torsion of H is p-divisible (in the sense of group schemes) and scheme-theoretically dense in H for every prime number p, then H is connected with trivial unipotent radical, and hence is an extension of an abelian variety by a torus. By Corollary 4.5.3.11, this is exactly the case for G, and hence the result follows. □

4.5.4 Proof of the Equivalences

Let $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*}$ denote the full subcategory of $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}$ formed by those objects in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*}$ satisfying Condition 4.5.1.6. So far we have constructed in Sections 4.5.2 and 4.5.3 a functor

$$\mathrm{M}_{\mathrm{ample}}^{\mathrm{split},*} : \mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*} \to \mathrm{DEG}_{\mathrm{ample}} : (A, \mathcal{M}, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, \psi) \mapsto (G, \mathcal{L})$$

following Mumford and Faltings-Chai, such that $\mathrm{F}_{\mathrm{ample}}$ sends $(G, \mathcal{L})$ to an object isomorphic to $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural = \pi^*\mathcal{M}, \tau, \psi)$. The goal of this section is to show that the functor $\mathrm{M}_{\mathrm{ample}}^*$ actually induces functors

$$\begin{aligned} \mathrm{M} &: \mathrm{DD} \to \mathrm{DEG} \\ \mathrm{M}_{\mathrm{ample}} &: \mathrm{DD}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}} \\ \mathrm{M}_{\mathrm{pol}} &: \mathrm{DD}_{\mathrm{pol}} \to \mathrm{DEG}_{\mathrm{pol}} \\ \mathrm{M}_{\mathrm{IS}} &: \mathrm{DD}_{\mathrm{IS}} \to \mathrm{DEG}_{\mathrm{IS}} \end{aligned}$$

compatible in the obvious sense, so that $\mathrm{M}_{\mathrm{ample}}$ gives a quasi-inverse to the associations

$$\begin{aligned} \mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}} : (G, \mathcal{L}) &\mapsto (A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \mathcal{L}^{\natural}, \tau, \psi) \\ \mathrm{F}_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}} \to \mathrm{DD}_{\mathrm{pol}} : (G, \lambda) &\mapsto (A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau) \end{aligned}$$

described respectively in Theorem 4.2.1.8 and Definition 4.4.9. Then $\mathrm{M}_{\mathrm{ample}}$, $\mathrm{M}_{\mathrm{pol}}$, $\mathrm{F}_{\mathrm{ample}}$, and $\mathrm{F}_{\mathrm{pol}}$ will all be equivalences of categories as claimed in Theorem 4.4.18.

Definition 4.5.4.1. *The category* $\mathrm{DD}^*_{\mathrm{ample}}$ *is the subcategory of* $\mathrm{DD}_{\mathrm{ample}}$ *consisting of objects* $(A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \mathcal{L}^{\natural}, \tau, \psi)$ *over* S *such that the following is true: Over a finite étale surjection* $S' \to S$ *where the étale sheaves* $\underline{X}$ *and* $\underline{Y}$ *are constant with values respectively* X *and* Y*, so that there exists a relatively ample invertible sheaf* $\mathcal{M}$ *on* A *such that* $\mathcal{L}^{\natural} \cong \pi^*\mathcal{M}$ *(by Corollary 3.2.5.7), the tuple* $(A, \mathcal{M}, \underline{X}, \underline{Y}, c, c^{\vee}, \tau, \psi)$ *satisfies Condition 4.5.1.6.*

Proposition 4.5.4.2. *Let* $(A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \mathcal{L}^{\natural}, \tau, \psi)$ *be an object in* $\mathrm{DD}^*_{\mathrm{ample}}$. *Then there is a functor*

$$\mathrm{M}^*_{\mathrm{ample}} : \mathrm{DD}^*_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}} : (A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \mathcal{L}^{\natural}, \tau, \psi) \to (G, \mathcal{L})$$

such that $\mathrm{F}_{\mathrm{ample}}$ *sends* $(G, \mathcal{L})$ *to an object isomorphic to*

$$(A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \mathcal{L}^{\natural} = \pi^*\mathcal{M}, \tau, \psi)$$

in $\mathrm{DD}_{\mathrm{ample}}$ *(which necessarily lands in* $\mathrm{DD}^*_{\mathrm{ample}}$*).*

Proof. By assumption, there exists a finite étale base extension $S' \to S$ over which both $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and over which $\mathcal{L}^{\natural}_{S'} \cong \pi^*\mathcal{M}'$ for some relatively ample invertible sheaf $\mathcal{M}'$ on $A_{S'}$. Then we obtain a split object in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ over S' (with this choice of $\mathcal{M}'$), and the construction in Section 4.5.2 gives us an object $(G', \mathcal{L}')$ in $\mathrm{DEG}_{\mathrm{ample}}$ over S'. Since $\mathcal{L}'$ is relatively ample on G' over S', the pair $(G', \mathcal{L}')$ descends uniquely to a pair $(G, \mathcal{L})$ over S by fpqc descent. (See [58, VIII, 7.8].) □

Remark 4.5.4.3. In [37, Ch. III, §6, p. 71], they did not use the fact that the pullback of $\mathcal{M}$ to T can be trivialized over a finite étale surjection over S, and hence the argument was more complicated. The reason that they emphasize finite étale descent is because they are working over $\mathrm{Spf}(R)$, and an étale surjection over $\mathrm{Spf}(R)$ (which is a compatible system of étale surjections over $R_i = R/I^{i+1}$) might not be the formal completion of an étale surjection over $\mathrm{Spec}(R)$. This is not a problem if we have a finite morphism over $\mathrm{Spf}(R)$, which does algebraize to a finite morphism over $\mathrm{Spec}(R)$.

Back to our construction. We would like to extend the domain of the functor $\mathrm{M}^*_{\mathrm{ample}} : \mathrm{DD}^*_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$ to the whole category $\mathrm{DD}_{\mathrm{ample}}$. By Lemma 4.5.4.4 below, which is a consequence of Corollary 4.5.1.8, we can achieve the goal if we can show that the construction is compatible with tensor operation of invertible sheaves.

Lemma 4.5.4.4. *Let* $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}^\natural_1, \tau, \psi_1)$ *be an object in* $\mathrm{DD}_{\mathrm{ample}}$. *Let* $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}^\natural_2, \tau, \psi_2)$ *be either an object in* $\mathrm{DD}_{\mathrm{ample}}$*, or an object in* $\mathrm{DD}_{\mathrm{IS}}$ *such that all of* ϕ_2*,* $\mathcal{L}^\natural_2$*, and* ψ_2 *are trivial. Then there is an integer* $n_0 > 0$ *such that both*

$$(A, \underline{X}, \underline{Y}, (2n+1)\phi_1, c, c^\vee, (\mathcal{L}^\natural_1)^{\otimes n+1} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n}, \tau, \psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A, \underline{X}, \underline{Y}, \phi_2 + 2n\phi_1, c, c^\vee, \mathcal{L}^\natural_2 \underset{\mathscr{O}_{G^\natural}}{\otimes} (\mathcal{L}^\natural_1)^{\otimes n} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n}, \tau, \psi_2\psi_1^n[-1]^*\psi_1^n)$$

define objects in $\mathrm{DD}^*_{\mathrm{ample}}$.

Proof. After a finite étale base extension such that both $\underline{X}$ and $\underline{Y}$ become constant with values respectively X and Y, and such that $\mathcal{L}^\natural_1 = \pi^*\mathcal{M}_1$ and $\mathcal{L}^\natural_2 = \pi^*\mathcal{M}_2$ for some invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ on A, Corollary 4.5.1.8 tells us that both

$$(A, \mathcal{M}_1^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_1^{\otimes n}, X, Y, (2n+1)\phi_1, c, c^\vee, \tau, \psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A, \mathcal{M}_2 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_1^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_1^{\otimes n}, X, Y, \phi_2 + 2n\phi_1, c, c^\vee, \tau, \psi_2\psi_1^n[-1]^*\psi_1^n)$$

define objects in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$. In other words, even before we perform the finite étale base extension, both

$$(A, \underline{X}, \underline{Y}, (2n+1)\phi_1, c, c^\vee, (\mathcal{L}^\natural_1)^{\otimes n+1} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n}, \tau, \psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A,\underline{X},\underline{Y},\phi_2+2n\phi_1,c,c^\vee,\mathcal{L}_2^\natural\underset{\mathscr{O}_{G^\natural}}{\otimes}(\mathcal{L}_1^\natural)^{\otimes n}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_2\psi_1^n[-1]^*\psi_1^n)$$

define objects in $\mathrm{DD}^*_{\mathrm{ample}}$. □

Construction 4.5.4.5. Given any object $(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)$ in $\mathrm{DD}_{\mathrm{IS}}$, by Lemma 4.4.16, there exists $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)$ in $\mathrm{DD}_{\mathrm{ample}}$ such that

$$\begin{aligned}&(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\\&\quad=(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)\otimes(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)^{\otimes -1}.\end{aligned}$$

By Lemma 4.5.4.4, there is an integer $n_0>0$ such that both the tuples

$$(A,\underline{X},\underline{Y},(2n+1)\phi_1,c,c^\vee,(\mathcal{L}_1^\natural)^{\otimes n+1}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A,\underline{X},\underline{Y},\phi_2+2n\phi_1,c,c^\vee,\mathcal{L}_2^\natural\underset{\mathscr{O}_{G^\natural}}{\otimes}(\mathcal{L}_1^\natural)^{\otimes n}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_2\psi_1^n[-1]^*\psi_1^n)$$

define objects in $\mathrm{DD}^*_{\mathrm{ample}}$. For simplicity, let us replace the tuples $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)$ respectively by the tuples

$$(A,\underline{X},\underline{Y},(2n+1)\phi_1,c,c^\vee,(\mathcal{L}_1^\natural)^{\otimes n+1}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A,\underline{X},\underline{Y},\phi_2+2n\phi_1,c,c^\vee,\mathcal{L}_2^\natural\underset{\mathscr{O}_{G^\natural}}{\otimes}(\mathcal{L}_1^\natural)^{\otimes n}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_2\psi_1^n[-1]^*\psi_1^n).$$

Then the functor $\mathrm{M}^*_{\mathrm{ample}}$ defines objects $(G,\mathcal{L}_1)$ and $(G,\mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}$ for the two tuples in $\mathrm{DD}^*_{\mathrm{ample}}$, and defines an object $(G,\mathcal{F}:=\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2^{\otimes -1})$ in $\mathrm{DEG}_{\mathrm{IS}}$. Certainly, the point is to show that the assignment

$$(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\mapsto(G,\mathcal{F})$$

does not depend on the choices.

Lemma 4.5.4.6. *Suppose we are given a tuple* $(A, \mathcal{M}_1, X, Y, \phi_1, c, c^\vee, \tau, \psi_1)$ *(resp.* $(A, \mathcal{M}_2, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \tau, \psi_2)$*) in* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ *that admits a relatively complete model* $(P_1^\natural, \mathcal{L}_1^\natural)$ *(resp.* $(P_2^\natural, \mathcal{L}_2^\natural)$*) extending* $(G^\natural, \mathcal{L}_1^\natural := \pi_* \mathcal{M}_1)$ *(resp.* $(G^\natural, \mathcal{L}_2^\natural := \pi_* \mathcal{M}_2)$*), with quotient* $(G, \mathcal{L}_1)$ *(resp.* $(G, \mathcal{L}_2)$*) by Mumford's construction in Section 4.5.2. Then the tensor product* $(A, \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2, X, Y, \phi_1 + \phi_2, c, c^\vee, \tau, \psi_1\psi_2)$ *also admits a relatively complete model* $(P^\natural, \mathcal{L}^\natural)$ *extending* $(G^\natural, \mathcal{L}^\natural := \mathcal{L}_1^\natural \underset{\mathscr{O}_{G^\natural}}{\otimes} \mathcal{L}_2^\natural)$*, which has quotient* $(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2)$ *by Mumford's construction.*

Proof. Consider the product object

$$(A \underset{S}{\times} A, \mathrm{pr}_1^* \mathcal{M}_1 \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathcal{M}_2, X \times X, Y \times Y, \phi \times \phi,$$
$$c \times c, c^\vee \times c^\vee, \tau_\times := \mathrm{pr}_{13}^* \tau \, \mathrm{pr}_{13} + \mathrm{pr}_{24}^* \tau \, \mathrm{pr}_{24}, \psi_1 \times \psi_2)$$

and the diagonal embedding of $(A, \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2, X, Y, \phi_1 + \phi_2, c, c^\vee, \tau, \psi_1\psi_2)$ into it. Note that here the definition of $\tau_\times$ makes sense because

$$\mathcal{P}_{A \underset{S}{\times} A} \cong \mathrm{pr}_{13}^* \mathcal{P}_A \underset{\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A^\vee \underset{S}{\times} A^\vee}}{\otimes} \mathrm{pr}_{24}^* \mathcal{P}_A.$$

Then $(P_1^\natural \underset{S}{\times} P_2^\natural, \mathrm{pr}_1^* \mathcal{L}_1^\natural \underset{\mathscr{O}_{P_1^\natural \underset{S}{\times} P_2^\natural}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2^\natural)$ is a relatively complete model for the product object, and by taking the closure $P^\natural$ of the image of $G^\natural \to G^\natural \underset{S}{\times} G^\natural \hookrightarrow P_1^\natural \underset{S}{\times} P_2^\natural$, and the restriction of $\mathrm{pr}_1^* \mathcal{L}_1^\natural \underset{\mathscr{O}_{P_1^\natural \underset{S}{\times} P_2^\natural}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2^\natural$ to $P^\natural$, we obtain a relatively complete model of $(A, \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2, X, Y, \phi_1 + \phi_2, c, c^\vee, \tau, \psi_1\psi_2)$. The quotient of $(G^\natural \underset{S}{\times} G^\natural, \mathrm{pr}_1^* \mathcal{L}_1^\natural \underset{\mathscr{O}_{G^\natural \underset{S}{\times} G^\natural}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2^\natural)$ by Mumford's construction is $(G \underset{S}{\times} G, \mathrm{pr}_1^* \mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2)$, and the construction preserves the diagonal embedding. Since the pullback of $\mathrm{pr}_1^* \mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2$ along $G \hookrightarrow G \underset{S}{\times} G$ is $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2$, we see that the quotient of $(G^\natural, \mathcal{L}^\natural)$ by Mumford's construction is $(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2)$, as desired. $\square$

Corollary 4.5.4.7. *The assignment*

$$(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta) \mapsto (G, \mathcal{F})$$

as in Construction 4.5.4.5 is independent of choices, and defines a functor

$$\mathrm{M}_{\mathrm{IS}} : \mathrm{DD}_{\mathrm{IS}} \to \mathrm{DEG}_{\mathrm{IS}}$$

that extends $\mathrm{M}^*_{\mathrm{ample}} : \mathrm{DD}^*_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$. *In particular, it defines a functor*

$$\mathrm{M}_{\mathrm{ample}} : \mathrm{DD}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}.$$

By extending and forgetting extra structures, this also defines a functor

$$\mathrm{M} : \mathrm{DD} \to \mathrm{DEG}.$$

Proof. Note that uniqueness can be verified under étale descent. Therefore, after a finite étale base extension if necessary, we may assume that both $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and that we have four objects $(A, \mathcal{M}_1, X, Y, \phi_1, c, c^\vee, \tau, \psi_1)$, $(A, \mathcal{M}_2, X, Y, \phi_2, c, c^\vee, \tau, \psi_2)$, $(A, \mathcal{M}'_1, X, Y, \phi'_1, c, c^\vee, \tau, \psi'_1)$, and $(A, \mathcal{M}'_2, X, Y, \phi'_2, c, c^\vee, \tau, \psi'_2)$ such that for $\mathcal{L}_1^\natural := \pi^*\mathcal{M}_1$, $\mathcal{L}_2^\natural := \pi^*\mathcal{M}_2$, $\mathcal{L}_1^{\natural\prime} := \pi^*\mathcal{M}'_1$, and $\mathcal{L}_2^{\natural\prime} := \pi^*\mathcal{M}'_2$, we have

$$\begin{aligned}
&(A, X, Y, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta) \\
&= (A, X, Y, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau, \psi_1) \otimes (A, X, Y, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau, \psi_2)^{\otimes -1} \\
&= (A, X, Y, \phi'_1, c, c^\vee, \mathcal{L}_1^{\natural\prime}, \tau, \psi'_1) \otimes (A, X, Y, \phi'_2, c, c^\vee, \mathcal{L}_2^{\natural\prime}, \tau, \psi'_2)^{\otimes -1}
\end{aligned}$$

Let $(G, \mathcal{L}_1)$, $(G, \mathcal{L}_2)$, $(G, \mathcal{L}'_1)$, and $(G, \mathcal{L}'_2)$ be the respective quotients defined by Mumford's construction of the four objects in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ that we have introduced. Then we have two possible definitions of $(G, \mathcal{F})$, which are respectively $(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2^{\otimes -1})$ and $(G, \mathcal{L}'_1 \underset{\mathscr{O}_G}{\otimes} (\mathcal{L}'_2)^{\otimes -1})$. We need to show that $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2^{\otimes -1} = \mathcal{L}'_1 \underset{\mathscr{O}_G}{\otimes} (\mathcal{L}'_2)^{\otimes -1}$, which is the same as $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}'_2 = \mathcal{L}_2 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}'_1$. This follows from Lemma 4.5.4.6 because we have by assumption an equality of tensor products of objects in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$:

$$\begin{aligned}
&(A, X, Y, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau, \psi_1) \otimes (A, X, Y, \phi'_2, c, c^\vee, \mathcal{L}_2^{\natural\prime}, \tau, \psi'_2) \\
&= (A, X, Y, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau, \psi_2) \otimes (A, X, Y, \phi'_1, c, c^\vee, \mathcal{L}_1^{\natural\prime}, \tau, \psi'_1).
\end{aligned}$$

The remaining statements of the corollary are clear. □

Let us construct a functor $\mathrm{M}_{\mathrm{pol}} : \mathrm{DD}_{\mathrm{pol}} \to \mathrm{DEG}_{\mathrm{pol}}$ as well. For this purpose, we need to construct the dual of G using Mumford's construction as well. Let $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ be a tuple in DD. Let $\underline{X} \underset{S}{\times} \underline{Y} \xrightarrow{\sim} \underline{Y} \underset{S}{\times} \underline{X}$ be the isomorphism switching the two factors. Let $A^\vee \times A \xrightarrow{\sim} A \times A^\vee$ be the isomorphism switching the two factors, over which we have an isomorphism $\mathcal{P}_{A^\vee} \xrightarrow{\sim} \mathcal{P}_A$ covering it. Let $\tau^\vee : \mathbf{1}_{\underline{X} \underset{S}{\times} \underline{Y}, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A^\vee, \eta}^{\otimes -1}$ be defined by switching the factors in $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c \times c^\vee)^* \mathcal{P}_{A, \eta}^{\otimes -1}$ using the above-mentioned isomorphisms.

Definition 4.5.4.8. *The* **dual tuple** *of a tuple* $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ *in* DD *is defined by the tuple* $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ *(which is not yet known to be a tuple in* DD *as we do not know its extendability to an object in* $\mathrm{DD}_{\mathrm{pol}}$*).*

Lemma 4.5.4.9. *The dual tuple* $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ *defined above is also an object in* DD*.*

Proof. We have to show that there is a tuple $(A^\vee, \lambda_{A^\vee}, \underline{Y}, \underline{X}, \phi^\vee, c^\vee, c, \tau^\vee)$ in $\mathrm{DD}_{\mathrm{pol}}$ extending $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$, so that $\tau^\vee$ satisfies the positivity condition defined using $\phi^\vee$.

Let $G^\natural$ (resp. $G^{\vee,\natural}$) denote respectively the extension defined by $c : \underline{X} \to A^\vee$ (resp. $c^\vee : \underline{Y} \to (A^\vee)^\vee = A$). Let us take any ϕ and λ_A in addition to $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, \tau)$ that altogether define an object in $\mathrm{DD}_{\mathrm{pol}}$, and let us take an integer $N > 0$ sufficiently large such that there exists a map $\lambda^{\vee,\natural} : G^{\vee,\natural} \to G^\natural$ such that $\lambda^{\vee,\natural}\lambda^\natural = [N]$ and $\lambda^\natural\lambda^{\vee,\natural} = [N]$, the multiplications by N on respectively $G^\natural$ and $G^{\vee,\natural}$. (Do not confuse $\lambda^{\vee,\natural}$ as defined by the dual of any morphism. The notation simply means we are defining an morphism on the dual objects.) Then there exist an embedding $\phi^\vee : \underline{X} \hookrightarrow \underline{Y}$ and an isogeny $\lambda_{A^\vee} : A^\vee \to A$ with compatibility $c^\vee\phi^\vee = \lambda_{A^\vee}c)$ defining $\lambda^{\vee,\natural}$ such that $\phi'\phi = [N]$, $\phi\phi' = [N]$, $\lambda_{A^\vee}\lambda_A = [N]$, and $\lambda_A\lambda_{A^\vee} = [N]$ are multiplications by N on respectively $\underline{Y}$, $\underline{X}$, A, and $A^\vee$. Then $\lambda_{A^\vee}$ is necessarily a polarization by Corollary 1.3.2.25.

After an étale base extension so that $\underline{X}$ and $\underline{Y}$ become constant with values respectively X and Y, the positivity condition is satisfied for τ over $Y \times \phi(Y) \supset \phi'(X) \times \phi(\phi'(X)) = \phi'(X) \times NX$, and hence for $\tau^\vee$ over $NX \times \phi'(X)$, or equivalently over $X \times \phi'(X)$ by the bimultiplicativity. This shows that the tuple $(A^\vee, \lambda_{A^\vee}, \underline{Y}, \underline{X}, \phi^\vee, c^\vee, c, \tau^\vee)$ is indeed in $\mathrm{DD}_{\mathrm{pol}}$, as desired. □

Note that $\tau^\vee$ corresponds to a period map $\iota^\vee : \mathbf{1}_{\underline{X},\eta} \xrightarrow{\sim} G_\eta^{\vee,\natural}$.

By applying the functor M to the dual tuple $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ of $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$, we obtains an object G' in DEG. We claim that this G' is simply the dual semi-abelian scheme $G^\vee$ of G defined in Theorem 3.4.3.1.

To show that G' is dual to G, we need to construct an invertible sheaf $\mathcal{P}$ on $G \underset{S}{\times} G'$ whose restriction $\mathcal{P}_\eta$ to the fiber $G_\eta \underset{\eta}{\times} G'_\eta$ over η makes G'_η isomorphic to the dual abelian variety of G_η and serves as the Poincaré invertible sheaf on $G_\eta \underset{\eta}{\times} G'_\eta$ under this identification. More precisely, since $\mathcal{P}_\eta$ is birigidified, it will suffice to show that $\chi(\mathcal{P}_\eta) = \pm 1$, which sets up the *divisorial correspondence* between G_η and G'_η.

Remark 4.5.4.10. The rigidification properties of $\mathcal{P}_\eta$ and the universal property of $G_\eta^\vee$ induces a canonical morphisms $G'_\eta \to G_\eta^\vee$. To show that this morphism is an isomorphism, it suffices to check it over a geometric point $\bar{\eta}$ over η. Hence the result follows from the treatment of divisorial correspondences in the well-known reference [99, §8, Prop. 2] for abelian varieties defined over an algebraically closed field.

Construction 4.5.4.11. Let us first construct $(G \underset{S}{\times} G', \mathcal{P})$ as an object in $\mathrm{DEG}_{\mathrm{IS}}$. Since G is given by $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$, and G' is given by $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$, their product $G \underset{S}{\times} G'$ can be given by

$$(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, c \times c^\vee, c^\vee \times c, \tau_\times := \mathrm{pr}_{13}^* \tau \,\mathrm{pr}_{13} + \mathrm{pr}_{24}^* \tau^\vee \,\mathrm{pr}_{24}).$$

Here the definition of $\tau_\times$ makes sense because

$$\mathcal{P}_{A \underset{S}{\times} A^\vee} \cong \mathrm{pr}_{13}^* \mathcal{P}_A \underset{\mathscr{O}_{A \underset{S}{\times} A^\vee \underset{S}{\times} A^\vee \underset{S}{\times} A}}{\otimes} \mathrm{pr}_{24}^* \mathcal{P}_{A^\vee}.$$

Let $\pi : G^\natural \to A$ and $\pi^\vee : G^{\vee,\natural} \to A^\vee$ denote the canonical projections. The extra data we need for giving an invertible sheaf $\mathcal{P}_G$ on $G \underset{S}{\times} G'$ are as follows:

1. The switching isomorphism $f_{A \underset{S}{\times} A^\vee} : A \underset{S}{\times} A^\vee \xrightarrow{\sim} A^\vee \underset{S}{\times} A$.
2. The switching isomorphism $f_{\underline{Y} \underset{S}{\times} \underline{X}} : \underline{Y} \underset{S}{\times} \underline{X} \xrightarrow{\sim} \underline{X} \underset{S}{\times} \underline{Y}$.
3. The invertible sheaf $\mathcal{P}^\natural := (\pi \times \pi^\vee)^* \mathcal{P}_A$, with compatibility relation verified by
$$(c \times c^\vee) f_{\underline{Y} \underset{S}{\times} \underline{X}} = f_{A \underset{S}{\times} A^\vee} (c^\vee \times c).$$

4. The cubical trivialization $\psi_{\mathcal{P}} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (\iota \times \iota^\vee)^* \mathcal{P}^\natural_\eta$ defined by

$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c \times c^\vee)^* \mathcal{P}_{A,\eta} = (\iota \times \iota^\vee)^* (\pi \times \pi^\vee)^* \mathcal{P}_{A,\eta} = (\iota \times \iota^\vee)^* \mathcal{P}^\natural_\eta.$$

Note that

$$\mathcal{D}_2(\mathcal{P}_A) \cong \operatorname{pr}_{14}^* \mathcal{P}_A \underset{\mathscr{O}_{A \underset{S}{\times} A^\vee \underset{S}{\times} A \underset{S}{\times} A^\vee}}{\otimes} \operatorname{pr}_{23}^* \mathcal{P}_{A^\vee} \cong (\mathrm{Id}_{A \underset{S}{\times} A^\vee} \times f_{A \underset{S}{\times} A^\vee})^* \mathcal{P}_{A \underset{S}{\times} A^\vee}.$$

To see this, we use the biextension structure of $\mathcal{P}_A$ and evaluate $\mathcal{D}_2(\mathcal{P}_A)$ at any functorial point (a_1, a'_1, a_2, a'_2) of $A \underset{S}{\times} A^\vee \underset{S}{\times} A \underset{S}{\times} A^\vee$. Then we obtain

$$\begin{aligned} &\mathcal{P}_A|_{(a_1+a_2, a'_1+a'_2)} \underset{\mathscr{O}_S}{\otimes} (\mathcal{P}_A|_{(a_1,a_2)})^{\otimes -1} \underset{\mathscr{O}_S}{\otimes} (\mathcal{P}_A|_{(a'_1,a'_2)})^{\otimes -1} \\ &\quad \cong \mathcal{P}_A|_{(a_1,a'_2)} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(a_2,a'_1)} \cong \mathcal{P}_A|_{(a_1,a'_2)} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_{A^\vee}|_{(a'_1,a_2)}, \end{aligned}$$

as desired. Moreover,

$$\mathcal{D}_2(\psi_{\mathcal{P}}) = \mathcal{D}_2(\tau) = (\mathrm{Id}_{\underline{Y} \underset{S}{\times} \underline{X}} \times f_{\underline{Y} \underset{S}{\times} \underline{X}})^* \tau_\times,$$

because at any functorial point $(y_1, \chi_1, y_2, \chi_2)$ of $\underline{Y} \underset{S}{\times} \underline{X} \underset{S}{\times} \underline{Y} \underset{S}{\times} \underline{X}$, we have

$$\begin{aligned} \mathcal{D}_2(\tau)(y_1, \chi_1, y_2, \chi_2) &= \tau(y_1 + y_2, \chi_1 + \chi_2) \tau(y_1, \chi_1)^{-1} \tau(y_2, \chi_2)^{-1} \\ &= \tau(y_1, \chi_2) \tau(y_2, \chi_1) = \tau(y_1, \chi_2) \tau^\vee(\chi_1, y_2). \end{aligned}$$

This is the compatibility between $\psi_{\mathcal{P}}$ and $\tau_\times$ that we need.

Thus $(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, f_{\underline{Y} \underset{S}{\times} \underline{X}}, c, c^\vee, \mathcal{P}^\natural, \tau_\times, \psi_{\mathcal{P}})$ defines an object in $\mathrm{DD}_{\mathrm{IS}}$, and we obtain by applying M_{IS} an object $(G \underset{S}{\times} G', \mathcal{P})$ in $\mathrm{DEG}_{\mathrm{IS}}$.

As we explained before, we need to show that $\chi(\mathcal{P}_\eta) = \pm 1$ to show that $\mathcal{P}_\eta$ establishes a divisorial correspondence between G_η and G'_η.

We shall prove a more general result that computes $\chi(\mathcal{F}_\eta)^2$ when $(G, \mathcal{F})$ is associated to some tuple $(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$ in $\mathrm{DD}_{\mathrm{IS}}$ under M_{IS}. Since this is a question about equalities, we can always make an étale base extension and assume that both $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and that $\mathcal{F} \cong \pi^* \mathcal{N}$ for some invertible sheaf $\mathcal{N}$ on A inducing $f_A : A \to A^\vee$.

Theorem 4.5.4.12 (cf. [37, Ch. III, Thm. 6.1])**.** *Assume that we have a tuple* $(A, \mathcal{N}, X, Y, f_Y, c, c^\vee, \tau, \zeta)$ *as above such that* $(A, X, Y, f_Y, c, c^\vee, \mathcal{F}^\natural = \pi^*\mathcal{N}, \tau, \zeta)$ *is an object in* $\mathrm{DD}_{\mathrm{IS}}$*. Then* $\chi(\mathcal{F}_\eta)^2 = \chi(\mathcal{N}_\eta)^2 \cdot \deg(f_Y)^2 = \deg(f_A) \cdot \deg(f_Y)^2$*.*

Proof. The relation $\chi(\mathcal{N}_\eta) = \deg(f_A)$ follows from the Riemann-Roch theorem for abelian varieties. (See [99, §16].) Therefore it suffices to prove the first equality. Starting with the given object $(A, \mathcal{N}, X, Y, f_Y, c, c^\vee, \tau, \zeta)$, there is always an object $(A, \mathcal{M}_0, X, Y, \phi_0, c, c^\vee, \tau, \psi_0)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ such that for some integer $n_0 > 0$, $(A, \mathcal{N} \underset{\mathcal{O}_A}{\otimes} \mathcal{M}_0^{\otimes n}, X, Y, f_A + n\phi_0, c, c^\vee, \tau, \zeta\psi_0^n)$ is in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}$ for all $n \geq n_0$, because we only need the injectivity of $f_A + n\phi_0$ and the relative ampleness of $\mathcal{N} \underset{\mathcal{O}_A}{\otimes} \mathcal{M}_0^{\otimes n}$. This is still true if we replace $\mathcal{M}_0$ by $\mathcal{M}_1 := \mathcal{M}_0 \underset{\mathcal{O}_A}{\otimes} [-1]^*\mathcal{M}_0$, ϕ_0 by $\phi_1 := \phi_0 + [-1]^*\phi_0$, and ψ_0 by $\psi_1 := \psi_0[-1]^*\psi_0$. By Corollary 4.5.1.8, there is another $n_1 \geq n_0$ such that $(A, \mathcal{N} \underset{\mathcal{O}_A}{\otimes} \mathcal{M}_1^{\otimes n}, X, Y, f_A + n\phi_1, c, c^\vee, \tau, \zeta\psi_1^n)$ is in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$ for all $n \geq n_1$. Let $(G, \mathcal{L}_1)$ be the pair in $\mathrm{DEG}_{\mathrm{ample}}$ associated to $(A, X, Y, \phi_1, c, c^\vee, \mathcal{L}_1^\natural = \pi_*\mathcal{M}_1, \tau, \psi_1)$ in $\mathrm{DD}_{\mathrm{ample}}$ by $\mathrm{M}_{\mathrm{ample}}$. Let us consider $\chi(\mathcal{F}_\eta \underset{\mathcal{O}_{G,\eta}}{\otimes} \mathcal{L}_{1,\eta}^{\otimes n})^2$ and $\chi(\mathcal{N}_\eta \underset{\mathcal{O}_{A,\eta}}{\otimes} \mathcal{M}_{1,\eta}^{\otimes n})^2 \cdot \deg(f_Y + n\phi_1)$, which are both polynomials in n by the Riemann-Roch theorem for abelian varieties. (See [99, §16] again.) Therefore, if the equality holds for all $n \geq n_1$, then it actually holds for all n, and in particular $n = 0$. Therefore it suffices to deal with the case that we have an arbitrary object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}$ that admits a relatively complete model $(P^\natural, \mathcal{L}^\natural)$ with quotient $(P, \mathcal{L})$ by Mumford's construction.

Recall (from Section 4.3) that the partial Fourier expansion of a section $s \in \Gamma(G, \mathcal{L})$ is obtain by considering the image of s in $\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})$, which has an T_{for}-action making the space an I-adic complete direct sum of weight spaces $\Gamma(G_{\mathrm{for}}, \mathcal{O}_{G,\mathrm{for}}) = \widehat{\bigoplus_{\chi \in X}} \Gamma(A, \mathcal{O}_\chi)$. Then we can write $s = \sum_{\chi \in X} \sigma_\chi(s)$, where $\sigma_\chi(s) \in \Gamma(A, \mathcal{O}_\chi)$. Since we have the relatively complete model $(P^\natural, \mathcal{L}^\natural)$, and since $P_\eta = G_\eta$, there is a nonzero element $a \in R$ such that $a \cdot s \in \Gamma(P, \mathcal{L})$. Since P_{for} is the quotient of $P^\natural_{\mathrm{for}}$ under Y-action, we have an embedding

$$\Gamma(P, \mathcal{L}) \to \Gamma(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})^Y \subset \Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}).$$

Since the Y-translates of $G^\natural_{\text{for}}$ cover $P^\natural_{\text{for}}$, the composition of $\Gamma(P^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}})^Y \subset \Gamma(P^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}})$ with the restriction map $\Gamma(P^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}}) \to \Gamma(G^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}})$ remains to be injective. Therefore we have an embedding

$$\Gamma(P, \mathcal{L}) \subset \Gamma(G^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}}),$$

from which we obtain a Fourier expansion

$$a \cdot s = \sum_{\chi \in X} a \cdot \sigma_\chi(s)$$

with Y-invariance described by

$$\sigma_{\chi+\phi(y)}(s) = \tilde{S}_y(T^*_{c^\vee(y)}\sigma_\chi(s)).$$

This shows that we may identify $\Gamma(G_\eta, \mathcal{L}_\eta)$ with the K-subspace V of $\Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \underset{R}{\otimes} K = \Gamma(G^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}}) \underset{R}{\otimes} K$ spanned by the infinite sums

$$\{\sum_{\chi \in X} \theta_\chi : \theta_\chi \in \Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}),\ \theta_{\chi+\phi(y)} = \tilde{S}_y(T^*_{y^\vee}(\theta_\chi))\}.$$

We claim that $\Gamma(G_\eta, \mathcal{L}_\eta)$ span the whole of V. It suffices to show that for every element θ_χ in $\Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta})$, there is a nonzero element $a \in R$ such that $a \sum_{y \in Y} \tilde{S}_y(T^*_{y^\vee}(\theta_\chi))$ lies in $\Gamma(P^\natural_{\text{for}}, \mathcal{L}^\natural_{\text{for}})^Y = \Gamma(P_{\text{for}}, \mathcal{L}_{\text{for}})$. The only thing we need to show is that this sum converges I-adically. This follows because there is an G-invariant open subscheme $U \subset P^\natural$ of finite type over S such that $\underset{y \in Y}{\cup} S_y(U) = P^\natural$. Then the proof of the claim follows from the following:

Lemma 4.5.4.13. *Let χ be a fixed element in X. For any $n > 0$, we have $I_y \cdot I_{y,\chi} \subset I^n$ for all but finitely any $y \in Y$.*

Proof of Lemma 4.5.4.13. As in 4.2.3, let us denote by Υ_1 the set of valuations of K defined by height one primes of R, and let us denote by Υ_I the set of discrete valuations v of K having centers on $S_0 = \text{Spec}(R_0)$.

For convenience, let us denote the functions $Y \to \text{Inv}(R) : y \mapsto I_y$ and $Y \times X \to \text{Inv}(R) : (y, \chi) \mapsto I_{y,\chi}$ respectively by a and b.

Let us first show that $I_y \cdot I_{y,\chi} \subset R$ for all but finitely many $y \notin Q$. Note that $I_y \subset R$ for all but finitely many $y \in Y$. So it suffices to consider the finitely many valuations υ in Υ_1 for which $\upsilon(I_{y,\chi}) < 0$ can happen. As in the proof of Lemma 4.5.1.7, by forming quotient by the radical of the associated pairing for each υ, we may assume that $\upsilon(b(\,\cdot\,,\phi(\,\cdot\,)))$ is positive definite. Let $\|y\|_\upsilon := \upsilon(b(y,\phi(y)))^{1/2}$ be the associated norm on the real vector space $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, in which we have two lattices X and Y with X embedded in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ via the embedding $\phi : Y \hookrightarrow X$ with finite cokernel. Since the quadratic function $\upsilon(a(\,\cdot\,))$ has associated bilinear pairing $\upsilon(b(\,\cdot\,,\phi(\,\cdot\,)))$, which is positive semi-definite, there are constants $A_\upsilon, B_\upsilon, C_\upsilon \in \mathbb{R}$, with $A_\upsilon > 0$, such that

$$\upsilon(a(y)) > A_\upsilon \|y\|_\upsilon^2 + B_\upsilon \|y\|_\upsilon + C_\upsilon$$

for all $y \in Y$. Now $\upsilon(b(y,\chi)) > \|y\|_\upsilon \|\chi\|_\upsilon$, and hence we have

$$\upsilon(a(y)) > A_\upsilon \|y\|_\upsilon^2 + (B_\upsilon - \|\chi\|_\upsilon)\|y\|_\upsilon + C_\upsilon$$

with $A_\upsilon > 0$. In particular, $\upsilon(a(y) \cdot b(y,\chi)) \geq 0$ for all but finitely many y. Since the number of υ to consider is finite, by Lemma 4.2.4.2, we have $I_y \cdot I_{y,\chi} \subset R$ for all but finitely many $y \in Y$.

Next, let us show that for any $n > 0$, we have $I_y \cdot I_{y,\chi} \subset I^n$ for all but finitely many $y \in Y$. Let us first exclude those $y \in Y$ such that $I_y \cdot I_{y,\chi} \subset R$ is not true. Note that by the positivity condition for ψ, $I_y \subset I^n$ for all but finitely $y \in Y$. As a result, we only need to consider those finitely $\upsilon \in \Upsilon_I$ for which $\upsilon(b(y,\chi)) < 0$ can happen. Moreover, for any $\upsilon \in \Upsilon_I$, there exists again constants $A_\upsilon, B_\upsilon, C_\upsilon \in \mathbb{R}$, with $A_\upsilon > 0$, such that

$$\upsilon(a(y)) > A_\upsilon \|y\|_\upsilon^2 + B_\upsilon \|y\|_\upsilon + C_\upsilon$$

for all $y \in Y$. Now $\upsilon(b(y,\chi)) > \|y\|_\upsilon \|\chi\|_\upsilon$, and hence we have

$$\upsilon(a(y)) > A_\upsilon \|y\|_\upsilon^2 + (B_\upsilon - \|\chi\|_\upsilon)\|y\|_\upsilon + C_\upsilon$$

with $A_\upsilon > 0$. In particular, $\upsilon(a(y) \cdot b(y,\chi)) \geq n$ for all but finitely many y. Since the number of υ to consider is finite, by Lemma 4.2.4.4, we have $I_y \cdot I_{y,\chi} \subset I^n$ for all but finitely many $y \in Y$, as desired. □

Back to the proof of Theorem 4.5.4.12. Now that we have proved Lemma 4.5.4.13, we can identify $\Gamma(G_\eta, \mathcal{L}_\eta^\natural)$ with V. Using the fact that $\dim_K \Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) = \dim_K \Gamma(A_\eta, \mathcal{M}_\eta) = \chi(\mathcal{M}_\eta)$ for all $\chi \in X$, we see that $\chi(\mathcal{L}_\eta^\natural) = \chi(\mathcal{M}_\eta) \cdot \deg(\phi)$, which proves the theorem. □

Corollary 4.5.4.14. *Given any tuple* $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ *in* DD *that defines* G *by* M : DD → DEG*, the dual tuple* $(\lambda_A, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ *as defined in Definition 4.5.4.8 defines* $G^\vee$ *by* M*.*

As a corollary of proof of Theorem 4.5.4.12:

Corollary 4.5.4.15. *The functor* $\mathrm{F}_{\mathrm{ample}} : \mathrm{DEG}_{\mathrm{ample}} \to \mathrm{DD}_{\mathrm{ample}}$ (*given by Theorem 4.2.1.8*) *is a quasi-inverse to* $\mathrm{M}_{\mathrm{ample}}$*, and hence both* $\mathrm{F}_{\mathrm{ample}}$ *and* $\mathrm{M}_{\mathrm{ample}}$ *are equivalences of categories.*

Proof. Indeed, by reduction to the case for objects in $\mathrm{DD}^*_{\mathrm{ample}}$, we see by Proposition 4.5.4.2 that the composition $\mathrm{F}_{\mathrm{ample}} \mathrm{M}_{\mathrm{ample}}$ is canonically isomorphic to the identity. Then $\mathrm{F}_{\mathrm{ample}} \mathrm{M}_{\mathrm{ample}} \mathrm{F}_{\mathrm{ample}}$ is also canonically isomorphic to $\mathrm{F}_{\mathrm{ample}}$. As we already know from Theorem 4.2.1.8 that $\mathrm{F}_{\mathrm{ample}}$ detects isomorphisms, $\mathrm{M}_{\mathrm{ample}} \mathrm{F}_{\mathrm{ample}}$ must be also isomorphic to the identity. This shows that $\mathrm{F}_{\mathrm{ample}}$ and $\mathrm{M}_{\mathrm{ample}}$ are quasi-inverse to each other, and hence are both equivalences of categories. □

Finally, we are ready to define $\mathrm{M}_{\mathrm{pol}}$.

Construction 4.5.4.16. For any object $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$, we obtain a morphism from $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ to $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ in DD given by $\lambda_A : A \to A^\vee$, $\phi : \underline{Y} \hookrightarrow \underline{X}$, and $\phi : \underline{Y} \hookrightarrow \underline{X}$. By Corollary 4.5.4.14, the tuple $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ defines $G^\vee$ by M. As a result, we obtain a morphism $\lambda : G \to G^\vee$.

Lemma 4.5.4.17. *This morphism* λ *is a polarization.*

Proof. By Definition 1.3.2.20, we shall verify the condition in Proposition 1.3.2.18 that the pullback $(\mathrm{Id}_G, \lambda)^* \mathcal{P}$ is ample over G_η. To do this, note that the embedding

$$(\mathrm{Id}_G, \lambda) : G \to G \underset{S}{\times} G^\vee$$

is given in terms of degeneration data by the morphism from $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ to $(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, c \times c^\vee, c^\vee \times c, \tau_\times)$, where $\tau_\times := \mathrm{pr}_{13}^* \tau \, \mathrm{pr}_{13} + \mathrm{pr}_{24}^* \tau^\vee \, \mathrm{pr}_{24}$ as in Construction 4.5.4.11, defined by the morphisms

$$(\mathrm{Id}_A, \lambda_A) : A \to A \underset{S}{\times} A^\vee,$$

$$\mathrm{Id}_{\underline{X}} + \phi : \underline{X} \underset{S}{\times} \underline{Y} \to \underline{X},$$

and

$$(\mathrm{Id}_{\underline{Y}}, \phi) : \underline{Y} \hookrightarrow \underline{Y} \underset{S}{\times} \underline{X}.$$

Recall that the tuple $(G \underset{S}{\times} G^\vee, \mathcal{P})$ is given by the tuple

$$(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, f_{\underline{Y} \underset{S}{\times} \underline{X}}, c \times c^\vee, c^\vee \times c, \mathcal{P}^\natural := (\pi \times \pi^\vee)^* \mathcal{P}_A, \tau_\times, \psi_{\mathcal{P}})$$

in Construction 4.5.4.11. To obtain the pullback of $\mathcal{P}$ under (Id_G, λ), we need the pullbacks of $f_{\underline{Y} \underset{S}{\times} \underline{X}}$, $(\pi \times \pi^\vee)^* \mathcal{P}_A$, and $\psi_{\mathcal{P}}$ as well. The pullback of $f_{\underline{Y} \underset{S}{\times} \underline{X}}$ is the composition $(\mathrm{Id}_{\underline{X}} + \phi)(f_{\underline{Y} \underset{S}{\times} \underline{X}})\phi$, which is simply 2ϕ. The pullback of the invertible sheaf $(\pi \times \pi^\vee)^* \mathcal{P}_A$ is $\pi^*(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$, because $(\pi \times \pi^\vee)(\mathrm{Id}_{G^\natural}, \lambda^\natural) = (\pi, \lambda^\natural \pi) = (\mathrm{Id}_A, \lambda_A)\pi$. Note that $(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ is relatively ample, because λ_A is a polarization. Therefore $\pi^*(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ is also relatively ample. The pullback of

$$(\psi_{\mathcal{P}} := \tau) : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$$

is

$$\begin{aligned}(\psi := (\mathrm{Id}_{\underline{Y}}, \phi)^* \tau) : \mathbf{1}_{\underline{Y}, \eta} &\xrightarrow{\sim} (\mathrm{Id}_{\underline{Y}}, \phi)^* (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1} \\ &= (c^\vee)^* (\mathrm{Id}, \lambda_A)^* \mathcal{P}_{A,\eta}^{\otimes -1} = \iota^* \pi^* (\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_{A,\eta}^{\otimes -1}\end{aligned}$$

The compatibility $\mathcal{D}_2(\psi) = (\mathrm{Id}_{\underline{Y}} \times 2\phi)^* \tau$ is satisfied, because for any functorial points $y_1, y_2 \in Y$ we have

$$\begin{aligned}\mathcal{D}_2(\psi)(y_1, y_2) &= \psi(y_1 + y_2)\psi(y_1)^{-1}\psi(y_2)^{-1} \\ &= \tau(y_1 + y_2, \phi(y_1 + y_2))\tau(y_1, \phi(y_1))^{-1}\tau(y_2, \phi(y_2))^{-1} \\ &= \tau(y_1, 2\phi(y_2)).\end{aligned}$$

As a result, the pullback tuple $(A, \underline{X}, \underline{Y}, 2\phi, c, c^\vee, \pi^*(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A, \tau, \psi)$, which is a priori an object in $\mathrm{DD}_{\mathrm{IS}}$, defines an object in $\mathrm{DD}_{\mathrm{ample}}$ because 2ϕ is injective and the invertible sheaf $\pi^*(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ on $G^\natural$ is relatively ample over S. ☐

Corollary 4.5.4.18. *The assignment*

$$(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \mapsto (G, \lambda)$$

in Construction 4.5.4.16 defines a functor

$$\mathrm{M}_{\mathrm{pol}} : \mathrm{DD}_{\mathrm{pol}} \to \mathrm{DEG}_{\mathrm{pol}},$$

which is compatible with all the other functors M, $\mathrm{M}_{\mathrm{ample}}$, *and* M_{IS} *we have constructed so far.*

Corollary 4.5.4.19. *The functor* $\mathrm{F}_{\mathrm{pol}} : \mathrm{DEG}_{\mathrm{pol}} \to \mathrm{DD}_{\mathrm{pol}}$ *(defined in Definition 4.4.9) is a quasi-inverse to* $\mathrm{M}_{\mathrm{pol}}$*, and hence both* $\mathrm{F}_{\mathrm{pol}}$ *and* $\mathrm{M}_{\mathrm{pol}}$ *are equivalences of categories.*

Proof. This follows from Corollary 4.5.4.15 and the construction of $\mathrm{M}_{\mathrm{pol}}$. ☐

By construction, we have:

Corollary 4.5.4.20. *The functors* $\mathrm{M}_{\mathrm{ample}} : \mathrm{DD}_{\mathrm{ample}} \to \mathrm{DEG}_{\mathrm{ample}}$*,* $\mathrm{M}_{\mathrm{pol}} : \mathrm{DD}_{\mathrm{pol}} \to \mathrm{DEG}_{\mathrm{pol}}$*,* $\mathrm{M}_{\mathrm{IS}} : \mathrm{DD}_{\mathrm{IS}} \to \mathrm{DEG}_{\mathrm{IS}}$*, and* $\mathrm{M} : \mathrm{DD} \to \mathrm{DEG}$ *are compatible with each other under the natural forgetful functors.*

Remark 4.5.4.21. This finishes the proof of Theorem 4.4.18.

4.6 Kodaira-Spencer Maps

In this section we associate Kodaira-Spencer maps to degeneration data of the form $(G^{\natural}, \iota : \underline{Y} \to G^{\natural}_{\eta})$ in DD over a base scheme $S = \mathrm{Spec}(R)$ as in Section 4.1, and compare the map we constructed with the Kodaira-Spencer map of G_{η} (defined as in Definition 2.1.7.8), where G is the object in DEG associated to $(G^{\natural}, \iota : \underline{Y} \to G^{\natural})$ under the functor M (given by Theorem 4.4.18). The notations R_i and S_i in the previous sections will no longer have their meanings in this section.

Let us fix a choice of a universal base scheme U in this section. For the moment let us take U to be locally noetherian. Later in Theorem 4.6.3.30 we shall require U to be excellent normal, mainly for retaining the noetherian normality after passing to completions of étale localizations.

4.6.1 Definition for Semi-Abelian Schemes

Let S be any scheme locally of finite presentation over U such that $\Omega^1_{S/\mathsf{U}}$ is *locally free of finite rank* over $\mathscr{O}_S$. This is the case, for example, when S is smooth over U.

Suppose that we are given a semi-abelian scheme $G^{\natural}$ of the form

$$0 \to T \to G^{\natural} \to A \to 0$$

over S associated to a homomorphism $c : \underline{X} = \underline{X}(T) \to A^{\vee}$. By general argument in Section 2.1.7 applied to the smooth scheme $G^{\natural}$, we know that there is a Kodaira-Spencer class

$$\mathsf{KS}_{G^\natural/S/\mathsf{U}} \in \underline{H}^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \cong \underline{H}^1(G^\natural, \mathscr{O}_{G^\natural}) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$$

describing the deformation of $G^\natural$.

Since $G^\natural$ is a smooth group scheme, we know by Lemma 2.1.5.12 that both $\underline{\mathrm{Der}}_{G^\natural/S}$ and $\Omega^1_{G^\natural/S}$ are constant, and $\underline{\mathrm{Lie}}^\vee_{G^\natural/S} := e^*_{G^\natural}\Omega^1_{G^\natural/S}$ is the dual of $\underline{\mathrm{Lie}}_{G^\natural/S} = e^*_{G^\natural}\underline{\mathrm{Der}}_{G^\natural/S}$. (Beware that taking $\underline{H}^0(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S})$ does not necessarily give $\underline{\mathrm{Lie}}_{G^\natural/S}$ when $G^\natural$ is not an abelian scheme over S.) However, although $\underline{H}^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$ is the space describing the deformation of $G^\natural$ as a smooth scheme, there is not enough rigidity for semi-abelian schemes to force any such deformation to have a structure as a *commutative group extension* of an abelian scheme by a torus as $G^\natural$ does. Therefore we would like to single out a submodule of $\underline{H}^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$ that does describe the deformation of $G^\natural$ as a commutative group extension.

According to Proposition 3.1.5.1, the deformation of $G^\natural$ as a commutative group extension is the same as the deformation of the pair (A, c). Let $S \hookrightarrow \tilde{S}$ be an embedding defined by a sheaf of ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Recall that the liftings of A as an abelian variety to $\tilde{S}$, if nonempty, is a torsor under the group $\underline{H}^1(A, \underline{\mathrm{Der}}_{A/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \cong \underline{H}^1(A, \underline{\mathrm{Lie}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A) \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \cong \underline{H}^1(A, \mathscr{O}_A) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \cong \underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$. (See Propositions 2.1.2.2 and 2.2.2.3, and Lemma 2.1.5.12.)

Proposition 4.6.1.1. *Liftings of the pair (A, c) to $\tilde{S}$, if nonempty, is a torsor under the group*

$$\underline{H}^1(A, \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A) \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \cong \underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}.$$

Moreover, the forgetful map from the liftings of (A, c) to liftings of A, if the source is nonempty, is equivariant with the canonical map

$$\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \to \underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$$

induced by the canonical map $\underline{\mathrm{Lie}}_{G^\natural/S} \to \underline{\mathrm{Lie}}_{A/S}$ given by the structural map $G^\natural \to A$ over S.

Proof. Let $(\tilde{A}_i, \tilde{c}_i)$, $i = 1, 2$, be any two liftings of (A, c) to $\tilde{S}$. Let $\tilde{G}^{\natural}{}_i$ be the extension of $\tilde{A}_i$ by $\tilde{T} := \mathrm{Hom}_{\tilde{S}}(X, \mathbf{G}_{\mathrm{m},\tilde{S}})$ defined by $\tilde{c}_i$.

Take an affine open covering $\{U_\alpha\}$ of A such that U_α is étale over the affine r-space $\mathbb{A}^r_S$ over S for some integer $r \geq 0$. By refining this open covering is necessary, we may assume that $G^\natural$ is trivialized as a T-bundle over each U_α. Note that T is an affine subset of some affine r'-space $\mathbb{A}^{r'}_S$ for $r' \geq 0$ equal to the relative dimension of T over S, and hence $U_\alpha \underset{S}{\times} T$ is étale over $\mathbb{A}^{r+r'}_S$. Then the open coverings $\{U_\alpha\}$ and $\{U_\alpha \underset{S}{\times} T\}$ are lifted to open coverings of respectively $\tilde{A}_i$ and $\tilde{G}^{\natural}{}_i$ over $\tilde{S}$, for $i = 1, 2$. Now we may conclude as in the proof of Proposition 2.1.3.2. ☐

If the embedding $S \hookrightarrow \tilde{S}$ is given by the first infinitesimal neighborhood of the diagonal morphism $\Delta : S \to S \underset{\mathsf{U}}{\times} S$, then $\mathscr{I} \cong \Omega^1_{S/\mathsf{U}}$. By pulling back along the two projections, we obtain two liftings $\tilde{G}^{\natural}{}_i := \mathrm{pr}_i^* G^\natural$ of $G^\natural$, and hence an element $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ in $\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$ by Proposition 4.6.1.1, which sends $\tilde{G}^{\natural}{}_1$ to $\tilde{G}^{\natural}{}_2$. By duality, $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ can be interpreted as a morphism

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S} \to \Omega^1_{S/\mathsf{U}},$$

which we denote by $\mathrm{KS}_{(A,c)/S/\mathsf{U}}$.

Definition 4.6.1.2. *The class* $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ (*resp. the map* $\mathrm{KS}_{(A,c)/S/\mathsf{U}}$) *above is called the* **Kodaira-Spencer class** (*resp. the* **Kodaira-Spencer map**) *for* (A, c).

Since the liftings of $G^\natural$ as a commutative group scheme extension to $\tilde{S}$ are also liftings of the underlying smooth scheme, the natural map

$$\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \to \underline{H}^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \tag{4.6.1.3}$$

induced by

$$\underline{\mathrm{Lie}}_{A^\vee/S} \cong \underline{H}^1(A, \mathscr{O}_A) \to \underline{H}^1(G, \mathscr{O}_G),$$

sends $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ (up to a difference in sign) to $\mathsf{KS}_{G^\natural/S/\mathsf{U}}$, by the very definitions of the cohomology classes.

Lemma 4.6.1.4. *The natural map* $\underline{H}^i(A, \mathscr{O}_A) \to \underline{H}^i(G, \mathscr{O}_G)$ *is an embedding for any integer* $i \geq 0$.

Proof. Let us fix any integer $i \geq 0$ as in the statement of the lemma. Denote the structural map $G^\natural \to A$ by π. By the fact that π is relative affine (with fibers locally isomorphic to T), so that $R^j \pi_* \mathscr{O}_{G^\natural}$ vanishes for any $j > 0$, the Leray spectral sequence (see [44, Ch. II, Thm. 4.17.1], or [64, Ch. III, Exer. 8.1]) shows that there is an isomorphism

$$\underline{H}^i(A, \pi_* \mathscr{O}_{G^\natural}) \cong \underline{H}^i(G^\natural, \mathscr{O}_{G^\natural}).$$

By an étale base extension that splits T if necessary, we may assume that $\underline{X}$ is constant with value X. Then we have a decomposition

$$\pi_* \mathscr{O}_{G^\natural} \cong \underset{\chi \in \underline{X}}{\oplus} \mathscr{O}_\chi,$$

where $\mathscr{O}_\chi$ is the rigidified invertible sheaf corresponding to the point $c(\chi) \in A^\vee$, and where $c : \underline{X} \to A^\vee$ is the map describing the structure of $G^\natural$ as an extension of A by T as above. This gives a corresponding decomposition

$$\underline{H}^i(A, \pi_* \mathscr{O}_{G^\natural}) \cong \underline{H}^i(A, \underset{\chi \in \underline{X}}{\oplus} \mathscr{O}_\chi) \cong \underset{\chi \in \underline{X}}{\oplus} \underline{H}^i(A, \mathscr{O}_\chi).$$

In particular, we have an inclusion

$$\underline{H}^i(A, \mathscr{O}_A) \hookrightarrow \underline{H}^i(G^\natural, \mathscr{O}_{G^\natural})$$

corresponding to the term $\mathscr{O}_0 = \mathscr{O}_A$. This inclusion is independent of the trivialization of $\underline{X}$ over the étale base extension of S. Hence étale descent applies and shows that the inclusion is defined over S. Moreover, it has to agree with the natural maps $\underline{H}^i(A, \mathscr{O}_A) \to \underline{H}^i(G, \mathscr{O}_G)$. In particular, the natural maps are injections. ☐

As a consequence:

Proposition 4.6.1.5. *The natural map (4.6.1.3) above is an injection, and identifies* $\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$ *as a subspace of* $\underline{H}^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$. *By choosing the sign convention compatibly, the Kodaira-Spencer class* $\mathsf{KS}_{G^\natural/S/\mathsf{U}}$ *lies in this subspace and agrees with the Kodaira-Spencer class* $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ *of* (A, c). *If we interpret the class* $\mathsf{KS}_{G^\natural/S/\mathsf{U}}$ *by duality as a map*

$$\mathrm{KS}_{G^\natural/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S} \to \Omega^1_{S/\mathsf{U}},$$

then it agrees with the Kodaira-Spencer map $\mathrm{KS}_{(A,c)/S/\mathsf{U}}$ *of* (A, c) *as in Definition 4.6.1.2.*

4.6.2 Definition for Period Maps

Let S be any scheme locally of finite presentation over U such that $\Omega^1_{S/\mathsf{U}}$ is *locally free of finite rank* over $\mathscr{O}_S$ and such that all connected components of S are integral. Let $G^\natural$ be an extension of an abelian scheme A by a torus T over S, so that the character group of T is $\underline{X}$ and so that the extension class of $G^\natural$ is described by some group homomorphism $c : \underline{X} \to A^\vee$. Let $\underline{Y}$ be the character group of some torus $T^\vee$ that has the same dimension as T, and let $c^\vee : \underline{Y} \to A$ be a group homomorphism that characterizes an extension $G^{\vee,\natural}$ of $A^\vee$ by $T^\vee$. Let S_1 be an open dense subscheme of S over which we have a group homomorphism $\iota : \underline{Y}_{S_1} \to G^\natural_{S_1}$. We shall investigate how liftings of the pair $(G^\natural, \iota)$ to an embedding $S \hookrightarrow \tilde{S}$ defined by a sheaf of ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ should be classified, under some suitable assumptions if necessary.

Let $\tilde{S}_1$ be an open subscheme of $\tilde{S}$ lifting S_1, and let $\mathscr{I}_1 := (\tilde{S}_1 \hookrightarrow \tilde{S})^*\mathscr{I} \cong \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$ be the ideal defining the embedding $S_1 \hookrightarrow \tilde{S}_1$. The map $c^\vee : \underline{Y} \to A$ and $\iota : \underline{Y}_{S_1} \to G^\natural_{S_1}$ can be interpreted as *actions* of $\underline{Y}$ on respectively A and $G^\natural_{S_1}$. Hence the question is about the liftings of actions of $\underline{Y}$. For technical simplicity, let us assume for the time being that $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y.

Definition 4.6.2.1. *Let H be any scheme with Y-action over some base scheme Z. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two sheaves of $\mathscr{O}_H$-modules admitting actions of Y covering the given Y-action on H. A (relative) Y*-**equivariant extension** *of $\mathcal{E}_1$ by $\mathcal{E}_2$ (over Z) is a sheaf $\mathcal{E}$ of $\mathscr{O}_H$-modules admitting an Y-action (covering the action of Y on Z), which fits into an Y-equivariant exact sequence*

$$0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}_1 \to 0.$$

Two such extensions $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if there is an isomorphism (between usual extensions)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_2 \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, dotted, "\wr"] & \mathcal{E}_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}_2 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}_1 \arrow[r] & 0
\end{tikzcd}$$

that makes this whole diagram equivariant with respect to Y-actions. We denote the $\mathscr{O}_Z$-sheaf of Y-**equivariant extension classes** *of $\mathcal{E}_1$ by $\mathcal{E}_2$ by $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_Z}(\mathcal{E}_1, \mathcal{E}_2)$.*

Lemma 4.6.2.2. *$\underline{H}^i(A, \mathscr{O}_A)^Y = \underline{H}^i(A, \mathscr{O}_A)$ for all $i \geq 0$.*

Proof. By Proposition 2.1.5.15, it suffices to treat the case $i = 1$. By Corollary 2.1.5.10, we know that $\underline{H}^1(A, \mathscr{O}_A) \cong \underline{\mathrm{Lie}}_{A^\vee/S}$ is the tangent space to $A^\vee \cong \underline{\mathrm{Pic}}^0(A/S)$. Since Y acts on A by translation, and since translation action is trivial on $\underline{\mathrm{Pic}}^0(A/S)$ by the theorem of square, we see that Y acts trivially on $\underline{H}^1(A, \mathscr{O}_A)$, as desired. $\square$

Corollary 4.6.2.3. $\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)^Y \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A) \cong \underline{\mathrm{Lie}}_{A^\vee/S}$.

Lemma 4.6.2.4. *$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}}, \mathscr{O}_{A_{S_1}})$ is canonically isomorphic to $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}$ as extensions of $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1}$ by $\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}$.*

Proof. For simplicity of notations, let us assume that $S = S_1$, and so $\tilde{S} = \tilde{S}_1$ and $\mathscr{I} = \mathscr{I}_1$, as their differences are irrelevant to our goal.

First let us exhibit a realization of the canonical isomorphism $\underline{\mathrm{Lie}}_{A^\vee/S} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$ using the Poincaré invertible sheaf $\mathcal{P}_A$. Let us denote by $A^\vee_{(1)}$ the first infinitesimal neighborhood of the identity section $e_{A^\vee} : S \to A^\vee$. Let $S_\varepsilon := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$, which contains S as a subscheme defined by $\varepsilon = 0$. Then we have a canonical bijection between $\underline{\mathrm{Lie}}_{A^\vee/S}$ and morphisms $v : S_\varepsilon \to A^\vee_{(1)}$ such that $v|_S = e_A$. For any morphism $v : S_\varepsilon \to A^\vee_{(1)}$ such that $v|_S = e_A$, consider the pullback $(\mathrm{Id}_A \times v)^*\mathcal{P}_A$ of $\mathcal{P}_A$ to $A \underset{S}{\times} S_\varepsilon$. By Proposition 2.1.5.4 and Corollary 2.1.5.10, and by the identification $\underline{H}^1(A, \mathscr{O}_A) \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$, we see that $(\mathrm{Id}_A \times v)^*\mathcal{P}_A$ corresponds to an extension $\mathcal{E}_v$ of $\mathscr{O}_A$ by $\mathscr{O}_A$. This identification can be given rather explicitly: Since A and $A \underset{S}{\times} S_\varepsilon$ have the same underlying topological spaces, we may view $(\mathrm{Id}_A \times v)^*\mathcal{P}_A$ as an $\mathscr{O}_A$-module, together with a surjective map to $\mathscr{O}_A$ with kernel $\varepsilon \cdot ((\mathrm{Id}_A \times v)^*\mathcal{P}_A) \cong \mathscr{O}_A$. Let us write this as $\mathcal{E}_v \cong (\mathrm{Id}_A \times v)^*\mathcal{P}_A$. Then the universal property of $\mathcal{P}_A$ shows that the association $v \mapsto \mathcal{E}_v$ realizes the canonical isomorphism $\underline{\mathrm{Lie}}_{A^\vee/S} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$.

Now let us we consider the Y-actions as well. Let us construct a canonical morphism $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S} \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$ as follows:

Consider the structural projection $\pi^\vee : G^{\vee,\natural} \to A^\vee$, and consider the pullback $\tilde{\mathcal{P}} := (\mathrm{Id} \times \pi^\vee)^*\mathcal{P}_A$ of $\mathcal{P}_A$ to $A \underset{S}{\times} G^{\vee,\natural}$. Let $G^{\vee,\natural}_{(1)}$ (resp. $T^\vee_{(1)}$) be the first infinitesimal neighborhood of $e_{G^{\vee,\natural}} : S \to G^{\vee,\natural}$ (resp. $e_{T^\vee} : S \to T^\vee$), and let S_ε and $A^\vee_{(1)}$ be as above. Then we may identify $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$ with morphisms $v : S_\varepsilon \to G^{\vee,\natural}_{(1)}$ such that $v|_S = e_{G^{\vee,\natural}}$. For any such v, we obtain the pullback

$(\mathrm{Id}\times v)^*\tilde{\mathcal{P}}$ to $A\underset{S}{\times}S_\varepsilon$, which can be identified as an extension $\mathcal{E}_v$ of $\mathscr{O}_A$ by $\mathscr{O}_A$. However, we claim that there is a canonical action of Y on $\mathcal{E}_v$, or rather on $\tilde{\mathcal{P}}$. Let $\mathscr{O}_y := \mathcal{P}_A|_{\{c^\vee(y)\}\times A^\vee}$. Then the construction of $G^{\vee,\natural}$ from $c^\vee$ shows that $\pi^\vee_*\mathscr{O}_{G^{\vee,\natural}} \cong \underset{y\in Y}{\oplus}\mathscr{O}_y$. By the biextension structure of $\mathcal{P}_A$, we see that $(T_{c^\vee(y)}\times\mathrm{Id}_A)^*\mathcal{P}_A \cong \mathcal{P}_A \underset{\mathscr{O}_{A\times A^\vee}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_y$. Therefore there is a canonical isomorphism $(T_{c^\vee(y)}\times\mathrm{Id}_A)^*\tilde{\mathcal{P}}\cong\tilde{\mathcal{P}}$ covering the translation action of Y on the first factor A of $A\underset{S}{\times}G^{\vee,\natural}$, or in other words a canonical action of Y on $\tilde{\mathcal{P}}$. This proves the claim. It is clear from the proof of the claim that $\tilde{\mathcal{P}}$ is universal for such a property. As a result, we see that the association $v\mapsto\mathcal{E}_v$ (with its Y-action) gives a canonical isomorphism $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}\to\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathscr{O}_A,\mathscr{O}_A)$, as desired. $\square$

Proposition 4.6.2.5. *Liftings of the tuple $(A_{S_1},c,c^\vee,\iota)$ to $\tilde{S}_1$, if nonempty, is a torsor under the group*

$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_{A_{S_1}},\mathscr{I}_1\underset{\mathscr{O}_{S_1}}{\otimes}\mathscr{O}_{A_{S_1}})\cong\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I}.$$

Moreover, the forgetful map from the liftings of $(A_{S_1},c,c^\vee,\iota)$ to the liftings of (A_{S_1},c), if the source is nonempty, is equivariant with the canonical map

$$\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I}\to\underline{\mathrm{Lie}}_{A^\vee/S}\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I}$$

induced by the canonical map $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}\to\underline{\mathrm{Lie}}_{A^\vee/S}$ given by the structural map $G^{\vee,\natural}\to A^\vee$ over S.

Proof. For simplicity of notations, let us assume that $S=S_1$. Let $(\tilde{A}_i,\tilde{c}_i,c_i^\vee,\iota_i)$, $i=1,2$, be any two liftings of $(A,c,c^\vee,\iota)$ to $\tilde{S}$. Let $\tilde{G^\natural}_i$ be the extension of $\tilde{A}_i$ by $\tilde{T}:=\mathrm{Hom}_{\tilde{S}}(X,\mathbf{G}_{\mathrm{m},\tilde{S}})$ defined by $\tilde{c}_i$.

Take open coverings $\{U_\alpha\}$ and $\{U_\alpha\underset{S}{\times}T\}$ that are lifted to open coverings of respectively $\tilde{A}_i$ and $\tilde{G^\natural}_i$ over $\tilde{S}$, for $i=1,2$, as in the proof of Proposition 4.6.1.1. By Lemma 2.1.1.7 and by abuse of notations, let us use the same notations $\tilde{U}_\alpha$ and $\tilde{U}_\alpha\underset{\tilde{S}}{\times}\tilde{T}$ for the liftings of open subschemes to respectively $\tilde{A}_i$ and $\tilde{G^\natural}_i$ over $\tilde{S}$, for $i=1,2$. As in the proof of Proposition 2.1.2.2, for $i=1,2$, the scheme $\tilde{G^\natural}_i$ is given by a collection of gluing maps $\xi_{\alpha\beta,i}:(\tilde{U}_\alpha\underset{\tilde{S}}{\times}\tilde{T})|_{U_{\alpha\beta}}\xrightarrow{\sim}(\tilde{U}_\beta\underset{\tilde{S}}{\times}\tilde{T})|_{U_{\alpha\beta}}$ such that $\xi_{\alpha\gamma,i}=\xi_{\beta\gamma,i}\circ\xi_{\alpha\beta,i}$, and their difference is (up to a choice of sign convention) given by a collection of automorphisms $\{\xi_{\alpha\beta,2}^{-1} \circ \xi_{\alpha\beta,1}\}$, each member lying in

$$\operatorname{Aut}_{\tilde{S}}((\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T})|_{U_{\alpha\beta}}, S) \cong \underline{H}^0(U_{\alpha\beta}, \underline{\operatorname{Hom}}_A(\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A)),$$

which altogether defines a 1-cocycle in $\underline{H}^1(A, \underline{\operatorname{Hom}}_A(\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A))$, or rather a collection of splittings of a global extension

$$0 \to \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A \to \mathcal{E} \to \underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A \to 0$$

over $\{U_\alpha\}$.

Let us now also consider the difference between ι_1 and ι_2, which can be interpreted as actions of Y. The Y action is given by collection of maps $\eta(y)_{\alpha,i} : (\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T}) \xrightarrow{\sim} T_{\iota_i(y)}(\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T})$, which has to satisfy the compatibility $\eta(y)_{\beta,i} \circ \xi_{\alpha\beta,i} = T^*_{\iota_i(y)}(\xi_{\alpha\beta,i}) \circ \eta(y)_{\alpha,i}$. In other words, the classes of the two 1-cocycles $\{\xi_{\alpha\beta}\}$ and $\{T_{\iota_i(y)}(\xi_{\alpha\beta})\}$ differ by a 1-coboundary, and so they are equivalent. This shows that we only need to consider Y-equivariant extensions $\mathcal{E}$ above. Moreover, the difference between the two actions are (up to a choice of sign convention) measured by a collection of elements $\{\eta(y)_{\alpha,2}^{-1} \circ \eta(y)_{\alpha,1}\}$, each member lying in

$$\operatorname{Hom}_S(Y, \operatorname{Aut}_{\tilde{S}}(\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T}, S) \cong \underline{H}^0(U_\alpha, \underline{\operatorname{Hom}}_{\mathscr{O}_{G^\natural}}(\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A))),$$

which altogether defines a global section of

$$\operatorname{Hom}_S(Y, \underline{\operatorname{Hom}}_A(\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A)) \cong \operatorname{Hom}_S(Y, \underline{\operatorname{Hom}}_S(\underline{\operatorname{Lie}}^\vee_{G^\natural/S}, \mathscr{I})).$$

This corresponds to modifying the Y-action on $\mathcal{E}$ by maps from $\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A$ to $\mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A$ for each Y. Conversely, any two different Y-actions on $\mathcal{E}$ are compared by such a difference. Hence we see that the liftings of $(A, c, c^\vee, \iota)$ to $\tilde{S}$ form a torsor under $\underline{\operatorname{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\operatorname{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}})$, this is the first statement.

As for the second statement, simply note that the forgetful map $(A, c, c^\vee, \iota) \to (A, c)$ corresponds to forgetting the Y-actions on the extensions $\mathcal{E}$ above. $\square$

Remark 4.6.2.6. By étale descent, if we state only $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$ (without $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}))$ in Proposition 4.6.2.5, then we do not need to assume that $\underline{X}$ and $\underline{Y}$ are constant.

For simplicity, let us call the liftings of the tuple $(A_{S_1}, c, c^{\vee}, \iota)$ simply liftings of the pair $(G^{\natural}_{S_1}, \iota)$, with the understanding that we will only consider liftings of $G^{\natural}_{S_1}$ that are group extensions of abelian schemes by tori. If the embedding $S \hookrightarrow \tilde{S}$ is given by the first infinitesimal neighborhood of the diagonal morphism $\Delta : S \to S \underset{\mathsf{U}}{\times} S$, then $\mathscr{I} \cong \Omega^1_{S/\mathsf{U}}$, and $\mathscr{I}_1 \cong \Omega^1_{S_1/\mathsf{U}}$. By comparing the pullbacks along the two projections, we obtain an element $\mathsf{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ in $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ by Proposition 4.6.2.5, which sends $\mathrm{pr}_1^*(G^{\natural}_{S_1}, \iota)$ to $\mathrm{pr}_2^*(G^{\natural}_{S_1}, \iota)$. By duality, $\mathsf{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ can be interpreted as a morphism

$$\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}},$$

which we denote by $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$.

Definition 4.6.2.7. *The class* $\mathsf{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ (*resp. the map* $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$) *above is called the* **Kodaira-Spencer class** (*resp. the* **Kodaira-Spencer map**) *for* $(G^{\natural}_{S_1}, \iota)$.

According to Lemma 4.2.1.2 (or rather its proof in Section 4.2.2), the map $\iota : Y \to G^{\natural}_{S_1}$ can be identified with a trivialization $\tau : \mathbf{1}_{(Y \times X)_{S_1}} \xrightarrow{\sim} (c^{\vee} \times c)^* \mathcal{P}^{-1}_{A_{S_1}}$ of biextensions, and hence a map $\iota^{\vee} : X \to G^{\vee,\natural}_{S_1}$ lifting $c : X \to A^{\vee}_{S_1}$. Applying Proposition 4.6.2.5 to the dual situation of $(A^{\vee}_{S_1}, c^{\vee}, c, \iota^{\vee})$ as well, we see that the various Kodaira-Spencer classes fit into the natural commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[r] \arrow[d] & \underline{\mathrm{Lie}}_{A^{\vee}_{S_1}/S_1} \underset{\mathscr{O}_{S}}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[d] \\
\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[r] & \underline{\mathrm{Lie}}_{A^{\vee}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}
\end{tikzcd},$$

so that

$$\begin{tikzcd}
\mathsf{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}} \arrow[r, mapsto] \arrow[d, mapsto] & \mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}} \arrow[d, mapsto] \\
\mathsf{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}} \arrow[r, mapsto] & \mathsf{KS}_{A_{S_1}/S_1/\mathsf{U}}
\end{tikzcd}$$

as a result of forgetting the structures. (Note that here we should view $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ as $\underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ when considering $\mathsf{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}}$ as an element in it.)

Now suppose that the connected components of S are all integral, so that $\mathscr{O}_S$ is naturally a subsheaf of $(S_1 \hookrightarrow S)^*\mathscr{O}_{S_1}$, or simply $\mathscr{O}_{S_1}$ by abuse of language. If we consider the space $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S}$ as a subspace of $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1}$, and consider the restriction of the Kodaira-Spencer map $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ to this subspace, then we obtain a map

$$\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S} \to (S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}.$$

For our purpose, it is desirable to replace the codomain $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ by some subsheaf of it with better finiteness property. Since A, c, and $c^{\vee}$ are all defined over the whole base scheme S, the maps $\mathrm{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$, $\mathrm{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}}$, and $\mathrm{KS}_{A_{S_1}/S_1/\mathsf{U}}$ all extend over S. This shows that the images of both $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{A^{\vee}/S}$ and $\underline{\mathrm{Lie}}^{\vee}_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S}$ are contained in $\Omega^1_{S/\mathsf{U}}$. Hence the question is about the codomain of the induced map

$$\underline{\mathrm{Lie}}^{\vee}_{T/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/S} \to ((S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}})/\Omega^1_{S/\mathsf{U}}.$$

Since the answer is related only to ι, let us fix any particular lifting of the triple $(A, c, c^{\vee})$ to $\tilde{S}$, and investigate the different liftings of ι to $\tilde{S}_1$. (Note that such liftings always exist because $G^{\natural}_{S_1}$ is smooth over S_1.)

As already mentioned above, the map $\iota : \underline{Y}_{S_1} \to G^{\natural}_{S_1}$ is equivalent to a trivialization $\tau : \mathbf{1}_{(Y \times X)_{S_1}} \overset{\sim}{\to} (c^{\vee} \times c)^*\mathcal{P}^{-1}_{A_{S_1}}$ of biextensions, which we can interpret as a collection of isomorphisms $\tau(y,\chi) : \mathscr{O}_{S_1} \overset{\sim}{\to} (c^{\vee}(y), c(\chi))^*\mathcal{P}^{\otimes -1}_{A_{S_1}}$, (satisfying certain bimultiplicative relations) for $y \in Y$ and $\chi \in X$. This collection $\{\tau(y,\chi)\}_{y \in Y, \chi \in X}$ defines a collection of $\mathscr{O}_S$-invertible submodules $I_{y,\chi}$ of $\mathscr{O}_{S_1}$, with $\mathscr{O}_S$-module isomorphisms from $I_{y,\chi}$ to $(c^{\vee}(y), c(\chi))^*\mathcal{P}^{\otimes -1}_{A}$

induced by $\tau(y, \chi)$ as in the proof of Lemma 4.2.1.6 in Section 4.2.4. If we interpret these isomorphisms as maps between $\mathbf{G}_{\mathrm{m}}$-torsors over S_1, then over each particular lifting of $(A, c, c^\vee)$, the liftings of the sections $\{\tau(y, \chi)\}$ to $\tilde{S}_1$ form a torsor under the space of maps

$$Y \otimes X \to \mathscr{I}_1,$$

which can be identified with

$$\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{T_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{I}_1.$$

If the embedding $S \hookrightarrow \tilde{S}$ is the first infinitesimal neighborhood of the diagonal morphism $\Delta : S \to S \underset{\mathsf{U}}{\times} S$, then $\mathscr{I} \cong \Omega^1_{S/\mathsf{U}}$ and $\mathscr{I}_1 \cong \Omega^1_{S_1/\mathsf{U}}$. For any local generator q of $I_{y,\chi}$, the difference between the two pullbacks of the section $\tau(y, \chi)$ is given additively by $dq = \mathrm{pr}_2^*(q) - \mathrm{pr}_1^*(q)$ at the tangent space of $\mathbf{G}_{\mathrm{m},S_1}$ at q, or rather $d\log(q) := q^{-1}dq$ if we move to the tangent space of $\mathbf{G}_{\mathrm{m},S_1}$ at 1 by multiplication by q^{-1}. Such *logarithmic differentials* define an invertible $\mathscr{O}_S$-submodule $d\log(I_{y,\chi})$ of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$, which is independent of the choice of the local generators q.

Definition 4.6.2.8. *With the setting as above, the $\mathscr{O}_S$-sheaf $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ is the subsheaf of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ spanned by $\Omega^1_{S/\mathsf{U}}$ and all the $d\log(I_{y,\chi})$ arisen from the above construction. We call this $\mathscr{O}_S$-sheaf $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ the* **sheaf of logarithmic 1-differentials** *generated by $d\log(I_{y,\chi})$.*

Then the different liftings of ι (over each particular lifting of $(A, c, c^\vee)$) form a torsor under a subspace of the space of maps

$$Y \otimes X \to \Omega^1_{S/\mathsf{U}}[d\log\infty],$$

which can be identified with a submodule of

$$\underline{\mathrm{Lie}}_{T^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{T/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}[d\log\infty].$$

Hence we may replace the image $((S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}})/\Omega^1_{S/\mathsf{U}}$ in (4.6.2) by $(\Omega^1_{S/\mathsf{U}}[d\log\infty])//\Omega^1_{S/\mathsf{U}}$, and consequently the image $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ in (4.6.2) by $\Omega^1_{S/\mathsf{U}}[d\log\infty]$. Note that at this final stage we no longer need to assume that $\underline{X}$ and $\underline{Y}$ are constant, if they become constant after an étale localization. Let us summarize this as:

Proposition 4.6.2.9. *With the setting as above, the Kodaira-Spencer map*

$$\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}$$

for $(G^{\natural}_{S_1}, \iota)$ *over* S_1 *can be extended to a map*

$$\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_{S}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S} \to \Omega^1_{S/\mathsf{U}}[d\log\infty]$$

over S.

Definition 4.6.2.10. *The map* $\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}}$ *defined above is called the* **extended Kodaira-Spencer map** *for* $(G^{\natural}, \iota)$.

Remark 4.6.2.11. Certainly, the sheaf $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ has a much better meaning in the case where S is smooth over U and where the complement of S_1 in S is a relative Cartier divisor of normal crossings. But we do not need to know these properties of S and S_1 when defining the maps.

4.6.3 Compatibility with Mumford's Construction

Let $S = \mathrm{Spec}(R)$ be any affine scheme fitting into the setting of Section 4.1 such that it is of finite presentation over U and such that $\Omega^1_{S/\mathsf{U}}$ is *locally free* (*of finite rank*) over $\mathscr{O}_S$. Suppose we have an object (G, λ) in $\mathrm{DEG}_{\mathrm{pol}}$ mapped to an object $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau)$ in $\mathrm{DD}_{\mathrm{pol}}$ by the theory of degeneration data. Let $(A, \underline{X}, \underline{Y}, c, c^{\vee}, \tau)$ or equivalently $(G^{\natural}, \iota : \underline{Y}_{\eta} \to G^{\natural}_{\eta})$ be the underlying object in DD. By Mumford's construction, if we introduce the notion of relatively complete models and pass to the category of formal schemes, we could interpret G as a quotient of $G^{\natural}$ by the period map ι (although it is not a quotient in the category of schemes). Let us suppose that G is an abelian scheme G_{S_1} over some nonempty open subscheme S_1 of S. Such an open subscheme exists because G_{η} is an abelian scheme over η. Then, as in Definition 2.1.7.8, G_{S_1} admits a Kodaira-Spencer map

$$\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}.$$

On the other hand, the restriction of the map $\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}}$ (defined in Definition 4.6.2.10) to S_1 gives the map

$$\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}}|_{S_1} = \mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}$$

(defined in Definition 4.6.2.7), where no nontrivial logarithmic 1-differentials is needed by construction. Note that there is a canonical isomorphism $\underline{\mathrm{Lie}}^{\vee}_{G/S} \cong \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S}$ because they have the same formal completion $\underline{\mathrm{Lie}}^{\vee}_{G_{\mathrm{for}}/S_{\mathrm{for}}} = \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}}$ over S_{for}. (See Corollary 2.3.1.3.) Similarly, there is a canonical isomorphism $\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} \cong \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S}$. Therefore the two maps $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ and $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ have the same source and target, and it is natural to compare them.

Since the identification between two maps can be achieved locally in the étale topology, we shall assume until we finish the proof of Theorem 4.6.3.13 that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y for simplicity.

The elements in

$$\begin{aligned}\underline{H}^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S_1/\mathsf{U}} &\cong \underline{H}^1(G_{S_1}, \underline{\mathrm{Hom}}_{\mathscr{O}_{G_{S_1}}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})) \\ &\cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})\end{aligned}$$

parameterize extensions $\mathcal{E}$ of the form

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}} \to \mathcal{E} \to \Omega^1_{G_{S_1}/S_1} \to 0. \tag{4.6.3.1}$$

By Proposition 2.1.7.3, $\mathsf{KS}_{G_{S_1}/S_1/\mathsf{U}} \in \underline{H}^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$, interpreted as an element of $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$ as above, is the extension class of the first exact sequence of G_{S_1}:

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}} \to \Omega^1_{G_{S_1}/\mathsf{U}} \to \Omega^1_{G_{S_1}/S_1} \to 0. \tag{4.6.3.2}$$

Similarly, the elements in

$$\underline{H}^1(G^{\natural}_{S_1}, \underline{\mathrm{Der}}_{G^{\natural}_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^{\natural}_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^{\natural}_{S_1}}).$$

parameterize extensions $\mathcal{E}^{\natural}$ of the form

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^{\natural}_{S_1}} \to \mathcal{E}^{\natural} \to \Omega^1_{G^{\natural}_{S_1}/S_1} \to 0, \tag{4.6.3.3}$$

and $\mathsf{KS}_{G^{\natural}_{S_1}/S_1/\mathsf{U}} \in \underline{H}^1(G^{\natural}_{S_1}, \underline{\mathrm{Der}}_{G^{\natural}_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ corresponds by Proposition 2.1.7.3 to the extension class of the first exact sequence of $G^{\natural}$:

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^{\natural}_{S_1}} \to \Omega^1_{G^{\natural}_{S_1}/\mathsf{U}} \to \Omega^1_{G^{\natural}_{S_1}/S_1} \to 0. \tag{4.6.3.4}$$

Since Y acts equivariantly on all three terms in (4.6.3.4), the sequence also defines an Y-equivariant extension class in $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$.

Assume for the moment (in this paragraph) that we have the *hypothetical situation* that G is a scheme-theoretic quotient of $G^\natural$ by the translation action of $\iota : \underline{Y}_{S_1} \to G^\natural_{S_1}$. Let us denote the quotient map by $p : G^\natural_{S_1} \to G_{S_1}$. Assume moreover that the action is sufficiently nice so that the action is free and that there is an integer $k \geq 1$ such that every point of $G^\natural$ admits an affine open neighborhood U such that $\cup_{y \in kY}(\iota(y))(U)$ is a disjoint union. Then we can factor $p : G^\natural_{S_1} \to G_{S_1}$ as a composition $p = p_1 \circ p_2$ of two quotient maps, so that p_2 is a local isomorphism and p_1 is a quotient by a finite group action. Under this hypothesis, there is a natural map

$$\underline{H}^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \to \underline{H}^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$$

defined by pullback, which can be interpreted as the natural map

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}),$$

which pulls back the extension class of $\mathcal{E}$ as in (4.6.3.1) to the extension class of $\mathcal{E}^\natural$ as in (4.6.3.3), and which necessarily pulls back the extension class $\mathsf{KS}_{G_{S_1}/S_1/\mathsf{U}}$ of $\Omega^1_{G_{S_1}/\mathsf{U}}$ to the extension class $\mathsf{KS}_{G^\natural_{S_1}/S_1/\mathsf{U}}$ of $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$. Moreover, the image is exactly those extension classes that are invariant under the action of Y. Each representative $\mathcal{E}^\natural$ of such an extension class admits some Y-action. Then theory of descent (for a quotient map of a group action that is free and moreover a local isomorphism on a subgroup of finite index) implies that there is an equivalence between Y-equivariant extension classes of $\mathcal{E}^\natural$ over $G^\natural_{S_1}$ and the usual extension classes of $\mathcal{E}$ over G_{S_1} that pulls back to $\mathcal{E}^\natural$ over $G^\natural_{S_1}$. Explicitly, this means the natural pullback morphism

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$$

is an isomorphism. Moreover, this isomorphism is compatible with the natural surjections of the two sides to the Y-invariants $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y$ in $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$.

Let us restore our hypotheses before assuming the hypothetical situation. Note that the definition of $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$ and

$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y$ do not involve the hypothetical situation and the additional hypothesis following it.

Lemma 4.6.3.5. *The natural embeddings* $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}}) \hookrightarrow \underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})$ *has image in* $\underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y$. *The induced morphism* $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}}) \to \underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y$ *is an isomorphism under the assumption that* τ *(when restricted to* η*) satisfies the positivity condition in Definition 4.2.1.4.*

Proof. According to the argument as in (4.2.3.2), for each $y \in Y$, the translation by $\iota(y)$ on $G^\natural_{S_1}$ can be analyzed as an isomorphism

$$\tau(y, \chi) : T^*_{c^\vee(y)} \mathscr{O}_{\chi, S_1} \cong \mathscr{O}_{\chi, S_1}(c^\vee(y))_{S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{\chi, S_1} \overset{\sim}{\to} \mathscr{O}_{\chi, S_1}.$$

Therefore, according to the proof of Lemma 4.6.1.4, we may decompose

$$\underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}}) \cong \underset{\chi \in \underline{X}}{\oplus} \underline{H}^i(A_{S_1}, \mathscr{O}_{\chi, S_1})$$

and the action of $\iota(y)$ on $\underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})$ componentwise as

$$\underline{H}^i(A_{S_1}, \mathscr{O}_{\chi, S_1}) \overset{T^*_{c^\vee(y)}}{\overset{\sim}{\to}} \underline{H}^i(A_{S_1}, T^*_{c^\vee(y)} \mathscr{O}_{\chi, S_1}) \overset{\tau(y,\chi)}{\overset{\sim}{\to}} \underline{H}^i(A_{S_1}, \mathscr{O}_{\chi, S_1}).$$

If $\chi = 0$, then we know that $\tau(y, \chi) = 1$ in the sense of (4.3.1.5), namely it is the canonical isomorphism given by translations. Hence, by Lemma 4.6.2.2, the action of Y is trivial on $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}})$. This proves the first statement of the lemma.

Now suppose that τ satisfies the positive condition, and that $\chi \neq 0$. Take an integer $N \geq 1$ such that $N\chi \in \phi(Y)$. Let $y \in Y$ be such that $N\chi = \phi(y)$. Then $y \neq 0$, and we know that $I_{y, N\chi} \subset \underline{I}$, where $\underline{I}$ is the invertible $\mathscr{O}_S$-subsheaf of $\mathscr{O}_S$ corresponding to the ideal $I \subset R$. Equivalently, we know that $I_{Ny, \chi} \subset \underline{I}$. Suppose there is a section x in $\underline{H}^i(A_{S_1}, \mathscr{O}_{\chi, S_1})$ that is invariant under the action of Ny. By multiplying by a nonzero scalar in R, we may assume that $x \in \underline{H}^i(A, \mathscr{O}_\chi)$. By applying the Ny-action repeatedly, we see that $x \in \underline{I}^k \cdot \underline{H}^i(A, \mathscr{O}_\chi)$ for any $k \geq 0$. Since $\underline{H}^i(A, \mathscr{O}_\chi)$ is finitely generated as an $\mathscr{O}_S$-module, and since $S = \mathrm{Spec}(R)$ is the spectrum of a noetherian ring, this is possible only when $x = 0$. This proves the second statement of the lemma. □

Corollary 4.6.3.6. *The embedding of* $\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ *as a subspace of* $\underline{H}^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ *in Proposition 4.6.1.5 induces a canonical isomorphism*

$$\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y$$

sending the Kodaira-Spencer class $\mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$ *to the extension class of* $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$ *in (4.6.3.4).*

Proof. The map is the composition of the following canonical isomorphisms

$$\begin{aligned}&\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \cong \underline{H}^1(A_{S_1}, \mathscr{O}_{A_{S_1}}) \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}\\ &\xrightarrow{\sim} \underline{H}^1(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}/S_1})^Y \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}\\ &\cong (\underline{H}^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}})^Y \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y,\end{aligned}$$

where the second isomorphism is the one given by Lemma 4.6.3.5. ☐

Lemma 4.6.3.7. *The diagram (of canonical morphisms)*

$$\begin{tikzcd} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}}, \mathscr{O}_{A_{S_1}}) \arrow[r] \arrow[d] & \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}}) \arrow[d] \\ \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}}, \mathscr{O}_{A_{S_1}})^Y \arrow[r] & \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})^Y \end{tikzcd} \tag{4.6.3.8}$$

is commutative, and its horizontal rows are isomorphisms.

Proof. The commutativity follows from their definitions. The bottom row is an isomorphism by Lemma 4.6.3.5. To show that the top row is an isomorphism, we claim that the induced map between the kernels of vertical arrows is an isomorphism. Isomorphism classes of possible actions that we can define on a particular extension in $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})^Y$ form a torsor under

$$\underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(G^\natural_{S_1}, \underline{\mathrm{Hom}}_{\mathscr{O}_{G^\natural_{S_1}}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})^Y)) \cong \underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y),$$

while the isomorphism classes of possible actions on a particular extension in $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}},\mathscr{O}_{A_{S_1}})^Y$ is a torsor under $\underline{\mathrm{Hom}}_{S_1}(Y,\underline{H}^0(A_{S_1},\mathscr{O}_{A_{S_1}})^Y)$. Hence the question is whether the canonical map

$$\underline{H}^0(A_{S_1},\mathscr{O}_{A_{S_1}})^Y \to \underline{H}^0(G^\natural_{S_1},\mathscr{O}_{G^\natural_{S_1}})^Y$$

defined by pullback is an isomorphism. This again follows from Lemma 4.6.3.5. ☐

Corollary 4.6.3.9. *$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}},\mathscr{O}_{G^\natural_{S_1}})$ is canonically isomorphic to $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}$ as an extension of $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1}$ by $\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}$.*

Proof. This is because we have

$$\underline{\mathrm{Hom}}_{S_1}(Y,\underline{H}^0(A_{S_1},\mathscr{O}_{A_{S_1}})^Y)\cong\underline{\mathrm{Hom}}_{S_1}(Y,\mathscr{O}_{S_1})\cong\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}$$

in proof of Lemma 4.6.3.7. ☐

Corollary 4.6.3.10. *There is a canonical isomorphism*

$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1},\Omega^1_{S_1/\mathsf{U}}\underset{\mathscr{O}_{S_1}}{\otimes}\mathscr{O}_{G^\natural_{S_1}})\cong\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\mathrm{Lie}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}} \tag{4.6.3.11}$$

respecting their structures of extensions of $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\mathrm{Lie}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}}$ by $\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\mathrm{Lie}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}}$.

Proof. By Corollary 4.6.3.9 and Proposition 4.6.2.5, the composition of canonical isomorphisms

$$\begin{aligned}\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1},\Omega^1_{S_1/\mathsf{U}}\underset{\mathscr{O}_{S_1}}{\otimes}\mathscr{O}_{G^\natural_{S_1}})&\cong\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}},\mathscr{O}_{G^\natural_{S_1}})\underset{\mathscr{O}_{S_1}}{\otimes}\underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}}\\&\cong\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}}\end{aligned}$$

respects their canonical maps to $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\mathrm{Lie}_{G^\natural_{S_1}/S_1}\underset{\mathscr{O}_{S_1}}{\otimes}\Omega^1_{S_1/\mathsf{U}}$, and hence gives the map we want. ☐

Proposition 4.6.3.12. *The map (4.6.3.11) sends the Y-equivariant extension class of $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$ in (4.6.3.4) to the Kodaira-Spencer class $\mathsf{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$.*

Proof. If we reproduce the argument of the proof of Proposition 2.1.7.3 using open coverings of $G^\natural_{S_1}$ as in the proof of Proposition 4.6.2.5, then we see that the map

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \cong \underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathrm{Lie}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$$

induced from (4.6.3.4) (by forgetting Y-actions of the parameterized objects) sends the Y-equivariant extension class of $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$ in (4.6.3.4) to the Kodaira-Spencer class $\mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$. Let we also take the Y-actions into account. Again, for simplicity of notations, let us assume that $S = S_1$. Let us choose a basis dx_i of $\Omega^1_{G^\natural/S}$ over U_α using coordinates of some $\mathbb{A}^r_S$'s as in the proof of Proposition 2.1.7.3, and write the Y-action on $\Omega^1_{G^\natural/S}$ as $\eta(y)_\alpha : dx_i \mapsto dT_{\iota(y)}(dx_i)$. Note that these differentials are taken over S. Take bases of $\Omega^1_{G^\natural/\mathsf{U}}$ including the dx_i above. Now the differentials are taken over U, and hence the Y action might no longer send dx_i to $dT_{\iota(y)}(dx_i)$. The difference is given by an element in $\underline{\mathrm{Lie}}_{G^\natural/S} \underset{S}{\otimes} \Omega^1_{S/\mathsf{U}}$ for each y. On the other hand, the differentials over U, by definition, can be explicitly obtained by comparing the two pullbacks $\mathrm{pr}_1^*(\eta(y)_\alpha)$ and $\mathrm{pr}_2^*(\eta(y)_\alpha)$ to the first infinitesimal neighborhood $\tilde{S}$ of the diagonal embedding $S \hookrightarrow S \underset{\mathsf{U}}{\times} S$. This is (up to a choice of sign conventions) how we compared the difference between liftings of Y-actions in the proof of Proposition 4.6.2.5. Hence the result follows. $\square$

Now that we have reinterpreted $\underline{\mathrm{Lie}}_{G^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ and $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ as respectively $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$ and $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$, the question is whether we can find an isomorphism between them that matches the extension class of $\Omega^1_{G_{S_1}/\mathsf{U}}$ with the Y-equivariant extension class of $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$. Although we could not realize G_{S_1} as a quotient of $G^\natural_{S_1}$ in the category of schemes, the theory of Mumford's construction using relatively complete models does realize the formal completion of some proper model P of G_{S_1} as a quotient. This enables us to prove the following:

Theorem 4.6.3.13. *The two maps* $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ *and* $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ *are identified under the restriction of the two canonical isomorphisms* $\underline{\mathrm{Lie}}^\vee_{G/S} \cong \underline{\mathrm{Lie}}^\vee_{G^\natural/S}$ *and* $\underline{\mathrm{Lie}}^\vee_{G^\vee/S} \cong \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/S}$ *to* S_1.

Proof. First note that we are allowed to prove the theorem under any finite étale base extension, as we are just comparing two existing maps. Therefore, we may assume that both $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y. According to Lemma 4.5.4.4 (which uses Corollary 4.5.1.8) and Proposition 4.5.1.14, we know that if we start with a relatively ample invertible sheaf $\mathcal{M}$ over A, pull it back to a cubical invertible sheaf $\mathcal{L}$ over $G^\natural$, and replace it by a sufficiently high tensor power, then we may assume that we have a relatively complete model $(P^\natural, \mathcal{L}^\natural)$ over S for the pair $(G^\natural, \mathcal{L}^\natural)$, together with an Y-action over the whole S extending the one on $G^\natural_\eta$. Let us denote by $(P, \mathcal{L})$ the "quotient" of it by Y defined by Mumford's construction.

Since P is proper over S, we have a natural isomorphism

$$\left[H^1(P, \underline{\mathrm{Der}}_{P/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}\right] \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \to H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1}) \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}.$$

On the other hand, since the restriction $\Omega^1_{G/S}$ of $\Omega^1_{P/S}$ to the smooth group scheme G is trivial, the local higher extension classes of $\Omega^1_{P/S}$ by $\Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P$ are all supported on $C = (P - G)_{\mathrm{red}}$ (see [56] or [64, Ch. III, Exer. 2.3]), which has empty fiber over S_1. As a result, the local-to-global spectral sequence tells us that the kernel and cokernel of the canonical morphism

$$\underline{H}^1(P, \underline{\mathrm{Der}}_{P/S}) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P)$$

are not supported over S_1. This shows that the canonical morphism

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}). \tag{4.6.3.14}$$

is an isomorphism.

Since P is proper over S, and since the sheaves of differentials have proper supports over S as well, we have a canonical morphism

$$\left[\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P)\right]_{\mathrm{for}} \xrightarrow{\sim} \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{\mathrm{for}}}). \tag{4.6.3.15}$$

by Proposition 2.3.1.1. By construction, P_{for} is indeed a quotient of $P^\natural_{\mathrm{for}}$ as in the hypothetical situation mentioned before, together with the additional nice properties that the action is free and that the quotient can be realized as a local isomorphism followed by a quotient by finite group action. Hence the natural map

$$\begin{aligned}&\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{\mathrm{for}}}) \\ &\quad\to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\end{aligned} \tag{4.6.3.16}$$

defined by pulling back extensions from P_{for} to $P^\natural_{\mathrm{for}}$ is an isomorphism.

On the other hand, since $P^\natural$ is locally of finite type but not necessarily of finite type over S, the canonical morphisms from the formal completion of the cohomology groups of $P^\natural$ to the cohomology groups of the formal completion $P^\natural_{\mathrm{for}}$ might not necessarily be isomorphisms. Since we care only about Y-invariant classes together with Y-actions on them, and since we only care if there is an isomorphism after we kill torsion (which are not supported on S_1), we might circumvent this trouble by passing to cohomology groups on A, as follows:

Let $G^{\natural,*} = \underset{y\in Y}{\cup} S_y(G^\natural) \subset P^\natural$ and $C^\natural = (P^\natural - G^{\natural,*})_{\mathrm{red}}$ as before, which are both invariant under Y-action. Then $C^\natural$ has empty fiber over S_1, and the formal completion C_{for} of C is the quotient of the formal completion $C^\natural_{\mathrm{for}}$ of $C^\natural$ by Y. (See Construction 4.5.2.15.) Let us consider the two sheaves $\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}$ and $\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}$. Since the two modules are both canonically isomorphic to $\Omega^1_{G^{\natural,*}_{\mathrm{for}}/S_{\mathrm{for}}}$ when restricted to $G^{\natural,*}_{\mathrm{for}}$, their difference is given by sections supported on C_{for}. Moreover, since the actions of a finite index subgroup of Y on P_{for} and hence on $G^{\natural,*}_{\mathrm{for}}$ and on C_{for} are given by local isomorphisms, the sheaves $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})$ and $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})$ are coherent. In particular, it makes sense to consider their algebraizations, and it makes sense to say that the difference between their algebraizations is not supported on S_1. Let us denote the algebraization of a coherent sheaf by the subscript "alg". Then we have an isomorphism

$$\begin{aligned}&\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \\ &\quad\to \Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}.\end{aligned} \tag{4.6.3.17}$$

(Here we are directing the morphism as if there *were* a morphism from $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}}$ to $\Omega^1_{P^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}}$.)

Let us denote the structural map $P^{\natural} \to A$ by π, and denote its formal completion by π_{for}. Then the Leray spectral sequence (see [44, Ch. II, Thm. 4.17.1], or [64, Ch. III, Exer. 8.1]) shows that the kernel and cokernel of the natural map

$$\underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}}) \to \underline{H}^i(P^{\natural}_{\mathrm{for}}, \mathscr{O}_{P^{\natural}_{\mathrm{for}}})$$

are subquotients of $\underline{H}^{i-j}(A_{\mathrm{for}}, R^j\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})$ with $j > 0$. If we take Y-invariants, then for the same reason as before we obtain coherent modules that algebraize. Hence it makes sense to say that the kernel and cokernel of

$$\underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y \to \underline{H}^i(P^{\natural}_{\mathrm{for}}, \mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y \tag{4.6.3.18}$$

(as coherent $\mathscr{O}_S$-modules) are not supported on S_1. On the other hand, since T-action commutes with Y-action on $G^{\natural,*}$, we see that there is a decomposition

$$\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}} \cong \underset{\chi\in X}{\oplus} (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi}$$

into T_{for}-weight spaces. By construction of $P^{\natural}$, the Y-action on $P^{\natural}$ is defined componentwise by the canonical isomorphisms

$$\begin{aligned}\tau(y,\chi) : T^*_{c^{\vee}(y)}(\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi} \cong (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi}(c^{\vee}(y)) &\underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi} \\ &\xrightarrow{\sim} (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi}.\end{aligned}$$

Accordingly, the action

$$S(y)^* : \underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y \xrightarrow{\sim} \underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y$$

decomposes componentwise as

$$\tau(y,\chi) : \underline{H}^i(A_{\mathrm{for}}, (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi})^Y \xrightarrow{\sim} \underline{H}^i(A_{\mathrm{for}}, (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})_{\chi})^Y.$$

By coherence of $\underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y$, and hence the coherence of each component $\underline{H}^i(A_{\mathrm{for}}, (\pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}}))_{\chi})^Y$, the positivity condition of τ implies as in the proof of Lemma 4.6.3.5 that

$$\underline{H}^i(A_{\mathrm{for}}, \pi_{\mathrm{for},*}\mathscr{O}_{P^{\natural}_{\mathrm{for}}})^Y \cong \underline{H}^i(A_{\mathrm{for}}, \mathscr{O}_{A_{\mathrm{for}}})^Y. \tag{4.6.3.19}$$

Here the Y-action on A_{for} is induced by the group homomorphism $c^\vee : Y \to A$, and we know by Lemma 4.6.2.2 that $\underline{H}^i(A_{\mathrm{for}}, \mathscr{O}_{A_{\mathrm{for}}})^Y \cong \underline{H}^i(A_{\mathrm{for}}, \mathscr{O}_{A_{\mathrm{for}}})$. Nevertheless, it is suggestive to consider the Y-equivariant extensions $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}})$, although it is the same as the ones without Y-equivariant actions. The combination of our knowledge about (4.6.3.18) and (4.6.3.19) implies that the kernel and cokernel of the natural map

$$\begin{aligned} &\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}) \\ &\quad \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}}) \end{aligned} \tag{4.6.3.20}$$

(as coherent $\mathscr{O}_S$-modules) are not supported on S_1. As A is proper, there is a canonical isomorphism

$$\begin{aligned} &\left[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A)\right]_{\mathrm{for}} \\ &\quad \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}) \end{aligned} \tag{4.6.3.21}$$

of coherent $\mathscr{O}_{S_{\mathrm{for}}}$-modules. Moreover, there is a canonical isomorphism

$$\begin{aligned} &\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \\ &\quad \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}). \end{aligned} \tag{4.6.3.22}$$

Finally, Lemma 4.6.3.7 shows that there is a canonical isomorphism

$$\begin{aligned} &\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}) \\ &\quad \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}). \end{aligned} \tag{4.6.3.23}$$

Let us combine the results we have obtained. Denote the algebraization of a coherent sheaf or a morphism by the subscript "alg" (as above). Then we obtain a canonical isomorphism

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \tag{4.6.3.24}$$

as a composition of a series of canonical morphisms:

$$\begin{tikzcd}
\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \\
\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \arrow[u, "(4.6.3.14)", "\wr"'] \arrow[d, "(4.6.3.15)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}", "\wr"'] \\
\Big[\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \arrow[d, "(4.6.3.16)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}", "\wr"'] \\
\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \\
\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \arrow[u, "(4.6.3.17)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}", "\wr"'] \\
\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \arrow[u, "(4.6.3.20)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}", "\wr"'] \\
\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \arrow[u, "(4.6.3.21)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}", "\wr"'] \arrow[d, "(4.6.3.22)", "\wr"'] \\
\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}) \arrow[d, "(4.6.3.23)", "\wr"'] \\
\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})
\end{tikzcd}$$

Note that (4.6.3.24) is compatible with the canonical isomorphism $\underline{\mathrm{Lie}}_{G/S} \cong \underline{\mathrm{Lie}}_{G^\natural/S}$ because this is essentially the isomorphism $e^*_{G_{\mathrm{for}}}\underline{\mathrm{Der}}_{P_{\mathrm{for}}/S_{\mathrm{for}}} \xrightarrow{\sim} e^*_{G^\natural_{\mathrm{for}}}\underline{\mathrm{Der}}_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}$ that we have used implicitly in (4.6.3.16). The reason that (4.6.3.24) is compatible with the canonical isomorphism $\underline{\mathrm{Lie}}_{G^\vee/S} \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$ is more involving. First note that we can identify the isomorphism (4.6.3.24) with the isomorphism

$$\begin{aligned}&\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G_{S_1}/S_1}\\ &\quad\xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1}\end{aligned}$$

induced by an isomorphism

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$$

obtained by replacing $\Omega^1_{P/S}$ by $\mathscr{O}_P$ and $\Omega^1_{P^\natural/S}$ by $\mathscr{O}_{P^\natural}$ in what we have been doing. Using the Poincaré invertible sheaves as in the proof of Lemma 4.6.2.4, the result follows from the fact that $\mathcal{P}_G$ is a Mumford quotient of the pullback of $\mathcal{P}_A$ to $G^\natural \times G^{\vee,\natural}$ as in Construction 4.5.4.11.

Now consider the first exact sequence of P over S:

$$\Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P \to \Omega^1_{P/\mathsf{U}} \to \Omega^1_{P/S} \to 0. \tag{4.6.3.25}$$

The first morphism is not necessarily injective, but it becomes injective if we pullback the sequence to $\mathscr{O}_{S_1}$. In other words, the base change of it to S_1 defines an element in

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}. \tag{4.6.3.26}$$

Moreover, its image under (4.6.3.14) is the extension class of the sequence (4.6.3.2).

Similarly, consider the first exact sequence of $P^\natural$ over S:

$$\Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P^\natural} \to \Omega^1_{P^\natural/\mathsf{U}} \to \Omega^1_{P^\natural/S} \to 0. \tag{4.6.3.27}$$

This has a Y-equivariant structure because the Y-action on $G^\natural_\eta$ extends to the relatively complete model $P^\natural$ over the whole S. The base change of it to S_{for} defines an element in

$$\left[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\right]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}, \tag{4.6.3.28}$$

whose image under the complicated composition in the second half of the definition of (4.6.3.24) gives the Y-equivariant extension class of the sequence (4.6.3.4)

By Mumford's construction of P_{for} as a quotient of $P^{\natural}_{\mathrm{for}}$, the pullback of the exact sequence (4.6.3.25) to S_{for} can be realized as a quotient of the pullback of the extension class of the pullback of the exact sequence (4.6.3.27) to S_{for}. Therefore, the classes they define in (4.6.3.26) and in (4.6.3.28) are identified under the composition of $(4.6.3.15)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$ and $(4.6.3.16)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$. That is, the extension class of the sequence (4.6.3.2) is identified with the Y-equivariant extension class of the sequence (4.6.3.4) under the isomorphism (4.6.3.24). By taking duality, we obtain the identification between the Kodaira-Spencer maps $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ and $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$, as desired. ☐

Remark 4.6.3.29. The definition of $\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}}$ in [37, Ch. III, §9] involves some different formulations using universal extensions (see [93] and [92]), which might be equivalent to ours, but we do not guarantee. Otherwise there is no intentional difference between our Theorem 4.6.3.13 and [37, Ch. III, Thm. 9.4].

Let us globalize Theorem 4.6.3.13 as follows:

Theorem 4.6.3.30. *Let S be a smooth algebraic stack over an excellent normal base scheme U. Let G be a semi-abelian scheme over S. Suppose there is an open dense sub-algebraic stack S_1 of S, with complement $D_\infty := S - S_1$ a divisor of normal crossings, such that the restriction G_{S_1} of G to S_1 is an abelian scheme. Note that in this case there is a semi-abelian scheme $G^\vee$ (unique up to unique isomorphism) such that the restriction $G^\vee_{S_1}$ of $G^\vee$ to S_1 is the dual abelian scheme of G_{S_1}. Then there is a unique extension of the Kodaira-Spencer map*

$$\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}$$

to a morphism

$$\mathrm{KS}_{G/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to \Omega^1_{S/\mathsf{U}}[d\log D_\infty].$$

Here $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ is the sheaf of logarithmic 1-differentials, namely the subsheaf of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ generated locally by $\Omega^1_{S/\mathsf{U}}$ and those $d\log q$ where q is a local generator of a component of D_∞.

Proof. By local freeness of $\underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S}$ and by normality of S, there is always an extension

$$\underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to (S_1 \hookrightarrow S)_* \Omega^1_{S_1/\mathsf{U}}.$$

Therefore the question is whether the image of the extension lies in the subsheaf $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$. Since this question is local in nature, we may replace the smooth algebraic stack S by its completions of étale localizations, which are noetherian normal by our assumption of excellent normality on U and hence fit into the setting of Section 4.1. Let us also replace S_1 and D_∞ by their corresponding pullbacks. We may also assume that G_{S_1} is equipped with some particular polarization λ_{S_1}, because whether the extension lies in the subsheaf $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ does not depend on this choice of polarization. Then, by Theorem 4.6.3.13, we may simply take $\mathrm{KS}_{G/S/\mathsf{U}}$ to be the map $\mathrm{KS}_{(G^\natural,\iota)/S/\mathsf{U}}$, where $(G^\natural, \iota)$ is the object in DD underlying the degeneration datum $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{pol}}$ associated to (G, λ) in $\mathrm{DEG}_{\mathrm{pol}}$ via the functor $\mathrm{F}_{\mathrm{pol}}$ in Definition 4.4.9. By construction in Section 4.6.2, we need only logarithmic 1-differentials with poles supported on D_∞, because the period map ι can be defined over S_1. □

Remark 4.6.3.31. By Proposition 4.5.3.10, the proof of Theorem 4.6.3.30 shows that the logarithmic 1-differentials can be predicted by the character groups of the torus parts of the semi-abelian scheme. This observation will be essential in the proof of Proposition 6.3.2.13 below, which is the key to the gluing of boundary charts. (See the proof of Proposition 6.3.3.11 below.)

Definition 4.6.3.32. *The extended map* $\mathrm{KS}_{G/S/\mathsf{U}}$ *in Theorem 4.6.3.30 is called the* **extended Kodaira-Spencer map** *for G over S.*

Chapter 5

Degeneration Data for Additional Structures

In this chapter, we supply a theory for additional structures of PEL-type based on the theory of degeneration developed in Chapter 4. The running assumptions and notations in Chapter 4 will be continued in this chapter without further remark.

The main objective is to state and prove Theorem 5.3.1.17, with Theorem 5.3.3.1 and the discussion of cusp labels in Section 5.4 as byproducts. Technical results worth noting are Proposition 5.1.2.2 in Section 5.1; Propositions 5.2.2.23, 5.2.3.3, and 5.2.3.8, and Theorem 5.2.3.13 in Section 5.2; and Proposition 5.4.3.5 in Section 5.4. The discussions in Sections 5.2.4, 5.2.5 and 5.2.6 leading to the proof of Theorem 5.2.3.13 is the technical heart of this chapter.

5.1 Data for Endomorphism Structures

5.1.1 Analysis of Endomorphism Structures

Let S be a base scheme satisfying the assumptions in Section 4.1, with generic point η, and let $(G, \lambda) \to S$ be an object in $\mathrm{DEG}_{\mathrm{pol}}$ (defined as in Definition 4.4.2 and Remark 4.4.3).

Suppose moreover that (G_η, λ_η) is equipped with a ring homomorphism $i_\eta : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ defining an $\mathcal{O}$-endomorphism structure (with image in $\mathrm{End}_\eta(G_\eta)$) (as in Definition 1.3.3.1). Recall that our convention is to think of G_η as a left $\mathcal{O}$-module. (See Remark 1.3.3.3.) By Proposition 3.3.1.7, we know that the restriction $\mathrm{End}_S(G) \to \mathrm{End}_\eta(G_\eta)$ is an isomorphism under the noetherian normality assumption on the base scheme S, and so i_η defines by composition with the inverse of this isomorphism a ring homomorphism $i : \mathcal{O} \to \mathrm{End}_S(G)$.

Lemma 5.1.1.1. *Let C be any finite-dimensional semisimple algebra over $\mathbb{Q}$, and let C' be a finitely generated $\mathbb{Z}$-subalgebra of C. Then C' is generated as a $\mathbb{Z}$-algebra by finitely many elements in $C' \cap C^\times$.*

Proof. First we claim that, for any element $c \in C'$ but $c \notin C^\times$, there exists an integer $n_c \in \mathbb{Z}$ (depending on c) such that $c + n_c \in C^\times$. To see this claim, consider $C \hookrightarrow C \underset{\mathbb{Q}}{\otimes} \mathbb{C}$ and view every element of C as endomorphisms of complex vector spaces. Then an element c is invertible if all its finitely eigenvalues are nonzero, and the claim follows because addition by an integer n simply shift all the eigenvalues by n. Now that we have the claim, by adding integers to a set of generators of C' over $\mathbb{Z}$, we may assume that they are all in $C' \cap C^\times$, as desired. $\square$

By Lemma 5.1.1.1, we see that $\mathcal{O}$ is generated as an $\mathbb{Z}$-algebra by $\mathcal{O} \cap B^\times$. Since the elements of $\mathcal{O} \cap B^\times$ correspond to *isogenies* under i, they define morphisms between objects in $\mathrm{DEG}_{\mathrm{pol}}$. As a result, the $\mathcal{O}$-endomorphism structure $i : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ of $(G_\eta, \lambda_\eta) \to \eta$ corresponds by functoriality of $\mathrm{F}_{\mathrm{pol}}$ and $\mathrm{M}_{\mathrm{pol}}$ in Theorem 4.4.18 to the following data on the tuple $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$:

1. A ring homomorphism $i_A : \mathcal{O} \to \mathrm{End}_S(A)$ compatible with λ_A in the sense that it satisfies the *Rosati condition* $i_A(b)^\vee \circ \lambda_A = \lambda_A \circ i_A(b^\star)$ for any $b \in \mathcal{O}$. This defines an $\mathcal{O}$-endomorphism structure (with image in $\mathrm{End}_S(A)$) of $(A, \lambda_A) \to S$.

2. A ring homomorphism $i_T : \mathcal{O} \to \mathrm{End}_S(T)$ giving a left $\mathcal{O}$-module structure of T, which is equivalent to a ring homomorphism $i_{\underline{X}}^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(\underline{X})$ giving a right $\mathcal{O}$-module structure of $\underline{X}$, and a ring homomorphism $i_{T^\vee}^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(T^\vee)$ giving a right $\mathcal{O}$-module structure of $T^\vee$, which is equivalent to a ring homomorphism $i_{\underline{Y}} : \mathcal{O} \to \mathrm{End}_S(\underline{Y})$ giving a left $\mathcal{O}$-module structure of $\underline{Y}$.

   Note that the module structures above satisfy the compatibility ${}^t i_{\underline{X}}^{\mathrm{op}}(b) = i_T(b^\star)$ (resp. ${}^t i_{\underline{Y}}(b) = i_{T^\vee}^{\mathrm{op}}(b^\star)$) under the natural transposition defined by the pairing between X and T (Y and $T^\vee$), and the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}} : b \mapsto b^\star$.

Hence the two $\mathcal{O}$-module structures on Y and X make ϕ an anti-linear $\mathcal{O}$-module morphism, in the sense that

$$\begin{tikzcd}
\underline{Y} \arrow[r, "i_{\underline{Y}}(b^\star)"] \arrow[d, "\phi"'] & \underline{Y} \arrow[d, "\phi"] \\
\underline{X} \arrow[r, "i^{\mathrm{op}}_{\underline{X}}(b)"'] & \underline{X}
\end{tikzcd} ,$$

is commutative, or simply $i_{\underline{X}}(b) \circ \phi = \phi \circ i_{\underline{Y}}(b^\star)$ for any $b \in \mathcal{O}$.

If we view X (resp. $T^\vee$) as a left $\mathcal{O}$-module via the ring morphism $i_{\underline{X}} : \mathcal{O} \to \mathrm{End}_S(X)$ (resp. $i_{T^\vee} : \mathcal{O} \to \mathrm{End}_S(T^\vee)$) defined by composing the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}} : b \mapsto b^\star$ with $i^{\mathrm{op}}_{\underline{X}}$ (resp. $i^{\mathrm{op}}_{T^\vee}$), then we can view $\phi : Y \hookrightarrow X$ as an *$\mathcal{O}$-linear* morphism between *left* $\mathcal{O}$-modules. We will adopt this convention whenever possible.

3. The $\mathcal{O}$-equivariances of $c : \underline{X} \to A^\vee$ and $c^\vee : \underline{Y} \to A$. Here we endow $A^\vee$ with a left $\mathcal{O}$-module structure by $i_{A^\vee} : \mathcal{O} \to \mathrm{End}_S(A^\vee)$ defined by $i_{A^\vee}(b) := i_A(b^\star)^\vee$ for every $b \in \mathcal{O}$. Alternatively, we may define a natural right $\mathcal{O}$-module structure $i^{\mathrm{op}}_{A^\vee} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(A^\vee)$ on $A^\vee$ by $i^{\mathrm{op}}_{A^\vee}(b) := i_A(b)^\vee$, and then set $i_{A^\vee}$ to be the composition of the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}}$ with $i^{\mathrm{op}}_{A^\vee}$.

   (The data so far together with the compatibility $\lambda_A c^\vee = c\phi$ corresponds to a ring homomorphism $i^\natural : \mathcal{O} \to \mathrm{End}_S(G^\natural)$ compatible with $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ in the sense that $i^\natural(b)^\vee \circ \lambda^\natural = \lambda^\natural \circ i^\natural(b^\star)$ for all $b \in \mathcal{O}$.)

4. The $\mathcal{O}$-equivariance of the *period map* $\iota : \underline{Y} \to G^\natural_\eta$, given by the condition on the trivialization $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{A,\eta}$ that

$$(i_{\underline{Y}}(b) \times \mathrm{Id}_{\underline{X}})^* \tau = (\mathrm{Id}_{\underline{Y}} \times i^{\mathrm{op}}_{\underline{X}}(b))^* \tau = (\mathrm{Id}_{\underline{Y}} \times i_{\underline{X}}(b^\star))^* \tau$$

   for all $b \in \mathcal{O}$, which makes sense because

$$(f \times \mathrm{Id}_{A^\vee})^* \mathcal{P}_A \cong (\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A$$

   for any endomorphism $f \in \mathrm{End}_S(A)$, and because c and $c^\vee$ are both $\mathcal{O}$-equivariant.

Remark 5.1.1.2. The only nontrivial part is 4, which requires the reasoning behind the equivalence between a period map $\iota : \underline{Y} \to G_\eta^\natural$ and a trivialization of biextension $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$.

Lemma 5.1.1.3. *The underlying groups X and Y of the étale sheaves $\underline{X}$ and $\underline{Y}$ are $\mathcal{O}$-lattices with their (left) $\mathcal{O}$-module structure.*

Proof. It suffices to know that X and Y are $\mathbb{Z}$-lattices (by Definition 1.1.1.18), which is the case because T and $T^\vee$ are tori (by Definition 3.1.1.5). ☐

Definition 5.1.1.4. *Assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ *has objects of the form (G, λ, i) (over S), where (G, λ) defines an object in* $\mathrm{DEG}_{\mathrm{pol}}$*, and where $i : \mathcal{O} \to \mathrm{End}_S(G)$ defines by restriction an $\mathcal{O}$-structure $i_\eta : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ of (G_η, λ_η). Note that by Proposition 3.3.1.7 the restriction morphism $\mathrm{End}_S(G) \to \mathrm{End}_\eta(G_\eta)$ is an isomorphism.*

Definition 5.1.1.5. *Assumptions as in Section 4.1, the category* $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$ *has objects of the form $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$, with $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ defining an object in* $\mathrm{DD}_{\mathrm{pol}}$*, and with additional $\mathcal{O}$-structure compatibilities described as follows:*

1. *$i_A : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure for (A, λ_A).*
2. *The underlying modules X and Y of respectively $\underline{X}$ and $\underline{Y}$ have structures of $\mathcal{O}$-lattices of the same multi-rank (defined as in Definition 1.2.1.20), given respectively by the ring homomorphisms $i_{\underline{X}} : \mathcal{O} \to \mathrm{End}_S(X)$ and $i_{\underline{Y}} : \mathcal{O} \to \mathrm{End}_S(Y)$. The embedding $\phi : \underline{Y} \hookrightarrow \underline{X}$ is $\mathcal{O}$-linear with respect to these $\mathcal{O}$-module structures.*
3. *The homomorphisms $c : \underline{X} \to A^\vee$ and $c^\vee : \underline{Y} \to A$ are both $\mathcal{O}$-linear.*
4. *The trivialization $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$ satisfy the compatibility*

$$(i_{\underline{Y}}(b) \times \mathrm{Id}_{\underline{X}})^* \tau = (\mathrm{Id}_{\underline{Y}} \times i_{\underline{X}}(b^\star))^* \tau$$

   for all $b \in \mathcal{O}$, which gives rise to the $\mathcal{O}$-linearity of the **period map** *$\iota : \underline{Y} \to G_\eta^\natural$.*

Then our result can be summarized as follows:

Theorem 5.1.1.6. *There is an equivalence of categories*

$$\mathrm{M}_{\mathrm{PE},\mathcal{O}} : \mathrm{DD}_{\mathrm{PE},\mathcal{O}} \to \mathrm{DEG}_{\mathrm{PE},\mathcal{O}} : (A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \mapsto (G, \lambda, i).$$

5.1.2 Analysis of Lie Algebra Conditions

Let V_0 and V_0^c be the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module defined in Section 1.2.1, which have signatures respectively (p_τ) and (q_τ) (defined as in Definition 1.2.5.1).

Proposition 5.1.2.1. *Suppose V_0 has signatures (p_τ) as above, and suppose $V_0^c \cong V_0^\vee(1)$ has signatures (q_τ), so that $p_\tau = q_{\tau \circ c}$. Let W be the unique integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-module of multi-rank $r_{[\tau]}$ (defined as in Definition 1.2.1.20 and Lemma 1.2.1.23). Then there is a totally isotropic embedding $W \underset{\mathbb{Z}}{\otimes} \mathbb{R} \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ if we have $p_\tau \geq r_{[\tau]}$ and $q_\tau \geq r_{[\tau]}$ for all $\tau : F \to \mathbb{C}$. (The notations make sense because each $\tau : F \to \mathbb{C}$ determines a unique factorization $[\tau] : F \to \mathbb{Q}_{[\tau]} \hookrightarrow \mathbb{R}_{[\tau]} \hookrightarrow \mathbb{C}$.)*

Proof. Let $m = (m_{[\tau]})$ be the multi-rank of L as an $\mathcal{O}$-module. For any nontrivial morphism $\tau : F \to \mathbb{C}$, let $\mathbb{R}_{[\tau]}$ be either $\mathbb{R}$ or $\mathbb{C}$, depending on whether τ is real or complex. By Corollary 1.1.2.5, there is a decomposition

$$W \cong \underset{[\tau]:F \hookrightarrow \mathbb{R}_{[\tau]}}{\oplus} W_{[\tau]}^{r_{[\tau]}}$$

and similarly a decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \underset{[\tau]:F \hookrightarrow \mathbb{R}_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus\, m_{[\tau]}},$$

where $[\tau] : F \hookrightarrow \mathbb{R}_{[\tau]}$ runs through the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits of nontrivial morphisms $\tau : F \to \mathbb{C}$. By definition, two morphisms τ_1 and τ_2 are in the same $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit exactly when either $\tau_1 = \tau_2$ or $\tau_1 = \tau_2 \circ c$. This $W_{[\tau]}$ is the unique irreducible $\mathcal{O} \underset{F,[\tau]}{\otimes} \mathbb{R}_{[\tau]}$-module, which is considered as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-module via the canonical projection from $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ to $\mathcal{O} \underset{F,[\tau]}{\otimes} \mathbb{R}_{[\tau]}$. If $\mathbb{R}_{[\tau]}$ is real, then $W_{[\tau]} \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong W_\tau$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules for the unique representative $\tau : F \to \mathbb{C}$ of $[\tau]$. If $\mathbb{R}_{[\tau]}$ is complex, then $W_{[\tau]} \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong W_\tau \oplus W_{\tau \circ c}$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules for any of the two representatives $\tau : F \to \mathbb{C}$ of $[\tau]$. The self-dual pairing $\langle \cdot , \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is equivalent to an isomorphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \xrightarrow{\sim} \mathrm{Hom}_\mathbb{R}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}(1))$, which necessarily sends $W_{[\tau]}^{\oplus\, m_{[\tau]}}$ in the decomposition to $(W_{[\tau]}^\vee)^{\oplus\, m_{[\tau]}}(1) := \mathrm{Hom}_\mathbb{R}(W_{[\tau]}^{\oplus\, m_{[\tau]}}, \mathbb{R}(1))$, as the complexified map necessarily sends $W_{\tau}^{\oplus\, m_{[\tau]}}$ to $({W_{\tau\circ c}}^{\vee})^{\oplus\, m_{[\tau]}}(1) := \mathrm{Hom}_{\mathbb{C}}(W_{\tau\circ c}^{\oplus\, m_{[\tau]}}, \mathbb{C}(1))$ because the action of the center is twisted by the involution $\star$. The decomposition $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c \cong V_0 \oplus V_0^{\vee}(1)$ corresponds to the decomposition $W_{[\tau]}^{\oplus\, m_{[\tau]}} \underset{\mathbb{R}}{\otimes} \mathbb{C} = W_{\tau}^{\oplus\, p_\tau} \oplus W_{\tau}^{\oplus\, q_\tau} = W_{\tau}^{\oplus\, p_\tau} \oplus (W_{\tau}^{\vee})^{\oplus\, p_\tau}(1)$ when $\mathbb{R}_{[\tau]}$ is real, and to the decomposition $W_{[\tau]}^{\oplus\, m_{[\tau]}} \underset{\mathbb{R}}{\otimes} \mathbb{C} = (W_{\tau}^{\oplus\, p_\tau} \oplus W_{\tau\circ c}^{\oplus\, p_{\tau\circ c}}) \oplus (W_{\tau}^{\oplus\, q_\tau} \oplus W_{\tau\circ c}^{\oplus\, q_{\tau\circ c}}) \cong (W_{\tau}^{\oplus\, p_\tau} \oplus W_{\tau\circ c}^{\oplus\, p_{\tau\circ c}}) \oplus ((W_{\tau}^{\vee})^{\oplus\, p_\tau} \oplus (W_{\tau\circ c}^{\vee})^{\oplus\, p_{\tau\circ c}})(1)$ when $\mathbb{R}_{[\tau]}$ is complex, as $p_\tau + q_\tau = p_\tau + p_{\tau\circ c} = m_{[\tau]}$.

To prove the lemma, it suffices to prove the following statement: For any representative $\tau : F \to \mathbb{C}$ of $[\tau] : F \to \mathbb{R}$, there is a totally isotropic embedding $W_{[\tau]}^{\otimes r_{[\tau]}} \hookrightarrow W_{[\tau]}^{\otimes m_{[\tau]}}$ if we have $p_\tau \geq r_{[\tau]}$ and $p_{\tau\circ c} \geq r_{[\tau]}$. When $\mathbb{R}_{[\tau]}$ is real, we take any embedding $\varepsilon_1 : W_{[\tau]}^{\oplus\, r_{[\tau]}} \cong W_{\tau}^{\oplus\, r_{[\tau]}} \hookrightarrow W_{\tau}^{\oplus\, p_\tau}$, and consider the composition ε_2 of this embedding by with the map $W_{\tau}^{\oplus\, p_\tau} \xrightarrow{\sim} (W_{\tau}^{\vee})^{\oplus\, p_\tau}(1)$ induced by the pairing. Then we embed $W_{[\tau]}^{\oplus\, r_{[\tau]}}$ into $W_{[\tau]}^{\oplus\, m_{[\tau]}} \cong W_{\tau}^{\oplus\, p_\tau} \oplus (W_{\tau}^{\vee})^{\oplus\, p_\tau}(1)$ by $x \mapsto \varepsilon(x) := (\varepsilon_1(x), \varepsilon_2(x))$. This embedding is totally isotropic because $\langle \varepsilon(x), \varepsilon(y) \rangle = \langle (\varepsilon_1(x), \varepsilon_2(x)), (\varepsilon_1(y), \varepsilon_2(y)) \rangle = \varepsilon_2(y)(\varepsilon_1(x)) - \varepsilon_1(y)(\varepsilon_2(x)) = \langle x, y \rangle - \langle x, y \rangle = 0$, and it is defined over $\mathbb{R}$ because the complex conjugation simply switches the two factors. When $\mathbb{R}_{[\tau]}$ is complex, we take any embedding $\varepsilon_1 : W_{[\tau]}^{\oplus\, r_{[\tau]}} \cong W_{\tau}^{\oplus\, r_{[\tau]}} \oplus W_{\tau\circ c}^{\oplus\, r_{[\tau]}} \hookrightarrow W_{\tau}^{\oplus\, p_\tau} \oplus W_{\tau\circ c}^{\oplus\, p_{\tau\circ c}}$, consider the composition ε_2 of this embedding induced by the pairing, and embed $W_{[\tau]}^{\oplus}$ into $W_{[\tau]}^{\oplus\, m_{[\tau]}} \cong (W_{\tau}^{\oplus\, p_\tau} \oplus W_{\tau\circ c}^{\oplus\, p_{\tau\circ c}}) \oplus ((W_{\tau}^{\vee})^{\oplus\, p_\tau} \oplus (W_{\tau\circ c}^{\vee})^{\oplus\, p_{\tau\circ c}})(1)$ by $x \mapsto \varepsilon(x) := (\varepsilon_1(x), \varepsilon_2(x))$ as above. Then, again, this embedding is defined over $\mathbb{R}$ and totally isotropic, which concludes the proof. $\square$

Let B, $\mathcal{O}$, $(L, \langle \cdot\,, \cdot \rangle)$ and $\Box$ be chosen as in Section 1.4. Let us assume the setting of Section 4.1, and assume that the generic point η of the base scheme S is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. (We do not need the whole scheme S to be defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.)

Proposition 5.1.2.2. *Assumptions as above, let* (G, λ, i) *be an object in* $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$*, with associated degeneration datum* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$*. Then, for* $(G_\eta, \lambda_\eta, i_\eta)$ *to satisfy the Lie algebra condition defined by* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)$ *as in Definition 1.3.4.2, it is necessary that there exists a totally isotropic embedding* $\mathrm{Hom}_{\mathbb{R}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$*, where* X *is the underlying* $\mathcal{O}$*-lattice of* $\underline{X}$*.*

In this case, if we denote the image of this embedding by $\mathsf{Z}_{-2,\mathbb{R}}$, *and denote its annihilator by* $\mathsf{Z}_{-1,\mathbb{R}}$. *Let* $\mathsf{Z}_{0,\mathbb{R}}$ *be* $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *and* $\mathsf{Z}_{-3,\mathbb{R}}$ *be* 0. *Then we have an symplectic admissible filtration* $\mathsf{Z}_{\mathbb{R}} := \{\mathsf{Z}_{-i,\mathbb{R}}\}$ *on* $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *(by construction). The pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *induces a pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}$ *on* $\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}} := \mathsf{Z}_{-1,\mathbb{R}}/\mathsf{Z}_{-2,\mathbb{R}}$ *satisfying Condition 1.2.1.2. Now, for* $(G_\eta, \lambda_\eta, i_\eta)$ *to satisfy the Lie algebra condition defined by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$, *it is both necessary and sufficient that* $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ *satisfies the Lie algebra condition defined by* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$.

The actual choice of the embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \mathbb{R}) \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *is immaterial, because the isomorphism class of* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$ *(as a symplectic module over* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$*) is independent of this choice.*

Proof. Let $(r_{[\tau]})$ be the multi-rank of X (defined as in Definition 1.2.1.20). By Proposition 5.1.2.1, to show that there is a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \mathbb{R}) \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, it suffices to show that we have $p_\tau \geq r_{[\tau]}$ and $q_\tau \geq r_{[\tau]}$ for all $\tau : F \to \mathbb{C}$. (Here the notations make sense because each $\tau : F \to \mathbb{C}$ factorizes uniquely as $[\tau] : F \to \mathbb{Q}_{[\tau]} \hookrightarrow \mathbb{R}_{[\tau]} \hookrightarrow \mathbb{C}$.)

By assumption above that η is define over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, the characteristic p of the generic point η is unramified in F. Let $k(\eta)$ denote the field defining η, and let $k(\eta)^{\mathrm{sep}}$ denote a separable closure of $k(\eta)$. By Remark 1.2.5.14, the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}k(\eta)^{\mathrm{sep}}$-modules also admit the notions of classification and signatures that can be matched with the corresponding notions for $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules. If we take the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathcal{O}_{F_0}$-module L_0 defined in Lemma 1.2.5.10, then the Lie algebra condition for $\mathcal{O}\underset{\mathbb{Z}}{\otimes}k(\eta)$-modules (which is defined because η is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$) is given by comparing the isomorphism class of $L_0 \underset{\mathcal{O}_{F_0}}{\otimes} k(\eta)$ with the corresponding Lie algebras. We shall arrange that $L_0\underset{\mathbb{Z}}{\otimes}k(\eta)$ has signatures (p_τ) when considered as just $\mathcal{O}\underset{\mathbb{Z}}{\otimes}k(\eta)^{\mathrm{sep}}$-modules.

The (separable) polarization $\lambda_\eta : G_\eta \to G_\eta^\vee$ defines an $\mathcal{O}$-anti-linear isomorphism $d\lambda_\eta : \mathrm{Lie}_{G_\eta/\eta} \xrightarrow{\sim} \mathrm{Lie}_{G_\eta^\vee/\eta}$. Depending on whether $F = F^+$ or not, the action of the center F on $\mathrm{Lie}_{G_\eta/\eta}$ and on $\mathrm{Lie}_{G_\eta^\vee/\eta}$ either agree or differ by a nontrivial involution (which is the restriction of the complex conjugation to the center under any embedding $\tau : F \hookrightarrow \mathbb{C}$). Therefore, we know that if $\mathrm{Lie}_{G_\eta/\eta}$ has signatures (p_τ), then $\mathrm{Lie}^\vee_{G_\eta^\vee/\eta}$ must have signatures (q_τ) because

$q_\tau = p_{\tau\circ c}$.

On the other hand, by Corollary 2.3.1.3, we have a canonical isomorphism $\underline{\mathrm{Lie}}_{G/S} \cong \underline{\mathrm{Lie}}_{G^\natural/S}$, as both of them have the same formal completion $\underline{\mathrm{Lie}}_{G_{\mathrm{for}}/S_{\mathrm{for}}}$ along S_0. Similarly, we have a canonical isomorphism $\underline{\mathrm{Lie}}_{G^\vee/S} \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$. Both of the canonical isomorphisms are $\mathcal{O}$-linear, by the functoriality of Corollary 2.3.1.3. We know that there is an $\mathcal{O}$-linear embedding $\phi : Y \hookrightarrow X$ of finite index. In particular, X and Y have the same multi-rank $(r_{[\tau]})$. By definition, $T \cong \mathrm{Hom}_S(X, \mathbf{G}_{\mathrm{m},S})$ and $T^\vee \cong \mathrm{Hom}_S(Y, \mathbf{G}_{\mathrm{m},S})$, and hence there are isomorphisms $\mathrm{Lie}_{T_\eta/\eta} \cong \mathrm{Hom}(X, k(\eta))$ and $\mathrm{Lie}_{T^\vee_\eta/\eta} \cong \mathrm{Hom}(Y, k(\eta))$. As a result, the signatures of $\mathrm{Lie}_{T_\eta/\eta}$, $\mathrm{Lie}_{T^\vee_\eta/\eta}$, and hence $\mathrm{Lie}^\vee_{T^\vee_\eta/\eta}$ all consist of integers $\{r_\tau\}$ such that $r_{\tau_1} = r_{\tau_2}$ if τ_1 and τ_2 determine the same $[\tau_1] = [\tau_2]$, as all the simple $\mathcal{O}\underset{\mathbb{Z}}{\otimes} k(\eta)$-modules must occur with the same multiplicities. Since $\mathrm{Lie}_{T_\eta/\eta}$ and $\mathrm{Lie}^\vee_{T^\vee_\eta/\eta}$ are $\mathcal{O}\underset{\mathbb{Z}}{\otimes} k(\eta)$-submodules of respectively $\mathrm{Lie}_{G^\natural_\eta/\eta}$ and $\mathrm{Lie}^\vee_{G^{\vee,\natural}_\eta/\eta}$, we must have $p_\tau \geq r_{[\tau]}$ and $q_\tau \geq r_{[\tau]}$, as desired. (The remaining statements of the proposition are just necessary consequences of the existence of a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \mathbb{R}) \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$.) $\square$

Definition 5.1.2.3. *Assumptions as above, the category* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ *has objects of the form* (G, λ, i) *(over* S*), each defining an object in* $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ *such that* $(G_\eta, \lambda_\eta, i_\eta)$ *satisfies the Lie algebra condition defined by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition 1.3.4.2.*

Definition 5.1.2.4. *Assumptions as above, the category* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ *has objects of the form* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$*, each tuple defining an object in* $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$*, such that there exists a totally isotropic embedding* $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \mathbb{R}) \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$*, where* X *is the underlying* $\mathcal{O}$*-lattice of* $\underline{X}$*, and such that* $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ *satisfies the Lie algebra condition defined by* $(\mathrm{Gr}^{\mathrm{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$*.*

Then our Theorem 5.1.1.6 can be strengthened as follows:

Theorem 5.1.2.5. *There is an equivalence of categories*

$$\begin{aligned}\mathrm{M}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)} : \mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)} &\to \mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)} \\ (A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) &\mapsto (G, \lambda, i).\end{aligned}$$

5.2 Data for Principal Level Structures

5.2.1 The Setting

Let B, $\mathcal{O}$, $(L, \langle \cdot\,, \cdot \rangle)$ and $\Box$ be chosen as in Section 1.4. Let $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ be an open compact subgroup. Let the moduli problem $\mathsf{M}_{\mathcal{H}}$ be defined over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.4.

For technical reasons regarding the existence of splittings of filtrations, we shall assume that L is chosen so that it satisfies Condition 1.4.3.9. (See Lemma 5.2.2.4 below.) Practically, this means we might have to replace L by a larger lattice L' so that the action of $\mathcal{O}$ extends to a maximal order $\mathcal{O}'$ containing $\mathcal{O}$. By Corollary 1.4.3.7, although this assumption does impose a restriction on the order $\mathcal{O}$ and the $\mathcal{O}$-lattice L that we could working with, it does not affect our purpose of studying and compactifying the moduli problem $\mathsf{M}_{\mathcal{H}}$ if $\mathcal{H}$ can still be chosen to be contained in $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ under this assumption. (See Remark 1.4.3.8.)

With the same setting of Section 4.1, assume moreover that the generic point $\eta = \mathrm{Spec}(K)$ of the base scheme $S = \mathrm{Spec}(R)$ is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. (We do not need the whole scheme S to be defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.) Fix a choice of a geometric point $\bar{\eta} = \mathrm{Spec}(\bar{K})$ over η. Let $K \hookrightarrow \tilde{K}$ be any finite separable subextension of $K \hookrightarrow \bar{K}$ defining a finite étale morphism $\tilde{\eta} = \mathrm{Spec}(\tilde{K}) \to \eta = \mathrm{Spec}(K)$. In this case, the inclusion $\tilde{K} \hookrightarrow \bar{K}$ defines a canonical lifting of $\bar{\eta}$ to $\tilde{\eta}$. Recall that we have the following:

Lemma 5.2.1.1. *Let R_1 be any noetherian normal integral domain with field of fractions K_1. Suppose K_2 is a finite separable extension of K_1, and let R_2 be the integral closure of R_1 in K_2. Then R_2 is a finite R_1-module. In particular, R_2 is again noetherian.*

This convenient fact can be found, for example, in [91, §33, Lem. 1] or [36, Prop. 13.14]. (For general extensions $K_1 \hookrightarrow K_2$, it may not be true that R_2 is noetherian. It is nevertheless true if R_1 is excellent or more generally Nagata. See [90, §31 – §34] for discussions of this issue.)

5.2.2 Analysis of Principal Level Structures

In this section we study the construction of (principal) level-n structures using the theory of degeneration, assuming the theory in Section 5.1.2 for

Lie algebra conditions.

With the setting as in Section 5.2.1, consider any triple (G, λ, i) defining an object in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ over S. By Theorem 5.1.2.5, we have an associated degeneration datum $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$. Assume for simplicity that $\underline{Y}$ and $\underline{X}$ are constant with values respectively Y and X. Then Y and X are $\mathcal{O}$-lattices of the same multi-rank (defined as in Definition 1.2.1.20), as we have seen in Lemma 5.1.1.3. Let $\phi : Y \hookrightarrow X$ be the $\mathcal{O}$-linear embedding defined by λ. We know that the map $\lambda : G \to G^\vee$ induces a map between Raynaud extensions $\lambda^\natural : G^\natural \to G^{\vee,\natural}$, and induces a polarization $\lambda_A : A \to A^\vee$ on the abelian parts because λ extends a polarization $\lambda_\eta : G_\eta \to G_\eta^\vee$.

By Corollary 4.5.3.11, together with the general theorem of orthogonality in mind (see for example [59, IX, 2.4] and [96, IV, 2.4], or Theorem 3.4.2.6), the structure of $G[n]_\eta$ can be described as follows:

Proposition 5.2.2.1. *With the setting as above, we have a canonical $\mathcal{O}$-equivariant exact sequence of finite étale group schemes*

$$0 \to G^\natural[n]_\eta \to G[n]_\eta \to \tfrac{1}{n}Y/Y \to 0.$$

over $\mathrm{Spec}(R)$*, which induces by taking limit over n with $\Box \nmid n$*

$$0 \to \mathrm{T}^\Box\, G^\natural_{\bar{\eta}} \to \mathrm{T}^\Box\, G_{\bar{\eta}} \to Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to 0.$$

Moreover, we have a canonical $\mathcal{O}$-equivariant exact sequence of finite étale group schemes

$$0 \to T[n] \to G^\natural[n] \to A[n] \to 0$$

over $\mathrm{Spec}(R)$*, which induces by taking limit over n with $\Box \nmid n$*

$$0 \to \mathrm{T}^\Box\, T_{\bar{\eta}} \to \mathrm{T}^\Box\, G^\natural_{\bar{\eta}} \to \mathrm{T}^\Box\, A_{\bar{\eta}} \to 0.$$

Under the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$, the submodules $\mathrm{T}^\Box\, T_{\bar{\eta}}$ and $\mathrm{T}^\Box\, G^\natural_{\bar{\eta}}$ of $\mathrm{T}^\Box\, G_{\bar{\eta}}$ are identified as the annihilators of each other, which induce the λ_A-Weil pairing $\mathrm{e}^{\lambda_A}(\,\cdot\,,\,\cdot\,)$ on $\mathrm{T}^\Box\, A_{\bar{\eta}}$, and a pairing

$$\mathrm{e}^{\phi}(\,\cdot\,,\,\cdot\,) : \mathrm{T}^\Box\, T_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \to \mathrm{T}^\Box\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$$

which is the canonical one

$$\begin{aligned}\mathrm{T}^{\Box}\, T_{\tilde{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) &\overset{\text{can.}}{\xrightarrow{\sim}} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}) \times Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \\ &\overset{\mathrm{Id}\times\phi}{\to} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}) \times X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \overset{\text{can.}}{\to} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}\end{aligned}$$

with the sign convention that $\mathrm{e}^{\phi}(t,y) = t(\phi(y)) = (\phi(y))(t)$ *for any* $t \in \mathrm{T}^{\Box}\, T_{\tilde{\eta}}$ *and any* $y \in Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$*. (This is compatible with the sign convention that we will adopt later in Sections 5.2.4 and 5.2.6.)*

Remark 5.2.2.2. By taking the dual objects and the conditions on pairings into account, we see that we actually have a perfect pairing between $\mathrm{T}^{\Box}\, G_{\tilde{\eta}} / \mathrm{T}^{\Box}\, G^{\natural}_{\tilde{\eta}}$ and $\mathrm{T}^{\Box}\, T^{\vee}_{\tilde{\eta}}$, with value in $\mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}$. This justifies the identification between $\mathrm{T}^{\Box}\, G_{\tilde{\eta}} / \mathrm{T}^{\Box}\, G^{\natural}_{\tilde{\eta}}$ and $Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, and hence the corresponding version mod n.

Now let $\tilde{\eta} \to \eta$ be any finite étale morphism defined by a field extension as in Section 5.2.1. (The reason to consider such étale localizations of η is for the applicability of the theory to the study of level structures that are not principal, as defined in Section 1.3.7.)

Suppose there is a level-n structure of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ (defined over $\tilde{\eta}$), namely an $\mathcal{O}$-equivariant isomorphism $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ that can be lifted (noncanonically) to an $\mathcal{O}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\tilde{\eta}}$, together with an isomorphism $\nu(\alpha_n) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}$, which carry the chosen pairing $\langle \cdot , \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to the λ_{η}-Weil pairing. Then for each choice of the lifting $\hat{\alpha}$, the $\mathcal{O}$-invariant filtration

$$0 \subset \mathrm{T}^{\Box}\, T_{\tilde{\eta}} \subset \mathrm{T}^{\Box}\, G^{\natural}_{\tilde{\eta}} \subset \mathrm{T}^{\Box}\, G_{\tilde{\eta}}$$

of $\mathrm{T}^{\Box}\, G[n]_{\tilde{\eta}}$ as described in Proposition 5.2.2.1 induces an $\mathcal{O}$-invariant filtration

$$0 \subset \mathtt{Z}_{-2} \subset \mathtt{Z}_{-1} \subset \mathtt{Z}_{0} := L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$$

of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, together with isomorphisms on the graded pieces:

$$\mathrm{Gr}^{\mathtt{Z}}_{-2} := \mathtt{Z}_{-2} \xrightarrow{\sim} \mathrm{T}^{\Box}\, T_{\tilde{\eta}}$$

$$\begin{aligned} \mathrm{Gr}^{\mathsf{Z}}_{-1} &:= \mathsf{Z}_{-1}/\mathsf{Z}_{-2} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{\eta}} \\ \mathrm{Gr}^{\mathsf{Z}}_{0} &:= \mathsf{Z}_{0}/\mathsf{Z}_{-1} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}. \end{aligned}$$

Here the first isomorphism can be given more precisely by the composition

$$\mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{\eta}}) \xrightarrow{\sim} \mathrm{T}^{\Box} T_{\bar{\eta}},$$

in which the first is the essential datum, in which the second is given by the isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{\eta}}$ given by $\hat{\alpha}$, and in which the third is canonical. Different choices of $\hat{\alpha}$ might induce different filtrations of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, but the reductions mod n are the same. Moreover, the isomorphism between the filtrations is *symplectic*. Namely, Z_{-2} and Z_{-1} are the annihilator of each other under the pairing $\langle \cdot, \cdot \rangle$ of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, as are $\mathrm{T}^{\Box} T_{\bar{\eta}}$ and $\mathrm{T}^{\Box} G^{\natural}_{\bar{\eta}}$, and the induced isomorphisms

$$\mathrm{Gr}^{\mathsf{Z}}_{-2} \times \mathrm{Gr}^{\mathsf{Z}}_{0} \xrightarrow{\sim} \mathrm{T}^{\Box} T_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$$

and

$$\mathrm{Gr}^{\mathsf{Z}}_{-1} \times \mathrm{Gr}^{\mathsf{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^{\Box} A_{\bar{\eta}} \times \mathrm{T}^{\Box} A_{\bar{\eta}}$$

on the graded pieces respect the pairings on both sides under the same unique isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{\eta}}$ given by $\hat{\alpha}$, induced respectively by the pairing $\langle \cdot, \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ and the λ-Weil pairing on $\mathrm{T}^{\Box} G_{\bar{\eta}}$.

On the other hand, having isomorphisms on the graded pieces alone is not sufficient for recovering the isomorphism between the whole spaces. Let us introduce some noncanonical choices in this setting, namely splittings of the underlying lattices. (See Section 1.2.6.)

Now suppose that we are given some filtration

$$0 \subset \mathsf{Z}_{-2} \subset \mathsf{Z}_{-1} \subset \mathsf{Z}_{0} = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}.$$

It does not make sense to consider arbitrary filtrations, as the filtrations on $\mathrm{T}^{\Box} G_{\bar{\eta}}$ do satisfy some special conditions.

Lemma 5.2.2.3. *Any filtration* $\mathsf{Z} = \{\mathsf{Z}_{-i}\}$ *coming from* $\mathrm{T}^{\Box} G_{\bar{\eta}}$ *as above is* **integrable** *and* **symplectic** (*defined as in Definitions 1.2.6.1 and 1.2.6.8*).

Proof. The fact that $\mathtt{Z}$ is symplectic follows from Proposition 5.2.2.1 and the discussion above. Let us denote by $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ the induced symplectic module. By Proposition 5.1.2.2 we also have a filtration $\mathtt{Z}_{\mathbb{R}}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, with an induced symplectic module $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$. Let us show that $\mathtt{Z}$ is integrable. From the above discussion it is clear that $\mathtt{Z}_{-2}$ and $\mathtt{Z}_0$ are integrable. Therefore it remains to show that $\mathtt{Z}_{-1}$ is integrable. Consider the *abelian part* $(A_{\tilde{\eta}}, \lambda_{A_{\tilde{\eta}}}, i_{i,\tilde{\eta}}, \varphi_{-1,n})$ (over $\tilde{\eta}$) of the data we have, which defines a point of the smooth moduli problem defined by $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$ and $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$. Then, as explained in Remark 1.4.3.13, there exists (noncanonically) a PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathtt{Z}})$ as in Definition 1.2.1.3 such that there exist symplectic isomorphisms $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11}) \xrightarrow{\sim} (L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathtt{Z}})$ and $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}) \xrightarrow{\sim} (L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathtt{Z}})$. In particular, $\mathtt{Z}_{-1}$ is integrable, as desired. $\Box$

By Corollary 1.2.6.5, we know that any integrable filtration is automatically split when $\mathcal{O}$ is *maximal.* For general $\mathcal{O}$ (which might not be maximal), we have the following:

Lemma 5.2.2.4. *Under the assumption that the PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ satisfies Condition 1.4.3.9, any filtration $\mathtt{Z}$ of $\mathtt{Z}_0 = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ that could be realized as a pullback of the filtration $0 \subset \mathrm{T}^{\Box}\, T_{\tilde{\eta}} \subset \mathrm{T}^{\Box}\, G^{\natural}_{\tilde{\eta}} \subset \mathrm{T}^{\Box}\, G_{\tilde{\eta}}$ by a symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\tilde{\eta}}$ is necessarily* **split***.*

Proof. Condition 1.4.3.9 means that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$. Via the symplectic isomorphism $\hat{\alpha}$, we obtain an action of $\mathcal{O}'$ on $\mathrm{T}^{\Box}\, G_{\tilde{\eta}}$ extending the one of $\mathcal{O}$. By Lemma 5.1.1.1, $\mathcal{O}'$ is generated as a $\mathbb{Z}$-algebra by $\mathcal{O}' \cap B^{\times}$. The elements in $\mathcal{O}' \cap B^{\times}$ define isogenies under Lemma 1.3.5.2. The condition that they acts on $\mathrm{T}^{\Box}\, G_{\tilde{\eta}}$ shows that they define only $\Box$-primary isogenies, namely isogenies with degree divisible by only primes in $\Box$. Let us think of these isogenies as quotients maps from $G_{\tilde{\eta}} \to G'_{\tilde{\eta}}$.

Since $\mathcal{O}' \cap B^{\times}$ is finitely generated, there is a point $\eta' = \mathrm{Spec}(K')$ finite étale over $\tilde{\eta}$ such that the $\Box$-primary quotient maps are defined over η'. Let R' be the normalization of R in K'. By Lemma 5.2.1.1, R' with $I' := \sqrt{I \cdot R'}$ satisfies the requirements in Section 4.1. Let $S' = \mathrm{Spec}(R')$, and let $G' = G \underset{S}{\times} S'$. By mimicking the construction of $G^{\vee}$ in Theorem 3.4.3.1 using

Lemma 3.4.3.3, the quotients of $G_{\eta'} = (G')_{\eta'}$ defined by $\mathcal{O}' \cap B^\times$ extends to quotients of G' over S', which induce also quotients of $(G')^\natural := G^\natural \underset{S}{\times} S'$ (resp. $T' := T \underset{S}{\times} S'$). As a result, we see that the action of $\mathcal{O}'$ on $\mathrm{T}^\Box G_{\bar\eta}$ maps $\mathrm{T}^\Box G^\natural_{\bar\eta}$ (resp. $\mathrm{T}^\Box T_{\bar\eta}$) to itself.

Now that $0 \subset \mathrm{T}^\Box T_{\bar\eta} \subset \mathrm{T}^\Box G^\natural_{\bar\eta} \subset \mathrm{T}^\Box G_{\bar\eta}$ is an integrable filtration of $\mathcal{O}$-modules, with each of the graded pieces of the form $M \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ for some $\mathcal{O}$-lattices M. Writing $\hat{\mathbb{Z}}^\Box = (\prod\limits_{p|\mathrm{Disc}} \mathbb{Z}_p) \times (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$, we see that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p) = \mathcal{O}' \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$. In particular, $M \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$ is projective also as an $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$-module. On the other hand, for each of the finitely many $p \mid \mathrm{Disc}$, $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is torsion free as a $\mathbb{Z}_p$-module. By Definition 1.1.1.18, $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is a $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattice. By Proposition 1.1.1.20, $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is projective as an $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-module. Putting the *finite* product together again, we see that the filtration is projective (defined as in Definition 1.2.6.2), and is therefore split (by Lemma 1.2.6.4). Since $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ contains $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$, the splitting is $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$-linear as well. Hence the result follows. ☐

As a result, for the purpose of studying level structures, we only need to consider filtrations Z of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ that are integrable, symplectic, and *split*. Recall that (in Definition 1.2.6.6) a filtration of an integrable module is called *admissible* if it is both integrable and split.

Definition 5.2.2.5. *The* **multi-rank** *of a symplectic admissible filtration* Z *of* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ *is the multi-rank* (*defined as in Definition 1.2.1.20*) *of* Z_{-2} *as an integrable* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$*-module.*

Let us investigate the possible splittings of an admissible filtration of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. Let us define $\mathrm{Gr}^{\mathsf{Z}}_{-i} := \mathsf{Z}_{-i}/\mathsf{Z}_{-i-1}$ as before.

If there is a first splitting, then we obtain a direct sum decomposition

$$\hat{\delta} : \mathrm{Gr}^{\mathsf{Z}} := \mathrm{Gr}^{\mathsf{Z}}_{-2} \oplus \mathrm{Gr}^{\mathsf{Z}}_{-1} \oplus \mathrm{Gr}^{\mathsf{Z}}_{0} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box.$$

The pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$ can thus be expressed in *matrix form* as

$$\begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle_{11} & \langle\,\cdot\,,\,\cdot\,\rangle_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle_{02} & \langle\,\cdot\,,\,\cdot\,\rangle_{01} & \langle\,\cdot\,,\,\cdot\,\rangle_{00} \end{pmatrix},$$

where the pairings

$$\langle\,\cdot\,,\,\cdot\,\rangle_{ij} : \mathrm{Gr}^{\mathsf{Z}}_{-i} \times \mathrm{Gr}^{\mathsf{Z}}_{-j} \to \hat{\mathbb{Z}}^\square(1)$$

satisfy $\langle\,\cdot\,,\,\cdot\,\rangle_{ij} = -\,{}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{ji}$ for any i and j. Namely, they satisfy $\langle x, by\rangle_{ij} = \langle b^\star x, y\rangle_{ij} = -\langle y, b^\star x\rangle_{ji}$ for any $x \in \mathrm{Gr}^{\mathsf{Z}}_{-i}$, $y \in \mathrm{Gr}^{\mathsf{Z}}_{-j}$, and $b \in \mathcal{O}$. Here we have nothing on the three upper-left blocks because Z_{-2} and Z_{-1} are the annihilators of each other.

If there is now a second splitting $\hat{\delta}' : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$, then there is *change of basis* $\hat{\mathsf{z}} : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathsf{Z}}$ such that $\hat{\delta}' = \hat{\delta} \circ \hat{\mathsf{z}}$. In matrix form, we can write

$$\hat{\mathsf{z}} = \begin{pmatrix} 1 & \mathsf{z}_{21} & \mathsf{z}_{20} \\ & 1 & \mathsf{z}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\mathsf{z}_{ij} : \mathrm{Gr}^{\mathsf{Z}}_{-j} \to \mathrm{Gr}^{\mathsf{Z}}_{-i}$$

are $\mathcal{O}$-equivariant maps between the graded pieces. The matrix of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$ using the second splitting can be expressed by

$$\begin{aligned} &\begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle'_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle'_{11} & \langle\,\cdot\,,\,\cdot\,\rangle'_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{02} & \langle\,\cdot\,,\,\cdot\,\rangle'_{01} & \langle\,\cdot\,,\,\cdot\,\rangle'_{00} \end{pmatrix} \\ &\quad := \begin{pmatrix} 1 & & \\ {}^t\mathsf{z}^\star_{21} & 1 & \\ {}^t\mathsf{z}^\star_{20} & {}^t\mathsf{z}^\star_{10} & 1 \end{pmatrix} \begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle_{11} & \langle\,\cdot\,,\,\cdot\,\rangle_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle_{02} & \langle\,\cdot\,,\,\cdot\,\rangle_{01} & \langle\,\cdot\,,\,\cdot\,\rangle_{00} \end{pmatrix} \begin{pmatrix} 1 & \mathsf{z}_{21} & \mathsf{z}_{20} \\ & 1 & \mathsf{z}_{10} \\ & & 1 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle'_{20} &= \langle\,\cdot\,,\,\cdot\,\rangle_{20}, \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{11} &= \langle\,\cdot\,,\,\cdot\,\rangle_{11}, \end{aligned}$$

$$\begin{aligned}
\langle\,\cdot\,,\,\cdot\,\rangle'_{10} &= \langle\,\cdot\,,\,\cdot\,\rangle_{10} + {}^t\mathsf{z}^\star_{21}\langle\,\cdot\,,\,\cdot\,\rangle_{20} + \langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathsf{z}_{10},\\
\langle\,\cdot\,,\,\cdot\,\rangle'_{00} &= \langle\,\cdot\,,\,\cdot\,\rangle_{00} + ({}^t\mathsf{z}^\star_{20}\langle\,\cdot\,,\,\cdot\,\rangle_{20} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{20}\mathsf{z}_{20})\\
&\qquad + ({}^t\mathsf{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{10} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{10}\mathsf{z}_{10}) + {}^t\mathsf{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathsf{z}_{10},
\end{aligned}$$

and the symbolic notations have the meanings

$$\begin{aligned}
{}^t\mathsf{z}^\star_{ki}\langle x, y\rangle_{kj} &= \langle \mathsf{z}_{ki}(x), y\rangle_{kj},\\
\langle x, y\rangle_{ik}\mathsf{z}_{kj} &= \langle x, \mathsf{z}_{kj}(y)\rangle_{ik},\\
{}^t\mathsf{z}^\star_{ki}\langle x, y\rangle_{kl}\mathsf{z}_{lj} &= \langle \mathsf{z}_{ki}(x), \mathsf{z}_{lj}(y)\rangle_{kl},
\end{aligned}$$

for any $x \in \mathrm{Gr}^{\mathsf{Z}}_{-i}$, $y \in \mathrm{Gr}^{\mathsf{Z}}_{-j}$. The notations thus designed then satisfies the symbolic relation

$${}^t({}^t\mathsf{z}^\star_{ki}\langle\,\cdot\,,\,\cdot\,\rangle_{kj})^\star = {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{kj}\mathsf{z}_{ki} = -\langle\,\cdot\,,\,\cdot\,\rangle_{jk}\mathsf{z}_{ki}.$$

Definition 5.2.2.6. *Two pairs of pairings* $(\langle\,\cdot\,,\,\cdot\,\rangle_{10}, \langle\,\cdot\,,\,\cdot\,\rangle_{00})$ *and* $(\langle\,\cdot\,,\,\cdot\,\rangle'_{10}, \langle\,\cdot\,,\,\cdot\,\rangle'_{00})$ *as above are defined to be equivalent under* $(\langle\,\cdot\,,\,\cdot\,\rangle_{20}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$*, denoted simply as*

$$(\langle\,\cdot\,,\,\cdot\,\rangle_{10}, \langle\,\cdot\,,\,\cdot\,\rangle_{00}) \sim (\langle\,\cdot\,,\,\cdot\,\rangle'_{10}, \langle\,\cdot\,,\,\cdot\,\rangle'_{00}),$$

if there are some z_{21}*,* z_{10}*, and* z_{20} *such that*

$$\begin{aligned}
\langle\,\cdot\,,\,\cdot\,\rangle'_{10} &= \langle\,\cdot\,,\,\cdot\,\rangle_{10} + {}^t\mathsf{z}^\star_{21}\langle\,\cdot\,,\,\cdot\,\rangle_{20} + \langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathsf{z}_{10},\\
\langle\,\cdot\,,\,\cdot\,\rangle'_{00} &= \langle\,\cdot\,,\,\cdot\,\rangle_{00} + ({}^t\mathsf{z}^\star_{20}\langle\,\cdot\,,\,\cdot\,\rangle_{20} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{20}\mathsf{z}_{20})\\
&\qquad + ({}^t\mathsf{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{10} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{10}\mathsf{z}_{10}) + {}^t\mathsf{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathsf{z}_{10}.
\end{aligned}$$

As a result, we see that $\langle\,\cdot\,,\,\cdot\,\rangle_{20}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{11}$ are independent of the splitting $\hat{\delta} : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, and $\langle\,\cdot\,,\,\cdot\,\rangle_{10}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{00}$ are well-defined only up to equivalence.

Now suppose that we are given any splitting $\hat{\delta} : \mathrm{Gr}^{\mathsf{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. By reduction mod n, we obtain an admissible filtration

$$0 \subset \mathsf{Z}_{-2,n} \subset \mathsf{Z}_{-1,n} \subset \mathsf{Z}_{0,n} := L/nL,$$

where $\mathsf{Z}_{-i,n} := \mathsf{Z}_{-i}/n\mathsf{Z}_{-i}$, with graded pieces $\mathrm{Gr}^{\mathsf{Z}}_{-i,n} := \mathsf{Z}_{-i,n}/\mathsf{Z}_{-i-1,n} \cong \mathrm{Gr}^{\mathsf{Z}}_{-i}/n\,\mathrm{Gr}^{\mathsf{Z}}_{-i}$. Here we shall always follow the convention that $i \leq 0$ and $n \geq 1$, and so there could be no ambiguity when we write

$$\delta_n : \mathrm{Gr}^{\mathsf{Z}}_n := \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \oplus \mathrm{Gr}^{\mathsf{Z}}_{-1,n} \oplus \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} L/nL$$

as the reduction mod n of $\hat{\delta}$.

Remark 5.2.2.7. To keep the information after reduction mod n, we shall equip the filtration $\mathtt{Z}_n := \{\mathtt{Z}_{-i,n}\}$ with the notion of multi-ranks, given by the multi-ranks (defined as in Definition 5.2.2.5) of the (admissible) filtrations $\mathtt{Z}$ we started with. Therefore, even if $n = 1$, in which case L/nL is trivial, there might still be different filtrations $\mathtt{Z}_n$ of L/nL because there might be different multi-ranks. This convention will be tacitly assumed in all our arguments.

Definition 5.2.2.8. *An admissible filtration*

$$0 \subset \mathtt{Z}_{-2,n} \subset \mathtt{Z}_{-1,n} \subset \mathtt{Z}_{0,n} = L/nL$$

of L/nL of a prescribed multi-rank is called **symplectic-liftable** *if it is the reduction mod n of some symplectic admissible filtration*

$$0 \subset \mathtt{Z}_{-2} \subset \mathtt{Z}_{-1} \subset \mathtt{Z}_0 = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$$

(*defined as in Definitions 1.2.6.6 and 1.2.6.8*) *of multi-rank equal to the prescribed one.*

Equivalently, a symplectic-liftable filtration $\mathtt{Z}_n$ of L/nL is an equivalence class of symplectic admissible filtrations $\mathtt{Z}$ of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$, where two symplectic admissible filtrations $\mathtt{Z}$ and $\mathtt{Z}$ are defined to be equivalent if their multi-ranks are the same, and if their reductions mod n are the same.

Definition 5.2.2.9. *A splitting $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ for a symplectic-liftable filtration $\{\mathtt{Z}_{-i,n}\}$ of L/nL is called* **liftable** *if it is the reduction mod n of some splitting $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$.*

On the other hand, we would like to perform the same analysis for $\mathrm{T}^\Box\, G_{\bar{\eta}}$. In particular, we shall investigate the possible splittings of the filtration

$$0 \subset \mathrm{T}^\Box\, T_{\bar{\eta}} \subset \mathrm{T}^\Box\, G^\natural_{\bar{\eta}} \subset \mathrm{T}^\Box\, G_{\bar{\eta}}$$

(when it is admissible). In analogous notations, let us define

$$\mathtt{W}_{-2} := \mathrm{T}^\Box\, T_{\bar{\eta}}, \qquad \mathtt{W}_{-1} := \mathrm{T}^\Box\, G^\natural_{\bar{\eta}}, \qquad \mathtt{W}_0 := \mathrm{T}^\Box\, G_{\bar{\eta}},$$
$$\mathrm{Gr}^{\mathtt{W}}_{-i} := \mathtt{W}_{-i}/\mathtt{W}_{-i-1}, \qquad \mathrm{Gr}^{\mathtt{W}} := \mathrm{Gr}^{\mathtt{W}}_{-2} \oplus \mathrm{Gr}^{\mathtt{W}}_{-1} \oplus \mathrm{Gr}^{\mathtt{W}}_0$$

as in the case of $\mathrm{Gr}^{\mathtt{Z}}$. Then we know that $\mathrm{Gr}^{\mathtt{W}}_{-2} = \mathrm{T}^\Box\, T_{\bar{\eta}}$, $\mathrm{Gr}^{\mathtt{W}}_{-1} \cong \mathrm{T}^\Box\, A_{\bar{\eta}}$ and $\mathrm{Gr}^{\mathtt{W}}_0 \cong Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. A splitting of this filtration corresponds to an isomorphism

$$\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} = \mathrm{T}^\Box\, T_{\bar{\eta}} \oplus \mathrm{T}^\Box\, A_{\bar{\eta}} \oplus (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \xrightarrow{\sim} \mathrm{T}^\Box\, G_{\bar{\eta}}.$$

Let us denote the multiplication and inversion in $\mathrm{T}^{\square}\,\mathbf{G}_{\mathrm{m},\bar{\eta}}$ *additively* by the notations $+$ and $-$ when we talk about matrix entries. Then the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ on $\mathrm{T}^{\square}\,G_{\bar{\eta}}$ can be expressed in *matrix form* as

$$\begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix}$$

where the pairings

$$\mathrm{e}_{ij} : \mathrm{Gr}^{\mathtt{W}}_{-i} \times \mathrm{Gr}^{\mathtt{W}}_{-j} \to \mathrm{T}^{\square}\,\mathbf{G}_{\mathrm{m},\bar{\eta}}$$

satisfy $\mathrm{e}_{ij} = -\,{}^t\mathrm{e}^{\star}_{ji}$ for any i and j. Namely, they satisfy $\mathrm{e}_{ij}(x, by) = \mathrm{e}_{ij}(b^{\star}x, y) = -\mathrm{e}_{ji}(y, b^{\star}x)$ for any $x \in \mathrm{Gr}^{\mathtt{W}}_{-i}$, $y \in \mathrm{Gr}^{\mathtt{W}}_{-j}$, and $b \in \mathcal{O}$. Here we have nothing on the three upper-left blocks because $\mathtt{W}_{-2}$ and $\mathtt{W}_{-1}$ are the annihilators of each other. Note that $\mathrm{e}_{20} = \mathrm{e}^{\phi}$ and $\mathrm{e}_{11} = \mathrm{e}^{\lambda_A}$.

If there is now a second splitting $\hat{\varsigma}' : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square}\,G_{\bar{\eta}}$, then there is a *change of basis* $\hat{\mathtt{w}} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ such that $\hat{\varsigma}' = \hat{\varsigma} \circ \hat{\mathtt{w}}$. In matrix form, we have

$$\hat{\mathtt{w}} = \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\mathtt{w}_{ij} : \mathrm{Gr}^{\mathtt{W}}_{-j} \to \mathrm{Gr}^{\mathtt{W}}_{-i}$$

are $\mathcal{O}$-equivariant maps between the graded pieces. The matrix of the pairing $\mathrm{e}^{\lambda_\eta}$ on $\mathrm{T}^{\square}\,G_{\bar{\eta}}$ using the second splitting can be expressed by

$$\begin{pmatrix} & & \mathrm{e}'_{20} \\ & \mathrm{e}'_{11} & \mathrm{e}'_{10} \\ \mathrm{e}'_{02} & \mathrm{e}'_{01} & \mathrm{e}'_{00} \end{pmatrix} := \begin{pmatrix} 1 & & \\ {}^t\mathtt{w}^{\star}_{21} & 1 & \\ {}^t\mathtt{w}^{\star}_{20} & {}^t\mathtt{w}^{\star}_{10} & 1 \end{pmatrix} \begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix} \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\begin{aligned}
\mathrm{e}'_{20} &= \mathrm{e}_{20} = \mathrm{e}^{\phi}, \\
\mathrm{e}'_{11} &= \mathrm{e}_{11} = \mathrm{e}^{\lambda_A}, \\
\mathrm{e}'_{10} &= \mathrm{e}_{10} + {}^t\mathtt{w}^{\star}_{21}\mathrm{e}_{20} + \mathrm{e}_{11}\mathtt{w}_{10}, \\
\mathrm{e}'_{00} &= \mathrm{e}_{00} + (\,{}^t\mathtt{w}^{\star}_{20}\mathrm{e}_{20} - {}^t\mathrm{e}^{\star}_{20}) + (\,{}^t\mathtt{w}^{\star}_{10}\mathrm{e}_{10} - {}^t\mathrm{e}^{\star}_{10}) + {}^t\mathtt{w}^{\star}_{10}\mathrm{e}_{11}\mathtt{w}_{10},
\end{aligned}$$

and the symbolic notations such as ${}^t\mathtt{w}^{\star}_{21}\mathrm{e}_{20}$ are interpreted in the same way as before.

Definition 5.2.2.10. *Two pairs of pairings* $(\mathrm{e}_{10}, \mathrm{e}_{00})$ *and* $(\mathrm{e}'_{10}, \mathrm{e}'_{00})$ *as above are defined to be equivalent under* $(\mathrm{e}_{20}, \mathrm{e}_{11}) = (\mathrm{e}^{\phi}, \mathrm{e}^{\lambda_A})$*, denoted simply as*

$$(\mathrm{e}_{10}, \mathrm{e}_{00}) \sim (\mathrm{e}'_{10}, \mathrm{e}'_{00}),$$

if there are some $\mathtt{w}_{21}$*,* $\mathtt{w}_{10}$*, and* $\mathtt{w}_{20}$ *such that*

$$\begin{aligned} \mathrm{e}'_{10} &= \mathrm{e}_{10} + {}^t\mathtt{w}^{\star}_{21}\mathrm{e}_{20} + \mathrm{e}_{11}\mathtt{w}_{10}, \\ \mathrm{e}'_{00} &= \mathrm{e}_{00} + ({}^t\mathtt{w}^{\star}_{20}\mathrm{e}_{20} - {}^t\mathrm{e}^{\star}_{20}) + ({}^t\mathtt{w}^{\star}_{10}\mathrm{e}_{10} - {}^t\mathrm{e}^{\star}_{10}) + {}^t\mathtt{w}^{\star}_{10}\mathrm{e}_{11}\mathtt{w}_{10}. \end{aligned}$$

As a result, we see that $\mathrm{e}_{20} = \mathrm{e}^{\phi}$ and $\mathrm{e}_{11} = \mathrm{e}^{\lambda_A}$ are independent of the splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$, and e_{10} and e_{00} are well-defined only up to equivalence.

Let us denote by the reduction mod n of the filtration $\{\mathtt{W}_{-i}\}$ by $\{\mathtt{W}_{-i,n}\}$. Then we have the filtration

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\eta} \subset \mathtt{W}_{-1,n} = G^{\natural}_{\eta} \subset \mathtt{W}_{0,n} = G[n]_{\eta},$$

with $\mathrm{Gr}^{\mathtt{W}}_{-i,n} := \mathtt{W}_{-i,n}/\mathtt{W}_{-i-1,n}$ and $\mathrm{Gr}^{\mathtt{W}}_{n} := \mathrm{Gr}^{\mathtt{W}}_{-2,n} \oplus \mathrm{Gr}^{\mathtt{W}}_{-1,n} \oplus \mathrm{Gr}^{\mathtt{W}}_{0,n}$ as in the case of $\mathrm{Gr}^{\mathtt{Z}}_{n}$. We shall use the same notations $\mathtt{W}_{-i,n}$ for the pullbacks of the objects to $\tilde{\eta}$ or $\bar{\eta}$. We write

$$\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_{n} \xrightarrow{\sim} G[n]_{\bar{\eta}}$$

as the reduction mod n of $\hat{\varsigma}$. If it is defined over $\tilde{\eta}$, then we also denote it as $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_{n} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$.

Definition 5.2.2.11. *A splitting* $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_{n} \xrightarrow{\sim} G[n]_{\bar{\eta}}$ *for the filtration* $\{\mathtt{W}_{-i,n}\}$ *of* $G[n]_{\bar{\eta}}$ *is called* **liftable** *if it is the reduction mod* n *of some splitting* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$.

Now suppose that we have a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ defined over $\tilde{\eta}$, which by definition can be lifted to some symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$. The filtration $\{\mathtt{W}_{-i}\}$ on $\mathrm{T}^{\square} G_{\bar{\eta}}$ induces a filtration $\{\mathtt{Z}_{-i}\}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ by $\hat{\alpha}$, with isomorphisms $\mathrm{Gr}_{-i}(\hat{\alpha}) : \mathrm{Gr}^{\mathtt{Z}}_{-i} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i}$ on the graded pieces. Let $\mathrm{Gr}(\hat{\alpha}) := \oplus \mathrm{Gr}_{-i}(\hat{\alpha})$. A splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$ determines (and conversely is determined by) a splitting of $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ by the relation $\hat{\delta} = \hat{\alpha}^{-1} \circ \hat{\varsigma} \circ \mathrm{Gr}(\hat{\alpha})$.

$$\begin{tikzcd}
\mathrm{Gr}^{\mathtt{Z}} \arrow[r, dotted, "\hat{\delta}", "\sim"'] \arrow[d, "\mathrm{Gr}(\hat{\alpha})"', "\wr"] & L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box} \arrow[d, "\hat{\alpha}", "\wr"'] \\
\mathrm{Gr}^{\mathtt{W}} \arrow[r, "\sim", "\hat{\varsigma}"'] & \mathrm{T}^{\Box}\, G_{\bar{\eta}}
\end{tikzcd}$$

If the λ_{η}-Weil pairing $\mathrm{e}^{\lambda_{\eta}}(\,\cdot\,,\,\cdot\,)$ on $\mathrm{T}^{\Box}\, G_{\bar{\eta}}$ and the symplectic pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ are given respectively under the splittings in matrix forms

$$\begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle_{11} & \langle\,\cdot\,,\,\cdot\,\rangle_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle_{02} & \langle\,\cdot\,,\,\cdot\,\rangle_{01} & \langle\,\cdot\,,\,\cdot\,\rangle_{00} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix},$$

then we have to match e_{ij} with $\langle\,\cdot\,,\,\cdot\,\rangle_{ij}$ under $\mathrm{Gr}(\hat{\alpha})$.

Definition 5.2.2.12. *Given splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\bar{\eta}}$, *a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *is a* **symplectic isomorphism** *with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *if* $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\bar{\eta}}$ *is a symplectic isomorphism.* (*Note that then* $\hat{f}$ *comes together with an isomorphism* $\nu(\hat{f}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$, *as in Definition 1.1.4.11.*)

By definition, the $\mathrm{Gr}(\hat{\alpha})$ constructed above is symplectic. We shall view $\hat{\delta}$ and $\hat{\varsigma}$ as symplectic isomorphisms by simply setting the *similitudes* to be ones. Then the relation $\hat{\delta} = \hat{\alpha}^{-1} \circ \hat{\varsigma} \circ \mathrm{Gr}(\hat{\alpha})$ above is an identity of symplectic isomorphisms.

Lemma 5.2.2.13. *Suppose we are given splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$, *so that we have the induced pairings* $\langle\,\cdot\,,\,\cdot\,\rangle_{ij}$ *and* e_{ij} *defined between the graded pieces. Then a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ (*with an isomorphism* $\nu(\hat{f}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$) *defines a symplectic isomorphism with respective to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *in the sense of Definition 5.2.2.12 if and only if* $\hat{f}^{*}(\mathrm{e}_{ij}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{ij}$ *for any* i *and* j.

Definition 5.2.2.14. *A triple* $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ *as in Definition 5.2.2.12 such that* $\hat{f}$ *is symplectic with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *is called a* **symplectic triple**.

Note that the filtration $\{\mathtt{Z}_{-i}\}$ of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ defines by reduction mod n a filtration $\{\mathtt{Z}_{-i,n}\}$ of L/nL, which depends on α_n but not on the choice of the lifting $\hat{\alpha}$ of α_n. The isomorphisms $\mathrm{Gr}_{-i}(\hat{\alpha}) : \mathrm{Gr}^{\mathtt{Z}}_{-i} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i}$ induce by reduction mod n isomorphisms $\mathrm{Gr}_{-i,n}(\alpha_n) : \mathrm{Gr}^{\mathtt{Z}}_{-i,n} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i,n}$ on the graded pieces, and a symplectic isomorphism $\mathrm{Gr}_n(\alpha_n) := \oplus\, \mathrm{Gr}_{-i,n} : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$, both of which depend on α_n but not on the choice of $\hat{\alpha}$. This symplectic isomorphism $\mathrm{Gr}_n(\alpha_n)$ is the reduction mod n of the above symplectic isomorphism $\mathrm{Gr}(\hat{\alpha})$.

Definition 5.2.2.15. *Suppose we are given a symplectic-liftable filtration* $\{\mathtt{Z}_{-i,n}\}$ *of* $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$, *and liftable splittings* $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ *and* $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *(defined over* $\tilde{\eta}$*). A graded symplectic isomorphism* $f_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$ *defined over* $\tilde{\eta}$ *is called* **symplectic-liftable** *if there are splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\bar{\eta}}$ *lifting respectively* δ_n *and* ς_n*, such that* f_n *is the reduction mod* n *of a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *that is symplectic with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *(i.e.* $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1}$ *is a symplectic isomorphism), and such that* $\nu(f_n)$ *is the reduction mod* n *of* $\nu(\hat{f})$*.*

For simplicity, we shall call a symplectic-liftable graded symplectic isomorphism simply a symplectic-liftable graded isomorphism, omitting the second appearance of symplectic.

By definition, the $\mathrm{Gr}_n(\alpha_n)$ constructed above is *symplectic-liftable*.

Conversely, suppose we are given a symplectic-liftable filtration $\{\mathtt{Z}_{-i,n}\}$ of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$, and liftable splittings $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ and $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$. Suppose we are given a *symplectic-liftable* $\mathcal{O}$-equivariant graded isomorphism $f_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$ defined over $\tilde{\eta}$. Then there are splittings $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ and $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\bar{\eta}}$, together with an $\mathcal{O}$-equivariant graded isomorphism $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ lifting f_n, which is symplectic with respect to $\hat{\delta}$ and $\hat{\varsigma}$. From these we can produce an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ (defined over $\tilde{\eta}$) by setting $\alpha_n := \varsigma_n \circ f_n \circ \delta_n^{-1}$, which is symplectic-liftable because it is the reduction mod n of the symplectic isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\, G_{\bar{\eta}}$ defined by setting $\hat{\alpha} := \hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1}$. In other words, α_n

is level-n structure.

$$\begin{tikzcd}
\mathrm{Gr}_n^{\mathtt{Z}} \arrow[r, "\delta_n", "\sim"'] \arrow[d, "f_n"', "\wr"] & L/nL \arrow[d, dotted, "\alpha_n", "\wr"'] & \rightsquigarrow & \mathrm{Gr}^{\mathtt{Z}} \arrow[r, "\hat{\delta}", "\sim"'] \arrow[d, "\hat{f}"', "\wr"] & L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \arrow[d, dotted, "\hat{\alpha}", "\wr"'] \\
\mathrm{Gr}_n^{\mathtt{W}} \arrow[r, "\sim", "\varsigma_n"'] & G[n]_{\eta} & & \mathrm{Gr}^{\mathtt{W}} \arrow[r, "\hat{\varsigma}", "\sim"'] & \mathrm{T}^{\square} G_{\bar{\eta}}
\end{tikzcd}$$

Definition 5.2.2.16. *A triple* $(\delta_n, \varsigma_n, f_n)$ *defined over* $\tilde{\eta}$ *as in Definition 5.2.2.15 such that* f_n *is symplectic-liftable with respect to* δ_n *and* ς_n *is called a* **symplectic-liftable triple***.*

If we start with a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$, then we can recover α_n from $\mathrm{Gr}_n(\alpha_n)$ by applying the above process just described to $f_n = \mathrm{Gr}_n(\alpha_n)$.

Note that we could have produced different level-n structures α_n if we intentionally modify the splittings incompatibly. Indeed, this is equivalent to having a change of basis $\mathtt{z}_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{Z}}$ that is the reduction mod n of some change of basis $\hat{\mathtt{z}} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}$ given by a matrix of the form

$$\begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix}.$$

Then $\alpha_n = \varsigma_n \circ f_n \circ \delta_n^{-1}$ is replaced by $\alpha_n' = \varsigma_n \circ f_n \circ \mathtt{z}_n^{-1} \circ \delta_n^{-1}$.

$$\begin{tikzcd}
\mathrm{Gr}_n^{\mathtt{Z}} \arrow[r, "\mathtt{z}_n", "\sim"'] \arrow[d, "f_n"', "\wr"] & \mathrm{Gr}_n^{\mathtt{Z}} \arrow[r, "\delta_n", "\sim"'] & L/nL \arrow[d, dotted, "\alpha_n'", "\wr"'] \\
\mathrm{Gr}_n^{\mathtt{W}} \arrow[rr, "\sim", "\varsigma_n"'] & & G[n]_{\eta}
\end{tikzcd}$$

Definition 5.2.2.17. *Two symplectic triples* $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ *and* $(\hat{\delta}', \hat{\varsigma}', \hat{f}')$ *as in Definition 5.2.2.14 are said to be* **equivalent** *if* $\hat{f} \circ \hat{\mathtt{z}} = \hat{\mathtt{w}} \circ \hat{f}'$*, where* $\hat{\mathtt{z}}$ *is the change of basis such that* $\hat{\delta}' = \hat{\delta} \circ \hat{\mathtt{z}}$*, and where* $\hat{\mathtt{w}}$ *is the change of basis such* *that* $\hat{\varsigma}' = \hat{\varsigma} \circ \hat{\mathtt{w}}$.

$$\begin{tikzcd}
\mathrm{Gr}^{\mathtt{Z}} \arrow[rr, "\hat{\delta}", "\sim"'] \arrow[dd, "\hat{f}"', "\wr"] & & L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \arrow[dd, dotted, "\wr"', "?"] \\
 & \mathrm{Gr}^{\mathtt{Z}} \arrow[ul, "\sim", "\hat{\mathtt{z}}"'] \arrow[ur, "\sim", "\hat{\delta}'"'] \arrow[dd, "\wr"', "\hat{f}'"] & \\
\mathrm{Gr}^{\mathtt{W}} \arrow[rr, "\hat{\varsigma}", "\sim"'] & & \mathrm{T}^{\square}\, \mathbf{G}_{\mathrm{m},\bar{\eta}} \\
 & \mathrm{Gr}^{\mathtt{W}} \arrow[ul, "\sim", "\hat{\mathtt{w}}"'] \arrow[ur, "\sim", "\hat{\varsigma}'"'] &
\end{tikzcd}$$

Then necessarily $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1} = \hat{\varsigma}' \circ \hat{f}' \circ (\hat{\delta}')^{-1}$, *as the unique dotted arrow making the whole diagram commute.*

Definition 5.2.2.18. *A change of basis* $\mathtt{z}_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}_n$ *of the form*

$$\mathtt{z}_n = \begin{pmatrix} 1 & \mathtt{z}_{21,n} & \mathtt{z}_{20,n} \\ & 1 & \mathtt{z}_{10,n} \\ & & 1 \end{pmatrix}$$

is called **liftable** *if it is the reduction mod* n *of some change of basis* $\hat{\mathtt{z}} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}$ *of the form*

$$\hat{\mathtt{z}} = \begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix}.$$

Definition 5.2.2.19. *A change of basis* $\mathtt{w}_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$ *of the form*

$$\mathtt{w}_n = \begin{pmatrix} 1 & \mathtt{w}_{21,n} & \mathtt{w}_{20,n} \\ & 1 & \mathtt{w}_{10,n} \\ & & 1 \end{pmatrix}$$

is called **liftable** *if it is the reduction mod* n *of some change of basis* $\hat{\mathtt{w}} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *of the form*

$$\hat{\mathtt{w}} = \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix}.$$

Definition 5.2.2.20. *Two symplectic-liftable triples* $(\delta_n, \varsigma_n, f_n)$ *and* $(\delta'_n, \varsigma'_n, f'_n)$ *as in Definition 5.2.2.16 are said to be* **equivalent** *if*

$f_n \circ \mathtt{z}_n = \mathtt{w}_n \circ f'_n$, where $\mathtt{z}_n$ is the liftable change of basis such that $\delta'_n = \delta_n \circ \mathtt{z}_n$, and where $\mathtt{w}_n$ is the liftable change of basis such that $\varsigma'_n = \varsigma_n \circ \mathtt{w}_n$. Then necessarily $\varsigma_n \circ f_n \circ \delta_n^{-1} = \varsigma'_n \circ f'_n \circ (\delta'_n)^{-1}$.

We can summarize our analysis in this section as follows:

Proposition 5.2.2.21. *Suppose that we are given a symplectic admissible filtration $\mathtt{Z} := \{\mathtt{Z}_{-i}\}$ of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$. Then the symplectic isomorphisms $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box} G_{\bar{\eta}}$ matching the filtration $\mathtt{Z}$ with the filtration $\mathtt{W}$ are in bijection with equivalence classes of symplectic triples $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ as in Definition 5.2.2.17.*

Remark 5.2.2.22. The property of being symplectic for a triple $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ can be checked using Lemma 5.2.2.13, based on a particular choice of the splittings $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ and $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box} G_{\bar{\eta}}$.

Proposition 5.2.2.23. *Suppose that we are given a symplectic-liftable admissible filtration $\mathtt{Z}_n := \{\mathtt{Z}_{-i,n}\}$ of L/nL in the sense of Definition 5.2.2.8. Then the level-n structures $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot\,, \cdot \rangle)$ (as in Definition 1.3.6.1) matching the filtration $\mathtt{Z}_n$ with the filtration $\mathtt{W}_n$ are in bijection with equivalence classes of symplectic-liftable triples $(\delta_n, \varsigma_n, f_n)$ (defined over $\tilde{\eta}$) as in Definition 5.2.2.20.*

5.2.3 Analysis of Splittings for $G[n]_\eta$

Continue to suppose that we are in the setting of Section 5.2.1, suppose that we have a triple (G, λ, i) defining an object in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ over S, which by Theorem 5.1.1.6 corresponds to a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$. For simplicity, let us continue to assume that $\underline{Y}$ and $\underline{X}$ are both constant with values respectively Y and X.

Let $\tilde{\eta} \to \eta$ be a finite étale morphism defined by a field extension as in Section 5.2.2. Our eventual goal is to construct level-n structures $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}}, \alpha_n)$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot\,, \cdot \rangle)$ as in Definition 1.3.6.1, under the additional assumption that (G, λ, i) defines an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle)}$. As we have seen in Section 5.2.2, we need to construct splittings $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ that are liftable to splittings $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box} G_{\bar{\eta}}$. Let us first focus on splittings $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ regardless of the liftability condition. (For the purpose of studying the splittings ς_n and $\hat{\varsigma}$, it suffices to proceed with the assumption that (G, λ, i) defines an object in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$.)

To split the filtration

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde{\eta}} \subset \mathtt{W}_{-1,n} = G^{\natural}[n]_{\tilde{\eta}} \subset \mathtt{W}_{0,n} = G[n]_{\tilde{\eta}},$$

we need to split both the surjection $G^{\natural}[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$ and the surjection $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$.

If we have a splitting for the surjection $G^{\natural}[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$, then the image of the splitting gives a closed subgroup scheme of $G^{\natural}[n]_{\tilde{\eta}}$, which we again denote by $A[n]_{\tilde{\eta}}$. Thus this splitting defines an isogeny $G^{\natural}_{\tilde{\eta}} \twoheadrightarrow {G^{\natural}}'_{\tilde{\eta}} := G^{\natural}_{\tilde{\eta}}/A[n]_{\tilde{\eta}}$. The subgroup scheme $T_{\tilde{\eta}}$ of $G^{\natural}_{\tilde{\eta}}$ embeds into a subgroup scheme $T'_{\tilde{\eta}}$ of ${G^{\natural}}'_{\tilde{\eta}}$, because $T_{\tilde{\eta}} \cap A[n]_{\tilde{\eta}} = 0$. Hence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\tilde{\eta}} \arrow[r] \arrow[d, "\wr"] & G^{\natural}_{\tilde{\eta}} \arrow[r] \arrow[d] & A_{\tilde{\eta}} \arrow[r] \arrow[d, "\mathrm{mod}\ A[n]_{\tilde{\eta}}"] & 0 \\
0 \arrow[r] & T'_{\tilde{\eta}} \arrow[r] & {G^{\natural}}'_{\tilde{\eta}} \arrow[r] & A_{\tilde{\eta}} \arrow[r] & 0
\end{tikzcd} .$$

We can complete this to a diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\tilde{\eta}} \arrow[r] \arrow[d, "\wr"] & G^{\natural}_{\tilde{\eta}} \arrow[r] \arrow[d] & A_{\tilde{\eta}} \arrow[r] \arrow[d, "\mathrm{mod}\ A[n]_{\tilde{\eta}}"] & 0 \\
0 \arrow[r] & T'_{\tilde{\eta}} \arrow[r] \arrow[d, "\mathrm{mod}\ T'_{\tilde{\eta}}[n]"'] & {G^{\natural}}'_{\tilde{\eta}} \arrow[r] \arrow[d] & A_{\tilde{\eta}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_{\tilde{\eta}} \arrow[r] & G^{\natural}_{\tilde{\eta}} \arrow[r] & A_{\tilde{\eta}} \arrow[r] & 0
\end{tikzcd} ,$$

in which every composition of two vertical arrows is the multiplication by n map. Therefore, finding a splitting of $G^{\natural}[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$ is equivalent to finding an isogeny ${G^{\natural}}'_{\tilde{\eta}} \twoheadrightarrow G^{\natural}_{\tilde{\eta}}$ of the form

$$\begin{tikzcd}
0 \arrow[r] & T'_{\tilde{\eta}} \arrow[r] \arrow[d, "\mathrm{mod}\ T'_{\tilde{\eta}}[n]"'] & {G^{\natural}}'_{\tilde{\eta}} \arrow[r] \arrow[d] & A_{\tilde{\eta}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_{\tilde{\eta}} \arrow[r] & G^{\natural}_{\tilde{\eta}} \arrow[r] & A_{\tilde{\eta}} \arrow[r] & 0
\end{tikzcd} .$$

Let us identify the map $T'_{\tilde{\eta}} \twoheadrightarrow T_{\tilde{\eta}}$ as the dual of the inclusion map $\underline{X}_{\tilde{\eta}} \hookrightarrow \frac{1}{n}\underline{X}_{\tilde{\eta}}$. By Proposition 3.1.5.1, isogenies $G^{\natural}{}'_{\tilde{\eta}} \to G^{\natural}_{\tilde{\eta}}$ of the above form are equivalent to liftings $c_n : \frac{1}{n}\underline{X}_{\tilde{\eta}} \to A^{\vee}_{\tilde{\eta}}$ over $\tilde{\eta}$ of the map $c : \underline{X} \to A^{\vee}$ defining the extension structure of $0 \to T \to G^{\natural} \to A \to 0$. Since all the morphisms we consider above are $\mathcal{O}$-linear, the lifting c_n is also $\mathcal{O}$-linear by functoriality of Proposition 3.1.5.1. Let us summarize the result as:

Lemma 5.2.3.1. *Splittings of $G^{\natural}[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$ correspond bijectively to liftings $c_n : \frac{1}{n}\underline{X}_{\tilde{\eta}} \to A^{\vee}_{\tilde{\eta}}$ of $c : \underline{X} \to A^{\vee}$ over $\tilde{\eta}$.*

If we have a splitting for the surjection $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$, then the image of the splitting gives a closed subgroup scheme of $G[n]_{\tilde{\eta}}$, which we denote by $K_{\tilde{\eta}}$. Let $\tilde{S} \twoheadrightarrow S$ be the normalization of S over $\tilde{\eta} \to \eta$, which is noetherian normal by Lemma 5.2.1.1. Let $K_{\tilde{S}}$ be the schematic closure of $K_{\tilde{\eta}}$ in $G_{\tilde{S}}$, which is flat and quasi-finite over $\tilde{S}$. Then, mimicking the construction of $G^{\vee}$ in Theorem 3.4.3.1 using Lemma 3.4.3.3, this splitting defines an isogeny $G_{\tilde{S}} \twoheadrightarrow G' := G_{\tilde{S}}/K_{\tilde{S}}$. On the fiber over $\tilde{\eta}$, we have thus obtained an isogeny $G_{\tilde{\eta}} \twoheadrightarrow G'_{\tilde{\eta}}$, whose kernel $K_{\tilde{\eta}}$ is isomorphic to $\frac{1}{n}Y/Y$. Since the isogeny $G_{\tilde{S}} \twoheadrightarrow G'_{\tilde{S}}$ is determined by its restriction to the fiber over $\tilde{\eta}$ by Proposition 3.3.1.7, we see that splittings of $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$ are in bijection with the G' thus constructed. Since we cannot push-forward the pullback $\lambda_{\tilde{S}}$ of the polarization λ to G' by the quotient $G_{\tilde{S}} \twoheadrightarrow G'$, we prefer to consider $G_{\tilde{S}}$ as a quotient of G' instead. Using the fact that $K_{\tilde{S}} \subset G[n]_{\tilde{S}}$, there is a surjection $G' \twoheadrightarrow G_{\tilde{S}}$ such that the composition $G_{\tilde{S}} \twoheadrightarrow G' \twoheadrightarrow G_{\tilde{S}}$ is the multiplication by n on $G_{\tilde{S}}$. If we pullback $\lambda_{\tilde{S}} : G_{\tilde{S}} \to G^{\vee}_{\tilde{S}}$ to a morphism $\lambda' : G' \to (G')^{\vee}$ by composing $G' \to G_{\tilde{S}} \overset{\lambda_{\tilde{S}}}{\to} G^{\vee}_{\tilde{S}} \to (G')^{\vee}$, then we obtain a morphism $(G', \lambda', i') \to (G_{\tilde{S}}, \lambda_{\tilde{S}}, i_{\tilde{S}})$ of objects in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ over $\tilde{S}$. By Theorem 5.1.1.6, this corresponds to a morphism from some tuple $(A', \lambda_{A'}, i_{A'}, \underline{X}', \underline{Y}', \phi', c', (c^{\vee})', \tau')$ to the pullback of $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau)$ to $\tilde{S}$.

Let us consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & G^{\natural}[n]_{\tilde{\eta}} \arrow[r] \arrow[d, equal] & G[n]_{\tilde{\eta}} \arrow[r] \arrow[d] & \frac{1}{n}Y/Y \arrow[r] \arrow[d] & 0 , \\
0 \arrow[r] & G^{\natural}[n]_{\tilde{\eta}} \arrow[r] \arrow[d, "n"'] & G'[n]_{\tilde{\eta}} \arrow[r] \arrow[d] & \frac{1}{n}Y'/Y' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G^{\natural}[n]_{\tilde{\eta}} \arrow[r] & G[n]_{\tilde{\eta}} \arrow[r] & \frac{1}{n}Y/Y \arrow[r] & 0
\end{tikzcd}$$

on the n-torsion points on the fibers over $\tilde{\eta}$, in which every composition of two vertical arrows is the multiplication by n, and in which the arrow $\frac{1}{n}Y'/Y' \to \frac{1}{n}Y/Y$ is an isomorphism. Here we are assuming $\underline{Y}'$ to be constant with value Y' by passing to an étale base extension if necessary. Note that we are identifying $G^{\natural}[n]_{\tilde{\eta}}$ as a subgroup of $G'[n]_{\tilde{\eta}}$ because $G^{\natural}[n]_{\tilde{\eta}} \cap K_{\tilde{\eta}} = 0$. In particular, the isogeny $G_{\tilde{S}} \twoheadrightarrow G'$ induces an isomorphism $G^{\natural}_{\tilde{S}} \overset{\sim}{\to} G^{\natural\prime}$ between the Raynaud extensions, and hence $A' = A_{\tilde{S}}$, $\underline{X}' = \underline{X}_{\tilde{S}}$, and $c' = c_{\tilde{S}}$. (Note that this means on the other hand that the isogeny $G' \twoheadrightarrow G_{\tilde{S}}$ induces the multiplication by n on $G^{\natural}_{\tilde{S}}$ if we identify $G^{\natural\prime}$ with $G^{\natural}_{\tilde{S}}$ using the isomorphism induced by $G_{\tilde{S}} \twoheadrightarrow G'$.) Moreover, the cokernel of $Y' \hookrightarrow Y$ is trivial, which means $Y' = Y$. This shows that the assumption that $\underline{Y}'$ is constant is redundant. Since the composition $Y \to Y' \to Y$ is dual to the multiplication by n on the torus part $T^{\vee}_{\tilde{S}}$ of $G^{\vee,\natural}_{\tilde{S}}$, which is the multiplication by n sending Y to $nY \subset Y$, we may identify Y' with $\frac{1}{n}Y$ and identify the above composition as $Y \hookrightarrow \frac{1}{n}Y \overset{n}{\overset{\sim}{\to}} Y$. In this case, the map $(c^{\vee})' : Y' = \frac{1}{n}Y \to A_{\tilde{S}}$ necessarily satisfies $(c^{\vee})'|_Y = c^{\vee}_{\tilde{S}}$. Following our previous convention of c_n, we would like to rename $(c^{\vee})'_{\tilde{\eta}}$ as $c^{\vee}_n$, which can be interpreted as a lifting of $c^{\vee}$ to $\frac{1}{n}Y$ over $\tilde{\eta}$. Note that all the morphisms we consider, including $c^{\vee}_n$, are $\mathcal{O}$-linear by Theorem 5.1.1.6.

We would like to give an interpretation of the trivialization $\tau' : \mathbf{1}_{\frac{1}{n}Y \times X,\tilde{\eta}} \overset{\sim}{\to} (c^{\vee}_n \times c)^*\mathcal{P}^{\otimes -1}_{A,\tilde{\eta}}$, which satisfies $\tau'|_{\mathbf{1}_{Y \times X,\tilde{\eta}}} = \tau_{\tilde{\eta}}$. By the same convention as c_n and $c^{\vee}_n$, we would like to rename τ' as τ_n. Then τ_n corresponds to a period map $\iota_n : \frac{1}{n}Y \to G^{\natural}_{\tilde{\eta}}$ lifting $\iota : Y \to G^{\natural}_{\eta}$, such that $\iota_n|_Y = \iota_{\eta}$. Note that the tuple $(A, \underline{X}, \underline{Y}, c, c^{\vee}, \tau)$ in DD is part of the tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$, with a given choice of ϕ and λ_A, so that c, $c^{\vee}$, and τ satisfy the compatibility relations $c\phi = \lambda_A c^{\vee}$ and $\tau(y_1, \phi(y_2)) = \tau(y_2, \phi(y_1))$ under the symmetry isomorphism of $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ for every y_1 and y_2. By construction of λ', the polarization $\lambda_{A'}$ is the pullback of the composition $A \overset{[n]}{\to} A \overset{\lambda_A}{\to} A^{\vee} \overset{[n]}{\to} A^{\vee}$ to $\tilde{S}$, which is simply $n^2\lambda_{A,\tilde{S}}$, and ϕ' is the composition $\frac{1}{n}Y \overset{[n]}{\overset{\sim}{\to}} Y \overset{\phi}{\to} X \overset{[n]}{\to} X$, which can be interpreted as $n^2\phi_n : \frac{1}{n}Y \to nX \subset \frac{1}{n}X$ if we extend $\phi : Y \to X$ naturally to $\phi_n : \frac{1}{n}Y \to \frac{1}{n}X$. As a result, we see that the counterpart for $\lambda_A c^{\vee} = c\phi$ is $(n^2\lambda_{A,\tilde{\eta}})c^{\vee}_n = c_{\tilde{\eta}}(n^2\phi_n)$, which is true because $(n^2\lambda_{A,\tilde{\eta}})c^{\vee}_n(\frac{1}{n}y) = n\lambda_{A,\tilde{\eta}}nc^{\vee}_n(\frac{1}{n}y) = n\lambda_{A,\tilde{\eta}}c^{\vee}(y) = nc_{\tilde{\eta}}\phi(y) = c_{\tilde{\eta}}n^2\phi_n(\frac{1}{n}y)$ for any $y \in Y$. On the other hand, the counterpart for $\tau(y_1, \phi(y_2)) = \tau(y_2, \phi(y_1))$

for every $y_1, y_2 \in Y$ is given by $\tau_n(\frac{1}{n}y_1, n^2\phi_n(\frac{1}{n}y_2)) = \tau_n(\frac{1}{n}y_2, n^2\phi_n(\frac{1}{n}y_1)$ for every $y_1, y_2 \in Y$, which is true because

$$\begin{aligned}&\tau_n(\tfrac{1}{n}y_1, n^2\phi_n(\tfrac{1}{n}y_2)) = \tau_n(\tfrac{1}{n}y_1, n\phi(y_2)) = \tau_n(y_1, \phi(y_2))\\&= \tau(y_1, \phi(y_2)) = \tau(y_2, \phi(y_1)) = \tau_n(y_2, \phi(y_1))\\&= \tau_n(\tfrac{1}{n}y_2, n\phi(y_1)) = \tau_n(\tfrac{1}{n}y_2, n^2\phi_n(\tfrac{1}{n}y_1)).\end{aligned}$$

It is convenient to replace the relation

$$\tau_n(\tfrac{1}{n}y_1, n^2\phi_n(\tfrac{1}{n}y_2)) = \tau_n(\tfrac{1}{n}y_2, n^2\phi_n(\tfrac{1}{n}y_1))$$

for every $y_1, y_2 \in Y$ by the equivalent relation

$$\tau_n(y_1, \phi(y_2)) = \tau_n(y_2, \phi(y_1))$$

for every $y_1, y_2 \in Y$, whose validity is more transparent because the restriction of τ_n to $\mathbf{1}_{Y \times X, \tilde{\eta}}$ is $\tau_{\tilde{\eta}}$. Certainly, there is also the compatibility $\tau_n(b\frac{1}{n}y, \chi) = \tau_n(\frac{1}{n}y, b^\star\chi)$ for any $\frac{1}{n}y \in \frac{1}{n}Y$, $\chi \in X$, and $b \in \mathcal{O}$ by Theorem 5.1.2.5.

To summarize:

Lemma 5.2.3.2. *Let $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ be a tuple in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$ corresponding to a tuple (G, λ, i) in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ via Theorem 5.1.1.6. Then ($\mathcal{O}$-linear) splittings of $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$ correspond bijectively to ($\mathcal{O}$-linear) liftings $\iota_n : \frac{1}{n}Y \to G^\natural_{\tilde{\eta}}$ of $\iota : Y \to G^\natural_\eta$, and hence bijectively to liftings $(c_n^\vee, \tau_n)$ of $(c^\vee, \tau)$ over $\tilde{\eta}$, in the sense that $c_n^\vee : \frac{1}{n}Y \to A_{\tilde{\eta}}$ and $\tau_n : \mathbf{1}_{\frac{1}{n}Y \times X, \tilde{\eta}} \xrightarrow{\sim} (c_n^\vee \times c_{\tilde{\eta}})^*\mathcal{P}^{\otimes -1}_{A,\tilde{\eta}}$ respect the $\mathcal{O}$-structures as $c^\vee$ and τ do, and satisfy respectively $c_n^\vee|_Y = c^\vee_{\tilde{\eta}}$ and $\tau_n|_{\mathbf{1}_{Y \times X, \tilde{\eta}}} = \tau_{\tilde{\eta}}$. In this case, τ_n satisfies the symmetry condition $\tau_n(y_1, \phi(y_2)) = \tau_n(y_2, \phi(y_1))$ for every $y_1, y_2 \in Y$*

Proposition 5.2.3.3. *Let $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ be a tuple in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$ corresponding to a triple (G, λ, i) in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}$ via Theorem 5.1.1.6. Let $\{\mathtt{W}_{-i,n}\}$ denote the filtration*

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde{\eta}} \subset \mathtt{W}_{-1,n} = G^\natural[n]_{\tilde{\eta}} \subset \mathtt{W}_{0,n} = G[n]_{\tilde{\eta}}$$

of $G[n]_{\tilde{\eta}}$, with graded pieces $\mathrm{Gr}^{\mathtt{W}}_{-i,n} := \mathtt{W}_{-i,n}/\mathtt{W}_{-i-1,n}$. Then splittings $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n = \oplus \mathrm{Gr}^{\mathtt{W}}_{-i,n} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of the filtration are in bijection with triples $(c_n, c_n^\vee, \tau_n)$ lifting $(c, c^\vee, \tau)$ over $\tilde{\eta}$, in the sense that the maps $c_n : \frac{1}{n}X \to A^\vee_{\tilde{\eta}}$, $c_n^\vee : \frac{1}{n}Y \to A_{\tilde{\eta}}$, and $\tau_n : \mathbf{1}_{\frac{1}{n}Y \times X, \tilde{\eta}} \xrightarrow{\sim} (c_n^\vee \times c_{\tilde{\eta}})^*\mathcal{P}^{\otimes -1}_{A,\tilde{\eta}}$ respect the $\mathcal{O}$-structures as c, $c^\vee$, and τ do, and satisfy respectively $c_n|_X = c_{\tilde{\eta}}$, $c_n^\vee|_Y = c^\vee_{\tilde{\eta}}$, and $\tau_n|_{\mathbf{1}_{Y \times X, \tilde{\eta}}} = \tau_{\tilde{\eta}}$.

Definition 5.2.3.4. *A triple* $(c_n, c_n^\vee, \tau_n)$ *as in Proposition 5.2.3.3 is called* **liftable** *if it is étale locally over* $\tilde{\eta}$ *liftable to some* $(c_m, c_m^\vee, \tau_m)$ *as in Proposition 5.2.3.3 for all* m *such that* $n|m$ *and* $\Box \nmid m$. *We shall write a compatible system of such liftings* $\{(c_m, c_m^\vee, \tau_m)\}_{n|m, \Box\nmid m}$ *symbolically as a triple* $(\hat{c}, \hat{c}^\vee, \hat{\tau})$, *where the maps are written respectively as* $\hat{c} : X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to A_{\tilde{\eta}}^\vee$, *as* $\hat{c}^\vee : Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to A_{\tilde{\eta}}$, *and as* $\hat{\tau} : \mathbf{1}_{(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times Y, \tilde{\eta}} \xrightarrow{\sim} (\hat{c}^\vee \times c_{\tilde{\eta}})^* \mathcal{P}_{A,\tilde{\eta}}^{\otimes -1}$.

Remark 5.2.3.5. The étale base extensions over η we need for the definition of each $(c_m, c_m^\vee, \tau_m)$ could depend on m, and the triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ could be defined only over a projective system of finite étale morphisms $\tilde{\eta} \to \eta$ given by field extensions as in Section 5.2.1. We will never need to use $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ directly.

Corollary 5.2.3.6. *With the setting as in Proposition 5.2.3.3, the* ς_n *is liftable (defined as in Definition 5.2.2.11) if and only if the triple* $(c_n, c_n^\vee, \tau_n)$ *is liftable (defined as in Definition 5.2.3.4). In this case, a lifting* $\hat{\varsigma}$ *corresponds to a symbolic triple* $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ *as in Definition 5.2.3.4.*

It is natural to ask if a lifting $(c_n, c_n^\vee, \tau_n)$ determines the original triple $(c, c^\vee, \tau)$. To answer this question, let us introduce some new notations:

Definition 5.2.3.7. *Let* U *be a scheme. Let* $\underline{N}$ *be any étale sheaf of left* $\mathcal{O}$*-modules that becomes constant over a finite étale surjection of* U, *with underlying module* N *a finitely generated* $\mathcal{O}$*-module. Let* (Z, λ_Z) *be any polarized abelian scheme over* U *with left* $\mathcal{O}$*-module structure given by some* $i_Z : \mathcal{O} \to \mathrm{End}_U(Z)$. *Then we denote by* $\underline{\mathrm{Hom}}_{\mathcal{O}}(\underline{N}, Z)$ *the group functor of* $\mathcal{O}$*-linear group homomorphisms from the group functor* $\underline{N}$ *to the group functor* Z.

Proposition 5.2.3.8. *With the setting as in Definition 5.2.3.7, suppose* $\underline{N}$ *is constant with value some finitely generated* $\mathcal{O}$*-module* N. *Then the following are true:*

1. *The group functor* $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ *is representable by a proper subscheme of an abelian scheme over* U.
2. *If* N *is torsion with number of elements prime to the residue characteristic of* U, *then* $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ *is finite étale over* U.
3. *If* N *is projective as an* $\mathcal{O}$*-module, then* $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ *is representable by an abelian scheme.*

4. *If N is an $\mathcal{O}$-lattice, and the residue characteristics of U are unramified in $\mathcal{O}$, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is representable by a proper smooth group scheme which is an extension of a finite étale group scheme, whose rank has no prime factors other than those of* Disc, *by an abelian scheme over U.*

   In particular, if we denote by $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)^\circ$ the (fiber-wise) **identity component** *of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ (see [51, IV, 15.6.5], or [20, pp. 154–155]), then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)^\circ$ is an abelian scheme.*

This is the so-called *Serre's construction.*

Proof. Since $\mathcal{O}$ is (left) noetherian, and since N is finitely generated, we see that N is finitely presented in the sense that there is a *free resolution*

$$\mathcal{O}^{\oplus s} \to \mathcal{O}^{\oplus r} \to N \to 0.$$

By taking $\underline{\mathrm{Hom}}_{\mathcal{O}}(\,\cdot\,, Z)$, we obtain a sequence

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z) \to Z^r \xrightarrow{f} Z^s, \tag{5.2.3.9}$$

where Z^r and Z^s stands for the fiber products of respectively r and s copies of Z over U, which represents $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ as the kernel of the morphism f from the abelian scheme Z^r to the abelian scheme Z^s.

To show that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is proper, note that the first morphism in (5.2.3.9) is a closed immersion because Z^s is separated, and a closed subscheme of Z^r is certainly proper. This proves statement 1 of Proposition 5.2.3.8.

Suppose N is torsion with number of elements prime to the residue characteristics of U. Let $m \geq 1$ be an integer that is invertible in $\mathscr{O}_U$ and kills every element in N. Then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is a proper subscheme of the finite *étale* group scheme $\underline{\mathrm{Hom}}(N, Z[m])$. The dimension of the fibers $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ over U is *constant*, because there exists a constant module M such that $Z[m]_{\bar{s}} \cong M$ (forgetting Galois module structures) for any geometric point $\bar{s}$ of U. This forces $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ to be finite étale over U. This proves statement 2 of Proposition 5.2.3.8.

If N is projective, then it is in particular flat (by [112, Cor. 2.16]). This is the same for its dual (right) $\mathcal{O}$-module $N^\vee$. Hence, for any embedding $U \hookrightarrow \tilde{U}$ defined by an ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, the surjectivity of the map $Z(\tilde{U}) \to Z(U)$ of $\mathcal{O}$-modules implies the surjectivity of the map

$(N^\vee \underset{\mathcal{O}}{\otimes} Z)(\tilde{U}) = N^\vee \underset{\mathcal{O}}{\otimes} Z(\tilde{U}) \to (N^\vee \underset{\mathcal{O}}{\otimes} Z)(U) = N^\vee \underset{\mathcal{O}}{\otimes} Z(U)$. This shows that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is formally smooth, and hence smooth because it is of finite type. Moreover, since N is projective, there exists some projective $\mathcal{O}$-module N' such that $N \oplus N' \cong \mathcal{O}^{\oplus r}$ for some $r \geq 0$. Then we have $\underline{\mathrm{Hom}}(N, Z) \underset{U}{\times} \underline{\mathrm{Hom}}(N', Z) \cong Z^r$, which shows that it is $\underline{\mathrm{Hom}}(N, Z)$ is geometric connected. Hence we see by definition that $\underline{\mathrm{Hom}}(N, Z)$ is an abelian scheme. This proves statement 3 of Proposition 5.2.3.8.

Finally, suppose that N is an $\mathcal{O}$-lattice, and that the residue characteristics of U are unramified in $\mathcal{O}$.

Let us first treat the case that the $\mathcal{O}$-action on N extends to a maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$. Since $\mathcal{O}'$ is maximal, N is projective as an $\mathcal{O}'$-module by Proposition 1.1.1.20. By statement 3 of Proposition 5.2.3.8 proved above, we know that $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z)$ is an abelian scheme. Hence it suffices to show that the cokernel of the natural inclusion $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is finite étale and with rank having no prime factors other than those of Disc. By Proposition 1.1.1.17, there exists an integer $m \geq 1$, with no prime factors other than those of Disc, such that $m\mathcal{O}' \subset \mathcal{O}$. Consider the following diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z) \arrow[r, hook] \arrow[dr, "{[m]}"'] & \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z) \arrow[d, dotted] \arrow[dr, "{[m]}"] & \\
& \underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z) \arrow[r, hook] & \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)
\end{tikzcd},$$

where $[m]$ denotes the map induces by the multiplication by m on Z. The dotted morphisms exists because $m\mathcal{O}' \subset \mathcal{O}$, or more precisely because the compatibility

$$f(bx) = b(f(x)),$$

for any $b \in \mathcal{O}$, $x \in N$, and $f \in \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$, implies in particular that

$$(nf)(b'x) = f((nb')x) = (nb')(f(x)) = b'((nf)(x)), \forall$$

for any $b' \in \mathcal{O}'$, $x \in N$, and $f \in \underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z)$. Viewing $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ as a map of schemes finite over $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z)$, we may identify its cokernel with the quotient of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)[m]$ by the image of $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N, Z)[m]$. Note that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)[m]$ is finite étale because it can be identified as a finite subgroup of the m-torsion subgroup of the abelian scheme used in the construction of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ above. In particular, the cokernel of $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N,Z) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is finite étale and with rank having no prime factors other than those of Disc, as desired.

For the general case, let $\mathcal{O}'$ be any maximal order in B containing $\mathcal{O}$. Take any integer $m \geq 1$ such that $m\mathcal{O}' \subset \mathcal{O}$. By assumption, m can be taken to have no prime factors other than those of Disc. Let L be the $\mathcal{O}'$-span of N in $N \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Then multiplication by m annihilates all elements of the finite group L/N. In other words, we have $L' := mL \subset N \subset L$, with N/L' also finite and killed by m. Consider the sequences

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}}(L/N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(L,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$$

and

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N/L',Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(L',Z).$$

The first shows that the identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is an abelian scheme, because it contains the isogenous quotient of $\underline{\mathrm{Hom}}_{\mathcal{O}}(L,Z)$ by $\underline{\mathrm{Hom}}_{\mathcal{O}}(L/N,Z)$, which is an abelian scheme and hence geometrically connected. The second shows that the component group of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is killed if we form the quotient of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ by the finite étale subgroup $\underline{\mathrm{Hom}}_{\mathcal{O}}(N/L',Z)$. In particular, the component group of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is isomorphic to a quotient of $\underline{\mathrm{Hom}}_{\mathcal{O}}(L/N,Z)$, which is necessarily finite étale of rank having no prime factors other than those of Disc, as desired. □

As a consequence of Proposition 5.2.3.8, we see that the group functors such as $\underline{\mathrm{Hom}}_{\mathcal{O}}(\underline{X},A^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(\underline{Y},A)$ are representable by *proper schemes* over S. In particular:

Corollary 5.2.3.10. *The morphisms c and $c^\vee$ are determined respectively (as unique extensions) by their pullbacks c_η and $c_\eta^\vee$ to η by noetherian normality of S, and hence by c_n and $c_n^\vee$ if $c_n|_X$ and $c_n^\vee|_Y$ descend to η.*

Note that $\lambda_A c^\vee = c\phi$ implies that $(\lambda_{A,\tilde{\eta}} c_n^\vee - c_n\phi_n)(\frac{1}{n}y)$ is n-torsion in $A_{\tilde{\eta}}^\vee$ for any $y \in Y$. Also, the relation $\tau_n(y_1,\phi(y_2)) = \tau_n(y_2,\phi(y_1))$ implies that $\tau_n(\frac{1}{n}y_1,\phi(y_2))\tau_n(\frac{1}{n}y_2,\phi(y_1))^{-1}$ is an n-torsion in $\mathbf{G}_{\mathrm{m},\tilde{\eta}}$ for any $y_1,y_2 \in Y$. Here the comparison between $\tau_n(\frac{1}{n}y_1,\phi(y_2))$ and $\tau_n(\frac{1}{n}y_2,\phi(y_1))^{-1}$ makes sense because we have a canonical isomorphism

$$\begin{aligned}\mathcal{P}_{A,\tilde{\eta}}|_{(c_n^\vee(\frac{1}{n}y_1),c\phi(y_2))} &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}_{\tilde{\eta}}^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y_1),c_n^\vee(\frac{1}{n}y_2))} \\ &\overset{\text{sym.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}_{\tilde{\eta}}^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y_2),c_n^\vee(\frac{1}{n}y_1))} \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_{A,\tilde{\eta}}|_{(c_n^\vee(\frac{1}{n}y_2),c\phi(y_1))}.\end{aligned}$$

Let us record this observation as:

Lemma 5.2.3.11. *With the setting as in Proposition 5.2.3.3, any splitting* $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *that corresponds to a lifting* $(c_n, c_n^\vee, \tau_n)$ *of* $(c, c^\vee, \tau)$ *over* $\tilde{\eta}$ *defines two pairings:*

1. *The first pairing*

$$\mathtt{d}_{10,n} : \mathrm{Gr}^{\mathtt{W}}_{-1,n} \times \mathrm{Gr}^{\mathtt{W}}_{0,n} = A[n]_{\tilde{\eta}} \times (\tfrac{1}{n}Y/Y) \to \boldsymbol{\mu}_{n,\tilde{\eta}}$$

   is defined by sending $(a, \frac{1}{n}y)$ *to* $\mathrm{e}_{A[n]}(a, (\lambda_{A,\tilde{\eta}} c_n^\vee - c_n\phi_n)(\frac{1}{n}y))$*, where* $\mathrm{e}_{A[n]} : A[n] \times A^\vee[n] \to \boldsymbol{\mu}_{n,S}$ *is the natural pairing between* $A[n]$ *and its Cartier dual* $A^\vee[n] = \underline{\mathrm{Hom}}_S(A[n], \mathbf{G}_{\mathrm{m},S}) = \underline{\mathrm{Hom}}_S(A[n], \boldsymbol{\mu}_{n,S})$*. (For simplicity, we use the same notation for its pullback to* $\tilde{\eta}$*.)*

2. *The second pairing*

$$\mathtt{d}_{00,n} : \mathrm{Gr}^{\mathtt{W}}_{0,n} \times \mathrm{Gr}^{\mathtt{W}}_{0,n} = (\tfrac{1}{n}Y/Y) \times (\tfrac{1}{n}Y/Y) \to \boldsymbol{\mu}_{n,\tilde{\eta}}$$

   is defined by sending $(\frac{1}{n}y_1, \frac{1}{n}y_2)$ *to* $\tau_n(\frac{1}{n}y_1, \phi(y_2))\tau_n(\frac{1}{n}y_2, \phi(y_1))^{-1}$*.*

Then $-{}^t\mathtt{d}^\star_{00,n} = \mathtt{d}_{00,n}$ *as in the context of Section 5.2.2.*

Corollary 5.2.3.12. *With the setting as in Corollary 5.2.3.6, any splitting* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box} G_{\bar{\eta}}$ *that corresponds to a lifting* $(\hat{c}, \hat{c}^\vee, \hat{\tau}) = \{(c_n, c_n^\vee, \tau_n)\}_{n|m, \Box\nmid m}$ *of* $(c, c^\vee, \tau)$ *defines two pairings:*

1. *The first pairing*

$$\mathtt{d}_{10} : \mathrm{Gr}^{\mathtt{W}}_{-1} \times \mathrm{Gr}^{\mathtt{W}}_{0} = \mathrm{T}^{\Box} A_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{\eta}}$$

   defined by the pairings $\{\mathtt{d}_{10,m}\}_{n|m, \Box\nmid m}$*. We can interpret this pairing as defined by sending* (a, y) *to* $\mathrm{e}_A(a, (\lambda_{A,\bar{\eta}}\hat{c}^\vee - \hat{c}\hat{\phi})(y))$*, where* $\mathrm{e}_A : \mathrm{T}^{\Box} A_{\bar{\eta}} \times \mathrm{T}^{\Box} A^\vee_{\bar{\eta}} \to \mathrm{T}^{\Box} \mathbf{G}_{\mathrm{m},\bar{\eta}}$ *is the natural pairing defined by* $\{\mathrm{e}_{A[m]}\}_{\Box\nmid m}$*.*

2. *The second pairing*

$$\mathtt{d}_{00} : \mathrm{Gr}^{\mathtt{W}}_{0} \times \mathrm{Gr}^{\mathtt{W}}_{0} = (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \mathbf{G}_{\mathrm{m},\bar{\eta}}$$

   defined by the pairings $\{\mathtt{d}_{00,m}\}_{n|m, \Box\nmid m}$*.*

Then $-{}^t\mathtt{d}^\star_{00} = \mathtt{d}_{00}$ *as in the context of Section 5.2.2.*

On the other hand, the splitting $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ allows us to write the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ on $G[n]_{\tilde{\eta}}$ in matrix form

$$\begin{pmatrix} & & \mathrm{e}_{20,n} \\ & \mathrm{e}_{11,n} & \mathrm{e}_{10,n} \\ \mathrm{e}_{02,n} & \mathrm{e}_{01,n} & \mathrm{e}_{00,n} \end{pmatrix},$$

where

$$\mathrm{e}_{ij,n} : \mathrm{Gr}^{\mathtt{W}}_{-i,n} \times \mathrm{Gr}^{\mathtt{W}}_{-j,n} \to \boldsymbol{\mu}_{n,\tilde{\eta}}$$

are the pairings induced by ς_n. We know that $\mathrm{e}_{20,n} = \mathrm{e}^{\phi}$ and $\mathrm{e}_{11,n} = \mathrm{e}^{\lambda_A}$ (in their natural mod n versions). The point is to identify $\mathrm{e}_{10,n}$ and $\mathrm{e}_{00,n}$ with something we can parameterize.

The following calculation is the key to the generalization of Faltings-Chai theory:

Theorem 5.2.3.13. *With the setting as above, we have* $\mathrm{e}_{10,n} = \mathtt{d}_{10,n}$ *and* $\mathrm{e}_{00,n} = \mathtt{d}_{00,n}$. (*Note that we use additive notations when we write matrix entries.*)

Corollary 5.2.3.14. *With the setting as above, we have* $\mathrm{e}_{10} = \mathtt{d}_{10}$ *and* $\mathrm{e}_{00} = \mathtt{d}_{00}$*.*

Remark 5.2.3.15. The importance of Theorem 5.2.3.13 and Corollary 5.2.3.14 is that we can define the degeneration data leading to an expected pairing without invoking any of the degeneration theorems we have developed so far (namely Theorems 4.2.1.8, 4.4.18, 5.1.1.6, and 5.1.2.5). The precise formulation will be given in Definitions 5.2.7.9 and 5.2.7.14.

5.2.4 Weil Pairings in General

Before we present the proof of Theorem 5.2.3.13, let us review the calculation of Weil pairings in general. (In particular, let us make clear our choice of sign conventions.)

In this section, we shall assume that we are given an abelian scheme A over an arbitrary base scheme S. Whenever possible, we shall intentionally confuse the notion of $\mathbf{G}_{\mathrm{m}}$-torsors and invertible sheaves. (See Corollary 3.1.2.13.) In particular, we shall use $\mathscr{O}_A$ to denote both the trivial invertible sheaf and the trivial $\mathbf{G}_{\mathrm{m}}$-bundle on A. Moreover, $\mathscr{O}_A$ will also be used to mean the structural sheaf of A. We hope that the convenience of such a confusion will outweigh the confusion it incurs.

Let us denote by $\mathcal{P}_A$ the Poincaré biextension of $A \underset{S}{\times} A^\vee$ by $\mathbf{G}_{\mathrm{m}}$, and denote by $\mathcal{N}_h$ the invertible sheaf $\mathcal{P}_A|_{A \underset{S}{\times} \{h\}}$, where $h : S \to A^\vee$ is a point of $A^\vee$. (Note that we are using the confusion alluded above here.) Then the tautological rigidification $\mathcal{P}_A|_{e \underset{S}{\times} A^\vee} \overset{\sim}{\to} \mathscr{O}_{A^\vee}$ gives a rigidification $\mathcal{N}_h(e) \overset{\sim}{\to} \mathscr{O}_{A^\vee}(h) \overset{\sim}{\to} \mathscr{O}_S$ for any $h \in A^\vee$. Here the last isomorphism $\mathscr{O}_{A^\vee}(h) := h^* \mathscr{O}_{A^\vee} \overset{\sim}{\to} \mathscr{O}_S$ is the tautological one by the structural morphism $h^* : \mathscr{O}_{A^\vee} \to \mathscr{O}_S$ of $h : S \to A^\vee$.

Let us first explain how to calculate the canonical perfect pairing

$$\mathrm{e}_{A[n]} : A[n] \times A^\vee[n] \to \boldsymbol{\mu}_{n,S}$$

for any integer $n \geq 1$. More generally, suppose that we have an isogeny $f : A \to A'$ with kernel K. Suppose that the dual isogeny $f^\vee : (A')^\vee \to A^\vee$ has kernel $K^\vee$. Then the kernels K and $K^\vee$ are related by a canonical perfect pairing

$$\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S},$$

and $\mathrm{e}_{A[n]}$ is simply the special case of e_K with $K = A[n]$. We shall explain how to calculate e_K for general K.

For any point $h \in K^\vee$, the invertible sheaf $\mathcal{N}_h := \mathcal{P}_{A'}|_{A' \underset{S}{\times} \{h\}}$ satisfies

$$\begin{aligned} f^* \mathcal{N}_h = f^*(\mathcal{P}_{A'}|_{A' \underset{S}{\times} \{h\}}) &\overset{\text{can.}}{\overset{\sim}{\to}} ((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{A \underset{S}{\times} \{h\}} \\ &\overset{\text{can.}}{\overset{\sim}{\to}} ((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{A \underset{S}{\times} \{h\}} \overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{P}_A|_{A \underset{S}{\times} \{f^\vee(h)\}} \overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{P}_A|_{A \underset{S}{\times} \{e\}} \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_A, \end{aligned}$$

because of the canonical isomorphism $(f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'} \overset{\text{can.}}{\overset{\sim}{\to}} (\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A$ given by Lemma 1.3.2.13, and because of the rigidification along the first factor of the Poincaré biextension. On the other hand, we shall interpret $f : A \to A'$ as identifying A' with the quotient A/K of A by the finite flat group scheme K. Therefore, by the theory of descent, the isomorphism $f^* \mathcal{N}_h \overset{\sim}{\to} \mathscr{O}_A$ corresponds to the descent datum on $\mathscr{O}_A$ describing $\mathcal{N}$ as a descended form of $\mathscr{O}_A$. The descent datum is given by an action of K on $\mathscr{O}_A$, which is a compatible collection of isomorphisms

$$\kappa(a) : T_a^* \mathscr{O}_A \overset{\sim}{\to} \mathscr{O}_A$$

for each $a \in K$. On the other hand, the structural map of the translation isomorphism $T_a : A \overset{\sim}{\to} A$ gives another isomorphism

$$\mathrm{str.}(a) : T_a^* \mathscr{O}_A \overset{\sim}{\to} \mathscr{O}_A.$$

Since A is an abelian scheme, which satisfies Assumption 3.1.2.7, the two isomorphisms can only differ on the rigidifications. That is, we would compare there pullbacks to the identity section $e : S \to A$, namely the maps

$$\mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_A(e) \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_S.$$

in both cases. We shall now redefine $\kappa(a)$ (resp. $\mathrm{str.}(a)$) to be the isomorphism $\mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_S$ such that the composition

$$T_a^* \mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_A$$

gives the original $\kappa(a)$ (resp. $\mathrm{str.}(a)$) above. Their difference is going to be a section

$$\tilde{h}(a) := \mathrm{str.}(a) \circ \kappa(a)^{-1} : \mathscr{O}_S \xrightarrow{\sim} \mathscr{O}_S,$$

which can be interpreted as a function

$$\tilde{h} : K \to \mathbf{G}_{\mathrm{m},S}.$$

The multiplicative structure of A gives a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_A(a+a') \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\mathrm{str.}(a+a')"', "\wr"] & \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \arrow[d, "\wr"', "\mathrm{str.}(a)\otimes\mathrm{str.}(a')"] \\
\mathscr{O}_S \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S
\end{tikzcd} . \tag{5.2.4.1}$$

The compatibility of the action, which compares

$$T_{a+a'}^* \mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a')) \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \xrightarrow{\sim} \mathscr{O}_A$$

using $\kappa(a) : \mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_S$ and $\kappa(a') : \mathscr{O}_A(a') \xrightarrow{\sim} \mathscr{O}_S$ with

$$T_{a+a'}^* \mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_S$$

using $\kappa(a+a') : \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_A$ (resp. $\mathrm{str.}(a+a') : \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_A$), gives another commutative diagram

$$\begin{tikzcd}
\mathscr{O}_A(a+a') \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\kappa(a+a')"', "\wr"] & \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \arrow[d, "\wr"', "\kappa(a)\otimes\kappa(a')"] \\
\mathscr{O}_S \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S
\end{tikzcd} . \tag{5.2.4.2}$$

The comparison of (5.2.4.1) and (5.2.4.2) shows that $\tilde{h}$ is a homomorphism. That is, $\tilde{h}$ is an element of $\underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$, the Cartier dual of K. This sets up an identification between $h \in K^\vee$ and $\tilde{h} \in \underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$, which defines an isomorphism $K^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$. Hence we have a perfect pairing

$$\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S}.$$

This is essentially the argument of [99, §15, proof of Thm. 1].

We would like to summarize the above argument as follows:

Lemma 5.2.4.3. *The following diagram is commutative:*

$$\begin{tikzcd}
((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{(a,h)} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] \arrow[dr, "\text{can.}", "\sim"'] & \mathcal{P}_{A'}|_{(f(a),h)} \arrow[r, equal] & \mathcal{P}_{A'}|_{(e,h)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, equal] \\
((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{(a,h)} \arrow[d, "\text{can.}"', "\wr"] & (f^*(\mathcal{N}_h))(a) \arrow[rr, "\sim"] \arrow[d, "\text{can.}"', "\wr"] & & \mathscr{O}_S \arrow[d, "\wr"', "\tilde{h}(a) = \mathrm{e}_K(a,h)"] \\
\mathcal{P}_A|_{(a,f^\vee(h))} \arrow[d, equal] & \mathscr{O}_A(a) \arrow[rr, "\sim", "\text{str.}(a)"'] \arrow[urr, "\kappa(a)", "\sim"'] & & \mathscr{O}_S \arrow[d, equal] \\
\mathcal{P}_A|_{(a,e)} \arrow[rrr, "\sim", "\text{rig.}"'] \arrow[ur, "\sim", "\text{can.}"'] & & & \mathscr{O}_S
\end{tikzcd}$$

Proof. The key point is to observe that the action map

$$\kappa(a) : T_a^* \mathscr{O}_A \xrightarrow{\sim} \mathscr{O}_A$$

is the composition of

$$T_a^* \mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} T_a^* f^* \mathcal{N}_h \overset{\text{can.}}{\xrightarrow{\sim}} f^* T_{f(a)}^* \mathcal{N}_h = f^* T_e^* \mathcal{N}_h = f^* \mathcal{N}_h \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A.$$

If we pullback to the identity section $e : S \to A$, and compose with the rigidification $\mathscr{O}_A(e) \xrightarrow{\sim} \mathscr{O}_S$, then

$$\kappa(a) : \mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_A(e) \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_S$$

is the composition

$$\mathscr{O}_A(a) \xrightarrow{\sim} (f^*(\mathcal{N}_h))(a) \xrightarrow{\sim} \mathcal{N}_h(f(a)) = \mathcal{P}_{A'}|_{(f(a),h)} = \mathcal{P}_{A'}|_{(e,h)} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_S,$$

which is exactly the upper half of the diagram. The commutativity of the remaining diagram is clear. $\square$

Note that this diagram suggests that, in order to compute $\mathrm{e}_K(a,h)$, it is not necessary to know anything about those isomorphisms such as $\kappa(a)$ and $\mathrm{str.}(a)$ in the diagram. Let us rephrase the essential result we need as:

Proposition 5.2.4.4. *The canonical pairing* $\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S}$ *gives the isomorphism* $\mathrm{e}_K(a,h)$*, for any* $a \in K$ *and any* $h \in K^\vee$*, which makes the following diagram commute:*

$$\begin{tikzcd}
((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{(a,h)} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, "\mathrm{can.}"', "\wr"] & \mathcal{P}_{A'}|_{(f(a),h)} \arrow[r, equal] & \mathcal{P}_{A'}|_{(e,h)} \arrow[r, "\mathrm{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\wr"', "\mathrm{e}_K(a,h)"] & . \\
((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{(a,h)} \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{P}_A|_{(a,f^\vee(h))} \arrow[r, equal] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_S &
\end{tikzcd}$$

That is, $\mathrm{e}_K(a,h)$ *measures the difference between the two rigidifications.*

Now suppose that we are given a polarization $\lambda_A : A \to A^\vee$ of A (defined as in Definition 1.3.2.20). Then the λ_A-Weil pairing e^{λ_A} (on $A[n]$) is defined to be the composition

$$A[n] \times A[n] \overset{\mathrm{Id}_A \times \lambda_A}{\to} A[n] \times A^\vee[n] \overset{\mathrm{e}_{A[n]}}{\to} \boldsymbol{\mu}_{n,S}.$$

That is, $\mathrm{e}^{\lambda_A}(a,a') = \mathrm{e}_{A[n]}(a, \lambda_A(a'))$ for any $a, a' \in A[n]$.

Suppose that (after an étale localization if we would like to) λ_A is of the form $\lambda_{\mathcal{M}}$ for some relatively ample invertible sheaf $\mathcal{M}$ on A. (See Construction 1.3.2.10 and Proposition 1.3.2.18.) Then $(\mathrm{Id}_A \times \lambda_A)^* \mathcal{P}_A \cong \mathcal{D}_2(\mathcal{M})$, and we have:

Corollary 5.2.4.5. *The* λ_A*-Weil pairing* $\mathrm{e}^{\lambda_A} : A[n] \times A[n] \to \boldsymbol{\mu}_{n,S}$ *gives the isomorphism* $\mathrm{e}^{\lambda_A}(a,a')$*, for any* $a, a' \in A[n]$*, which makes the following diagram commute:*

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, equal] & (([n]_A \times \lambda_A)^* \mathcal{P}_A)|_{(a,a')} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, "\mathrm{can.}"', "\wr"] & \mathcal{P}_A|_{(e,\lambda_A(a'))} \arrow[r, "\mathrm{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\wr"', "\mathrm{e}^{\lambda_A}(a,a')"] & . \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\mathrm{can.}"'] & ((\mathrm{Id}_A \times n\lambda_A)^* \mathcal{P}_A)|_{(a,a')} \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_S &
\end{tikzcd}$$

That is, $\mathrm{e}^{\lambda_A}(a,a')$ *measures the difference between the two rigidifications.*

Now let us relate this calculation to the so-called *Riemann form* defined by a relatively ample invertible sheave $\mathcal{M}$ on A. Suppose $K = K(\mathcal{M}) := \ker(\lambda_{\mathcal{M}})$ is defined as in (3.2.4.1). Then for any $a \in K$, we have an isomorphism

$$\mathcal{D}_2(\mathcal{M})|_{A \times \{a\}} = \mathcal{P}_A|_{A \underset{S}{\times} \{e\}} \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_A$$

given by one of the rigidifications of the Poincaré biextension. This gives a canonical isomorphism

$$T_a^*\mathcal{M} \overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(a)$$

as always. Therefore any section $\tilde{a} \in \mathcal{M}(a)$, or rather $\tilde{a} : \mathscr{O}_S \overset{\sim}{\to} \mathcal{M}(a)$, gives an isomorphism $\tilde{a}^{-1} : \mathcal{M}(a) \overset{\sim}{\to} \mathscr{O}_S$, and hence an isomorphism

$$\tilde{a}^{-1} : T_a^*\mathcal{M} \overset{\sim}{\to} \mathcal{M}.$$

Let $\mathcal{G}(\mathcal{M}) := \mathcal{M}|_K$. Then $\tilde{a}$ defines by restriction an isomorphism

$$\tilde{a}^{-1} : T_a^*\mathcal{G}(\mathcal{M}) \overset{\sim}{\to} \mathcal{G}(\mathcal{M})$$

and hence a group structure on $\mathcal{G}(\mathcal{M})$ covering the group structure on K. For any $a, a' \in K$, $\tilde{a} \in \mathcal{M}(a)$, and $\tilde{a}' \in \mathcal{M}(a')$, the group structure defines a composition $\tilde{a} * \tilde{a}' \in \mathcal{M}(a + a')$ such that its *inverse*

$$(\tilde{a} * \tilde{a}')^{-1} : T_{a+a'}^*\mathcal{M} \overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(a + a') \overset{\sim}{\to} \mathcal{M}$$

makes the following diagram

$$\begin{tikzcd}
T_a^*T_{a'}^*\mathcal{M} \arrow[r, "T_a^*((\tilde{a}')^{-1})", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & T_a^*\mathcal{M} \arrow[d, "\wr"', "\tilde{a}^{-1}"] \\
T_{a+a'}^*\mathcal{M} \arrow[r, dotted, "\sim", "(\tilde{a}' * \tilde{a})^{-1}"'] & \mathcal{M}
\end{tikzcd}$$

commute. In other words, we define

$$(\tilde{a} * \tilde{a}')^{-1} := \tilde{a}^{-1} \circ T_a^*((\tilde{a}')^{-1}).$$

Remark 5.2.4.6. This is sensible because we interpret the morphisms at the right-hand side as right multiplications, but we should not forget that this is a substantial *choice* with actually no justification.

In alternative language, the isomorphism

$$\mathcal{D}_2(\mathcal{M})|_{(a,a')} \overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{P}_A|_{(a,\lambda_{\mathcal{M}}(a'))} = \mathcal{P}_A|_{(a,e)} \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_S$$

gives an isomorphism

$$\mathcal{M}(a+a') \otimes \mathcal{M}(a)^{\otimes -1} \otimes \mathcal{M}(a')^{\otimes -1} \overset{\sim}{\to} \mathscr{O}_S,$$

or equivalently an isomorphism

$$\mathcal{M}(a)^{\otimes -1} \otimes \mathcal{M}(a')^{\otimes -1} \overset{\sim}{\to} \mathcal{M}(a+a')^{\otimes -1},$$

which sends $\tilde{a}^{-1} \otimes (\tilde{a}')^{-1}$ to $(\tilde{a} * \tilde{a}')^{-1}$ according to our definition.

Definition 5.2.4.7. *The Riemann form* $\mathrm{e}^{\mathcal{M}} : K \times K \to \mathbf{G}_{\mathrm{m},S}$ *is defined by setting* $\mathrm{e}^{\mathcal{M}}(a,a')$ *to be the difference between the two sections* $\tilde{a} * \tilde{a}'$ *and* $\tilde{a}' * \tilde{a}$ *of* $\mathcal{M}(a+a')$*, for any* $\tilde{a} \in \mathcal{M}(a)$ *and any* $\tilde{a}' \in \mathcal{M}(a')$*, so that the following diagram*

$$\begin{tikzcd}
\mathscr{O}_S \arrow[r, "\tilde{a}*\tilde{a}'", "\sim"'] \arrow[d, equal] & \mathcal{M}(a+a') \\
\mathscr{O}_S \arrow[r, "\sim", "\tilde{a}'*\tilde{a}"'] & \mathcal{M}(a+a') \arrow[u, "\mathrm{e}^{\mathcal{M}}(a,a')"', "\wr"]
\end{tikzcd}$$

is commutative, or equivalently so that the following diagram

$$\begin{tikzcd}
T_a^* T_{a'}^* \mathcal{M} \arrow[r, "T_a^*((\tilde{a}')^{-1})", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & T_a^* \mathcal{M} \arrow[r, "\tilde{a}^{-1}", "\sim"'] & \mathcal{M} \\
T_{a'}^* T_a^* \mathcal{M} \arrow[r, "\sim", "T_{a'}^*(\tilde{a}^{-1})"'] & T_{a'}^* \mathcal{M} \arrow[r, "\sim", "(\tilde{a}')^{-1}"'] & \mathcal{M} \arrow[u, "\mathrm{e}^{\mathcal{M}}(a,a')"', "\wr"]
\end{tikzcd}$$

is commutative. (*Note that it is alternating by definition.*)

The construction above of $\tilde{a} * \tilde{a}'$ using one of the rigidifications of $\mathcal{P}_A$ implies that:

Proposition 5.2.4.8. *The Riemann form* $\mathrm{e}^{\mathcal{M}} : K \times K \to \mathbf{G}_{\mathrm{m}}$ *gives the isomorphism* $\mathrm{e}^{\mathcal{M}}(a,a')$*, for any* $a, a' \in K$*, which makes the following diagram commute:*

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M})|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{D}_2(\mathcal{M})|_{(a',a)} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S \arrow[u, "\mathrm{e}^{\mathcal{M}}(a,a')"', "\wr"]
\end{tikzcd} .$$

That is, $\mathrm{e}^{\mathcal{M}}(a,a')$ *measures the difference between the two rigidifications.* (*Here the symmetry map is the canonical one as in Lemma 3.2.2.1.*)

The same argument applies when we replace $\mathcal{M}$ by $\mathcal{M}^{\otimes n}$. Suppose moreover that $a \in A[n]$ and $a' \in K(\mathcal{M}^{\otimes n}) = [n]^{-1}K(\mathcal{M}) = \lambda_{\mathcal{M}}^{-1}(A^\vee[n])$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, leftrightarrow, "\mathrm{sym.}"', "\wr"] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\mathrm{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a',a)} \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_S \arrow[u, "\wr", "\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')"']
\end{tikzcd}$$

On the other hand, for formal reasons, we also have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a',a)} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, leftrightarrow, "\mathrm{sym.}"', "\wr"] & ((\mathrm{Id}_A \times n\lambda_{\mathcal{M}})^*\mathcal{P}_A)|_{(a',a)} \arrow[r, "\mathrm{can.}", "\sim"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\mathrm{rig.}", "\sim"'] \arrow[d, leftrightarrow, "\mathrm{sym.}"', "\wr"] & \mathscr{O}_S \arrow[d, equal] \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\mathrm{can.}"'] & (([n]_A \times \lambda_{\mathcal{M}})^*\mathcal{P}_A)|_{(a,a')} \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{P}_A|_{(e,\lambda_{\mathcal{M}}(a'))} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_S
\end{tikzcd}$$

Comparing the diagrams, we obtain:

Corollary 5.2.4.9. *The restriction of* $\mathrm{e}^{\mathcal{M}^{\otimes n}} : K(\mathcal{M}^{\otimes n}) \times K(\mathcal{M}^{\otimes n}) \to \mathbf{G}_{\mathrm{m},S}$ *to* $A[n] \times K(\mathcal{M}^{\otimes n})$ *gives the isomorphism* $\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$*, for any* $a, a' \in K$*, which makes the following diagram commute:*

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, equal] & \mathcal{P}_A|_{(e,\lambda_{\mathcal{M}}(a'))} \arrow[r, "\mathrm{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\wr"', "\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')"] \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_S
\end{tikzcd} \quad .$$

That is, $\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$ *measures the difference between the two rigidifications.*

Comparing Corollary 5.2.4.9 with Corollary 5.2.4.5, we arrive at the following important formula (cf. [99, §23, p. 228, (5)]):

Proposition 5.2.4.10. *If* $\mathcal{M}$ *is a relatively ample invertible sheaf on* A*, then*

$$\mathrm{e}^{\lambda_{\mathcal{M}}}(a,a') = \mathrm{e}_{A[n]}(a, \lambda_{\mathcal{M}}(a')) = \mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$$

for any $a, a' \in A[n]$.

Since any polarization λ_A is étale locally of the form $\lambda_{\mathcal{M}}$ for some relatively ample invertible sheaf $\mathcal{M}$ on A (by Definition 1.3.2.20 and Proposition 1.3.2.18), we have the following:

Corollary 5.2.4.11. *The* λ_A*-Weil pairing* e^{λ_A} *is alternating for any polarization* λ_A*.*

(This is the same argument used in [99, §23].)

Remark 5.2.4.12. Although Proposition 5.2.4.10 is the main tool people use for calculating e^{λ_A}, the realization of the Weil pairings or Riemann forms as differences between rigidifications of the Poincaré biextension will be crucial for us in later arguments.

5.2.5 Sheaf-Theoretic Realization of Splittings of $G[n]_\eta$

Now let us return to the context of Section 5.2.3, with the assumption that $\tilde{\eta} = \eta$ and $\tilde{S} = S$ for simplicity of notations. (This is certainly harmless for our purpose.)

As we have seen in Section 5.2.3, the splitting $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \overset{\sim}{\to} G[n]_\eta$ can be described by a triple $(c_n, c_n^\vee, \tau_n)$, where $c_n : \frac{1}{n}X \to A_\eta^\vee$ corresponds to a splitting of $G^\natural[n]_\eta \twoheadrightarrow A[n]_\eta$, and where the pair $(c_n^\vee, \tau_n)$, being equivalent to a lifting $\iota_n : \frac{1}{n}Y \to G_\eta^\natural$, corresponds to a splitting of $G[n]_\eta \twoheadrightarrow \frac{1}{n}Y/Y$.

Let us first describe the splitting $G^\natural[n]_\eta \twoheadrightarrow A[n]_\eta$ given by c_n. By abuse of notations, we shall denote the unique extension $\frac{1}{n}X \to A^\vee$ of c_n to S by the same notation, and describe instead the corresponding splitting $G^\natural[n] \twoheadrightarrow A[n]$. By a convention we have adopted since Chapter 4 (see in particular Section 4.2.2), we shall identify $\mathscr{O}_{G^\natural}$ with its push-forward under $\pi : G^\natural \to A$, and write $\mathscr{O}_{G^\natural}$ as a sum $\mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$ of weight spaces under T-action. Let us also use similar abuse of notations without further explanation. Then the map $c : X \to A^\vee$ is defined by the isomorphism $\mathscr{O}_\chi \cong \mathcal{N}_{c(\chi)} := \mathcal{P}_A|_{A \underset{S}{\times} \{c(\chi)\}}$ in $\underline{\mathrm{Pic}}_e^0(A/S)$ (respecting rigidifications) for every $\chi \in X$.

The subgroup scheme $G^\natural[n]$ of $G^\natural$ can be understood by considering the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G^\natural[n] \arrow[r] \arrow[d] & G^\natural|_{A[n]} \arrow[r, "{[n]_{G^\natural}}"] \arrow[d] & G^\natural|_{\{e\}} \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & A[n] \arrow[r] & A[n] \arrow[r, "{[n]_A}"'] & \{e\} \arrow[r] & 0
\end{tikzcd}$$

In terms of relative affine group schemes over $A[n]$, we have

$$\begin{tikzcd} 0 \arrow[r] & G^\natural[n] \arrow[r] & G^\natural|_{A[n]} \arrow[r, "\text{``[n]''}"] & {[n]_A^*(G^\natural|_{\{e\}})} \arrow[r] & 0 \, . \end{tikzcd}$$

If we translate this into the language of $\mathscr{O}_{A[n]}$-sheaves of algebras, we see that $\mathscr{O}_{G^\natural[n]}$ should be given by the cokernel of the injection

$$[n]_A^*(\underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{A[n]}) \hookrightarrow \underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{A[n]}.$$

Let us write this as

$$0 \to \underset{\chi\in X}{\oplus}[n]_A^*\mathscr{O}_\chi|_{A[n]} \to \underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{A[n]} \to \underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{\bar{\chi}} \to 0.$$

The point is that

$$\mathscr{O}_{n\chi}|_{A[n]} \overset{\text{can.}}{\overset{\sim}{\to}} ([n]^*\mathscr{O}_\chi)|_{A[n]} \overset{[n]^*(\text{rig.})}{\overset{\sim}{\to}} ([n]^*\mathscr{O}_A)|_{A[n]} = \mathscr{O}_A|_{A[n]},$$

where rig. is actually the composition

$$\text{rig.} : \mathscr{O}_\chi(e) \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_S \overset{\text{str.}}{\overset{\sim}{\to}} \mathscr{O}_A(e).$$

Hence there is an isomorphism

$$\mathscr{O}_{\chi+n\chi'}|_{A[n]} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_\chi|_{A[n]} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{n\chi'}|_{A[n]} \overset{\sim}{\to} \mathscr{O}_\chi|_{A[n]}$$

for any $\chi, \chi' \in X$. If we take a representative χ for each class $\bar{\chi}$ of X/nX, and define $\mathscr{O}_{\bar{\chi}}$ to be $\mathscr{O}_\chi|_{A[n]}$, then the algebra map of $G^\natural$ given by $\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_{\chi+\chi'}$ induces isomorphisms $\mathscr{O}_{\bar{\chi}} \underset{\mathscr{O}_{A[n]}}{\otimes} \mathscr{O}_{\bar{\chi}'} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_{\bar{\chi}+\bar{\chi}'}$ giving the algebra map of $\underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{\bar{\chi}}$. This gives a realization $\underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{\bar{\chi}}$ of the $\mathscr{O}_{A[n]}$-sheaf of algebra $\mathscr{O}_{G^\natural[n]}$ (depending on our choices of representatives of X/nX), which is unique up to unique isomorphism.

Similarly, suppose $G_n^\natural$ is the group scheme defined by $c_n : \frac{1}{n}X \to A^\vee$, which can be given in terms of sheaf of $\mathscr{O}_A$-algebras by $\mathscr{O}_{G_n^\natural} = \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus} \mathscr{O}_{\frac{1}{n}\chi}$. Then the subgroup scheme $G_n^\natural[n]$ of $G_n^\natural$ can be realized via the exact sequence

$$0 \to \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus}[n]_A^*\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \to \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus} \mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \to \underset{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X}{\oplus} \mathscr{O}_{\frac{1}{n}\bar{\chi}} \to 0$$

as $\bigoplus\limits_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}}$, where $\mathscr{O}_{\frac{1}{n}\bar{\chi}}$ is defined as before using the sheaves $\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]}$ defined by representatives $\frac{1}{n}\chi$ of $\frac{1}{n}\bar{\chi}$.

Note that there is a structure map $\mathscr{O}_A|_{A[n]} \xrightarrow{\sim} \mathscr{O}_{\frac{1}{n}\bar{0}} \hookrightarrow \bigoplus\limits_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}}$, because there is an isomorphism from $\mathscr{O}_A|_{A[n]} = \mathscr{O}_{\frac{1}{n}0}|_{A[n]}$ to $\mathscr{O}_{\frac{1}{n}\bar{0}} = \mathscr{O}_{\frac{1}{n}\chi_0}|_{A[n]}$, for whatever representative $\frac{1}{n}\chi_0$ we choose for $\frac{1}{n}\bar{0} \in \frac{1}{n}X/X$. (Here we are using the clumsy notations such as $\mathscr{O}_{\frac{1}{n}\bar{0}}$ to avoid identification with $\mathscr{O}_{\bar{0}}$. They should not be confused.)

Now the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & G^\natural_n[n] \arrow[r] \arrow[d] & G^\natural_n|_{A[n]} \arrow[r] \arrow[d] & {[n]^*(G^\natural_n|_{\{e\}})} \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & G^\natural[n] \arrow[r] & G^\natural|_{A[n]} \arrow[r] & {[n]^*(G^\natural|_{\{e\}})} \arrow[r] & 0
\end{tikzcd}$$

corresponds to the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{\chi\in X} [n]^*_A\mathscr{O}_\chi|_{A[n]} \arrow[r] \arrow[d] & \bigoplus\limits_{\chi\in X} \mathscr{O}_\chi|_{A[n]} \arrow[r] \arrow[d] & \bigoplus\limits_{\bar{\chi}\in X/nX} \mathscr{O}_{\bar{\chi}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus\limits_{\frac{1}{n}\chi\in\frac{1}{n}X} [n]^*_A\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \arrow[r] & \bigoplus\limits_{\frac{1}{n}\chi\in\frac{1}{n}X} \mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \arrow[r] & \bigoplus\limits_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}} \arrow[r] & 0
\end{tikzcd}$$

of sheaves. Then the upshot is that the image of $\bigoplus\limits_{\chi\in X} \mathscr{O}_\chi|_{A[n]}$ lies in $\mathscr{O}_{\frac{1}{n}\bar{0}}|_{A[n]}$ inside $\bigoplus\limits_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}}$. Therefore the induced map

$$\bigoplus_{\bar{\chi}\in X/nX} \mathscr{O}_{\bar{\chi}} \to \bigoplus_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}}$$

factors through

$$\bigoplus_{\bar{\chi}\in X/nX} \mathscr{O}_{\bar{\chi}} \twoheadrightarrow \mathscr{O}_{\frac{1}{n}\bar{0}} \hookrightarrow \bigoplus_{\frac{1}{n}\bar{\chi}\in\frac{1}{n}X/X} \mathscr{O}_{\frac{1}{n}\bar{\chi}}.$$

However, this shows that the structural map

$$\mathscr{O}_A|_{A[n]} \xrightarrow{\sim} \mathscr{O}_{\bar{0}} \hookrightarrow \bigoplus_{\bar{\chi}\in X/nX} \mathscr{O}_{\bar{\chi}}$$

for $G[n] \twoheadrightarrow A[n]$ has a right inverse given by

$$\bigoplus_{\bar{\chi}\in X/nX} \mathscr{O}_{\bar{\chi}} \twoheadrightarrow \mathscr{O}_{\frac{1}{n}\bar{0}} \xrightarrow{\sim} \mathscr{O}_A|_{A[n]}.$$

In other words, $G^\natural[n] \twoheadrightarrow A[n]$ splits. In particular, since every $\mathscr{O}_{\bar{\chi}}$ is isomorphic to $\mathscr{O}_{\bar{0}}$ because it is also mapped isomorphically to $\mathscr{O}_{\frac{1}{n}\bar{0}} \xrightarrow{\sim} \mathscr{O}_A|_{A[n]}$, the splitting defines an isomorphism

$$\underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{\bar{\chi}} \xrightarrow{\sim} \mathscr{O}_A|_{A[n]} \underset{\mathscr{O}_S}{\otimes} (\underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{S,\bar{\chi}}),$$

where $\mathscr{O}_{S,\bar{\chi}}$ is just a copy of $\mathscr{O}_S$ with the prescribed weight $\bar{\chi}$ under $T[n]$ action. In other words, $\underset{\bar{\chi}\in X/nX}{\oplus} \mathscr{O}_{S,\bar{\chi}}$ is a realization of $\mathscr{O}_{T[n]}$ as an $\mathscr{O}_S$-sheaf of algebras. This corresponds to the isomorphism $A[n] \underset{S}{\times} T[n] \xrightarrow{\sim} G[n]$ defined by the splitting, because the map $T[n] \hookrightarrow G|_{\{e\}}$ is given by the rigidification maps $\mathscr{O}_\chi(e) \xrightarrow{\sim} \mathscr{O}_S$ respected by the isomorphisms $\mathscr{O}_{\frac{1}{n}\bar{0}} \xrightarrow{\sim} \mathscr{O}_A|_{A[n]}$ above.

For each point $a \in A[n]$, the splitting gives a map $S \to G^\natural$, which can be described sheaf-theoretically by a surjection

$$\mathscr{O}_{G^\natural}(a) = \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(a) \to \mathscr{O}_S$$

whose restriction to each $\mathscr{O}_\chi$ is given by the map

$$\begin{aligned}\mathscr{O}_\chi(a) &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{(a,c(\chi))} = \mathcal{P}_A|_{(a,nc_n(\frac{1}{n}\chi))} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{(na,c_n(\frac{1}{n}\chi))} = \mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi))} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_S.\end{aligned}$$

Let us denote this map by $r(a, c_n(\frac{1}{n}\chi))$, to signify the choice of c_n involved in this definition.

Note that any other lifting $c'_n : \frac{1}{n}X \to A^\vee$ is necessarily of the form $c'_n = c_n + d_n$ for some $d_n : \frac{1}{n}X \to A^\vee[n]$, or rather $d_n : \frac{1}{n}X/X \to A^\vee[n]$, because we need to have $c'_n|_X = c_n|_X = c$. Let us investigate the effect of such a modification.

The splitting defined by $c'_n = c_n + d_n$ is defined by

$$\mathscr{O}_{G^\natural}(a) = \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(a) \to \mathscr{O}_S$$

with maps $r(a, c'_n(\frac{1}{n}\chi)) : \mathscr{O}_\chi(a) \xrightarrow{\sim} \mathscr{O}_S$ defined by

$$\begin{aligned}\mathscr{O}_\chi(a) &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{(a,c(\chi))} = \mathcal{P}_A|_{(a,n(c_n+d_n)(\frac{1}{n}\chi))} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{(na,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} = \mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_S.\end{aligned}$$

The comparison of $r(a, c_n(\frac{1}{n}\chi))$ with $r(a, c_n(\frac{1}{n}\chi) + d_n(\frac{1}{n}\chi))$ can be obtained from the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_A|_{(a,c(\chi))} \arrow[r, "\text{can.}", "\sim"'] & \mathscr{O}_\chi(a) \arrow[d, "r(a,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))", "\wr"'] \\
\mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} \arrow[u, "\text{can.}", "\wr"'] \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi))} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(e,d_n(\frac{1}{n}\chi))} \arrow[u, "\text{can.}", "\wr"'] \arrow[d, "\text{can.}", "\wr"'] \arrow[r, "\text{rig.} \otimes \text{rig.}", "\sim"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[u, "\text{can.}", "\wr"'] \\
\mathcal{P}_A|_{(a,c(\chi))} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\text{can.} \otimes \text{rig.}"'] & \mathscr{O}_\chi(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[u, "r(a,c_n(\frac{1}{n}\chi)) \otimes \mathrm{e}_{A[n]}(a,d_n(\frac{1}{n}\chi))^{-1}", "\wr"']
\end{tikzcd}$$

Note that we have used Proposition 5.2.4.4 to find the correct isomorphism $\mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$ in the diagram. As a result, we see that we have the formal relation

$$r(a, c_n(\tfrac{1}{n}\chi) + d_n(\tfrac{1}{n}\chi)) = r(a, c_n(\tfrac{1}{n}\chi))\mathrm{e}_{A[n]}(a, d_n(\tfrac{1}{n}\chi))^{-1}.$$

On the other hand, consider the canonical isomorphism

$$\underline{\mathrm{Hom}}_S(\tfrac{1}{n}X, A^\vee[n]) \cong \underline{\mathrm{Hom}}_S(A[n], T_n[n]) \cong \underline{\mathrm{Hom}}_S(A[n], T[n]),$$

which we denote by $d_n \mapsto {}^t d_n$, with sign convention determined by the relation $\chi({}^t d_n(a)) = \mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$ for any $\chi \in X$ and any $a \in A[n]$. Then, by definition of χ, the multiplication by ${}^t d_n(a)$ is given by multiplication by $\chi({}^t d_n(a)) = \mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$ on $\mathscr{O}_\chi$. As a result, the modified splitting

$$r(\chi, c_n(\tfrac{1}{n}\chi) + d_n(\tfrac{1}{n}\chi)) : \mathscr{O}_\chi(a) \overset{\sim}{\to} \mathscr{O}_S$$

can be interpreted as a composition

$$\mathscr{O}_\chi(a) \quad \overset{\chi(-\,{}^t d_n(a))}{\overset{\sim}{\to}} \quad \mathscr{O}_\chi(a) \quad \overset{r(\chi, c_n(\frac{1}{n}\chi))}{\overset{\sim}{\to}} \quad \mathscr{O}_S,$$

which is the same as multiplying the section $S \to G^\natural$ defined by c_n by ${}^t d_n(a) \in T[n]$. This completes the picture of splittings produced by c_n.

We will not need a sheaf theoretic description of the splitting $G[n]_\eta \twoheadrightarrow \frac{1}{n}Y/Y$ described by c_n and τ_n. What we will need is simply a sheaf theoretic description of the lifting $\iota_n : \frac{1}{n}Y \to G^\natural_\eta$ of $\iota : Y \to G^\natural_\eta$, which is given by the isomorphisms

$$\tau_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$$

that altogether form the map

$$\iota_n(\tfrac{1}{n}y)^* := \underset{\chi\in X}{\oplus} \tau_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_{G^\natural}(c_n^\vee(\tfrac{1}{n}y))_\eta = \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \to \mathscr{O}_{S,\eta}$$

of $\iota_n(\frac{1}{n}y) : \eta \to G^\natural_\eta$. This lifts the structure map

$$\iota(y)^* := \underset{\chi\in X}{\oplus} \tau(y, \chi) : \mathscr{O}_{G^\natural}(c^\vee(y))_\eta = \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}$$

of $\iota(y) : \eta \to G^\natural_\eta$ when we restrict to the subgroup Y of $\frac{1}{n}Y$.

To make the picture complete, we would like to see what happens when we have different liftings $((c_n^\vee)', \tau_n')$ of $(c^\vee, \tau)$. As in the case of c_n, any other lifting $(c_n^\vee)'$ is necessarily of the form $(c_n^\vee)' = c_n^\vee + d_n^\vee$ for some $d_n^\vee : \frac{1}{n}Y \to A[n]_\eta$, or rather $d_n^\vee : \frac{1}{n}Y/Y \to A[n]_\eta$. Using the splitting of $G^\natural[n] \twoheadrightarrow A[n]$ defined by c_n, which defines an isomorphism $A[n]_\eta \oplus T[n]_\eta \xrightarrow{\sim} G^\natural[n]_\eta$, a different lifting ι_n' is given by the difference $\iota_n' - \iota_n : \frac{1}{n}Y \to G^\natural[n]_\eta$ covering the difference $(c_n^\vee)' - c_n^\vee = d_n^\vee : \frac{1}{n}Y$ in the $A[n]_\eta$ component. Therefore the essential new information is the difference of $\iota_n' - \iota_n$ in the $T[n]_\eta$ component, given by a map $e_n : \frac{1}{n}Y \to T[n]_\eta$, or rather $e_n : \frac{1}{n}Y/Y \to T[n]_\eta$.

Note that the multiplication map on $G^\natural$, which covers the multiplication map on A, is given by isomorphisms

$$m_A^*\mathscr{O}_\chi \xrightarrow{\sim} \mathrm{pr}_1^*\,\mathscr{O}_\chi \otimes \mathrm{pr}_2^*\,\mathscr{O}_\chi$$

given by the theorem of square. (See Section 3.1.4.) In particular, for any $a \in A[n]$, the translation by a on $G^\natural$ (using the splitting) defined by c_n is given simply by the isomorphisms

$$T_a^*\mathscr{O}_\chi \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_\chi \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(a) \overset{r(a,c_n(\frac{1}{n}\chi))}{\xrightarrow{\sim}} \mathscr{O}_\chi.$$

We shall also denote this isomorphism as $r(a, c_n(\frac{1}{n}\chi))$ if there is no confusion.

As a result, the translation $\iota_n'(\frac{1}{n}(y))$ of $\iota_n(\frac{1}{n}y)$ by $(c_n^\vee(\frac{1}{n}y), e_n(\frac{1}{n}y))$ in $A[n]_\eta \times T[n]_\eta \xrightarrow{\sim} G^\natural[n]_\eta$ corresponds to the isomorphisms

$$\tau_n'(\tfrac{1}{n}y, \chi) : \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta},$$

each of which is as defined by the dotted arrow in the diagram by the composition of the other arrows:

$$\begin{tikzcd}
\mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y)+d_n^\vee(\tfrac{1}{n}y))_\eta \arrow[r, "\chi(e_n(\frac{1}{n}y))", "\sim"'] \arrow[dd, dotted, "\tau'_n(\frac{1}{n}y,\chi)"', "\wr"] & \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y)+d_n^\vee(\tfrac{1}{n}y))_\eta \arrow[d, "\wr"', "\mathrm{can.}"] \\
& \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(d_n^\vee(\tfrac{1}{n}y))_\eta \arrow[d, "\wr"', "\tau_n(\frac{1}{n}y,\chi)\otimes r(d_n^\vee(\frac{1}{n}y),c_n(\frac{1}{n}\chi))"] \\
\mathscr{O}_S & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[l, "\sim", "\mathrm{can.}"']
\end{tikzcd} \tag{5.2.5.1}$$

Symbolically, we can write this as

$$\tau'_n(\tfrac{1}{n}y,\chi)=\tau_n(\tfrac{1}{n}y,\chi)r(d_n^\vee(\tfrac{1}{n}y),c_n(\tfrac{1}{n}\chi))\chi(e_n(\tfrac{1}{n}y)).$$

5.2.6 Weil Pairings for $G[n]_\eta$ via Splittings

Let us continue with the notations and assumptions in Section 5.2.5. In particular, we shall keep the assumption that $\tilde{\eta}=\eta$ and $\tilde{S}=S$ for simplicity of notations. The goal in the section is to compute the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ on $G[n]_\eta$ and prove Theorem 5.2.3.13. Although the argument is elementary in nature, it is the main technical heart of this chapter.

By étale localization if necessary, let us assume moreover that the tuple $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}$ is an *split object*, in the sense that $\underline{Y}$ and $\underline{X}$ are both constant with values respectively Y and X, and that λ is induced by some relatively ample invertible sheaf $\mathscr{L}$ on G such that $\mathscr{L}^\natural$ is the pullback (via $\pi: G^\natural\to A$) of a relatively ample invertible sheaf $\mathscr{M}$ on A. This assumption is harmless for the calculation of Weil pairings.

By Proposition 5.2.4.10, we can compute $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ on $G[n]_\eta$ using the Riemann form defined by $\mathscr{L}_\eta^{\otimes n}$. Ideally, for two points g_1 and g_2 of $G[n]_\eta$, we shall find sections of $\mathscr{L}_\eta^{\otimes n}(g_1)$ and $\mathscr{L}_\eta^{\otimes n}(g_2)$, which are unique up to constants in $\mathbf{G}_{\mathrm{m},\eta}$ and can be realized as isomorphisms $T_{g_1}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}\mathscr{L}_\eta^{\otimes n}$ and $T_{g_2}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}\mathscr{L}_\eta^{\otimes n}$, where T_{g_1} and T_{g_2} are translations on G. Then the difference between the two compositions $T_{g_1}^*T_{g_2}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}T_{g_1}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}\mathscr{L}_\eta^{\otimes n}$ and $T_{g_1}^*T_{g_2}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}T_{g_2}^*\mathscr{L}_\eta^{\otimes n}\xrightarrow{\sim}\mathscr{L}_\eta^{\otimes n}$ gives us the constant $\mathrm{e}^{\lambda_\eta}(g_1,g_2)$ in $\mathbf{G}_{\mathrm{m},\eta}$. Note that this constant can be found by comparing the effects of the isomorphisms on any of the nonzero global sections. To proceed further, let us make use of the full invertible sheaf $\mathcal{L}$ over S, and pass to the formal completion $\mathcal{L}_{\mathrm{for}} = \mathcal{L}^{\natural}_{\mathrm{for}}$ on $G_{\mathrm{for}} = G^{\natural}_{\mathrm{for}}$. Then the sections of $\Gamma(G_\eta, \mathcal{L}_\eta)$ can be realized using its Fourier expansions (as defined in Section 4.3) as sections in $\Gamma(G^{\natural}_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \underset{R}{\otimes} K$ invariant under the Y-action defined by some ψ (by étale descent if necessary and by working as in the proof of Theorem 4.5.4.12).

Recall that we have been writing $\mathscr{O}_{G^\natural}$ as an $\mathscr{O}_A$-sheaf as $\underset{\chi \in X}{\oplus} \mathscr{O}_\chi$, so that sections of $\mathscr{O}_{G^\natural,\mathrm{for}}$ can be realized as the I-adic convergent sums in $\hat{\oplus}_{\chi\in X} \mathscr{O}_{\chi,\mathrm{for}}$. By fixing the choice of $\mathcal{M}$ on A such that $\mathcal{L}^\natural = \pi^*\mathcal{M}$ (via $\pi : G^\natural \to A$), we can write $\mathcal{L}^\natural$ as an $\mathscr{O}_A$-sheaf as $\underset{\chi\in X}{\oplus}(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$, and so that sections of $\mathcal{L}^\natural_{\mathrm{for}}$ are written as the I-adic convergent sums in $\hat{\oplus}_{\chi\in X}(\mathcal{M}_{\mathrm{for}} \underset{\mathscr{O}_{A,\mathrm{for}}}{\otimes} \mathscr{O}_{\chi,\mathrm{for}})$. To compute the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ using Riemann forms for $g_1, g_2 \in G[n]_\eta$, with choices of $g_1^\natural, g_2^\natural \in G^\natural_\eta$ to be made clear later, we shall replace the isomorphisms $T^*_{g_1}\mathcal{L}^{\otimes n}_\eta \overset{\sim}{\to} \mathcal{L}^{\otimes n}_\eta$ and $T^*_{g_2}\mathcal{L}^{\otimes n}_\eta \overset{\sim}{\to} \mathcal{L}^{\otimes n}_\eta$ by suitable isomorphisms $T^*_{g_1^\natural}(\mathcal{L}^\natural_\eta)^{\otimes n} \overset{\sim}{\to} (\mathcal{L}^\natural_\eta)^{\otimes n}$ and $T^*_{g_2^\natural}(\mathcal{L}^\natural_\eta)^{\otimes n} \overset{\sim}{\to} (\mathcal{L}^\natural_\eta)^{\otimes n}$, so that the effect of these isomorphisms to their formal completions are compatible with the Fourier expansion maps. Since we have a splitting $\mathrm{Gr}^{\mathtt{W}}_n \overset{\sim}{\to} G[n]_\eta$, it suffices to calculate the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ between pairs consisting of only elements in $T[n]_\eta$, $A[n]_\eta$, and $\frac{1}{n}Y/Y$. Then the choices of $g_i^\natural$ can be made as follows: For points of $T[n]_\eta$ and $A[n]_\eta$, they are already identified with points in $G^\natural[n]_\eta$ under the splitting. For points of $\frac{1}{n}Y/Y$, we take any representatives of them in $\frac{1}{n}Y$, and embed them into $G^\natural_\eta$ using the given lifting $\iota_n : \frac{1}{n}Y \hookrightarrow G^\natural_\eta$ of $\iota : Y \hookrightarrow G^\natural_\eta$. Note that the actions of all these elements can be described in terms of their effects on the weight spaces $\mathcal{M}^{\otimes n}_\eta \otimes \mathscr{O}_{\chi,\eta}$, and the resulting pairing $\mathrm{e}^{\lambda_\eta}(g_1, g_2)$, as a constant in $\mathbf{G}_{\mathrm{m},\eta}$, can be seen on any of the weight spaces we consider. Therefore we can disregard the notion of I-adic sums and formal completions from now on, and focus only on the weight spaces $(\mathcal{M} \otimes \mathscr{O}_\chi)$ as part of the sections of $\mathcal{L}^\natural_\eta$.

Let us summarize the information we have at this point:

1. The Y-action on $\mathcal{L}_\eta^\natural$ is given by the isomorphisms

$$\begin{tikzcd}
T^*_{c^\vee(y)}(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[d, "\wr"', "\text{can.}"] \\
\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \arrow[d, "\wr"', "\psi(y)\tau(y,\chi)"] \\
\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}
\end{tikzcd}$$

   covering the translation by $\iota(y)$ given by the isomorphisms

$$T^*_{c^\vee(y)}\mathscr{O}_{\chi,\eta} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \overset{\tau(y,\chi)}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta},$$

   for each $y \in Y$.

2. If we tensor n copies of $\mathcal{L}_\eta^\natural$ together, then we are forming the isomorphism

$$\begin{tikzcd}
(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_2,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} \cdots \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_n,\eta}) . \arrow[d, "\wr"'] \\
\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1+\chi_2+\ldots+\chi_n,\eta}
\end{tikzcd}$$

   for any $\chi_1, \chi_2, \ldots, \chi_n \in X$. Then the Y-action is given by

$$\begin{tikzcd}
T^*_{c^\vee(y)}(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} \cdots \underset{\mathscr{O}_{A,\eta}}{\otimes} T^*_{c^\vee(y)}(\mathcal{M}_\eta \otimes \mathscr{O}_{\chi_n,\eta}) , \arrow[d, "\wr"', "\psi(y)\tau(y,\chi_1)\otimes\ldots\otimes\psi(y)\tau(y,\chi_n)"] \\
(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1+\phi(y),\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} \cdots \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathcal{M}_\eta \otimes \mathscr{O}_{\chi_n+\phi(y),\eta}) \arrow[d, "\wr"'] \\
\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1+\ldots+\chi_n+\phi(ny),\eta}
\end{tikzcd}$$

or simply

$$\psi(y)^n\tau(y,\chi):T^*_{c^\vee(y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\xrightarrow{\sim}\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(ny),\eta},$$

for each $y\in Y$.

3. The translation by $T[n]$ on $G^\natural$ is given by the isomorphisms

$$\chi(t):\mathscr{O}_\chi\xrightarrow{\sim}\mathscr{O}_\chi,$$

for each $t\in T[n]$.

4. The translation by $A[n]$ on $G^\natural$, using the splitting defined by c_n, is given by the isomorphisms

$$r(a,c_n(\tfrac{1}{n}\chi)):T^*_a\mathscr{O}_\chi\overset{\text{can.}}{\xrightarrow{\sim}}\mathscr{O}_\chi\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_\chi(a)\xrightarrow{\sim}\mathscr{O}_\chi,$$

for each $a\in A[n]$.

5. The translation by $\frac{1}{n}Y$ on $G^\natural_\eta$, using the period map $\iota_n:\frac{1}{n}Y\to G^\natural_\eta$, is given by the isomorphisms

$$\tau_n(y,\chi):T^*_{c_n^\vee(\frac{1}{n}y)}\mathscr{O}_{\chi,\eta}\overset{\text{can.}}{\xrightarrow{\sim}}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta\xrightarrow{\sim}\mathscr{O}_{\chi,\eta}.$$

We would like to compute $\mathrm{e}^{\lambda_\eta}(t,t')$, $\mathrm{e}^{\lambda_\eta}(t,a)$, $\mathrm{e}^{\lambda_\eta}(t,\frac{1}{n}y)$, $\mathrm{e}^{\lambda_\eta}(a,a')$, $\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)$, and $\mathrm{e}^{\lambda_\eta}(\frac{1}{n}y,\frac{1}{n}y')$, for any $t,t'\in T[n]_\eta$, $a,a'\in A[n]$, and $\frac{1}{n}y,\frac{1}{n}y'\in\frac{1}{n}Y$. The reason to include the pairings on $A[n]_\eta\times A[n]_\eta$ and on $T[n]_\eta\times\frac{1}{n}Y/Y$ is to make sure that our previous claims are compatible with the sign convention we have chosen in Section 5.2.4.

To do this, we need to choose sections on $(\mathcal{L}^\natural_\eta)^{\otimes n}$ that define isomorphisms of the form $T^*_{g^\natural}(\mathcal{L}^\natural_\eta)^{\otimes n}\to(\mathcal{L}^\natural_\eta)^{\otimes n}$ covering the translation maps $T_{g^\natural}$ on $G^\natural$. Let us give the choices in each cases we need:

1. For $t\in T[n]$, we consider simply the isomorphisms

$$\chi(t):\mathcal{M}^{\otimes n}\underset{\mathscr{O}_A}{\otimes}\mathscr{O}_\chi\xrightarrow{\sim}\mathcal{M}^{\otimes n}\underset{\mathscr{O}_A}{\otimes}\mathscr{O}_\chi,$$

which makes sense because this is simply the action of $T[n]$ on $\mathcal{L}^\natural$ that makes the cubical torsor $\mathcal{L}^\natural$ descend to $\mathcal{M}$.

2. For $a \in A[n]$, which in particular satisfies $a \in K(\mathcal{M}^{\otimes n})$, we need to consider isomorphisms

$$\begin{aligned} T_a^*(\mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi \underset{\mathscr{O}_S}{\otimes} \mathcal{M}^{\otimes n}(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(a) \\ &\overset{\tilde{a}^{-1} \otimes r(a, c_n(\frac{1}{n}\chi))}{\xrightarrow{\sim}} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi, \end{aligned}$$

   where $\tilde{a}^{-1} : \mathcal{M}^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_S$ is an isomorphism given by some section $\tilde{a} \in \mathcal{M}^{\otimes n}(a)$.

3. For $\frac{1}{n}y \in \frac{1}{n}Y$, we need to make some additional choices.

   Note that

$$\begin{aligned} \psi(y)^n &= \psi(y)^{n-2}\psi(y)\psi(y) \\ &= \psi(y)^{n-2}\psi(2y)\tau(y, \phi(y))^{-1} \\ &= \psi(y)^{n-3}\psi(y)\psi(2y)\tau(y, \phi(y))^{-1} \\ &= \psi(y)^{n-3}\psi(3y)\tau(y, \phi(2y))^{-1}\tau(y, \psi(y))^{-1} \\ &= \dots \\ &= \psi(ny)\tau(y, \phi(\tfrac{1}{2}(n-1)ny))^{-1}, \end{aligned}$$

   under the canonical isomorphism

$$\begin{aligned} (c^\vee)^*\mathcal{M}^{\otimes n} &\xrightarrow{\sim} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_Y, \tfrac{1}{2}(n-1)n\phi)^*(c^\vee \times c)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)nc\phi)^*\mathcal{P}_A^{\otimes -1} \\ &= (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)n\lambda_A c^\vee)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)nc^\vee)^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee)^*([n]_A^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_A, [\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}), \end{aligned}$$

   which is the pullback by $c^\vee$ of the canonical isomorphism

$$\mathcal{M}^{\otimes n} \overset{\text{can.}}{\xrightarrow{\sim}} [n]_A^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_A, [\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}$$

given by repeated application of the canonical isomorphisms

$$\begin{aligned}[m_1]^*\mathcal{M}\otimes[m_2]^*\mathcal{M} &\overset{\text{can.}}{\overset{\sim}{\to}} [m_1+m_2]^*\mathcal{M}\underset{\mathscr{O}_A}{\otimes}(m_1,m_2)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}\\ &\overset{\text{can.}}{\overset{\sim}{\to}} [m_1+m_2]^*\mathcal{M}\underset{\mathscr{O}_A}{\otimes}(1,m_1m_2)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}.\end{aligned}$$

(The upshot is that there is nowhere in this isomorphism that we use one of the two rigidifications of $\mathcal{P}_A$.)

If we pullback the canonical isomorphism by $c_n^\vee$, then we get

$$\begin{aligned}(c_n^\vee)^*\mathcal{M}^{\otimes n} &\overset{\text{can.}}{\overset{\sim}{\to}} (c_n^\vee)^*([n]_A^*\mathcal{M}\underset{\mathscr{O}_A}{\otimes}(\mathrm{Id}_A,[\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1})\\ &\overset{\text{can.}}{\overset{\sim}{\to}} (c^\vee)^*\mathcal{M}\underset{\mathscr{O}_A}{\otimes}(c_n^\vee,\tfrac{1}{2}(n-1)nc_n^\vee)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}.\end{aligned}$$

Let us set $\epsilon=1$ when n is odd, and $\epsilon=2$ when n is even. Then the above canonical isomorphism can be rewritten as

$$(c_n^\vee)^*\mathcal{M}^{\otimes n} \overset{\text{can.}}{\overset{\sim}{\to}} (c^\vee)^*\mathcal{M}\underset{\mathscr{O}_A}{\otimes}(c_n^\vee,\tfrac{1}{2}(n-1)\epsilon c_\epsilon^\vee)^*\mathcal{D}_2(\mathcal{M})_A^{\otimes -1}.$$

Certainly, it would be desirable if the pullback

$$\begin{aligned}\tau_n\circ(\mathrm{Id}_Y\times\phi):\mathbf{1}_{\frac{1}{n}Y\times Y,\eta}\overset{\sim}{\to}(c_n^\vee,c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1} &= (c_n^\vee,\lambda_A c^\vee)^*\mathcal{P}_{A,\eta}^{\otimes -1}\\ &= (c_n^\vee,c^\vee)^*\mathcal{D}_2(\mathcal{M})_\eta^{\otimes -1}\end{aligned}$$

is liftable to some trivialization

$$\tilde{\tau}_{n,\epsilon}:=\mathbf{1}_{\frac{1}{n}Y\times\frac{1}{\epsilon}Y}\overset{\sim}{\to}(c_n^\vee,c_\epsilon^\vee)^*\mathcal{D}_2(\mathcal{M})^{\otimes -1},$$

where $c_\epsilon^\vee=c_n^\vee|_{\frac{1}{\epsilon}Y}:\frac{1}{\epsilon}Y\to A$ is the restriction. Then we can define ψ_n by

$$\psi_n(\tfrac{1}{n}y)=\psi(y)\tilde{\tau}_{n,\epsilon}(\tfrac{1}{n}y,\tfrac{n-1}{2}y)^{-1},$$

where $\psi_n(\frac{1}{n}y)$ is interpreted as a section of $\mathcal{M}^{\otimes n}(c_n^\vee(\frac{1}{n}y))^{\otimes -1}$. Note that we have in particular

$$\psi_n(y)=\psi(y)^n$$

for every $y \in Y$. Since Y is finitely generated, this is always true after a finite étale localization. Therefore, we may assume that all the $\psi_n(\frac{1}{n}y)$'s that we encounter exist and have been chosen. The value of the λ_η-Weil pairing is invariant under étale localizations and independent of choices.

We shall consider the isomorphisms

$$\psi_n(\tfrac{1}{n}y)\tau_n(\tfrac{1}{n}y, \chi) : T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \otimes \mathscr{O}_{\chi,\eta}) \to \mathcal{M}_\eta^{\otimes n} \otimes \mathscr{O}_{\chi+\phi(y),\eta},$$

which covers the translation by $\iota_n(\frac{1}{n}Y)$ on $G_\eta^\natural$.

Note that we do not check that these isomorphisms commute with Y-action on $(\mathcal{L}_\eta^\natural)^{\otimes n}$ defined by ψ. This statement will be a byproduct when we compute the commutators between these isomorphisms, because the restriction of ψ_n to Y can always be chosen to be ψ^n, and other choices only differ by an element in $\mathbf{G}_{\mathrm{m},\eta}$.

Let us start with the computation of pairings involving $T[n]_\eta$:

1. Suppose $t, t' \in T[n]_\eta$. Since we have an action of $T[n]$ on $\mathcal{L}^\natural$, the commutator between $\chi(t)$ and $\chi(t')$ is always zero, and we must have $\mathrm{e}^{\lambda_\eta}(t, t') = 0$ for all $t, t' \in T[n]_\eta$.

2. Suppose $t \in T[n]_\eta$ and $a \in A[n]_\eta$. Using the splitting defined by c_n, we have $\mathrm{e}^{\lambda_\eta}(t, a) = 0$ again, because the following diagram is always commutative:

$$\begin{tikzcd}
T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\tilde{a}^{-1}\otimes r(a,c_n(\frac{1}{n}\chi))", "\sim"'] \arrow[d, "\chi(t)"', "\wr"] & \mathcal{M}^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \arrow[d, "\wr"', "\chi(t)"] \\
T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\tilde{a}^{-1}\otimes r(a,c_n(\frac{1}{n}\chi))", "\sim"'] & \mathcal{M}^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}
\end{tikzcd}$$

3. Suppose $t \in T[n]_\eta$ and $\frac{1}{n}y \in \frac{1}{n}Y$. Then we have the following commu-

tative diagram:

$$\begin{tikzcd}
T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\psi_n(\frac{1}{n}y)\tau_n(\frac{1}{n}y,\chi)", "\sim"'] \arrow[dd, "\chi(t)"', "\wr"] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} . \arrow[d, "(\chi+\phi(y))(t)", "\wr"'] \\
 & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \\
T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\sim", "\psi_n(\frac{1}{n}y)\tau_n(\frac{1}{n}y,\chi)"'] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \arrow[u, "(\phi(y))(t)"', "\wr"]
\end{tikzcd}$$

Comparing this with the sign convention in Definition 5.2.4.7, and by Proposition 5.2.4.10, we see that $\mathrm{e}^{\lambda_\eta}(t,y) = (\phi(y))(t)$. (This is the same sign convention we have used in Proposition 5.2.2.1.)

In particular, we see that the $T[n]$-action commutes with Y-action on $\mathcal{L}_\eta^\natural$.

Let us calculate those pairings involving $A[n]_\eta$:

1. Suppose $a, a' \in A[n]_\eta$. Choose sections $\tilde{a} \in \mathcal{M}_\eta^{\otimes n}(a)$ and $\tilde{a}' \in \mathcal{M}_\eta^{\otimes n}(a')$, which define respectively isomorphisms $\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$ and $(\tilde{a}')^{-1} : \mathcal{M}_\eta^{\otimes n}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}$.

   Let us analyze the first combination

$$T_a^* T_{a'}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}.$$

   More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$\begin{aligned}
&T_a^* T_{a'}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a')) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(a) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \\
&\qquad\qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(a)
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
&\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&r(a, c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&(\tilde{a}')^{-1} : \mathcal{M}_\eta^{\otimes n}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&r(a', c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&\mathrm{str.}(a) : \mathscr{O}_{A,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}.
\end{aligned}$$

The second combination

$$T_{a'}^* T_a^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} T_{a'}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}$$

can be described analogously by switching the roles of a and a'. Therefore, the essential difference of the two combinations is given by the difference between the two maps str.(a) and str.(a'), and we have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{O}_{A,\eta}(a) \arrow[r, "\mathrm{can.}", "\sim"'] \arrow[d, "\mathrm{can.}"', "\wr"] & \mathcal{D}_2(\mathcal{M}_\eta)|_{(a,e)} \arrow[r, "\mathrm{rig.}", "\sim"'] \arrow[d, leftrightarrow, "\mathrm{sym.}"', "\wr"] & \mathscr{O}_{S,\eta} & . \\
\mathscr{O}_{A,\eta}(a') \arrow[r, "\sim", "\mathrm{can.}"'] & \mathcal{D}_2(\mathcal{M}_\eta)|_{(a',e)} \arrow[r, "\sim", "\mathrm{rig.}"'] & \mathscr{O}_{S,\eta} \arrow[u, "\wr"', "\mathrm{e}^{\lambda_\eta}(a,a')"] &
\end{tikzcd}$$

Comparing this with Corollary 5.2.4.5, we see that $\mathrm{e}^{\lambda_\eta}(a, a') = \mathrm{e}^{\lambda_A}(a, a')$, as expected.

2. Suppose $a \in A[n]_\eta$ and $\frac{1}{n}y \in \frac{1}{n}Y$. Choose a section $\tilde{a} \in \mathcal{M}_\eta^{\otimes n}(a)$ that defines an isomorphism $\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. On the other hand, we have $\psi_n(\frac{1}{n}y) \in \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\frac{1}{n}y))^{\otimes -1}$ chosen above.

   Let us analyze the first combination

$$\begin{aligned}
T_a^* T_{c_n^\vee(\frac{1}{n}y)}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) &\xrightarrow{\sim} T_a^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta})\\
&\xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}.
\end{aligned}$$

   More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$\begin{aligned}
&T_a^* T_{c_n^\vee(\frac{1}{n}y)}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} T_a^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} T_a^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta} \\
&\qquad\qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta}(a) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta} \\
&\qquad\qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta}(a) \\
&\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \\
&\qquad\qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\phi(y),\eta}(a)
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
&\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta} \\
&r(a, c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta} \\
&\psi_n(\tfrac{1}{n}y) : \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta} \\
&\tau_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta} \\
&r(a, c_n(\tfrac{1}{n}\phi(y))) : \mathscr{O}_{\phi(y),\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}.
\end{aligned}$$

On the other hand, the second combination

$$\begin{aligned}
T_{c_n^\vee(\frac{1}{n}y)}^* T_a^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) &\xrightarrow{\sim} T_{c_n^\vee(\frac{1}{n}y)}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \\
&\xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}.
\end{aligned}$$

is the composition of the following canonical isomorphisms

$$\begin{aligned}
&T^*_{c_n^\vee(\frac{1}{n}y)}T^*_a(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta})\\
&\overset{\text{can.}}{\xrightarrow{\sim}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a))\\
&\overset{\text{can.}}{\xrightarrow{\sim}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\qquad\qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
&\psi_n(\tfrac{1}{n}y) : \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&\tau_n(\tfrac{1}{n}y,\chi) : \mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&r(a,c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}\\
&\text{str.}(c_n^\vee(\tfrac{1}{n}y)) : \mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y)) \xrightarrow{\sim} \mathscr{O}_{S,\eta}.
\end{aligned}$$

The essential difference between these two combinations can be described by the comparison between

$$\begin{aligned}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(a,c_n^\vee(\frac{1}{n}y))} &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}_\eta)|_{(a,c^\vee(y))} \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_{A,\eta}|_{(a,\lambda_A c^\vee(y))}\\
&= \mathcal{P}_{A,\eta}|_{(a,c\phi(y))} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{\phi(y),\eta}(a) \overset{r(a,c_n(\frac{1}{n}\phi(y)))}{\xrightarrow{\sim}} \mathscr{O}_{S,\eta}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y),a)} &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}_\eta)|_{(c_n^\vee(\frac{1}{n}y),na)} = \mathcal{D}_2(\mathcal{M}_\eta)|_{(c_n^\vee(\frac{1}{n}y),e)}\\
&= \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y)) \overset{\text{str.}}{\xrightarrow{\sim}} \mathscr{O}_{S,\eta}.
\end{aligned}$$

Note that the last part of the composition

$$\mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y)) \overset{\text{str.}}{\xrightarrow{\sim}} \mathscr{O}_{S,\eta}$$

can be interpreted alternatively as

$$\mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)} \overset{\text{sym.}}{\overset{\sim}{\to}} \mathcal{P}_{A,\eta}|_{(e,\lambda_A c_n^\vee(\frac{1}{n}y))} \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_{S,\eta}.$$

Therefore, we see that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{O}_{\phi(y),\eta}(a) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(a,c\phi(y))} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{sym.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(e,c_n\phi_n(\frac{1}{n}y))} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_{S,\eta} \\
\mathscr{O}_{A,\eta}(c_n^\vee(\frac{1}{n}y)) \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)} \arrow[r, "\sim", "\text{sym.}"'] & \mathcal{P}_{A,\eta}|_{(e,\lambda_A c_n^\vee(\frac{1}{n}y))} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_{S,\eta} \arrow[u, "\wr", "\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)"']
\end{tikzcd}.$$

There are obviously some redundancies in the diagram. Let us define $b_n := \lambda_A c_n^\vee - c_n\phi_n$ as in Lemma 5.2.3.11. Then the above diagram implies the commutativity of the diagram:

$$\begin{tikzcd}
 & \mathcal{P}_{A,\eta}|_{(e,-\lambda_A c_n^\vee(\frac{1}{n}y))} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_{A,\eta}|_{(e,b_n(\frac{1}{n}y))} \arrow[d, "\wr"', "\text{rig.} \otimes \text{rig.}"] \\
\mathcal{P}_{A,\eta}^{\otimes n}|_{(a,-\lambda_A c_n^\vee(\frac{1}{n}y))} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_{A,\eta}^{\otimes n}|_{(a,b_n(\frac{1}{n}y))} \arrow[ur, bend left, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{S,\eta} \arrow[d, "\wr"', "\mathrm{Id} \otimes \mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)"] \\
\mathcal{P}_{A,\eta}|_{(e,-\lambda_A c_n^\vee(\frac{1}{n}y))} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_{A,\eta}|_{(a,e)} \arrow[r, "\sim", "\text{rig.} \otimes \text{rig.}"'] & \mathscr{O}_{S,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{S,\eta}
\end{tikzcd}.$$

The essential content of the commutativity is then given by the diagram:

$$\begin{tikzcd}
\mathcal{P}_{A,\eta}|_{(e,b_n(\frac{1}{n}y))} \arrow[r, "\text{rig.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta} \arrow[d, "\wr"', "\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)"] \\
\mathcal{P}_{A,\eta}|_{(a,e)} \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_{S,\eta}
\end{tikzcd}.$$

Comparing this with Proposition 5.2.4.4, we see that $\mathrm{e}^{\lambda_\eta}(a, \frac{1}{n}y) = \mathrm{e}_{A[n]}(a, b_n(\frac{1}{n}y))$. This proves the first part of Theorem 5.2.3.13.

Finally, let us calculate the pairing on $(\frac{1}{n}Y/Y) \times (\frac{1}{n}Y/Y)$. Suppose $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y/Y$. Let us analyze the first combination

$$\begin{aligned} T^*_{c_n^\vee(\frac{1}{n}y)} T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) &\overset{\sim}{\to} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y'),\eta}) \\ &\overset{\sim}{\to} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}. \end{aligned}$$

More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$\begin{aligned}
&T^*_{c_n^\vee(\frac{1}{n}y)}T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y')))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}\\
&\qquad\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}\\
&\qquad\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\qquad\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
&\psi_n(\tfrac{1}{n}y):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta}\\
&\tau_n(\tfrac{1}{n}y,\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta}\\
&\psi_n(\tfrac{1}{n}y'):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\overset{\sim}{\to}\mathscr{O}_{S,\eta}\\
&\tau_n(\tfrac{1}{n}y',\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\overset{\sim}{\to}\mathscr{O}_{S,\eta}\\
&\tau_n(\tfrac{1}{n}y,\phi(y')):\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta}.
\end{aligned}$$

As in the case of $\mathrm{e}^{\lambda_\eta}(a,a')$, the second combination

$$\begin{aligned}
T^*_{c_n^\vee(\frac{1}{n}y')}T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})&\overset{\sim}{\to}T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta})\\
&\overset{\sim}{\to}\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}.
\end{aligned}$$

can be described analogously by switching the roles of y and y'. Therefore, the essential difference of the two combinations is given by the difference between the two maps $\tau_n(\frac{1}{n}y,\phi(y'))$ and $\tau_n(\frac{1}{n}y',\phi(y))$, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y),c_n^\vee(\frac{1}{n}y'))} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),c\phi(y'))} \arrow[r, "\tau_n(\frac{1}{n}y,\phi(y'))", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta} \\
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y'),c_n^\vee(\frac{1}{n}y))} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y'),c\phi(y))} \arrow[r, "\sim", "\tau_n(\frac{1}{n}y',\phi(y))"'] & \mathscr{O}_{S,\eta} \arrow[u, "\wr", "\mathrm{e}^{\lambda_\eta}(\frac{1}{n}y,\frac{1}{n}y')"']
\end{tikzcd}.$$

Symbolically, we can write this as

$$\mathrm{e}^{\lambda_\eta}(\tfrac{1}{n}y,\tfrac{1}{n}y') = \tau_n(\tfrac{1}{n}y,\phi(y'))\tau_n(\tfrac{1}{n}y',\phi(y))^{-1}.$$

This proves the second part of Theorem 5.2.3.13.

Remark 5.2.6.1. It may be possible that one can calculate the Weil paring without going through the sheaf theoretic calculations, by using the dual abelian scheme $G_\eta^\vee$. The triple $(c_n, c_n^\vee, \tau_n)$ gives a triple $(c_n^\vee, c_n, t_n^\vee)$ splitting $G^\vee[n]_\eta$, and the difference between them is naturally measured by the pairings $\mathtt{d}_{10,n}$ and $\mathtt{d}_{00,n}$ we have proposed. The realization of this difference by the λ_η-Weil pairing could be a nontrivial statement, anyway.

5.2.7 Construction of Principal Level Structures

With the same the setting as in Section 5.2.1, assume that we have a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$. Then we know by Theorem 5.1.2.5 that there is an object (G, λ, i) in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ corresponding to the tuple above via $\mathrm{M}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$. Let X be the underlying $\mathcal{O}$-lattice of $\underline{X}$. One of the assumptions for having an object in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ is that there exists totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R},\mathbb{R}) \hookrightarrow (L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$. (See Definition 5.1.2.4.) In this case, there is an induced filtration $\mathtt{Z}_{\mathbb{R}} = \{\mathtt{Z}_{-i,\mathbb{R}}\}$ on $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ determining a unique isomorphism class of the induced symplectic module $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$ over $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$. (See Proposition 5.1.2.2.)

Motivated by Lemma 5.2.2.3 and its proof:

Definition 5.2.7.1. *We say that a symplectic admissible filtration* $\mathtt{Z}$ *of* $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ *is* **fully symplectic with respect to** $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *if there is a symplectic admissible filtration* $\mathtt{Z}_{\mathbb{A}^\Box} = \{\mathtt{Z}_{-i,\mathbb{A}^\Box}\}$ *of* $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\Box$ *that* **extends** $\mathtt{Z}$ *in the sense that* $\mathtt{Z}_{-i,\mathbb{A}^\Box} \cap (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) = \mathtt{Z}_{-i}$ *in* $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\Box$ *for all* i.

Remark 5.2.7.2. Implicit in Definition 5.2.7.1 is that $\mathsf{Z}_{-i,\mathbb{A}^\Box}$ is *integrable* for every i. In this case, there exists (noncanonically) a PEL-type $\mathcal{O}$-lattice $(L^{\mathsf{Z}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}})$ such that there exists a symplectic isomorphism $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{A}^\Box}, \langle \cdot, \cdot \rangle_{11}) \xrightarrow{\sim} (L^{\mathsf{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\Box, \langle \cdot, \cdot \rangle^{\mathsf{Z}})$. By modifying $(L^{\mathsf{Z}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}})$ if necessary, we may assume that there exists symplectic isomorphisms $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot, \cdot \rangle_{11}) \xrightarrow{\sim} (L^{\mathsf{Z}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot, \cdot \rangle^{\mathsf{Z}})$ and $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle \cdot, \cdot \rangle_{11,\mathbb{R}}) \xrightarrow{\sim} (L^{\mathsf{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle^{\mathsf{Z}})$, and that $[(L^{\mathsf{Z}})^\# : L^{\mathsf{Z}}]$ contains no prime factors other than those of $[L^\# : L]$.

Definition 5.2.7.3. *A symplectic-liftable admissible filtration* Z_n *of* L/nL *is called* **fully symplectic-liftable** *with respect to* $(L, \langle \cdot, \cdot \rangle)$ *if it is the reduction mod* n *of some admissible filtration* Z *of* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ *that is fully symplectic with respect to* $(L, \langle \cdot, \cdot \rangle)$ *as in Definition 5.2.7.1.*

Remark 5.2.7.4. As explained in Remark 5.2.2.7, even when $n = 1$, in which case the whole space L/nL is trivial, we shall still distinguish the filtrations by their equipped multi-ranks.

Lemma 5.2.7.5. *Let* Z_n *be an admissible filtration of* L/nL *that is fully symplectic-liftable with respect to* $(L, \langle \cdot, \cdot \rangle)$. *Let* $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot, \cdot \rangle_{11})$ *be some lifting of* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,n}, \langle \cdot, \cdot \rangle_{11,n})$, *and let* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle \cdot, \cdot \rangle_{11,\mathbb{R}})$ *be determined as in Proposition 5.1.2.2. Then there is associated* (*noncanonically*) *a PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$ *such that:*

1. $[(L^{\mathsf{Z}_n})^\# : L^{\mathsf{Z}_n}]$ *is prime-to-*$\Box$.
2. *There exists symplectic isomorphisms* $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot, \cdot \rangle_{11}) \xrightarrow{\sim} (L^{\mathsf{Z}_n} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$ *and* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle \cdot, \cdot \rangle_{11,\mathbb{R}}) \xrightarrow{\sim} (L^{\mathsf{Z}_n} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$.
3. *The moduli problem* $\mathsf{M}_n^{\mathsf{Z}_n}$ *defined by the non-canonical* $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$ *as in Definition 1.4.1.2 is canonical in the sense that it depends only on* Z_n, *but not on the choice of* $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$.

Proof. The existence of $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$ with the given properties is explained in Remark 5.2.7.2. As pointed out in Remark 1.4.3.13, the moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ is smooth and has at least one complex point. □

Remark 5.2.7.6. To avoiding introducing unnecessarily more noncanonical data, we shall suppress the choice of $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$, and say that $\mathsf{M}_n^{\mathsf{Z}_n}$ is defined by $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot, \cdot \rangle_{11})$ and $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle \cdot, \cdot \rangle_{11,\mathbb{R}})$.

Let us begin with a symplectic-liftable admissible filtration $\mathtt{Z}_n := \{\mathtt{Z}_{-i,n}\}$ of L/nL that is fully symplectic with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$. By Proposition 5.2.2.23, this is the most basic information we have about a level-n structure. Our goal is to describe the additional data for producing a level-n structure of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\,\cdot\,,\,\cdot\,\rangle)$ over $\tilde{\eta}$.

Being in particular symplectic-liftable, the admissible filtration $\mathtt{Z}_n$ is the reduction mod n of some symplectic admissible filtration $\mathtt{Z} := \{\mathtt{Z}_{-i}\}$ of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. The pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ induces the pairings

$$\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_{0} \to \hat{\mathbb{Z}}^\Box(1),$$

and

$$\langle\,\cdot\,,\,\cdot\,\rangle_{11} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_{-1} \to \hat{\mathbb{Z}}^\Box(1)$$

both satisfying

$$\langle bx, y\rangle_{ij} = \langle x, b^\star y\rangle_{ij}$$

for any $x \in \mathrm{Gr}^{\mathtt{Z}}_{-i}$, $y \in \mathrm{Gr}^{\mathtt{Z}}_{-j}$, and $b \in \mathcal{O}$.

For simplicity, let us assume for the moment that the étale sheaves $\underline{X}$ and $\underline{Y}$ in the datum $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ are constant with values respectively X and Y.

The data X, Y, and $\phi : Y \hookrightarrow X$ define a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle^\phi : \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)) \times Y \to \mathbb{Z}(1)$$

by sending (x, y) to $x(\phi(y))$ for any $x \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1))$ and $y \in Y$. The $\mathcal{O}$-module structure $\mathcal{O} \to \mathrm{End}_{\mathbb{Z}}(X)$ induces by transposition a right $\mathcal{O}^{\mathrm{op}}$-module structure $\mathcal{O}^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)))$, and hence an $\mathcal{O}$-module structure by pre-composition with the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}} : b \mapsto b^\star$. The $\mathcal{O}$-linearity of ϕ implies that

$$\langle bx, y\rangle^\phi = (bf)(\phi(y)) = x(b^\star \phi(y)) = \langle x, b^\star y\rangle^\phi$$

for any $x \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1))$, $y \in Y$, and $b \in \mathcal{O}$. By linearity, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle^\phi$ induces naturally a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle^\phi : \mathrm{Hom}_{\mathbb{Z}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \to \hat{\mathbb{Z}}^\Box(1).$$

Then it makes to consider pairs $(\varphi_{-2}, \varphi_0)$ of $\mathcal{O}$-linear isomorphisms

$$\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1))$$

and

$$\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square.$$

such that

$$\langle x, y\rangle^\phi = \langle \varphi_{-2}^{-1}(x), \varphi_0^{-1}(y)\rangle_{20} \tag{5.2.7.7}$$

for any $x \in \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $y \in Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. (In order to make later constructions compatible, the sign convention for $\langle \cdot, \cdot\rangle^\phi$ is chosen to be analogous to the one of $\mathrm{e}^\phi(\cdot, \cdot)$ in Proposition 5.2.2.1.)

On the other hand, $\mathrm{Gr}^{\mathtt{Z}}_{-1}$ is paired with itself under $\langle \cdot, \cdot\rangle_{11}$, and $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle \cdot, \cdot\rangle_{11})$ is symplectic isomorphic to $(L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \langle \cdot, \cdot\rangle^{\mathtt{Z}})$ for some PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}}, \langle \cdot, \cdot\rangle^{\mathtt{Z}})$ by Lemma 5.2.7.5. Therefore it makes sense to consider level-n structures $\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\bar{\eta}}$ of type $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle \cdot, \cdot\rangle_{11})$ as in Definition 1.3.6.1.

Suppose that we have chosen a liftable splitting $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$, which can be lifted to some $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. Then, as in Section 5.2.2, the pairing $\langle \cdot, \cdot\rangle$ can be expressed in matrix form as some

$$\begin{pmatrix} & & \langle \cdot, \cdot\rangle_{20} \\ & \langle \cdot, \cdot\rangle_{11} & \langle \cdot, \cdot\rangle_{10} \\ \langle \cdot, \cdot\rangle_{02} & \langle \cdot, \cdot\rangle_{01} & \langle \cdot, \cdot\rangle_{00} \end{pmatrix},$$

extending the pairings $\langle \cdot, \cdot\rangle_{20}$ and $\langle \cdot, \cdot\rangle_{11}$ above between the graded pieces. The level-n structure $\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\bar{\eta}}$ can be lifted noncanonically to some symplectic isomorphism $\varphi_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{\eta}}$, which we can take as $\hat{f}_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-1}$. The symplectic isomorphism φ_{-1} gives in particular an isomorphism $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^\square(1) \xrightarrow{\sim} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar{\eta}}$. Suppose we have a pair $(\varphi_{-2}, \varphi_0)$ as above satisfying (5.2.7.7). Then we can define isomorphisms

$$\begin{aligned} \hat{f}_{-2} : \mathtt{Z}_{-2} &\overset{\varphi_{-2}}{\xrightarrow{\sim}} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \\ &\overset{\nu(\varphi_{-1})}{\xrightarrow{\sim}} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \mathbf{G}_{\mathrm{m},\bar{\eta}}) \overset{\text{can.}}{\xrightarrow{\sim}} \mathrm{T}^\square T[n]_{\bar{\eta}} = \mathrm{Gr}^{\mathtt{W}}_{-2}. \end{aligned}$$

and $\hat{f}_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \overset{\varphi_0}{\xrightarrow{\sim}} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square = \mathrm{Gr}^{\mathtt{W}}_0$, and a graded isomorphism $\hat{f} := \oplus \hat{f}_i$. As soon as $\hat{f}$ is symplectic (as in Definition 5.2.2.12) with respect to some suitable choices of the isomorphism $\nu(\hat{f}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\,\mathbf{G}_{\mathrm{m},\bar{\eta}}$ and of the splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$, we will obtain a symplectic isomorphism $\hat{\alpha}$ such that $\mathrm{Gr}(\hat{\alpha}) = \hat{f}$ by applying Proposition 5.2.2.21.

To find the condition for $\hat{f}$ to be symplectic, or equivalently the condition for $\hat{\varsigma}$ to make $\hat{f}$ symplectic, let us assume that a liftable splitting $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ is given in terms of a liftable triple $(c_n, c_n^\vee, \tau_n)$ over $\tilde{\eta}$, which can be lifted to a splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ in terms of a triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$. Then we can write the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}(\,\cdot\,,\,\cdot\,)$ on $\mathrm{T}^{\Box}\,G_{\bar{\eta}}$ in matrix form as some

$$\begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix}.$$

By Lemma 5.2.2.13, the condition for $\hat{f}$ to be symplectic is the condition that $\hat{f}^*(\mathrm{e}_{ij}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{ij}$ for any i and j for the isomorphism $\nu(\hat{f})$ accompanying $\hat{f}$ (which we have not chosen yet). By the construction of $\hat{f} = \oplus \hat{f}_{-i}$, and by Proposition 5.2.2.1, we know that if we take $\nu(\hat{f}) = \nu(\varphi_{-1})$, then it is automatic that $\hat{f}^*(\mathrm{e}_{20}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{20}$ and $\hat{f}^*(\mathrm{e}_{11}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{11}$. This forces the choice of $\nu(\hat{f})$ if $\hat{f}$ is ever going to be symplectic. Therefore we see that the condition is that e_{10} and e_{00} must satisfy $\hat{f}^*(\mathrm{e}_{10}) = \nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{10}$ and $\hat{f}^*(\mathrm{e}_{00}) = \nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{00}$.

By Corollaries 5.2.3.14 and 5.2.3.6, we know that e_{10} and e_{00} must agree with respectively the pairings $\mathtt{d}_{10}$ and $\mathtt{d}_{00}$ defined using the triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ corresponding to the splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ that we have not specified yet. Therefore we must impose the condition on $(c_n, c_n^\vee, \tau_n)$ that it is liftable to some triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ that allows the existence of a splitting $\hat{\varsigma}$ satisfying the condition we have just found. Note that this condition does not depend on the liftings we choose.

Assume that this condition is achieved. Then the graded isomorphism $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ defined above is symplectic by Lemma 5.2.2.13. By Proposition 5.2.2.21, the symplectic triple $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ defines a symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ by $\hat{\alpha} := \hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1}$. Then $\hat{\alpha}$ necessarily respects the filtrations, and $\mathrm{Gr}(\hat{\alpha}) = \hat{f}$ necessarily induces the isomorphisms we have specified on the graded pieces. By reduction mod n, we obtain a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$, as desired. This α_n depends only on the reduction mod n of the above choices $(\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat{\delta}, \hat{c}, \hat{c}^\vee, \hat{\tau})$, which we denote as a tuple $(\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$.

This gives a recipe for producing level-n structures from tuples of the form above, which exhausts all possible level-n structures defined over $\tilde{\eta}$ by Proposition 5.2.2.21. In order to state the result in general, we shall assume from now that étale sheaves such as $\underline{Y}$ and $\underline{X}$ are not necessarily constant. Since the statements about Weil-pairings can always be verified by passing to étale localizations, our arguments remain valid in the more general setting by étale descent.

Remark 5.2.7.8. What is necessary for the construction of level-n structures is simply the condition that $(\underline{X}/n\underline{X})_{\tilde{\eta}}$ (resp. $(\underline{Y}/n\underline{Y})_{\tilde{\eta}}$) is constant, which means the corresponding action of the Galois group $\mathrm{Gal}(\bar{\eta}/\tilde{\eta})$ on X (resp. Y) is trivial mod n. This condition will be implicit as soon as we have isomorphisms of étale group functors between $(\underline{X}/n\underline{X})_{\tilde{\eta}}$ (resp. $(\underline{Y}/n\underline{Y})_{\tilde{\eta}}$) and some constant object.

Definition 5.2.7.9. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$. *A* **pre-level-$n$ structure datum of type** $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\,\cdot\,,\,\cdot\,\rangle)$ *over* $\tilde{\eta}$ *is an 8-tuple* $\alpha_n^\natural :=$ $(\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$ *consists of the following data:*

1. Z_n *is an admissible filtration of* L/nL *that is fully symplectic-liftable with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *(defined as in Definition 5.2.7.3). The admissible filtration* Z_n*, being in particular symplectic-liftable, is the reduction mod n of some symplectic admissible filtration* Z *of* $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$*. This determines the pairings* $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \times \mathrm{Gr}^{\mathsf{Z}}_0 \to \hat{\mathbb{Z}}^\Box(1)$ *and* $\langle\,\cdot\,,\,\cdot\,\rangle_{11} :$ $\mathrm{Gr}^{\mathsf{Z}}_{-1} \times \mathrm{Gr}^{\mathsf{Z}}_{-1} \to \hat{\mathbb{Z}}^\Box(1)$*, whose reduction mod n are pairings* $\langle\,\cdot\,,\,\cdot\,\rangle_{20,n}$ *and* $\langle\,\cdot\,,\,\cdot\,\rangle_{11,n}$ *depending only on* Z_n *but not on the choice of* Z.

2. $\varphi_{-1,n} : \mathrm{Gr}^{\mathsf{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\tilde{\eta}}$ *is a level-n structure of* $(A_{\tilde{\eta}}, \lambda_{A,\tilde{\eta}}, i_{A,\tilde{\eta}})$ *of type* $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ *over* $\tilde{\eta}$*. By definition (as in Definition 1.3.6.1),* $\varphi_{-1,n}$ *comes together with an isomorphism* $\nu(\varphi_{-1,n}) : (\mathbb{Z}/n\mathbb{Z})(1) \xrightarrow{\sim} \boldsymbol{\mu}_{n,\tilde{\eta}}$*, such that* $\varphi_{-1,n}$ *and* $\nu(\varphi_{-1,n})$ *are the reductions mod n of some isomorphisms* $\varphi_{-1} : \mathrm{Gr}^{\mathsf{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{\eta}}$ *and* $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^\Box(1) \xrightarrow{\sim} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$ *forming a symplectic isomorphism* $\varphi_{-1} : \mathrm{Gr}^{\mathsf{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{\eta}}$ *in the sense that they match the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_{11}$ *on* $\mathrm{Gr}^{\mathsf{Z}}_{-1}$ *with the* λ_A*-Weil pairing on* $\mathrm{T}^\Box A_{\bar{\eta}}$.

3. $\varphi_{-2,n} : \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\tilde{\eta}}((\underline{X}/n\underline{X})_{\tilde{\eta}}, (\mathbb{Z}/n\mathbb{Z})(1))$ *and* $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim}$ $(\underline{Y}/n\underline{Y})_{\tilde{\eta}}$ *are isomorphisms that are liftable to some* $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim}$

$\mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ *and* $\varphi_0 : \mathrm{Gr}_0^{\mathtt{Z}} \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ *over* $\bar{\eta}$*, such that the pairing* e^ϕ *is pulled back to the pairing* $\nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{20}$*, where* $\mathrm{e}^\phi : \mathrm{T}^\square\, T_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \to \mathrm{T}^\square\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$ *is the natural pairing defined as in Proposition 5.2.2.1.*

4. $\delta_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} L/nL$ *is a liftable splitting as in Definition 5.2.2.9, which is the reduction mod* n *of some splitting* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$*. This* $\hat{\delta}$ *determines the pairings* $\langle\,\cdot\,,\,\cdot\,\rangle_{10} : \mathrm{Gr}_{-1}^{\mathtt{Z}} \times \mathrm{Gr}_0^{\mathtt{Z}} \to \hat{\mathbb{Z}}^\square(1)$ *and* $\langle\,\cdot\,,\,\cdot\,\rangle_{00} : \mathrm{Gr}_0^{\mathtt{Z}} \times \mathrm{Gr}_0^{\mathtt{Z}} \to \hat{\mathbb{Z}}^\square(1)$*, whose reduction mod any integer* m *such that* $n|m$ *and* $\square \nmid m$ *are pairings* $\langle\,\cdot\,,\,\cdot\,\rangle_{10,m}$ *and* $\langle\,\cdot\,,\,\cdot\,\rangle_{00,m}$ *depending only on* $\mathtt{Z}_m$ *but not on the full* $\mathtt{Z}$*.*

5. *The maps* $c_n : \frac{1}{n}\underline{Y} \to A_{\tilde{\eta}}$*,* $c_n^\vee : \frac{1}{n}\underline{X} \to A_{\tilde{\eta}}^\vee$*, and* $\tau_n : \mathbf{1}_{\frac{1}{n}\underline{Y} \underset{S}{\times} \underline{X}, \tilde{\eta}} \xrightarrow{\sim} (c_n^\vee, c_{\tilde{\eta}})^* \mathcal{P}_{A,\tilde{\eta}}^{\otimes -1}$ *are respectively liftings of* c*,* $c^\vee$*, and* τ *over* $\bar{\eta}$*, such that the triple* $(c_n, c_n^\vee, \tau_n)$ *is liftable as in Definition 5.2.3.4 to some compatible system of liftings* $(\hat{c}, \hat{c}^\vee, \hat{\tau}) = \{(c_m, c_m^\vee, \tau_m)\}_{n|m, \square\nmid m}$*, which determines two systems of pairings*

$$\{\mathtt{d}_{10,m} : A[m]_{\bar{\eta}} \times (\tfrac{1}{m}Y/Y) \to \boldsymbol{\mu}_{m,\bar{\eta}}\}_{n|m,\square\nmid m}$$

and

$$\{\mathtt{d}_{00,m} : (\tfrac{1}{m}Y/Y) \times (\tfrac{1}{m}Y/Y) \to \boldsymbol{\mu}_{m,\bar{\eta}}\}_{n|m,\square\nmid m},$$

(*as in Lemma 5.2.3.11 and Corollary 5.2.3.12*) *by setting*

$$\mathtt{d}_{10,m}(a, \tfrac{1}{m}y) := \mathrm{e}_{A[m]}(a, (\lambda_A c_m^\vee - c_m \phi_m)(\tfrac{1}{m}y) \in \boldsymbol{\mu}_m(\bar{\eta})$$

for each $a \in A[m]_{\bar{\eta}}$ *and* $\frac{1}{m}y \in \frac{1}{m}Y$*, and by setting*

$$\mathtt{d}_{00,m}(\tfrac{1}{m}y, \tfrac{1}{m}y') = \tau_m(\tfrac{1}{m}y, \phi(y'))\tau_m(\tfrac{1}{m}y', \phi(y))^{-1} \in \boldsymbol{\mu}_m(\bar{\eta})$$

for each $\frac{1}{m}y, \frac{1}{m}y' \in \frac{1}{m}Y$*.*

Moreover, we say that the pre-level-n *structure datum* $\alpha_n^\natural$ *is* **symplectic-liftable***, and call it a* **level-n structure datum of type** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ *over* $\tilde{\eta}$*, if the following condition is satisfied: There exist some choice* $\hat{\alpha}^\natural := (\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat{\delta}, \hat{c}, \hat{c}^\vee, \hat{\tau})$ *lifting* $\alpha_n^\natural$ *as above, which is* **symplectic** *in the sense that*

$$(\varphi_{-1} \times \varphi_0)^*(\mathtt{d}_{10}) = \nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{10}$$

and

$$(\varphi_0 \times \varphi_0)^*(\mathtt{d}_{00}) = \nu(\varphi_{-1}) \circ \langle \,\cdot\, , \,\cdot\, \rangle_{00}.$$

(*See Lemma 5.2.2.13.*)

Proposition 5.2.7.10. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ *corresponding to a triple* (G, λ, i) *in* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$ *via Theorem 5.1.2.5, and a level-n structure datum* $\alpha_n^\natural$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\,\cdot\,,\,\cdot\,\rangle)$ *defined over* $\tilde{\eta}$ *as in Definition 5.2.7.9* (*without the assumption that* $\underline{X}$ *and* $\underline{Y}$ *are constant*). *Then the datum* $\alpha_n^\natural$ *gives in particular a splitting* $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *of the filtration*

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde{\eta}} \subset \mathtt{W}_{-1,n} = G^\natural[n]_{\tilde{\eta}} \subset \mathtt{W}_{0,n} = G[n]_{\tilde{\eta}},$$

and a graded symplectic isomorphism $f_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ *defined over* $\tilde{\eta}$*, so that* $\alpha_n = \varsigma_n \circ f_n \circ \delta_n^{-1} : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *defines a level-n structure of* $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition 1.3.6.1. Moreover, every level-n structure* $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *of* $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\,\cdot\,,\,\cdot\,\rangle)$ *arises from some* $\alpha_n^\natural$ (*defined over* $\tilde{\eta}$) *in this way.*

However, this association is not one to one. There could be different level-n structure data $\alpha_n^\natural$ and $\alpha_n^{\natural\,\prime}$ that produce the same level-n structure α_n. Therefore we would like to find equivalences on the level-n structure data, so that the equivalence classes of them will correspond bijectively to the level-n structures.

If we start with a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$, then we see that the filtration $\mathtt{Z}_n$ is necessarily determined by $\mathtt{W}_n$ under α_n, and moreover $\mathrm{Gr}_{-2,n}(\alpha_n) = \nu(\varphi_{-1,n}) \circ \varphi_{-2,n}$, $\mathrm{Gr}_{-1,n}(\alpha_n) = \varphi_{-1,n}$, and $\mathrm{Gr}_{0,n}(\alpha_n) = \varphi_{0,n}$ are all unique determined by $\mathrm{Gr}_n(\alpha_n)$. Hence the only possible difference between two object giving the same level-n structure are between the choices of $(\delta_n, c_n, c_n^\vee, \tau_n)$. Let us suppose that we have a different tuple $(\delta_n', c_n', (c_n^\vee)', \tau_n')$ giving the same level structure α_n.

On the one hand, as elaborated in Section 5.2.3, the liftable triple $(c_n, c_n^\vee, \tau_n)$ corresponds to a liftable splitting $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$, and the different choice $(c_n', (c_n^\vee)', \tau_n')$ corresponds to a different splitting $\varsigma_n' : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ that is related to ς_n by a liftable change of basis $\mathtt{w}_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ as in Definition 5.2.2.19, in the sense that $\varsigma_n' = \varsigma_n \circ \mathtt{w}_n$. Here the matrix entries are maps $\mathtt{w}_{21,n} : A[n]_{\tilde{\eta}} \to T[n]_{\tilde{\eta}}$, $\mathtt{w}_{10,n} : (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}} \to A[n]_{\tilde{\eta}}$, and $\mathtt{w}_{20,n} : (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}} \to T[n]_{\tilde{\eta}}$, which correspond respectively to a map $d_n : \frac{1}{n}\underline{X} \to A^\vee[n]_{\tilde{\eta}}$, a map $d_n^\vee : \frac{1}{n}\underline{Y} \to A[n]_{\tilde{\eta}}$, and a pairing $e_n : \frac{1}{n}\underline{Y}_{\tilde{\eta}} \times \underline{X}_{\tilde{\eta}} \to \boldsymbol{\mu}_{n,\tilde{\eta}}$. As we have seen in Section 5.2.5, the triples $(d_n, d_n^\vee, e_n)$ and $(c_n', (c_n^\vee)', \tau_n')$ can be related by $c_n' = c_n + d_n$, $(c_n^\vee)' = c_n^\vee + d_n^\vee$, and more elaborately (over an étale surjection over which both $\underline{Y}$ and $\underline{X}$ become constant)

$$\tau_n'(\tfrac{1}{n}y, \chi) = \tau_n(\tfrac{1}{n}y, \chi) r(d_n^\vee(\tfrac{1}{n}y), c_n(\tfrac{1}{n}\chi))\chi(e_n(\tfrac{1}{n}y))$$

for each $\frac{1}{n}y \in \frac{1}{n}Y$ and $\chi \in X$ as in (5.2.5.1). (See Section 5.2.5 for details on notations and arguments.)

Definition 5.2.7.11. *We say in this case that* $(d_n, d_n^\vee, e_n)$ **translates** $(c_n, c_n^\vee, \tau_n)$ *to* $(c_n', (c_n^\vee)', \tau_n')$.

Hence we have a dictionary between change of basis $\mathtt{w}_n$ and translations by triples $(d_n, d_n^\vee, e_n)$, both giving the difference between ς_n and ς_n'.

On the other hand, the two different choices of liftable splittings $\delta_n, \delta_n' : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} L/nL$ are related by a liftable change of basis $\mathtt{w}_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ as in Definition 5.2.2.18, in the sense that $\delta_n' = \delta_n \circ \mathtt{z}_n$.

By Proposition 5.2.2.23, we know that level-n structures are in bijection with equivalence classes of symplectic-liftable triples as in Definition 5.2.2.20. Therefore, the key point to the equivalence is the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Gr}_n^{\mathtt{Z}} \arrow[r, "\mathtt{z}_n", "\sim"'] \arrow[d, "\mathrm{Gr}_n(\alpha_n)"', "\wr"] & \mathrm{Gr}_n^{\mathtt{Z}} \arrow[d, "\mathrm{Gr}_n(\alpha_n)", "\wr"'] \\
\mathrm{Gr}_n^{\mathtt{W}} \arrow[r, "\sim", "\mathtt{w}_n"'] & \mathrm{Gr}_n^{\mathtt{W}}
\end{tikzcd},$$

or more explicitly the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{-2,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{-1,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{0,n}^{\mathtt{Z}} \arrow[r, "\mathtt{z}_n", "\sim"'] \arrow[d, "\mathrm{Gr}_n(\alpha_n)"', "\wr"] & \mathrm{Gr}_{-2,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{-1,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{0,n}^{\mathtt{Z}} \arrow[d, "\mathrm{Gr}_n(\alpha_n)", "\wr"'] \\
T[n]_{\tilde{\eta}} \oplus A[n]_{\tilde{\eta}} \oplus (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}} \arrow[r, "\sim", "\mathtt{w}_n"'] & T[n]_{\tilde{\eta}} \oplus A[n]_{\tilde{\eta}} \oplus (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}}
\end{tikzcd},$$

which is equivalent to the commutativity of the following three diagrams:

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{-1,n} \arrow[r, "\mathsf{z}_{21}"] \arrow[d, "\varphi_{-1,n}"', "\wr"] & \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \arrow[d, "\nu(\varphi_{-1,n})\circ\varphi_{-2,n}", "\wr"'] \\
A[n]_{\tilde{\eta}} \arrow[r, "\mathtt{w}_{21}"'] & T[n]_{\tilde{\eta}}
\end{tikzcd},$$

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{0,n} \arrow[r, "\mathsf{z}_{10}"] \arrow[d, "\varphi_{0,n}"', "\wr"] & \mathrm{Gr}^{\mathsf{Z}}_{-1,n} \arrow[d, "\varphi_{-1,n}", "\wr"'] \\
(\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}} \arrow[r, "\mathtt{w}_{10}"'] & A[n]_{\tilde{\eta}}
\end{tikzcd},$$

and

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{0,n} \arrow[r, "\mathsf{z}_{20}"] \arrow[d, "\varphi_{0,n}"', "\wr"] & \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \arrow[d, "\nu(\varphi_{-1,n})\circ\varphi_{-2,n}", "\wr"'] \\
(\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde{\eta}} \arrow[r, "\mathtt{w}_{20}"'] & T[n]_{\tilde{\eta}}
\end{tikzcd}.$$

Let us formulate this observation as follows:

Definition 5.2.7.12. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)}$. *Two level-n structure data* $\alpha_n^\natural = (\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$ *and* $(\alpha_n^\natural)' = (\mathsf{Z}'_n, \varphi'_{-2,n}, \varphi'_{-1,n}, \varphi'_{0,n}, \delta'_n, c'_n, (c_n^\vee)', \tau'_n)$ *of type* M_n *defined over* $\tilde{\eta}$ *as in Definition 5.2.7.9 are called* **equivalent** *if the following conditions are satisfied:*

1. *The following identifies hold:* $\mathsf{Z}_n = \mathsf{Z}'_n$, $\varphi_{-2,n} = \varphi'_{-2,n}$, $\varphi_{-1,n} = \varphi'_{-1,n}$, *and* $\varphi_{0,n} = \varphi'_{0,n}$.

2. *There is a liftable change of basis* $\mathsf{z}_n : \mathrm{Gr}^{\mathsf{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathsf{Z}}_n$, *given in matrix form by*

$$\mathsf{z}_n = \begin{pmatrix} 1 & \mathsf{z}_{21,n} & \mathsf{z}_{20,n} \\ & 1 & \mathsf{z}_{10,n} \\ & & 1 \end{pmatrix},$$

such that $\delta'_n = \delta_n \circ \mathsf{z}_n$.

3. *If we consider the maps*

$$d_n : \tfrac{1}{n}\underline{X} \to A^\vee[n]_{\tilde{\eta}}$$

$$d_n^\vee : \tfrac{1}{n}\underline{Y} \to A[n]_{\tilde{\eta}}$$
$$e_n : \tfrac{1}{n}\underline{Y}_{\tilde{\eta}} \to T[n]_{\tilde{\eta}}$$

defined respectively by the relations

$$\begin{aligned} \mathrm{e}_{A[n]}(a, d_n(\tfrac{1}{n}\chi)) &= \chi(\nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{21,n} \circ \varphi_{-1,n}^{-1}(a)) \\ d_n^\vee(\tfrac{1}{n}y) &= \varphi_{-1,n} \circ \mathsf{z}_{10,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y) \\ e_n(\tfrac{1}{n}y) &= \nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{20,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y) \end{aligned}$$

for any $a \in A[n]_{\tilde{\eta}}$, $\chi \in \underline{X}_{\tilde{\eta}}$, *and* $\frac{1}{n}y \in \frac{1}{n}\underline{Y}_{\tilde{\eta}}$, *then* $(d_n, d_n^\vee, e_n)$ *translates* $(c_n, c_n^\vee, \tau_n)$ *to* $(c_n', (c_n^\vee)', \tau_n')$ *in the sense of Definition 5.2.7.11. Note that here we are identifying* $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$ *implicitly as a map* $\mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} \frac{1}{n}Y/Y$ *by the canonical isomorphism* $Y/nY \xrightarrow{\sim} \frac{1}{n}Y/Y$.

Definition 5.2.7.13. *With the setting as in Section 5.2.1, the category* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}$ *has objects of the form* $(G, \lambda, i, \alpha_n)$ *(over* S*), where:*

1. (G, λ, i) *defines an object in* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$.
2. $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *is a level-*n *structure of* $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ *as in Definition 1.3.6.1.*

When $\tilde{\eta} = \eta$, *we shall denote* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\eta}$ *by simply* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}$.

Definition 5.2.7.14. *With the setting as in Section 5.2.1, the category* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}$ *has objects of the form* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$, *where:*

1. $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *defines an object in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$.
2. *The extra entry* $[\alpha_n^\natural]$ *is an equivalence class of level-*n *structure data* $\alpha_n^\natural$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ *defined over* $\tilde{\eta}$. *(See Definitions 5.2.7.9 and Definition 5.2.7.12.)*

When $\tilde{\eta} = \eta$, *we shall denote* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}$ *by simply* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$.

We can now replace Proposition 5.2.7.10 by the following theorem, which is our main result of Section 5.2:

Theorem 5.2.7.15. *There is an equivalence of categories*

$$\begin{aligned} \mathrm{M}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}} : \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}} &\to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}} \\ (A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural]) &\mapsto (G, \lambda, i, \alpha_n). \end{aligned}$$

When $\tilde{\eta} = \eta$, *we shall denote* $\mathrm{M}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}$ *by simply* $\mathrm{M}_{\mathrm{PEL},\mathsf{M}_n}$.

5.3 Data for General PEL-Structures

Let us continue with the same setting as in Section 5.2 in this section.

5.3.1 Formation of Étale Orbits and Main Result

With the setting as in Section 5.2.1, let (G, λ, i) be a triple defining an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$, which is associated to a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$ by the equivalence $\mathrm{M}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$ in Theorem 5.1.2.5.

Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, and let $n \geq 1$ be an integer such that $\Box \nmid n$ and $\mathcal{U}^\Box(n) \subset \mathcal{H}$. Let $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^\Box(n)$. By definition, an integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ of $(G_\eta, \lambda_\eta, i_\eta)$ is given by an $\mathcal{H}_n$-orbit $\alpha_{\mathcal{H}_n}$ of étale-locally-defined level-n structures. In other words, there exists an étale morphism $\tilde{\eta} \to \eta$ (which we may assume to be defined by a field extension as in Setting 5.2.1) such that the pullback of $\mathcal{H}_n$ to $\tilde{\eta}$ is the disjoint union of elements in the $\mathcal{H}_n$-orbit of some level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ defined over $\tilde{\eta}$. According to Theorem 5.2.7.15, α_n is associated to an equivalence class $[\alpha_n^\natural]$ of level-n structure data $\alpha_n^\natural = (\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$. Then the $\mathcal{H}_n$-orbit of α_n is naturally associated to the $\mathcal{H}_n$-orbit of $\alpha_n^\natural$, if we can explain how the action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of α_n's (defined by $\alpha_n \mapsto \alpha_n \circ g_n$) is translated into an action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of $\alpha_n^\natural$'s. Even better, we would like to work out a direct way to define an action of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ on the set of $\hat{\alpha}^\natural$'s, which induces the action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of $\alpha_n^\natural$'s by reduction mod n. This is the goal of this section.

Let us fix a choice of α_n that corresponds to $\alpha_n^\natural$ as above. Let $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box \xrightarrow{\sim} \mathrm{T}^\Box G_{\bar{\eta}}$ be any symplectic isomorphism lifting α_n. Let $\hat{\alpha}^\natural = (\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat{\delta}, \hat{c}, \hat{c}^\vee, \hat{\tau})$ be any element lifting $\alpha_n^\natural$ (as a reduction mod n). By construction of $\alpha_n^\natural$, we may arrange that $\mathtt{Z}$ is the pullback of $\mathtt{W}$ under $\hat{\alpha}$, and that φ_{-2}, φ_{-1}, and φ_0 are determined uniquely by $\mathrm{Gr}(\hat{\alpha})$. There is a freedom of making the choice of the splitting $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$, but then the pairing $\langle\cdot,\cdot\rangle$ and $\hat{\alpha}$ forces a unique choice of $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ for each particular choice of δ.

Let g_n be any element of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$, and let g be any element of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ lifting g_n. If we replace $\hat{\alpha}$ by $\hat{\alpha} \circ g$, then the relation $\mathtt{W} = \hat{\alpha}(\mathtt{Z})$ is replaced by $\mathtt{W} = (\hat{\alpha} \circ g)(g^{-1}(\mathtt{Z}))$. Hence we see that we should replace $\mathtt{Z}$ by $\mathtt{Z}' := g^{-1}(\mathtt{Z})$, which is related to $\mathtt{Z}$ by the induced maps $g : \mathtt{Z}' = g^{-1}(\mathtt{Z}_{-i}) \xrightarrow{\sim} \mathtt{Z}_{-i}$ for $i = 0, 1, 2$. This determines the maps $\mathrm{Gr}_{-i}(g) : \mathrm{Gr}^{\mathtt{Z}'}_{-i} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}_{-i}$ on the graded pieces, and suggests accordingly:

1. We shall replace $\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ by $\varphi'_0 := \varphi_0 \circ \mathrm{Gr}_0(g)$.

2. We shall replace $\varphi_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^{\Box}\, A_{\bar{\eta}}$ by $\varphi'_{-1} := \varphi_{-1} \circ \mathrm{Gr}_{-1}(g)$, and accordingly $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^{\Box}(1) \xrightarrow{\sim} \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$ by $\nu(\varphi'_{-1}) = \nu(\varphi_{-1}) \circ \nu(g)$.

3. We shall replace $\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ by $\varphi'_{-2} := \nu(g)^{-1} \circ \varphi_{-2} \circ \mathrm{Gr}_{-2}(g)$. This is because (when $X \neq \{0\}$ and hence $L \neq \{0\}$) the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_0 \arrow[r, "\mathrm{Gr}_{-2}(\hat{\alpha}) \times \mathrm{Gr}_0(\hat{\alpha})"] & \mathrm{T}^{\Box}\, T_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \arrow[r, "\mathrm{e}^{\phi}"] \arrow[d, equal] & \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}} \arrow[d, equal] \\
\mathrm{Gr}^{\mathtt{Z}'}_{-2} \times \mathrm{Gr}^{\mathtt{Z}'}_0 \arrow[u, "\mathrm{Gr}_{-2}(g) \times \mathrm{Gr}_0(g)"', "\wr"] \arrow[r, "\mathrm{Gr}_{-2}(\hat{\alpha}\circ g) \times \mathrm{Gr}_0(\hat{\alpha}\circ g)"] & \mathrm{T}^{\Box}\, T_{\bar{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \arrow[r, "\mathrm{e}^{\phi}"] & \mathrm{T}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{\eta}}
\end{tikzcd}$$

forces the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_0 \arrow[r, "\varphi_{-2} \times \varphi_0"] & \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \arrow[r, "\mathrm{can.}"] & \hat{\mathbb{Z}}^{\Box}(1) \quad , \\
\mathrm{Gr}^{\mathtt{Z}'}_{-2} \times \mathrm{Gr}^{\mathtt{Z}'}_0 \arrow[u, "\mathrm{Gr}_{-2}(g) \times \mathrm{Gr}_0(g)"', "\wr"] \arrow[r, "\varphi'_{-2} \times \varphi'_0"] & \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \arrow[u, "\nu(g) \times \mathrm{Id}"', "\wr"] \arrow[r, "\mathrm{can.}"] & \hat{\mathbb{Z}}^{\Box}(1) \arrow[u, "\nu(g)"', "\wr"]
\end{tikzcd}$$

   where the canonical pairing $\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1)$ is the composition of $\mathrm{Id} \times \phi$ with the canonical pairing $\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1)$.

4. Then $\hat{f}_{-2} := \nu(\varphi_{-1}) \circ \varphi_{-2}$, $\hat{f}_{-1} := \varphi_{-1}$, and $\hat{f}_0 := \varphi_0$ are replaced by respectively by $\hat{f}'_{-2} := \nu(f'_{-2}) \circ \varphi'_{-2} = \nu(f_{-2}) \circ \nu(g) \circ \varphi_{-2} \circ \mathrm{Gr}_{-2}(g) = \hat{f}_{-2} \circ (\nu(g) \circ \mathrm{Gr}_{-2}(g))$, $\hat{f}'_{-1} = \hat{f}_{-1} \circ \mathrm{Gr}_{-1}(g)$, and $\hat{f}'_0 = \hat{f}_0 \circ \mathrm{Gr}_0(g)$. This replaces the symplectic isomorphism $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ by the symplectic isomorphism $\hat{f}' : \mathrm{Gr}^{\mathtt{Z}'} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$, so that $\hat{f}' = \hat{f} \circ \mathrm{Gr}(g)$, with $\nu(\hat{(f)})$ replaced by $\nu(\hat{f}') \circ \nu(g)$.

5. Let $\hat{\varsigma} : \mathtt{W} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$ be the splitting determined by $(\hat{c}, \hat{c}^{\vee}, \hat{\tau})$. Then the relation $\hat{\alpha} = \varsigma \circ \hat{f} \circ \hat{\delta}^{-1}$ can be rewritten as $\hat{\alpha} \circ g = \hat{\varsigma} \circ \hat{f}' \circ (g^{-1} \circ \hat{\delta} \circ \mathrm{Gr}(g))^{-1}$. This suggests that, if we take $\hat{\delta}' := g^{-1} \circ \hat{\delta} \circ \mathrm{Gr}(g)$ as one of the possible ways to modifying it (as it is not canonical), then we may set $\hat{\varsigma}' := \hat{\varsigma}$ and retain the relation $\hat{\alpha} \circ g = \hat{\varsigma}' \circ \hat{f}' \circ (\hat{\delta}')^{-1}$. (Note that $\nu(\hat{\delta})$ and $\nu(\hat{\delta}')$ are both the identity because $\nu(g^{-1})$ and $\nu(\mathrm{Gr}(g))$ *canceled* each other.)

Let us summarize the result as follows:

Proposition 5.3.1.1. *There is a natural action of* $\mathrm{G}(\hat{\mathbb{Z}}^{\square})$ *on the tuples of the form* $\hat{\alpha}^{\natural} = (\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat{\delta}, \hat{c}, \hat{c}^{\vee}, \hat{\tau})$*, defined for each* $g \in \mathrm{G}(\hat{\mathbb{Z}}^{\square})$ *by sending* $\hat{\alpha}^{\natural}$ *as above to*

$$(\mathtt{Z}', \varphi'_{-2}, \varphi'_{-1}, \varphi'_0, \hat{\delta}', \hat{c}, \hat{c}^{\vee}, \hat{\tau})$$

where:

1. $\mathtt{Z}' := g^{-1}(\mathtt{Z})$.

2. $\varphi'_{-2} := \nu(g) \circ \varphi_{-2} \circ \mathrm{Gr}_{-2}(g)$.

3. $\varphi'_{-1} := \varphi_{-1} \circ \mathrm{Gr}_{-1}(g)$ *and accordingly* $\nu(\varphi'_{-1}) := \nu(\varphi_{-1}) \circ \nu(g)^{-1}$.

4. $\varphi'_0 := \varphi_0 \circ \mathrm{Gr}_0(g)$.

5. $\hat{\delta}' := g^{-1} \circ \hat{\delta} \circ \mathrm{Gr}(g)$.

6. $(\hat{c}, \hat{c}^{\vee}, \hat{\tau})$ *is unchanged.*

Proposition 5.3.1.2. *By taking reduction mod* n *of the action defined in Proposition 5.3.1.1, we obtain an action of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ *on the level-*n *structure data of the form* $\alpha_n^{\natural} = (\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^{\vee}, \tau_n)$*, defined for each* $g_n \in \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ *by sending* $\alpha_n^{\natural}$ *as above to*

$$(\mathtt{Z}'_n, \varphi'_{-2,n}, \varphi'_{-1,n}, \varphi'_{0,n}, \delta'_n, c_n, c_n^{\vee}, \tau_n),$$

where:

1. $\mathtt{Z}'_n := g_n^{-1}(\mathtt{Z}_n)$.

2. $\varphi'_{-2,n} := \nu(g_n) \circ \varphi_{-2,n} \circ \mathrm{Gr}_{-2}(g_n)$.

3. *$\varphi'_{-1,n} := \varphi_{-1,n} \circ \mathrm{Gr}_{-1,n}(g_n)$ and accordingly $\nu(\varphi'_{-1,n}) := \nu(\varphi_{-1,n}) \circ \nu(g_n)^{-1}$.*

4. *$\varphi'_{0,n} := \varphi_{0,n} \circ \mathrm{Gr}_{0,n}(g_n)$.*

5. *$\delta'_n := g_n^{-1} \circ \delta_n \circ \mathrm{Gr}_n(g_n)$.*

6. *$(c_n, c_n^\vee, \tau_n)$ is unchanged.*

This action respects the equivalence relation between level-n structure data, and hence induces an action on the equivalence classes $[\alpha_n^\natural]$ as well.

Proof. Everything is clear except the last statement. The last statement is essentially a tautology, because the equivalence classes of level-n structure data correspond bijectively to level-n structures, and the action is defined by the action on level-n structures. □

Nevertheless, it is instructive to explain explicitly why the last statement is true:

By Definition 5.2.7.12, two level-n structure data are equivalent only if one of them is of the form $\alpha_n^\natural = (\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$ and the other is of the form $(\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n \circ \mathsf{z}_n, c'_n, (c_n^\vee)', \tau'_n)$, where $\mathsf{z}_n : \mathrm{Gr}_n^{\mathsf{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathsf{Z}}$ is a liftable change of basis defining a triple $(d_n, d_n^\vee, e_n)$ that translates $(c_n, c_n^\vee, \tau_n)$ to $(c'_n, (c_n^\vee)', \tau'_n)$ in the sense of Definition 5.2.7.11. Let us write z_n in matrix form as $\mathsf{z}_n = \begin{pmatrix} 1 & \mathsf{z}_{21,n} & \mathsf{z}_{20,n} \\ & 1 & \mathsf{z}_{10,n} \\ & & 1 \end{pmatrix}$, where each $\mathsf{z}_{ij,n}$ is a map $\mathrm{Gr}_{-j,n}^{\mathsf{Z}} \to \mathrm{Gr}_{-i,n}^{\mathsf{Z}}$. Then $(d_n, d_n^\vee, e_n)$ is defined by z_n by the relations

$$\begin{aligned} \mathrm{e}_{A[n]}(a, d_n(\tfrac{1}{n}\chi)) &= \chi(\nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{21,n} \circ \varphi_{-1,n}^{-1}(a)) \\ d_n^\vee(\tfrac{1}{n}y) &= \varphi_{-1,n} \circ \mathsf{z}_{10,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y) \\ e_n(\tfrac{1}{n}y) &= \nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{20,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y) \end{aligned}$$

for any $a \in A[n]_{\tilde{\eta}}$, $\chi \in \underline{X}_{\tilde{\eta}}$, and $\frac{1}{n}y \in \frac{1}{n}\underline{Y}_{\tilde{\eta}}$.

The action of g_n sends δ_n to $\delta'_n := g_n^{-1} \circ \delta_n \circ \mathrm{Gr}_n(g_n)$, and sends $\delta_n \circ \mathsf{z}_n$ to $g_n^{-1} \circ (\delta_n \circ \mathsf{z}_n) \circ \mathrm{Gr}_n(g_n) = \delta'_n \circ \mathsf{z}'_n$ if we define $\mathsf{z}'_n := \mathrm{Gr}_n(g_n)^{-1} \circ \mathsf{z}_n \circ \mathrm{Gr}_n(g_n)$. By definition, this means z'_n has matrix form as $\mathsf{z}'_n = \begin{pmatrix} 1 & \mathsf{z}'_{21,n} & \mathsf{z}'_{20,n} \\ & 1 & \mathsf{z}'_{10,n} \\ & & 1 \end{pmatrix}$, where each $\mathsf{z}'_{ij,n} : \mathrm{Gr}_{-j,n}^{\mathsf{Z}'} \to \mathrm{Gr}_{-i,n}^{\mathsf{Z}'}$ is related to $\mathsf{z}_{ij,n}$ by $\mathsf{z}'_{ij,n} = \mathrm{Gr}_{-i,n}(g_n)^{-1} \circ \mathsf{z}_{ij,n} \circ \mathrm{Gr}_{-j,n}(g_n)$. Then we have the relations

$$\nu(\varphi'_{-1,n}) \circ \varphi'_{-2,n} \circ \mathsf{z}'_{21,n} \circ (\varphi'_{-1,n})^{-1} = \nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{21,n} \circ \varphi_{-1,n}^{-1}$$

$$\begin{aligned} \varphi'_{-1,n} \circ \mathsf{z}'_{10,n} \circ (\varphi'_{0,n})^{-1} &= \varphi_{-1,n} \circ \mathsf{z}_{10,n} \circ \varphi_{0,n}^{-1} \\ \nu(\varphi'_{-1,n}) \circ \varphi'_{-2,n} \circ \mathsf{z}'_{20,n} \circ (\varphi'_{0,n})^{-1} &= \nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathsf{z}_{20,n} \circ \varphi_{0,n}^{-1}, \end{aligned}$$

which show that the triple $(d_n, d_n^\vee, e_n)$ is unchanged under the action g_n, or equivalently that the equivalence relation is respected under the action g_n, as desired.

Now let $\mathcal{H}_n$ be a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ as above, which is $\mathcal{H}_n = \mathcal{H}/\mathcal{U}^\square(n)$ for some open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$. We would like to find the correct formulation of an $\mathcal{H}_n$-orbit of $[\alpha_n^\natural]$ over $\tilde{\eta}$, so that we can *descend* the orbit to some similar object over η.

Definition 5.3.1.3. *Let Z_n be a fully symplectic-liftable filtration of L/nL. Then we define the following subgroups of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$:*

$$\begin{aligned}
\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} &:= \{g_n \in \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z}) : g_n^{-1}(\mathsf{Z}_n) = \mathsf{Z}_n\}, \\
\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n} &:= \{g_n \in \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} : \mathrm{Gr}_{-1,n}(g_n) = \mathrm{Id}_{\mathrm{Gr}^{\mathsf{Z}}_{-1,n}} \text{ and } \nu(g_n) = 1\}, \\
\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} &:= \{g_n \in \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} : \mathrm{Gr}_n(g_n) = \mathrm{Id}_{\mathrm{Gr}^{\mathsf{Z}}_n}\}, \\
\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} &:= \left\{\begin{matrix} (r_n, g_{-1,n}) \in \mathbf{G}_{\mathrm{m}}(\mathbb{Z}/n\mathbb{Z}) \times \mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{-1,n}) : \\ \exists g_n \in \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} \text{ s.t. } \nu(g_n) = r_n \text{ and } \mathrm{Gr}_{-1,n}(g_n) = g_{-1,n} \end{matrix}\right\}, \\
\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n} &:= \left\{\begin{matrix} (g_{-2,n}, g_{0,n}) \in \mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{-2,n}) \times \mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{0,n}) : \\ \exists g_n \in \mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n} \text{ s.t. } \mathrm{Gr}_{-2,n}(g_n) = g_{-2,n} \text{ and } \mathrm{Gr}_{0,n}(g_n) = g_{0,n} \end{matrix}\right\}, \\
\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n} &:= \left\{\begin{matrix} g_{20,n} \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{0,n}, \mathrm{Gr}^{\mathsf{Z}}_{-2,n}) : \\ \exists g_n \in \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} \text{ s.t. } \delta_n^{-1} \circ g_n \circ \delta_n = \left(\begin{smallmatrix} 1 & & g_{20,n} \\ & 1 & \\ & & 1 \end{smallmatrix}\right) \end{matrix}\right\}, \\
\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n} &:= \left\{\begin{matrix} (g_{21,n}, g_{10,n}) \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{0,n}, \mathrm{Gr}^{\mathsf{Z}}_{-2,n}) \times \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{0,n}, \mathrm{Gr}^{\mathsf{Z}}_{-2,n}) : \\ \exists g_n \in \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} \text{ s.t. } \delta_n^{-1} \circ g_n \circ \delta_n = \left(\begin{smallmatrix} 1 & g_{21,n} & g_{20,n} \\ & 1 & g_{10,n} \\ & & 1 \end{smallmatrix}\right) \text{ for some } g_{20,n} \end{matrix}\right\}.
\end{aligned}$$

Note that the condition $\nu(g_n) = 1$ in the definition of $\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}$ is automatic if we interpret $\mathrm{Gr}_{-1,n}(g_n) = \mathrm{Id}_{\mathrm{Gr}^{\mathsf{Z}}_{-1,n}}$ as an identity of symplectic isomorphisms (defined as in Definition 1.1.4.11), which means implicitly that $\nu(\mathrm{Gr}_{-1,n}(g_n)) := \nu(g_n) = 1$.

Lemma 5.3.1.4. *By definition, there are natural inclusions*

$$\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n} \subset \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} \subset \mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n} \subset \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} \subset \mathrm{G}^{\mathrm{ess}}_{\mathsf{Z}_n}, \tag{5.3.1.5}$$

and natural exact sequences:

$$1 \to \mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n} \to \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n} \to \mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \to 1, \tag{5.3.1.6}$$

$$1 \to \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} \to \mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n} \to \mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n} \to 1, \tag{5.3.1.7}$$

$$1 \to \mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n} \to \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n} \to \mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n} \to 1. \tag{5.3.1.8}$$

Definition 5.3.1.9. *Let $\mathcal{H}_n$ be a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ as above. For any of the subgroups $*$ in (5.3.1.5), we define $\mathcal{H}_{n,*} := \mathcal{H}_n \cap *$. For any of the quotient of two groups $* = *_1/*_2$ in (5.3.1.5), (5.3.1.6), (5.3.1.7), or (5.3.1.8), we define $\mathcal{H}_{n,*} := (\mathcal{H}_n \cap *_1)/(\mathcal{H}_n \cap *_2)$. Thus we have defined the following groups: $\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$, and $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$, so that we have the natural inclusions*

$$\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_n$$

and natural exact sequences

$$1 \to \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \to 1,$$
$$1 \to \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}} \to 1,$$
$$1 \to \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \to 1.$$

Back to the question of the description of $\mathcal{H}_n$-orbits of $[\alpha_n^\natural]$. Note that the filtration Z_n in each equivalence class $[\alpha_n^\natural]$ is independent of the representative $\alpha_n^\natural$ we take for $[\alpha_n^\natural]$.

Given any particular fully symplectic-liftable filtration Z_n of L/nL, the elements in its $\mathcal{H}_n$-orbit can be parameterized by the left cosets $\mathcal{H}_{n,\mathrm{P}_{\mathsf{Z}_n}} \backslash \mathcal{H}_n$. Let us denote the $\mathcal{H}_n$-orbit of Z_n by $\mathsf{Z}_{\mathcal{H}_n}$. We would like to view this as a *constant scheme* over η.

Let us now work over $\mathsf{Z}_{\mathcal{H}_n}$. Over each η-point of $\mathsf{Z}_{\mathcal{H}_n}$, we have a representative Z_n in the orbit $\mathsf{Z}_{\mathcal{H}_n}$. Let us investigate the $\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbits of the remaining objects. The next natural object to study is the symplectic isomorphism $\varphi_{-1,n} : \mathrm{Gr}^{\mathsf{Z}}_{-1,n} \xrightarrow{\sim} \mathrm{T}^{\square} A_{\bar{\eta}}$, which corresponds to the group $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \cong \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}$. The $\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbit of $\varphi_{-1,n}$ is simply the $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$-orbit of $\varphi_{-1,n}$, as the action is realized by $\mathrm{Gr}^{\mathsf{Z}}_{-1,n}(g_n)$ for each $g_n \in \mathrm{P}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. As a result, we see that what we need is simply a level-$\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$-structure of $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ of type $(\mathrm{Gr}^{\mathsf{Z}}_{-1,n}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$, which we shall denote as $\varphi_{-1,\mathcal{H}_n}$. Note that $\varphi_{-1,\mathcal{H}_n}$ is defined over $\mathsf{Z}_{\mathcal{H}_n}$. We shall interpret it as a subscheme of the pullback of $\underline{\mathrm{Isom}}_\eta((L/nL)_\eta, A[n]_\eta) \underset{\eta}{\times} \underline{\mathrm{Isom}}_\eta(((\mathbb{Z}/n\mathbb{Z})(1))_\eta, \boldsymbol{\mu}_{n,\eta})$ to $\mathsf{Z}_{\mathcal{H}_n}$. Then $\varphi_{-1,\mathcal{H}_n} \twoheadrightarrow \eta$ is finite étale.

Over $\varphi_{-1,\mathcal{H}_n}$, we have étale locally over each point a choice of some $(\mathsf{Z}_n, \varphi_{-1,n})$ in its $\mathcal{H}_n$-orbit. The next natural object to study is the

$\mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbit of $(\varphi_{-2,n}, \varphi_{0,n})$. The natural group to consider is the subgroup $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$ of $\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n} \cong \mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}$. Let us denote by the $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$-orbit of a particular pair $(\varphi_{-2,n}, \varphi_{0,n})$ by $(\varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$. Although the notation of using a pair is misleading, we shall interpret it as a subscheme of the pullback of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{-2,n}, \underline{\mathrm{Hom}}_{\eta}((\underline{X}/n\underline{X})_{\eta}, ((\mathbb{Z}/n\mathbb{Z})(1))_{\eta})) \underset{\eta}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{Z}}_{0,n}, (\underline{Y}/n\underline{Y})_{\eta})$ to $\varphi_{-1,\mathcal{H}_n}$. Then $(\varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n}) \twoheadrightarrow \eta$ is finite étale.

Over $(\varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$, we have étale locally over each point a choice of some $(\mathsf{Z}_{\mathcal{H}_n}, \varphi_{-2,\mathcal{H}_n}, \varphi_{-1,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$ in its $\mathcal{H}_n$-orbit. Let us study the $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbit of the remaining objects $(\delta_n, c_n, c_n^{\vee}, \tau_n)$ *up to equivalence.*

If we use the action that we have defined so far on the representatives, then the action is realized on δ_n, but the objects $(c_n, c_n^{\vee}, \tau_n)$ over $\tilde{\eta}$ are never changed under the actions. This is not pertinent for our purpose because we want to have descended forms of them over η. On the other hand, if we allow ourselves to take equivalent objects, then we may modify the action of $\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}$ on the representatives as follows: The elements $g_n \in \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}$ satisfy $\mathrm{Gr}_n(g_n) = \mathrm{Id}_{\mathrm{Gr}^{\mathsf{Z}}_n}$ by definition. Hence the g_n-action sends $(\delta_n, c_n, c_n^{\vee}, \tau_n)$ to $(g_n^{-1} \circ \delta_n, c_n, c_n^{\vee}, \tau_n)$. Suppose $\delta_n^{-1} \circ g_n \circ \delta_n = \left(\begin{smallmatrix} 1 & g_{21,n} & g_{20,n} \\ & 1 & g_{10,n} \\ & & 1 \end{smallmatrix}\right)$. Let us write $g_n^{-1} \circ \delta_n = \delta_n \circ (\delta_n^{-1} \circ g_n^{-1} \circ \delta_n)$, where $\delta_n^{-1} \circ g_n^{-1} \circ \delta_n$ is now viewed as a change of basis. Then we know that d_n, $d_n^{\vee}$, and that e_n) are determined by explicit formulae by respectively $g_{21,n}$, $g_{10,n}$, and $g_{20,n}$. In particular, $(g_n^{-1} \circ \delta_n, c_n, c_n^{\vee}, \tau_n)$ is equivalent to $(\delta_n, c_n', (c_n^{\vee})', \tau_n')$, where $(c_n', (c_n^{\vee})', \tau_n')$ is the translation of $(c_n, c_n^{\vee}, \tau_n)$ by the $(d_n, d_n^{\vee}, e_n)$ determined by $\delta_n^{-1} \circ g_n^{-1} \circ \delta_n$.

By the explicit formulae, the modified action of $\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}$ on $(c_n, c_n^{\vee})$ factors through $\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n} = \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}$. Hence it makes sense to form $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$-orbits of $(c_n, c_n^{\vee})$ and denote it as $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n})$. Again, although the notation of using a pair is misleading, we shall may represent it as a subscheme of the pullback of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{X}_{\eta}, A^{\vee}_{\eta}) \times \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{Y}_{\eta}, A_{\eta})$ to $(\varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$. (Certainly, such an ambient space is not the parameter space of all pairs that we allow. A more precise construction will be given in Section 6.2.3.) Then $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n}) \twoheadrightarrow \eta$ is finite étale.

Over $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n})$, we have étale locally over each point a choice of some representative $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n})$ in its $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$-orbit. Then it only remains to understand the action of $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ on τ_n. Let us denote by $\tau_{n,\mathcal{H}_n}$ the $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$-orbit of τ_n, and denote by $\iota_{n,\mathcal{H}_n}$ the $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$-orbit of ι_n. Similar to the case of $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n})$ above, we shall represent it as a subscheme of the pullback of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{Y}_{\eta}, G^{\natural}_{\eta})$ to $(c_{\mathcal{H}_n}, c^{\vee}_{\mathcal{H}_n})$, although this ambient space is not the precise parameter space we wanted. Then $\tau_{n,\mathcal{H}_n} \twoheadrightarrow \eta$ is finite étale.

Finally, over $\tau_{n,\mathcal{H}_n}$, we have étale locally over each point a choice of all the data, including the unique choice of δ_n, which we shall denote by $\delta_{\mathcal{H}_n}$. If we replace δ_n by $\delta_n \circ \mathbf{z}_n$ before we construct $(c_{\mathcal{H}_n}, c^\vee_{\mathcal{H}_n}, \tau_{\mathcal{H}_n})$, then we shall replace accordingly the whole orbit $(c_{\mathcal{H}_n}, c^\vee_{\mathcal{H}_n}, \tau_{\mathcal{H}_n})$ by another triple $(c'_{\mathcal{H}_n}, (c^\vee_{\mathcal{H}_n})', \tau'_{\mathcal{H}_n})$ (which is another scheme finite étale over η). (Therefore the notion of equivalences carries over naturally to the context of $\mathcal{H}_n$-orbits.)

To facilitate the language, we shall often ignore the fact that the above objects are schemes finite étale over η, and denote them as tuples as if they are discrete objects:

Definition 5.3.1.10. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle)}$. *Let $n \geq 1$ be an integer such that $\Box \nmid n$. Let $\mathcal{H}_n$ be a subgroup of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. *By an* **$\mathcal{H}_n$-orbit of étale-locally-defined level-n structure data***, we mean a scheme*

$$\alpha^\natural_{\mathcal{H}_n} = (\mathrm{Z}_{\mathcal{H}_n}, \varphi_{-2,\mathcal{H}_n}, \varphi_{-1,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n}, \delta_{\mathcal{H}_n}, c_{\mathcal{H}_n}, c^\vee_{\mathcal{H}_n}, \tau_{\mathcal{H}_n})$$

(*or rather just* $\tau_{\mathcal{H}_n}$) *finite étale over η, which is étale locally* (*over η*) *the disjoint union of elements in some $\mathcal{H}_n$-orbit of level-n structure data* (*as in Definition 5.2.7.9*). *We use the same terminology* **$\mathcal{H}_n$-orbit of étale-locally-defined** *for each of the entries in* $\alpha^\natural_{\mathcal{H}_n}$.

Definition 5.3.1.11. *With the setting as in Definition 5.3.1.10, we say that two $\mathcal{H}_n$-orbits $\alpha^\natural_{\mathcal{H}_n}$ and $(\alpha^\natural_{\mathcal{H}_n})'$ are equivalent if there is a finite étale extension $\tilde{\eta} \to \eta$ over which $\alpha^\natural_{\mathcal{H}_n}$ contains some level-n structure datum that is equivalent* (*as in Definition 5.2.7.12*) *to some level-n structure datum in* $(\alpha^\natural_{\mathcal{H}_n})'$.

Definition 5.3.1.12. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle)}$. *Let $\mathcal{H}$ be any open compact subgroup of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$. *For any integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^\Box(n) \subset \mathcal{H}$, set $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^\Box(n)$ as always. Then a* **level-$\mathcal{U}$ structure datum of type** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot, \cdot \rangle)$ *over η is a collection $\alpha^\natural_{\mathcal{H}} = \{\alpha^\natural_{\mathcal{H}_n}\}$ labeled by integers $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^\Box(n) \subset \mathcal{H}$, with elements $\alpha^\natural_{\mathcal{H}_n}$ described as follows:*

1. *For any n in the index set, $\alpha^\natural_{\mathcal{H}_n}$ is an $\mathcal{H}_n$-orbit of étale-locally-defined level-n structure data as in Definition 5.3.1.10.*

2. *For any $n|m$ in the index set, the $\mathcal{H}_n$-orbit $\alpha^\natural_{\mathcal{H}_n}$ is determined by the $\mathcal{H}_m$-orbit $\alpha^\natural_{\mathcal{H}_m}$ by reduction mod n.*

It is customary to denote $\alpha^\natural_{\mathcal{H}}$ as a tuple

$$\alpha^\natural_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}}, \varphi_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^\vee_{\mathcal{H}}, \tau_{\mathcal{H}}),$$

each entry being a collection labeled by n as $\alpha^\natural_{\mathcal{H}}$ is, and use $\iota_{\mathcal{H}}$ to denote the collection corresponding to $\tau_{\mathcal{H}}$.

Convention 5.3.1.13. *To facilitate the language, we shall call $\alpha^\natural_{\mathcal{H}}$ a $\mathcal{H}$-orbit, with similar usages applied to other objects with subscript $\mathcal{H}$. If we have two open compact subgroups $\mathcal{H}' \subset \mathcal{H}$, and if we have an object $\alpha^\natural_{\mathcal{H}'}$ at level $\mathcal{H}'$, then there is a natural meaning of the object $\alpha^\natural_{\mathcal{H}}$ at level $\mathcal{H}$ determined by $\alpha^\natural_{\mathcal{H}'}$. We say in this case that $\alpha^\natural_{\mathcal{H}}$ is the $\mathcal{H}$-orbit of $\alpha^\natural_{\mathcal{H}'}$.*

Definition 5.3.1.14. *With the setting as in Definition 5.3.1.12, we say that two level-$\mathcal{U}$ structure data $\alpha^\natural_{\mathcal{H}} = \{\alpha^\natural_{\mathcal{H}_n}\}$ and $(\alpha^\natural_{\mathcal{H}})' = \{(\alpha^\natural_{\mathcal{H}_n})'\}$ are equivalent if there is a some n such that $\alpha^\natural_{\mathcal{H}_n}$ and $(\alpha^\natural_{\mathcal{H}_n})'$ are equivalent as in Definition 5.3.1.11.*

Definition 5.3.1.15. *With the setting as in Section 5.2.1, the category* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *has objects of the form $(G, \lambda, i, \alpha_{\mathcal{H}})$ (over S), where:*

1. *(G, λ, i) defines an object in* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$.
2. *$\alpha_{\mathcal{H}}$ is a level-$\mathcal{H}$ structure of $(G_\eta, \lambda_\eta, i_\eta)$ of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ as in Definition 1.3.7.8.*

Definition 5.3.1.16. *With the setting as in Section 5.2.1, the category* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *has objects of the form $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha^\natural_{\mathcal{H}}])$, where:*

1. *$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ defines an object in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle)}$.
2. *The extra entry $[\alpha^\natural_{\mathcal{H}}]$ is an equivalence class of level-$\mathcal{H}$ structure data $\alpha^\natural_{\mathcal{H}}$ of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ defined over η. (See Definitions 5.3.1.12 and 5.3.1.14.)*

Then we have the following consequence of Theorem 5.2.7.15:

Theorem 5.3.1.17. *There is an equivalence of categories*

$$\begin{aligned} \mathrm{M}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}} : \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}} &\to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}} \\ (A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha^\natural_{\mathcal{H}}]) &\mapsto (G, \lambda, i, \alpha_{\mathcal{H}}). \end{aligned}$$

5.3.2 Degenerating Families

For ease of later exposition, let us make the following definitions:

Definition 5.3.2.1. *Let S be a normal locally noetherian algebraic stack. A tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over S is called a* **degenerating family of type** $\mathsf{M}_{\mathcal{H}}$*, or simply a* **degenerating family***, if there exists a dense sub-algebraic stack S_1 of S, S_1 being defined over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*, such that:*

1. *G is a (relative) semi-abelian scheme over S whose restriction G_{S_1} to S_1 is an abelian scheme. In this case, the dual semi-abelian scheme $G^\vee$ exists (up to unique isomorphism), whose restriction $G^\vee_{S_1}$ to S_1 is the dual abelian scheme of G_{S_1}.*
2. *$\lambda : G \overset{\sim}{\to} G^\vee$ is a group homomorphism that induces by restriction a prime-to-$\Box$ polarization λ_{S_1} of G_{S_1}.*
3. *$i : \mathcal{O} \to \mathrm{End}_S(G)$ is a map that defines by restriction an $\mathcal{O}$-structure $i_{S_1} : \mathcal{O} \to \mathrm{End}_{S_1}(G_{S_1})$ of (G_{S_1}, λ_{S_1}).*
4. *$\underline{\mathrm{Lie}}_{G_{S_1}/S_1}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i_{S_1} satisfies the determinantal condition in Definition 1.3.4.2 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$.*
5. *$\alpha_{\mathcal{H}}$ is a level-n structure for $(G_{S_1}, \lambda_{S_1}, i_{S_1})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as in Definition 1.3.7.8, defined over S_1.*

In other words, we require $(G_{S_1}, \lambda_{S_1}, i_{S_1}, \alpha_{\mathcal{H}}) \to S_1$ to define a tuple parameterized by the moduli problem $\mathsf{M}_{\mathcal{H}}$.

Remark 5.3.2.2. Conditions 2, 3, 4, 5 are closed conditions for structures on *abelian schemes* defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Hence the rather weak condition on S_1 in Definition 5.3.2.1 is justified if we notice that S_1 can always be replaced by the largest sub-algebraic stack of S (which is open dense in S) over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ such that G_{S_1} is an abelian scheme.

Definition 5.3.2.3. *With the setting as in Definition 5.3.2.1, suppose $n \geq 1$ is an integer prime-to-$\Box$. Then a tuple $(G, \lambda, i, \alpha_n)$ over S is called a* **degenerating family of type** M_n *if it satisfies the same definitions as in Definition 5.3.2.1 except that condition 5 in Definition 5.3.2.1 can be replaced by:*

5′. α_n is a principal level-n structure for $(G_{S_1}, \lambda_{S_1}, i_{S_1})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in Definition 1.3.6.1, defined over S_1.

Remark 5.3.2.4. Since a principal level-n structure α_n can be canonically as a level-$\mathcal{U}^\square(n)$ structure, the notion of a degenerating family of type M_n can be canonically identified with the notion of a degenerating family of type $\mathsf{M}_{\mathcal{U}^\square(n)}$.

Definition 5.3.2.5. *We define a tuple (G, λ, i) over S to be a* **degenerating family of type $\mathsf{M}_\mathcal{H}$ (resp. M_n) without level structures***, or still simply a* **degenerating family***, if G, λ, i satisfies conditions 1, 2, 3, and 4 as in Definition 5.3.2.1, without a level structure as described by condition 5 in the definition of $\mathsf{M}_\mathcal{H}$ (resp. M_n).*

5.3.3 Criterion for Properness

Here is an interesting consequence of Theorem 5.3.1.17:

Theorem 5.3.3.1. *Let M' be a separated algebraic stack of finite-type over an arbitrary locally noetherian algebraic stack S'. (In particular, we allow any residue characteristic to appear in S'.) Suppose there is an open dense sub-algebraic stack S'_1 of S' such that the following conditions are satisfied:*

1. *$\mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ is open dense in M'.*

2. *There exists a morphism $\mathsf{S}'_1 \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$ and a morphism $\mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1 \xrightarrow{f} \mathsf{M}_\mathcal{H} \underset{\mathsf{S}_0}{\times} \mathsf{S}'_1$ such that, for any complete discrete valuation ring V with algebraically closed residue field, a morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ defining a tuple $(G, \lambda, i, \alpha_\mathcal{H})$ by composition with f extends to a morphism $\xi : \mathrm{Spec}(V) \to \mathsf{M}'$ whenever the underlying abelian scheme G extends to an abelian scheme over $\mathrm{Spec}(V)$.*

3. *Any admissible filtration Z of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ that is fully symplectic with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ has multi-rank zero. (See Definitions 5.2.7.1 and 5.2.2.5.)*

Then $\mathsf{M}' \to \mathsf{S}'$ is proper.

Proof. To show that $\mathsf{M}' \to \mathsf{S}'$ is proper, we need to verify the valuative criterion for it. By condition 1 (and by Remark A.6.1.10), it suffices to show that, for any $\mathrm{Spec}(V) \to \mathsf{S}'$ where V is a complete discrete valuation ring V with algebraically closed residue field, any morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ extends to a morphism $\xi : \mathrm{Spec}(V) \to \mathsf{M}'$. By composition with the morphism f in condition 2, any morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ induces a morphism $f \circ \xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{S}'_1$ defining an object $(G, \lambda, i, \alpha_{\mathcal{H}})$ of $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(\mathrm{Frac}(V))$. By Theorem 3.3.2.4, we know that G extends to a semi-abelian scheme over $\mathrm{Spec}(V)$. By Proposition 3.3.1.7, both λ and i extends uniquely over $\mathrm{Spec}(V)$. Note that $\mathrm{Spec}(\mathrm{Frac}(V))$ is defined over S_0 via the morphism $\mathsf{S}'_1 \to \mathsf{S}_0$ in condition 2. Therefore it makes sense to say that we have a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(V)$ (defined as in Definition 5.3.2.1) extending $(G, \lambda, i, \alpha_{\mathcal{H}})$. Since every torus over the algebraically closed residue field of V is trivial, this degenerating family verifies the isotriviality condition in Definition 4.2.1.1 and defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$. By Theorem 5.3.1.17, this corresponds to an object of $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$. In particular, we obtain an $\mathcal{H}$-orbit of admissible filtrations Z of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ that are fully symplectic with respect to $(L, \langle \cdot , \cdot \rangle)$. By condition 3, the multi-rank of any such Z must be zero. Therefore, the torus part of the semi-abelian scheme extending G must be trivial, which means G extends to an abelian scheme over $\mathrm{Spec}(V)$. (See Remarks 3.3.1.5 and 3.3.1.6.) Then the result follows from condition 2, as desired. □

Remark 5.3.3.2. Condition 3 in Theorem 5.3.3.1 is satisfied, for example, when the signatures (p_τ, q_τ) of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ (defined as in Definition 1.2.5.1) satisfy $\min(p_\tau, q_\tau) = 0$ for at least one τ. In this case the result follows from the much simpler Theorem 5.1.2.5. More generally, it is satisfied if the group G^{ad}, namely the quotient of the group G (defined as in Definition 1.2.1.5) by its center, is anisotropic at some finite or infinite place of $\mathbb{Q}$.

Remark 5.3.3.3. Theorem 5.3.3.1 is applicable, for example, to proving the properness of moduli problems defining integral models of Shimura varieties with reasonably mild *bad reductions*.

5.4 Notion of Cusp Labels

5.4.1 Principal Cusp Labels

Definition 5.4.1.1. *With the setting as in Section 5.2.1, the category* $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ *has objects of the form*

$$(\mathsf{Z}_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n)),$$

where:

1. $\underline{X}$ *and* $\underline{Y}$ *are constant with values respectively* X *and* Y*.*
2. *If we set*

$$\begin{aligned} \alpha_n^\natural &:= (\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n), \\ c &:= \text{unique extension of } c_n|_{\underline{X}} \text{ (by Corollary 5.2.3.10)}, \\ c^\vee &:= \text{unique extension of } c_n^\vee|_{\underline{Y}} \text{ (by Corollary 5.2.3.10), } \textit{and} \\ \tau &:= \tau_n|_{\mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta}}, \end{aligned}$$

then $\alpha_n^\natural$ *is a level-n structure datum as in Definition 5.2.7.9, and*

$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$$

is an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$ *as in Definition 5.2.7.14.*

Remark 5.4.1.2. In other words, we are just assuming that $\underline{X}$ and $\underline{Y}$ are constant, and regrouping the information in a representative

$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, \alpha_n^\natural)$$

of an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}$ into a different form with an emphasis on the fully symplectic-liftable admissible filtration Z_n and the liftable splitting $\delta_n : \mathrm{Gr}_n^{\mathsf{Z}} \xrightarrow{\sim} L/nL$. In particular, there is a natural map $\mathrm{DEG}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n} \to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}$ defined by associating the class $[\alpha_n^\natural]$ to the representative $\alpha_n^\natural$.

Now let us introduce the idea of cusp labels. We would like to define our cusp labels as equivalence classes of the tuples $(\mathsf{Z}_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$, because it is the part of the degeneration datum that is discrete in nature. Later our construction will produce for each cusp label a formal scheme over which we have tautological data of $(A, \lambda_A, i_A, \varphi_{-1,n})$ and $(c_n, c_n^\vee, \tau_n)$. These formal schemes should be interpreted as our boundary components, because their suitable algebraic approximations will be glued to the moduli problem M_n in the étale topology (and then form the desired compactification). Since the gluing process will be carried out in the étale topology, it suffices to assume (in the definition of cusp labels) that $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y.

Definition 5.4.1.3. *Given a fully symplectic admissible filtration* Z *of* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ *with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition 5.2.7.1, a* **torus argument** Φ *for* Z *is a tuple* $\Phi := (X, Y, \phi, \varphi_{-2}, \varphi_0)$*, where:*

1. X *and* Y *are* $\mathcal{O}$*-lattices of the same multi-rank (defined as in Definition 1.2.1.20), and* $\phi : Y \hookrightarrow X$ *is an* $\mathcal{O}$*-linear embedding.*
2. $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ *and* $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ *are isomorphisms such that the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \times \mathrm{Gr}^{\mathsf{Z}}_0 \to \hat{\mathbb{Z}}^{\Box}(1)$ *defined by* Z *is the pullback of the pairing*
$$\langle\,\cdot\,,\,\cdot\,\rangle^{\phi} : \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1)$$
defined by the composition
$$\begin{aligned} &\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \\ &\quad \xrightarrow{\mathrm{Id} \times \phi} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1), \end{aligned}$$
with the sign convention that $\langle\,\cdot\,,\,\cdot\,\rangle^{\phi}(x, y) = x(\phi(y)) = (\phi(y))(x)$ *for any* $x \in \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ *and any* $y \in Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$*.*

Definition 5.4.1.4. *Given a fully symplectic-liftable admissible filtration* Z_n *of* L/nL *with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition 5.2.7.3, a* **torus argument** Φ_n **at level** n *for* Z_n *is a tuple* $\Phi_n := (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*, where:*

1. X *and* Y *are* $\mathcal{O}$*-lattices of the same multi-rank (defined as in Definition 1.2.1.20), and* $\phi : Y \hookrightarrow X$ *is an* $\mathcal{O}$*-linear embedding.*
2. $\varphi_{-2,n} : \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ *and* $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$ *are isomorphisms that are reduction mod* n *of some isomorphisms* $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ *and* $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$*, such that* $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ *form a torus argument as in Definition 5.4.1.3.*

   We say in this case that Φ_n *is the reduction mod* n *of* Φ*.*

Definition 5.4.1.5. *Two torus arguments $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ and $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$ at level n are* **equivalent** *if and only if there exists a pair of isomorphisms $(\gamma_X : X' \overset{\sim}{\to} X, \gamma_Y : Y \overset{\sim}{\to} Y')$ such that:*

1. $\phi = \gamma_X \phi' \gamma_Y$.

$$\begin{tikzcd}
Y \arrow[r, "\phi"] \arrow[d, "\gamma_Y"', "\wr"] & X \\
Y' \arrow[r, "\phi'"'] & X' \arrow[u, "\gamma_X", "\wr"']
\end{tikzcd}$$

2. $\varphi'_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$.

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{-2,n} \arrow[r, "\varphi_{-2,n}"] \arrow[dr, "\varphi'_{-2,n}"'] & \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1)) \arrow[d, "{}^t\gamma_X", "\wr"'] \\
& \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1))
\end{tikzcd}$$

3. $\varphi'_{0,n} = \gamma_Y \varphi_{0,n}$.

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{0,n} \arrow[r, "\varphi_{0,n}"] \arrow[dr, "\varphi'_{0,n}"'] & Y/nY \arrow[d, "\gamma_Y", "\wr"'] \\
& Y'/nY'
\end{tikzcd}$$

In this case, we say that Φ_n and Φ'_n are equivalent under the isomorphism (γ_X, γ_Y), which we denote as $(\gamma_X, \gamma_Y) : \Phi_n \overset{\sim}{\to} \Phi'_n$.

Definition 5.4.1.6. *Let us denote by $\Gamma_\phi = \Gamma_{X,Y,\phi}$ the subgroup of elements in $\mathrm{GL}_{\mathcal{O}}(X)$ leaving invariant the image of Y under ϕ.*

We can think of Γ_ϕ as the group of pairs of isomorphisms $(\gamma_X : X \overset{\sim}{\to} X, \gamma_Y : Y \overset{\sim}{\to} Y) \in \mathrm{GL}_{\mathcal{O}}(Y) \times \mathrm{GL}_{\mathcal{O}}(X)$ such that $\phi = \gamma_X \phi \gamma_Y$. This suggests that we may interpret Γ_ϕ as the group of integral points of an algebraic group GL_ϕ defined as follows:

Definition 5.4.1.7. *Let GL_ϕ be the group functor associating to any $\mathbb{Z}$-algebra R the group*

$$\mathrm{GL}_\phi(R) := \{(g_X, g_Y) \in \mathrm{GL}_{\mathcal{O}}(X) \times \mathrm{GL}_{\mathcal{O}}(Y) : \phi = g_X \phi g_Y\}.$$

Then we have a natural map

$$\Gamma_\phi = \mathrm{GL}_\phi(\mathbb{Z}) \to \mathrm{GL}_\phi(R) \hookrightarrow \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} R) \times \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} R)$$

for any $\mathbb{Z}$-algebra R. Here, certainly, Γ_ϕ is understood as a set independent of R rather than a functor.

Then we have the following simple observation:

Lemma 5.4.1.8. *If two torus arguments* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ *and* $\Phi'_n = (X, Y, \phi, \varphi'_{-2,n}, \varphi'_{0,n})$ *at level* n *are equivalent under some* (γ_X, γ_Y)*, then necessarily* $(\gamma_X, \gamma_Y) \in \Gamma_\phi$*.*

Definition 5.4.1.9. *A* (**principal**) **cusp label at level** n *for a PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot, \cdot \rangle)$*, or a cusp label of the moduli problem* M_n*, is an equivalence class of triples* $(\mathsf{Z}_n, \Phi_n, \delta_n)$*, where:*

1. Z_n *is an admissible filtration of* L/nL *that is fully symplectic-liftable in the sense of Definition 5.2.7.3,*
2. Φ_n *is a torus argument at level* n *for* Z_n*.*
3. $\delta_n : \mathrm{Gr}_n^{\mathsf{Z}} \xrightarrow{\sim} L/nL$ *is a liftable splitting.*

Two triples $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *and* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *are equivalent if* Z_n *and* Z'_n *are identical, and* Φ_n *and* Φ'_n *are equivalent as in Definition 5.4.1.5.*

Convention 5.4.1.10. *To simplify the notations, we will often suppress* Z_n *from the notation* $(\mathsf{Z}_n, \Phi_n, \delta_n)$*, with the understanding that the data* Φ_n *and* δ_n *do not make sense without an a priori choice of a fully symplectic-liftable admissible filtration* Z_n*.*

Even if we suppress Z_n *from the notations in the representatives of cusp labels, we shall still maintain the notations* $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n})$ *and* $\mathsf{M}_n^{\mathsf{Z}_n}$ *as in Lemma 5.2.7.5.*

Remark 5.4.1.11. It may seem that it is redundant to have the splitting δ_n in the definition, and it may seem pointless to suppress Z_n while retaining the redundant information δ_n. However, the presence of δ_n reminds us of the admissibility of the filtration (defined as in Definition 1.2.6.6), and the choice of a noncanonical δ_n will be indispensable in our construction of boundary charts. By incorporating the choice in the notation, we do not have to mention it again and again.

Lemma 5.4.1.12. *Suppose we are given an a priori choice of a representative* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *of a cusp label at level* n*, with* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*. Then for any object*

$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$*, there is a unique object*

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}^{\text{fil.-spl.}}$ *such that*

$$\alpha_n^\natural = (\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$$

is a representative of $[\alpha_n^\natural]$*.*

Definition 5.4.1.13. *The* **multi-rank** *of a cusp label at level* n *represented by some* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *is simply the multi-rank of* Z_n *(defined as in Definitions 5.2.2.5, 5.2.2.8, and Remark 5.2.2.7).*

Lemma 5.4.1.14. *Suppose we have a cusp label at level* n *represented by some* $(\mathsf{Z}_n, \Phi_n, \delta_n)$*, with* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*. For any admissible surjection* $s_X : X \twoheadrightarrow X'$ *of* $\mathcal{O}$*-lattices (defined as in Definition 1.2.6.7), there is canonically determined cusp label at level* n *which can be represented by some* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *with* $\Phi'_n = (X', Y', \phi', \varphi_{-2,n}, \varphi_{0,n})$ *described as follows:*

1. X' *is the same* X' *we have in the surjection* s_X*.*

2. *An admissible surjection* $s_Y : Y \twoheadrightarrow Y'$ *of* $\mathcal{O}$*-lattices for some* Y' *determined by setting* $\ker(s_Y) = \phi^{-1}(\ker(s_X))$*.*

3. *The definitions of* s_Y *and* Y' *induce an embedding* $\phi' : Y' \to X'$ *making the following diagram commute:*

$$\begin{tikzcd} Y \arrow[r, hook, "\phi", "\sim"'] \arrow[d, two heads, "s_Y"'] & X \arrow[d, two heads, "s_X"] \\ Y' \arrow[r, hook, dotted, "\sim", "\phi'"'] & X' \end{tikzcd}$$

4. *The admissible surjection* $s_X : X \twoheadrightarrow X'$ *defines an admissible embedding*

$$s_X^* : \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1)) \hookrightarrow \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1)).$$

The image of this embedding is mapped to an admissible submodule $\mathsf{Z}'_{-2,n}$ *of* $\mathsf{Z}_{-2,n}$*, and defines an isomorphism* $\varphi'_{-2} : \mathrm{Gr}^{\mathsf{Z}'}_{-2,n} = \mathsf{Z}'_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1))$*.*

The composition of the admissible surjection $\mathsf{Z}_{0,n} \twoheadrightarrow \mathsf{Z}_{0,n}/\mathsf{Z}_{-1,0} = \mathrm{Gr}^{\mathsf{Z}}_{0,n}$ *with the isomorphism* $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$ *defines an surjection* $\mathsf{Z}_{0,n} \twoheadrightarrow Y/nY$*, and hence the admissible surjection* $s_Y : Y \twoheadrightarrow Y'$ *defines an admissible surjection* $\mathsf{Z}_{0,n} \twoheadrightarrow Y'/nY'$*, whose kernel defines an admissible submodule* $\mathsf{Z}'_{-1,n}$ *of* $\mathsf{Z}_{0,n} = L/nL$*. This defines an isomorphism* $\varphi'_{0,n} : \mathrm{Gr}^{\mathsf{Z}'}_{0,n} \xrightarrow{\sim} Y'/nY'$*.*

Then this defines a fully symplectic-liftable admissible filtration $\mathsf{Z}'_n = \{\mathsf{Z}'_{-i,n}\}$ *of* L/nL*.*

5. $\delta'_n : \mathrm{Gr}^{\mathsf{Z}}_n \xrightarrow{\sim} L/nL$ *is just any liftable splitting.*

Note that the construction determines a unique pair (Z'_n, Φ'_n)*, and hence a unique cusp label at level* n*.*

Proof. The upshot is to show that the tuple $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ we arrive at, with $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$, is a representative of a cusp label. In other words, the filtration Z'_n has to be fully symplectic-liftable and admissible, and the torus argument Φ'_n at level n has to be a reduction mod n of some torus argument Φ' for some fully symplectic lifting Z of Z_n.

Let us fix some symplectic lifting Z of Z_n, and for some liftings $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ of respectively $\varphi_{-2,n} : \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ and $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$.

The admissible surjection $s_X : X \twoheadrightarrow X'$ defines an admissible embedding

$$s_X^* : \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \hookrightarrow \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)).$$

By the choice of φ_{-2} above this defines an admissible submodule Z'_{-2} of Z_{-2}, whose reduction mod n is the admissible submodule $\mathsf{Z}'_{-2,n}$ of $\mathsf{Z}_{-2,n}$. This defines an isomorphism $\varphi'_{-2} : \mathrm{Gr}^{\mathsf{Z}'}_{-2} = \mathsf{Z}'_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ lifting the isomorphism $\varphi'_{-2,n} : \mathrm{Gr}^{\mathsf{Z}'}_{-2,n} = \mathsf{Z}'_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1))$.

On the other hand, the composition of the admissible surjection $\mathsf{Z}_0 \twoheadrightarrow \mathsf{Z}_0/\mathsf{Z}_{-1} = \mathrm{Gr}^{\mathsf{Z}}_0$ with the isomorphism $\varphi_0^{-1} : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ defines an admissible surjection $\mathsf{Z}_0 \twoheadrightarrow Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, and hence the admissible surjection $s_Y : Y \twoheadrightarrow Y'$

defines an admissible surjection $\mathsf{Z}_0 \twoheadrightarrow Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. The kernel of $\mathsf{Z}_0 \twoheadrightarrow Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ defines an admissible submodule Z'_{-1} of $\mathsf{Z}_0 = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, whose reduction mod n is the admissible submodule $\mathsf{Z}_{-1,n}$ of $\mathsf{Z}_{0,n} = L/nL$. This defines an isomorphism $\varphi'_0 : \mathrm{Gr}_0^{\mathsf{Z}'} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ lifting the isomorphism $\varphi'_{0,n} : \mathrm{Gr}_{0,n}^{\mathsf{Z}'} \xrightarrow{\sim} Y'/nY'$.

The filtration $\mathsf{Z}' = \{\mathsf{Z}'_{-i}\}$ thus defined is admissible by construction, and it is symplectic because of the definition of ϕ and ϕ'. Indeed, we can define Z'_{-1} simply as the annihilator of Z'_{-2}. Also, it is fully symplectic because Z'_{-2} is a submodule of Z_{-2}. Hence the filtration $\mathsf{Z}'_n = \{\mathsf{Z}'_{-i,n}\}$ is fully symplectic-liftable and admissible, as desired. □

The realization of Y' as an admissible quotient of Y is unique in the construction of Lemma 5.4.1.14. However, once we fix a particular realization of Y' as a quotient of Y, by possibly a different quotient map, the remain construct in the proof of Lemma 5.4.1.14 carries over without any necessary modification. Hence, for comparing two objects in general, it will be more convenient if we can allow a twist of the identification of Y' by an isomorphism, as long as the kernel of $s_Y : Y \twoheadrightarrow Y'$ remains the same.

Definition 5.4.1.15. *A* **surjection** $(\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *between representatives of cusp labels at level* n*, where* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ *and where* $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$*, is a surjection* $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ *such that:*

1. *Both* s_X *and* s_Y *are admissible surjections, and they are compatible with* ϕ *and* ϕ' *in the sense that* $s_X\phi = \phi' s_Y$*.*

2. $\mathsf{Z}'_{-2,n}$ *is an admissible submodule of* $\mathsf{Z}_{-2,n}$*, and the natural embedding* $\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}}_{-2,n}$ *and the maps* s_X*,* $\varphi_{-2,n}$ *and* $\varphi'_{-2,n}$ *make the following diagram commutative:*

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{-2,n} \arrow[r, "\varphi_{-2,n}", "\sim"'] & \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1)) \\
\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \arrow[r, "\sim", "\varphi'_{-2,n}"'] \arrow[u, hook, "\mathrm{can.}"] & \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1)) \arrow[u, hook, "s_X^*"']
\end{tikzcd} .$$

3. $\mathsf{Z}_{-1,n}$ *is an admissible submodule of* $\mathsf{Z}'_{-1,n}$*, and the natural surjection* $\mathrm{Gr}^{\mathsf{Z}}_{0,n} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n}$ *and the maps* s_Y*,* $\varphi_{0,n}$ *and* $\varphi'_{0,n}$ *make the following dia-*

gram commutative:

$$\begin{tikzcd}
\mathrm{Gr}^{\mathsf{Z}}_{0,n} \arrow[r, "\varphi_{0,n}", "\sim"'] \arrow[d, two heads, "\mathrm{can.}"'] & Y/nY \arrow[d, two heads, "s_Y"] \\
\mathrm{Gr}^{\mathsf{Z}'}_{0,n} \arrow[r, "\sim", "\varphi'_{0,n}"'] & Y'/nY'
\end{tikzcd} .$$

(In other words, Z'_n *and* $(\varphi'_{-2,n}, \varphi'_{0,n})$ *are associated to respectively* Z_n *and* $(\varphi_{-2,n}, \varphi_{0,n})$ *under* (s_X, s_Y) *as in Lemma 5.4.2.11.) In this case, we write simply* $(s_X, s_Y) : (\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$

Lemma 5.4.1.16. *Let* (Φ_n, δ_n)*,* (Φ'_n, δ'_n)*,* (Φ''_n, δ''_n)*, and* (Φ'''_n, δ'''_n) *be representatives of cusp labels at level* n *(see Convention 5.4.1.10), with* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*,* $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$*, etc. Suppose* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *defines an isomorphism* $(\Phi_n, \delta_n) \xrightarrow{\sim} (\Phi'_n, \delta'_n)$*,* $(s_{X'} : X' \twoheadrightarrow X'', s_{Y'} : Y' \twoheadrightarrow Y'')$ *defines a surjection* $(\Phi'_n, \delta'_n) \twoheadrightarrow (\Phi''_n, \delta''_n)$*, and* $(\gamma_{X''} : X''' \xrightarrow{\sim} X'', \gamma_{Y''} : Y'' \xrightarrow{\sim} Y''')$ *defines an isomorphism* $(\Phi''_n, \delta''_n) \xrightarrow{\sim} (\Phi'''_n, \delta'''_n)$*. Then* $(\gamma_{X''}^{-1} s_{X'} \gamma_X^{-1} : X \twoheadrightarrow X''', \gamma_{Y''} s_{Y'} \gamma_Y : Y \twoheadrightarrow Y''')$ *defines a surjection* $(\Phi_n, \delta_n) \twoheadrightarrow (\Phi'''_n, \delta'''_n)$*.*

This justifies the following:

Definition 5.4.1.17. *We say that there is a* **surjection** *from a cusp label at level* n *represented by some* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *to another cusp label at level* n *represented by some* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *if there is a surjection* (s_X, s_Y) *from* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *to* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$*.*

5.4.2 General Cusp Labels

In this section, we give analogues of the definitions in Section 5.4.1 whenever it is appropriate in the context.

Suppose we are given a collection of orbits $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{\mathcal{H}_n}\}$ as in Definition 5.3.1.12. For any integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, we have an $\mathcal{H}_n$-orbit $\mathsf{Z}_{\mathcal{H}_n}$ of fully symplectic-liftable admissible filtrations of L/nL, which we view as a constant scheme over the base scheme. Over each point, namely for each choice of Z_n in this orbit, it determines a subgroup $\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}$. Any element $g_n \in \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}$ acts on the torus arguments at level n by sending $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ to $(X, Y, \phi, \varphi'_{-2,n}, \varphi'_{0,n})$ with $\varphi'_{-2,n} := \nu(g_n)^{-1} \circ \varphi_{-2,n} \circ$

$\mathrm{Gr}_{-2,n}(g_n)$ and $\varphi'_{0,n} := \varphi_{0,n} \circ \mathrm{Gr}_{0,n}(g_n)$. Then it makes sense to talk about the $\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbits of torus arguments over Z_n, and hence about $\mathcal{H}_n$-orbits $\Phi_{\mathcal{H}_n}$ of étale-locally-defined torus arguments over $\mathsf{Z}_{\mathcal{H}_n}$.

We would like to focus on the situation when $\Phi_{\mathcal{H}_n}$ is *split* in the sense that it is formed by orbits that are already defined over the same base (without having to take étale localizations):

Definition 5.4.2.1. *Given a collection of orbits $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{\mathcal{H}_n}\}$ as in Definition 5.3.1.12, a* **torus argument $\Phi_{\mathcal{H}}$ at level $\mathcal{H}$** *for $\mathsf{Z}_{\mathcal{H}}$ is a tuple $\Phi_{\mathcal{H}} := (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ which is a collection $\Phi_{\mathcal{H}} = \{\Phi_{\mathcal{H}_n}\}$ of $\mathcal{H}_n$-orbits of torus arguments at level n with elements $\Phi_{\mathcal{H}_n}$ described as follows:*

1. *For each n in the index set, there is an element Z_n in $\mathsf{Z}_{\mathcal{H}_n}$ and a torus argument $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ at level n for Z_n, such that $\Phi_{\mathcal{H}_n}$ is the $\mathcal{H}_n$-orbit of $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$.*

2. *For any $n|m$ in the index set, the $\mathcal{H}_n$-orbit $\Phi_{\mathcal{H}_n}$ is determined by the $\mathcal{H}_m$-orbit $\Phi_{\mathcal{H}_m}$ by reduction mod n.*

By abuse of notations, we shall write $\Phi_{\mathcal{H}_n} = (X, Y, \phi, \varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$ and $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$.

Definition 5.4.2.2. *Two torus arguments $\Phi_{\mathcal{H}} = \{\Phi_{\mathcal{H}_n}\}$ and $\Phi'_{\mathcal{H}} = \{\Phi'_{\mathcal{H}_n}\}$ at level $\mathcal{H}$ are* **equivalent** *if and only if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ such that there exists some n in the index set so that $\Phi_{\mathcal{H}}$ contains some torus arguments at level n that is equivalent under (γ_X, γ_Y) (as in Definition 5.4.1.5) to some torus arguments at level n in $\Phi'_{\mathcal{H}}$.*

In this case, we say that $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ are equivalent under the isomorphism (γ_X, γ_Y), which we denote as $(\gamma_X, \gamma_Y) : \Phi_{\mathcal{H}} \xrightarrow{\sim} \Phi'_{\mathcal{H}}$.

A trivial reformulation of Lemma 5.4.1.8 is:

Lemma 5.4.2.3. *If two torus arguments $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ at level n are equivalent under some (γ_X, γ_Y), then necessarily $(\gamma_X, \gamma_Y) \in \Gamma_{\phi}$.*

On the other hand, over each point of $\mathsf{Z}_{\mathcal{H}_n}$, namely for each choice of Z_n in this orbit, we can make sense of a liftable splitting δ_n over it, and hence we can talk about $\mathcal{H}_n$-orbits of $(\mathsf{Z}_{\mathcal{H}_n}, \delta_{\mathcal{H}_n})$ with the understanding that $\delta_{\mathcal{H}_n}$ is defined pointwise over $\mathsf{Z}_{\mathcal{H}_n}$. We say in this case that $\delta_{\mathcal{H}_n}$ is a splitting of $\mathsf{Z}_{\mathcal{H}_n}$. Hence it also makes sense to talk about a splitting $\delta_{\mathcal{H}}$ of $\mathsf{Z}_{\mathcal{H}}$.

Definition 5.4.2.4. *A* **cusp label at level** $\mathcal{H}$ *for a PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot , \cdot \rangle)$*, or a cusp label of the moduli problem* $\mathsf{M}_{\mathcal{H}}$*, is an equivalence class of triples* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*, where:*

1. $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{\mathcal{H}_n}\}$ *is a collection of orbits of admissible filtration of* L/nL *that are fully symplectic-liftable in the sense of Definition 5.2.7.3.*
2. $\Phi_{\mathcal{H}}$ *is a torus argument at level* $\mathcal{H}$ *for* $\mathsf{Z}_{\mathcal{H}}$*.*
3. $\delta_{\mathcal{H}}$ *is a liftable splitting of* $\mathsf{Z}_{\mathcal{H}}$*.*

Two triples $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent if* $\mathsf{Z}_{\mathcal{H}}$ *and* $\mathsf{Z}'_{\mathcal{H}}$ *are identical, and* $\Phi_{\mathcal{H}}$ *and* $\Phi'_{\mathcal{H}}$ *are equivalent as in Definition 5.4.2.2.*

If $\Phi_{\mathcal{H}}$ *and* $\Phi'_{\mathcal{H}}$ *are equivalent under some particular isomorphism* (γ_X, γ_Y) *as in Definition 5.4.2.2, then we also say that* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent under* (γ_X, γ_Y)*.*

Convention 5.4.2.5. *As in Convention 5.4.1.10, to simplify the notations, we will often suppress* $\mathsf{Z}_{\mathcal{H}}$ *from the notation* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*, with the understanding that the data* $\Phi_{\mathcal{H}}$ *and* $\delta_{\mathcal{H}}$ *do not make sense without an a priori choice of* $\mathsf{Z}_{\mathcal{H}}$*.*

Let us nevertheless define the notations $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}})$ and $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ even if we suppress $\mathsf{Z}_{\mathcal{H}}$ from the notations in the representatives of cusp labels (see Convention 5.4.1.10 and Lemma 5.2.7.5):

Definition 5.4.2.6. *The PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}})$ *is any of the PEL-type* $\mathcal{O}$*-lattices determined by an element* Z_n *in any* $\mathsf{Z}_{\mathcal{H}_n}$ *in* $\mathsf{Z}_{\mathcal{H}}$ *in Lemma 5.2.7.5. (Note that this choice is not unique.) It determines some group functor* $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_n})}$*, such that there is an identification* $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \cong \mathrm{G}^{\mathrm{ess}}_{(L^{\Phi_n}, \langle \cdot , \cdot \rangle^{\Phi_n})}(\mathbb{Z}/n\mathbb{Z})$*. Let* $\mathcal{H}_h$ *be the preimage of* $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ *under the surjection* $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_n})} \twoheadrightarrow \mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}$*. Then we define* $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *to be the moduli problem defined by* $(L^{\mathsf{Z}_n}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_n})$ *with level-*$\mathcal{H}_h$ *structures as in Lemma 5.2.7.5. (Note that this final moduli problem is well-defined and independent of the choice of* $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot , \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}})$*.)*

Definition 5.4.2.7. *The* **multi-rank** *of a cusp label at level* $\mathcal{H}$ *represented by some* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *is simply the multi-rank (defined as in Definition 5.4.1.13) of any* Z_n *in any* $\mathsf{Z}_{\mathcal{H}_n}$ *in* $\mathsf{Z}_{\mathcal{H}}$*.*

Definition 5.4.2.8. *With the setting as in Section 5.2.1, the category* $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *has objects of the form*

$$(\mathsf{Z}_{\mathcal{H}}, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

such that:

1. $\underline{X}$ *and* $\underline{Y}$ *are constant with values respectively* X *and* Y*,*
2. $(X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ *is a torus argument at level* $\mathcal{H}$ *as in Definition 5.4.2.1 above.*
3. *If we set* $\alpha^{\natural}_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}}, \varphi_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}})$, $c =$ *unique extension of* $c_{\mathcal{H}}|_{\underline{X}}$ *(in the sense that we take the common induced element in the étale-locally-defined orbit and descent),* $c^{\vee} =$ *unique extension of* $c^{\vee}_{\mathcal{H}}|_{\underline{Y}}$ *(in the same sense),* $\tau = \tau_n|_{\mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta}}$*, then* $\alpha^{\natural}_{\mathcal{H}}$ *is a level-*$\mathcal{H}$ *structure datum as in Definition 5.3.1.12, and*

   $$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau, [\alpha^{\natural}_{\mathcal{H}}])$$

   is an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *as in Definition 5.3.1.16.*

Remark 5.4.2.9. There is a natural map $\mathrm{DEG}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}} \to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$, which is essentially associating the class $[\alpha^{\natural}_{\mathcal{H}}]$ to the representative $\alpha^{\natural}_{\mathcal{H}}$.

Lemma 5.4.2.10. *Suppose we are given an a priori choice of a representative* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of a cusp label at level* $\mathcal{H}$*, with* $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$*. Then for any object*

$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau, [\alpha^{\natural}_{\mathcal{H}}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$*, there is a unique object*

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *such that*

$$\alpha^{\natural}_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}}, \varphi_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}})$$

is a representative of $[\alpha^{\natural}_{\mathcal{H}}]$*.*

Lemma 5.4.2.11. *Suppose we have a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, with $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$. For any admissible surjection $s_X : X \twoheadrightarrow X'$ of $\mathcal{O}$-lattices (defined as in Definition 1.2.6.7), there is canonically determined cusp label at $\mathcal{H}$ which can be represented by some $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ with $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ described as follows:*

1. *X' is the same X' we have in the surjection s_X.*

2. *An admissible surjection $s_Y : Y \twoheadrightarrow Y'$ of $\mathcal{O}$-lattices for some Y' determined by setting* $\ker(s_Y) = \phi^{-1}(\ker(s_X))$.

3. *The definitions of s_Y and Y' induce an embedding $\phi' : Y' \to X'$ making the following diagram commute:*

$$\begin{tikzcd}
Y \arrow[r, hook, "\phi", "\sim"'] \arrow[d, two heads, "s_Y"'] & X \arrow[d, two heads, "s_X"] \\
Y' \arrow[r, hook, dotted, "\sim", "\phi'"'] & X'
\end{tikzcd}$$

4. *Let us write $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{\mathcal{H}_n}\}$, with indices given by integers $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$.*

   For any n as above, the recipe in Lemma 5.4.1.14 determines an association of Z'_n and $(\varphi'_{-2,n}, \varphi'_{0,n})$ to respectively Z_n and $(\varphi_{-2,n}, \varphi_{0,n})$ under (s_X, s_Y), which is compatible with the process of taking orbits. Hence we have an induced association of $\mathsf{Z}'_{\mathcal{H}_n}$ and $(\varphi'_{-2,\mathcal{H}_n}, \varphi'_{0,\mathcal{H}_n})$ to respectively $\mathsf{Z}_{\mathcal{H}_n}$ and $(\varphi_{-2,\mathcal{H}_n}, \varphi_{0,\mathcal{H}_n})$, and hence an induced association of $\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ to respectively $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ under (s_X, s_Y).

5. *$\delta'_{\mathcal{H}}$ is just any liftable splitting of $\mathsf{Z}'_{\mathcal{H}}$*

Note that the construction determines a unique pair $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}})$, and hence a unique cusp label at level $\mathcal{H}$.

Definition 5.4.2.12. *A* **surjection** $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *between representatives of cusp labels at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and where $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$, is a surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ such that:*

1. *Both s_X and s_Y are admissible surjections, and they are compatible with ϕ and ϕ' in the sense that $s_X \phi = \phi' s_Y$.*

2. *$\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are associated to respectively $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ under (s_X, s_Y) as in Lemma 5.4.2.11. (We do not need to know if s_Y is the canonically determined surjection as in Lemma 5.4.2.11 for the construction there.)*

In this case, we write simply $(s_X, s_Y) : (\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Alternatively, we can view such a surjection as collection of orbits of surjections described in Definition 5.4.1.15, in its natural sense.

Then a trivial analogue of Lemma 5.4.1.16 justifies the following:

Definition 5.4.2.13. *We say that there is a* **surjection** *from a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to another cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ if there is a surjection (s_X, s_Y) from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.*

5.4.3 Hecke Actions

With the setting as in the beginning of Section 5.4.1, suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, and suppose we have two open compact subgroups $\mathcal{H}' \subset \mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. In other words, we have $\mathcal{H}' \subset \mathcal{H} \cap (g\mathcal{H}g^{-1})$.

Suppose we have a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}'})$ of type $\mathsf{M}_{\mathcal{H}'}$ over S. By definition, this means its restriction $(G_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H}'})$ to the generic point η of S defines an object parameterized by $\mathsf{M}_{\mathcal{H}'}$. Let $\bar{\eta}$ be a geometric point over η. By Proposition 1.4.3.3, the object defined by $(G_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H}'})$ corresponds to an object defined by $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha}]_{\mathcal{H}'})$. For each representative $\hat{\alpha}$ of the $\mathcal{H}'$-orbit $[\hat{\alpha}]_{\mathcal{H}'}$, we may consider the composition $\hat{\alpha} \circ g$. If we take a different representative $\hat{\alpha}'$, which is by definition $\hat{\alpha} \circ u$ for some $u \in \mathcal{H}'$, then $\hat{\alpha}' \circ g = \hat{\alpha} \circ g \circ (g^{-1}ug)$. Since $g^{-1}\mathcal{H}'g \subset \mathcal{H}$, we obtain a well-defined $\mathcal{H}$-orbit $[\hat{\alpha} \circ g]_{\mathcal{H}}$. This defines an object $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha} \circ g]_{\mathcal{H}})$ parameterized by $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$. As mentioned in Remark 1.4.3.10, this defines an *action* of g on the tower $\mathsf{M}^{\Box} = \varprojlim\limits_{\mathcal{H}, \mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})} \mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$, or rather a morphism from $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}'} \to \mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$.

For the purpose of studying degenerations, it is desirable that we may translate the object represented by $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha} \circ g]_{\mathcal{H}})$ back to some object represented by some $(G'_\eta, \lambda'_\eta, i'_\eta, \alpha'_{\mathcal{H}})$ parameterized by $\mathsf{M}_{\mathcal{H}}$. As in the proof of Proposition 1.4.3.3, this is realized by a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f_\eta : G_\eta \to G'_\eta$ defined as follows: Take any representative $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box}\, G_{\bar{\eta}}$ of $[\hat{\alpha}]_{\mathcal{H}}$, which by construction sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}\, G_{\bar{\eta}}$. The composition $\hat{\alpha} \circ g : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box}\, G_{\bar{\eta}}$ sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to the $\mathcal{O}$-invariant open compact subgroup $\hat{\alpha}(g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}))$ of $\mathrm{V}^{\Box}\, G_{\bar{\eta}}$, which by Corollary 1.3.5.4 corresponds to some $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f_{\eta} : G_{\eta} \to G'_{\eta}$ to another abelian scheme. The inclusion $\hat{\alpha}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \hookrightarrow \hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ in $\mathrm{V}^{\Box}\, G_{\bar{\eta}}$ corresponds by Corollary 1.3.5.4 to the class of the polarization $\lambda_{\eta} : G_{\eta} \to G_{\eta}^{\vee}$. Note that there are many other $\mathcal{O}$-invariant open compact subgroups in $\mathrm{V}^{\Box}\, G_{\bar{\eta}}$ that are isomorphic to $\hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$, and it is the condition that $\hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ is dual to $\hat{\alpha}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ under the λ-Weil pairing $\mathrm{e}^{\lambda}(\,\cdot\,,\,\cdot\,)$, or rather the condition that $L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ is dual to $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, that characterizes this class of $\mathbb{Z}_{(\Box)}^{\times}$-isogeny. (Remember that the class of a $\mathbb{Z}_{(\Box)}^{\times}$-isogeny is defined only up to isomorphism on the target.) Since $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$ satisfies $\langle gx, gy \rangle = \nu(g)\langle x, y \rangle$ for any $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$, we see that the perfect duality $\langle\,\cdot\,,\,\cdot\,\rangle : (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \times (L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1)$ is carried to the perfect duality $\langle\,\cdot\,,\,\cdot\,\rangle : g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \times g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \to \nu(g)\hat{\mathbb{Z}}^{\Box}(1)$. In particular, the dual of $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ under $\langle\,\cdot\,,\,\cdot\,\rangle$ is $\nu(g)^{-1}g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$. Under $\hat{\alpha}$, this translates to the statement that the dual of the $\mathbb{Z}_{(\Box)}^{\times}$-isogeny f_{η} is the $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f_{\eta}^{\vee} : (G'_{\eta})^{\vee} \to G_{\eta}^{\vee}$ with source and target defined respectively by the open compact subgroups $\nu(g)^{-1}g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ and $L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$. The composition $(f_{\eta}^{\vee})^{-1} \circ \lambda_{\eta} \circ f_{\eta}^{-1} : G'_{\eta} \to (G'_{\eta})^{\vee}$ has source and target defined respectively by $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ and $\nu(g)^{-1}g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$, and we know by Corollary 1.3.2.25 that this $\mathbb{Z}_{(\Box)}^{\times}$-isogeny is positive. Using the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}_{(\Box),>0}^{\times} \cdot \hat{\mathbb{Z}}^{\Box,\times}$, there is a unique element $r \in \mathbb{Z}_{(\Box),>0}^{\times}$ such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}^{\Box,\times}$. Set $\lambda'_{\eta} := r^{-1}(f_{\eta}^{\vee})^{-1} \circ \lambda_{\eta} \circ f_{\eta}^{-1}$. Then the source and target of the class of this $\mathbb{Z}_{(\Box)}^{\times}$-isogeny correspond respectively to the open compact subgroups $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ and $g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) = r\nu(g)^{-1}g(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$. This defines a polarization $\lambda' : G'_{\eta} \to (G'_{\eta})^{\vee}$, which satisfies $f_{\eta}^{\vee} \circ \lambda'_{\eta} \circ f_{\eta} = r\lambda_{\eta}$. Let us remark that $i_{\eta} : \mathcal{O} \to \mathrm{End}_{\eta}(G_{\eta})$ induces an $\mathcal{O}$-endomorphism structure $i'_{\eta} : \mathcal{O} \to \mathrm{End}_{\eta}(G'_{\eta})$ (with image in $\mathrm{End}_{\eta}(G'_{\eta})$ instead of $\mathrm{End}_{\eta}(G'_{\eta}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$)

because $g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ is invariant under $\mathcal{O}$. Finally, the symplectic isomorphism $\hat{\alpha}' := \mathrm{V}^{\Box}(f_\eta)\circ\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box}\,G'_{\bar{\eta}}$ now carries $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}\,G'_{\bar{\eta}}$. This $\mathcal{H}$-orbit of $\hat{\alpha}'$ is independent of the choice of $\hat{\alpha}$, because $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. Hence it is necessarily defined over η, and defines a rational level-$\mathcal{H}$ structure $[\hat{\alpha}']_{\mathcal{H}}$. Thus we have constructed an explicit $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f_\eta : G_\eta \to G'_\eta$ that defines the equivalence $(G_\eta,\lambda_\eta,i_\eta,[\hat{\alpha}\circ g]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}\text{-isog.}} (G'_\eta,\lambda'_\eta,i'_\eta,[\hat{\alpha}']_{\mathcal{H}})$. By Corollary 1.3.7.11, $(G'_\eta,\lambda'_\eta,i'_\eta,[\hat{\alpha}']_{\mathcal{H}})$ comes from a unique $(G'_\eta,\lambda'_\eta,i'_\eta,\alpha'_{\mathcal{H}})$ under Construction 1.3.7.10.

Let us take any integer $N \geq 1$ prime-to-$\Box$ such that Nf_η is a prime-to-$\Box$ isogeny. Let us first extend the prime-to-$\Box$ isogeny Nf_η to a prime-to-$\Box$ isogeny over the whole base scheme S. The procedure is the following standard one (which we have used in the proofs of Theorem 3.4.3.1 and Lemma 5.2.3.2): First take $K_\eta := \ker(Nf_\eta)$, and take K_S to be the schematic closure of K_η in G over S, which is flat and quasi-finite over S. Then, mimicking the construction of $G^{\vee}$ in Theorem 3.4.3.1 using Lemma 3.4.3.3, this defines an isogeny $Nf : G \twoheadrightarrow G' := G/K_S$ whose restriction to η is the Nf_η we started with. By taking f to be symbolically $N^{-1}\circ(Nf)$, we obtain a "$\mathbb{Z}_{(\Box)}^{\times}$-isogeny" $f : G \to G'$. Note that this "$\mathbb{Z}_{(\Box)}^{\times}$-isogeny" is independent of the N we have chosen, in the sense that, if we have chosen any other N' with $N|N'$, then necessarily $(N'/N)\circ(Nf) = N'f$. By noetherian normality of S, we know that this G' is uniquely determined by G'_η, and the isogenies Nf are all uniquely determined by Nf_η.

Let us now investigate the effect of the action of g on the degeneration data. Suppose that $(G,\lambda,i,\alpha_{\mathcal{H}'})$ and $(G',\lambda',i',\alpha'_{\mathcal{H}})$ are objects of respectively $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}'}}$ and $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ over S, and let $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^{\vee},\tau,[\alpha^{\natural}_{\mathcal{H}'}])$ and $(A',\lambda'_A,i'_A,\underline{X}',\underline{Y}',\phi',c',(c^{\vee})',\tau',[(\alpha'_{\mathcal{H}})^{\natural}])$ be the associated degeneration data in respectively $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}'}}$ and $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$. For our purpose, we would like to assume that $\underline{X}$ and $\underline{Y}$ are constant with values respectively X and Y, and similarly for $\underline{X}'$ and $\underline{Y}'$. Let us suppose that $\alpha^{\natural}_{\mathcal{H}'} = (\mathsf{Z}_{\mathcal{H}'},\varphi_{-2,\mathcal{H}'},\varphi_{-1,\mathcal{H}'},\varphi_{0,\mathcal{H}'},\delta_{\mathcal{H}'},c_{\mathcal{H}'},c^{\vee}_{\mathcal{H}'},\tau_{\mathcal{H}'})$ is a representative of $[\alpha^{\natural}_{\mathcal{H}'}]$, and that $(\alpha'_{\mathcal{H}})^{\natural} = (\mathsf{Z}'_{\mathcal{H}},\varphi'_{-2,\mathcal{H}},\varphi'_{-1,\mathcal{H}},\varphi'_{0,\mathcal{H}},\delta'_{\mathcal{H}},c'_{\mathcal{H}},(c^{\vee}_{\mathcal{H}})',\tau'_{\mathcal{H}})$ is a representative of $[(\alpha'_{\mathcal{H}})^{\natural}]$. Let us assume moreover that $\Phi_{\mathcal{H}'} = (X,Y,\phi,\varphi_{-2,\mathcal{H}'},\varphi_{0,\mathcal{H}'})$ and $\Phi'_{\mathcal{H}} = (X',Y',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$ are *split* in the sense that they define torus arguments at respectively level $\mathcal{H}'$ and $\mathcal{H}$ (defined as in Definition 5.4.2.1). Then we obtain two cusp labels represented respectively by triples $(\mathsf{Z}_{\mathcal{H}'},\Phi_{\mathcal{H}'},\varphi_{0,\mathcal{H}'}),\delta_{\mathcal{H}'})$ and $(\mathsf{Z}'_{\mathcal{H}},\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$. Our goal is to find the relation between the two cusp labels.

Let $n, m \geq 1$ be integers such that $n|m$, $\mathcal{U}^{\Box}(m) \subset \mathcal{H}'$, and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. Let $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$ and let $\mathcal{H}'_m := \mathcal{H}'/\mathcal{U}^{\Box}(m)$. Let $\alpha^{\natural}_{\mathcal{H}'_m} = (\mathtt{Z}_{\mathcal{H}'_m}, \varphi_{-2,\mathcal{H}'_m}, \varphi_{-1,\mathcal{H}'_m}, \varphi_{0,\mathcal{H}'_m}, \delta_{\mathcal{H}'_m}, c_{\mathcal{H}'_m}, c^{\vee}_{\mathcal{H}'_m}, \tau_{\mathcal{H}'_m})$ be the $\mathcal{H}'_m$-orbit in $\alpha^{\natural}_{\mathcal{H}'}$, and let $(\alpha'_{\mathcal{H}_n})^{\natural} = (\mathtt{Z}'_{\mathcal{H}_n}, \varphi'_{-2,\mathcal{H}_n}, \varphi'_{-1,\mathcal{H}_n}, \varphi'_{0,\mathcal{H}_n}, \delta'_{\mathcal{H}_n}, c'_{\mathcal{H}_n}, (c^{\vee}_{\mathcal{H}_n})', \tau'_{\mathcal{H}_n})$ be the $\mathcal{H}_n$-orbit in $\alpha^{\natural}_{\mathcal{H}}$.

First let us determine the relation between $\mathtt{Z}_{\mathcal{H}'}$ and $\mathtt{Z}'_{\mathcal{H}}$, or equivalently the relation between $\mathtt{Z}_{\mathcal{H}'_m}$ and $\mathtt{Z}_{\mathcal{H}_n}$. Since this question is related only to the group schemes $G[m]$ and $G'[n]$ that are independent of the choice of N in Nf, we may take any N and consider the isogeny $Nf : G \twoheadrightarrow G'$ with kernel K_S as in the above construction. Let us extend the filtration $\mathtt{W}$ of $\mathrm{T}^{\Box}\, G_{\bar{\eta}}$,

$$0 \subset \mathtt{W}_{-2} = \mathrm{T}^{\Box}\, T_{\bar{\eta}} \subset \mathtt{W}_{-1} = \mathrm{T}^{\Box}\, G^{\natural}_{\bar{\eta}} \subset \mathtt{W}_0 = \mathrm{T}^{\Box}\, G_{\bar{\eta}},$$

to a filtration $\mathtt{W}_{\mathbb{A}^{\infty,\Box}}$ of $\mathrm{V}^{\Box}\, G_{\bar{\eta}}$:

$$0 \subset \mathtt{W}_{-2,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^{\Box}\, T_{\bar{\eta}} \subset \mathtt{W}_{-1,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^{\Box}\, G^{\natural}_{\bar{\eta}} \subset \mathtt{W}_{0,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^{\Box}\, G_{\bar{\eta}}.$$

Similarly, we have a filtration $\mathtt{W}'$ of $\mathrm{T}^{\Box}\, G'_{\bar{\eta}}$ that extends to a filtration $\mathtt{W}'_{\mathbb{A}^{\infty,\Box}}$ of $\mathrm{V}^{\Box}\, G'_{\bar{\eta}}$. According to Section 3.4.1, the quasi-finite flat group scheme K_η has a natural filtration

$$0 \subset K^{\mu}_{\eta} \subset K^{\mathrm{f}}_{\eta} \subset K_{\eta}.$$

Here K^{f}_{η} is the maximal subgroup scheme that extends uniquely to a finite flat subgroup scheme K^{f}_S of G, which is isomorphic to a finite flat subgroup scheme $K^{\natural}_S$ of $G^{\natural}$, and K^{μ}_{η} is the maximal subgroup scheme that extends to a finite flat subgroup scheme K^{μ}_S of K^{f}_S, which is identified with a finite flat subgroup scheme $K^{\flat}_S$ of T. Since $Nf_\eta : G_\eta \to G'_\eta$ is the quotient of G_η by K_η, and since the filtrations are compatible with this quotient, the isomorphism $\mathrm{V}^{\Box}(f_\eta) : \mathrm{V}^{\Box}\, G_{\bar{\eta}} \overset{\sim}{\to} \mathrm{V}^{\Box}\, G'_{\bar{\eta}}$ identifies the two filtrations $\mathtt{W}$ and $\mathtt{W}'$.

Let us take any symplectic lifting $\hat{\alpha}$ of the level-$\mathcal{H}'$ structure $\alpha_{\mathcal{H}'}$, namely any representative of the $\mathcal{H}'$-orbit $[\hat{\alpha}]_{\mathcal{H}'}$ associated to $\alpha_{\mathcal{H}'}$ that sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}\, G_{\bar{\eta}}$, and consider the induced isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \overset{\sim}{\to} \mathrm{V}^{\Box}\, G_{\bar{\eta}}$. Then the pullback of $\mathtt{W}$ by $\hat{\alpha}$ is a filtration $\mathtt{Z}$ of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ whose reduction mod m is a filtration $\mathtt{Z}_m$ in the $\mathcal{H}'_m$-orbit $\mathtt{Z}_{\mathcal{H}'_m}$, and the pullback of $\mathtt{W}_{\mathbb{A}^{\infty,\Box}}$ by $\hat{\alpha}$ is the natural filtration $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ extending $\mathtt{Z}$. In other words, we can interpret $\mathtt{Z}_{\mathcal{H}'}$ as the $\mathcal{H}'$-orbit of $\mathtt{Z}$. By construction of f_η, we have $\hat{\alpha}(g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})) =$

$\mathrm{V}^{\Box}(f_\eta)^{-1}(\mathrm{T}^{\Box}\, G'_{\bar{\eta}})$, or equivalently $\hat{\alpha}' = \mathrm{V}^{\Box}(f_\eta)\circ\hat{\alpha}\circ g$. Therefore the pullback of the filtration $\mathtt{W}'_{\mathbb{A}^{\infty,\Box}} = \mathrm{V}^{\Box}(f_\eta)(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})$ by $\hat{\alpha}'$ is $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}} := (\hat{\alpha}')^{-1}(\mathtt{W}'_{\mathbb{A}^{\infty,\Box}}) = g^{-1}(\hat{\alpha}^{-1}(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})) = g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$. Equivalently, this means $\mathtt{Z}' = g^{-1}(g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\cap \mathtt{Z}_{\mathbb{A}^{\infty,\Box}}) = (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$. By taking reduction mod n, we obtain a filtration $\mathtt{Z}'_n$ of L/nL in the $\mathcal{H}_n$-orbit $\mathtt{Z}_{\mathcal{H}_n}$. If we replace the lifting $\hat{\alpha}$ of $\alpha_{\mathcal{H}'}$ by $\hat{\alpha}\circ u$ for some $u\in\mathcal{H}'$, then the filtration $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}} = \hat{\alpha}^{-1}(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})$ is replaced by $u^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$, and hence the filtration $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}} = g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ is replace by $g^{-1}(u^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})) = (g^{-1}u^{-1}g)g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}) = (g^{-1}u^{-1}g)(\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}})$. Accordingly, the filtration $\mathtt{Z}' = (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ is replace by $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\cap (g^{-1}u^{-1}g)(g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}))$. Since $g^{-1}\mathcal{H}'g\subset\mathcal{H}$ by assumption, we see that its $\mathcal{H}$-orbit remains the same, and hence we see that the reduction mod n of $\mathtt{Z}'$ remains in the same $\mathcal{H}_n$-orbit. Hence the association of $\mathtt{Z}'_{\mathcal{H}_n}$ to $\mathtt{Z}_{\mathcal{H}'_m}$ as described above is well-defined and does not depend on the choice of $\hat{\alpha}$. (Reformulating what we have said, if we interpret alternatively $\mathtt{Z}_{\mathcal{H}'}$ (resp. $\mathtt{Z}_{\mathcal{H}}$) as a $\mathcal{H}'$-orbit (resp. $\mathcal{H}$-orbit) of some fully symplectic filtration $\mathtt{Z}$ (resp. $\mathtt{Z}'$) of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$, and if we set $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ (resp. $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}{}'}$) to be the fully symplectic filtrations of $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$ induced by $\mathtt{Z}$ (resp. $\mathtt{Z}'$), then the $\mathcal{H}'$-orbit of $g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ is contained in the $\mathcal{H}$-orbit of $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}$.)

Next let us investigate the relation between $\Phi_{\mathcal{H}'}$ and $\Phi'_{\mathcal{H}}$, or equivalently the relation between $\Phi_{\mathcal{H}'_m}$ and $\Phi_{\mathcal{H}_n}$. Consider the exact sequence

$$0\to\mathrm{T}^{\Box}\,G_{\bar{\eta}}\to\mathrm{V}^{\Box}\,G_{\bar{\eta}}\to(G_{\bar{\eta}})^{\Box}_{\mathrm{tors}}\to 0,$$

which was first introduced in Section 1.3.5. Then the above-chosen isomorphism $\hat{\alpha}: L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}\xrightarrow{\sim}\mathrm{T}^{\Box}\,G_{\bar{\eta}}$ and its natural extension $\hat{\alpha}: L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\xrightarrow{\sim}\mathrm{V}^{\Box}\,G_{\bar{\eta}}$ defines an isomorphism $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})/(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\xrightarrow{\sim}(G_{\bar{\eta}})^{\Box}_{\mathrm{tors}}$. Let us still denote this isomorphism by $\hat{\alpha}$. Then we may identify $K_\eta = \hat{\alpha}(g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})/(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}))$, with filtration given by $K^{\mathrm{f}}_\eta = \hat{\alpha}(g(\mathtt{Z}'_{-1})/\mathtt{Z}_{-1})$ and $K^{\mu}_\eta = \hat{\alpha}(g(\mathtt{Z}'_{-2})/\mathtt{Z}_{-2})$. Here we are abusing the difference between a finite étale group scheme over a geometric point, and its group of closed geometric points.

The group scheme K^{μ}_η extends uniquely to a finite flat subgroup scheme K^{μ}_S of G over S, which is isomorphic to a finite flat subgroup scheme $K^{\natural}_S$ of T. The group scheme $K^{\natural}_S$ is the kernel of the isogeny $Nf_T: T\twoheadrightarrow T'$ induced by Nf, and therefore is dual to the cokernel of the inclusion $Nf_X: X'\hookrightarrow X$ of character groups. In particular, we may identify X' as an $\mathcal{O}$-sublattice in $X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, and define an isomorphism $f_X : X'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \xrightarrow{\sim} X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ by $f_X = N^{-1} \circ Nf_X$. This is dual to the "$\mathbb{Z}_{(\Box)}^{\times}$-isogeny" $f_T : T \to T'$.

On the other hand, the group scheme K_η^{f} extends uniquely to a finite flat subgroup scheme K_S^{f} of G over S, which is isomorphic to a finite flat subgroup scheme $K_S^\natural$ of $G^\natural$. The group scheme $K_S^\natural$ is the kernel of the isogeny $Nf^\natural : G^\natural \twoheadrightarrow (G')^\natural$ induced by Nf. The quotient $K_S/K_S^\natural$ of K_S by $K_S^\natural$ is an étale group scheme (which is not necessarily finite) over S, and its restriction $(K_S/K_S^\natural)_\eta$ to η is isomorphic to the cokernel of the inclusion $Nf_Y : Y \hookrightarrow Y'$ of character groups. Then we may identify Y' as an $\mathcal{O}$-sublattice in $Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, and define an isomorphism $f_Y : Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ by $f_Y = N^{-1} \circ Nf_Y$. We may also interpret this f_Y as the dual of some "$\mathbb{Z}_{(\Box)}^{\times}$-isogeny" $(T^\vee)' \to T^\vee$, as follows: By repeating the construction of $f : G \to G'$ for the dual objects, we obtain the "dual $\mathbb{Z}_{(\Box)}^{\times}$-isogeny" $f^\vee : (G')^\vee \to G^\vee$ that extends $f_\eta^\vee : (G'_\eta)^\vee \to G_\eta^\vee$. Then the restriction of $f^\vee$ to the torus part $(T^\vee)'$ of G' gives the above $(T^\vee)' \to T^\vee$.

Under the isomorphism $\mathrm{V}^\Box(f_\eta) : \mathrm{V}^\Box G_{\bar{\eta}} \xrightarrow{\sim} \mathrm{V}^\Box G'_{\bar{\eta}}$, and under the two isomorphisms $\mathrm{V}^\Box(f_{T,\eta}) : \mathrm{V}^\Box T_{\bar{\eta}} \xrightarrow{\sim} \mathrm{V}^\Box T'_{\bar{\eta}}$ and $f_Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} : Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \xrightarrow{\sim} Y'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$, the canonical pairing $\mathrm{e}^\phi(\,\cdot\,,\,\cdot\,) : \mathrm{V}^\Box T_{\bar{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) \to \mathrm{V}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$ induced by the λ-Weil pairing $\mathrm{e}^\lambda(\,\cdot\,,\,\cdot\,)$ (as in Proposition 5.2.2.1) can be rewritten as a pairing $\mathrm{V}^\Box T'_{\bar{\eta}} \times (Y'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) \to \mathrm{V}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$, which we may denote by $\mathrm{e}^{f_X^{-1}\circ\phi\circ f_Y^{-1}}(\,\cdot\,,\,\cdot\,)$. The restriction of this pairing gives a pairing $\mathrm{T}^\Box T'_{\bar{\eta}} \times (Y'\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \to r\,\mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$, where $r \in \mathbb{Z}_{(\Box),>0}^{\times}$ is the unique number such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}^{\Box,\times}$ in the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}_{(\Box),>0}^{\times} \cdot \hat{\mathbb{Z}}^{\Box,\times}$. Comparing this with the canonical pairing $\mathrm{e}^{\phi'}(\,\cdot\,,\,\cdot\,) : \mathrm{T}^\Box T'_{\bar{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \to \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$ induced by the λ'-Weil pairing $\mathrm{e}^{\lambda'}(\,\cdot\,,\,\cdot\,)$, and taking into account the relation $\lambda' = r^{-1}(f_\eta^\vee)^{-1} \circ \lambda \circ f_\eta$, we see that we must have $\phi' = r^{-1} f_X^{-1} \circ \phi \circ f_Y^{-1}$, or equivalently $f_X \circ \phi' \circ f_Y = r\phi$.

Let us give a more intrinsic interpretation of f_X and f_Y via a comparison between $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'_{\mathcal{H}'}})$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}_{\mathcal{H}}})$. First note that $\mathrm{Gr}(\hat{\alpha})$ determines a well-defined pair $(\varphi_{-2}, \varphi_0)$, so that $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1))$ differ from $\mathrm{Gr}_{-2}(\hat{\alpha})$ by the isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^\Box(1) \xrightarrow{\sim} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\bar{\eta}}$, and so that $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ is simply

$\mathrm{Gr}_0(\hat{\alpha})$. Similarly, $\mathrm{Gr}(\hat{\alpha}')$ determines a well-defined pair $(\varphi'_{-2}, \varphi'_0)$. Then $(\varphi_{-2,\mathcal{H}'_m}, \varphi_{0,\mathcal{H}'_m})$ is the $\mathcal{H}'_m$-orbit of the reduction mod m of $(\varphi_{-2}, \varphi_0)$, and $(\varphi'_{-2,\mathcal{H}_n}, \varphi'_{0,\mathcal{H}_n})$ is the $\mathcal{H}_n$-orbit of the reduction mod n of $(\varphi'_{-2}, \varphi'_0)$. Let us also denote by $\varphi_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} : \mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}} \overset{\sim}{\to} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1))$, etc, the induced morphisms between $\mathbb{A}^{\infty,\Box}$-modules. The isomorphism $g : L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \overset{\sim}{\to} L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$ sends the filtration $\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}} = g^{-1}(\mathsf{Z}_{\mathbb{A}^{\infty,\Box}})$ to $\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}$, which induces an isomorphism $\mathrm{Gr}_{-i}(g) : \mathrm{Gr}_{-i}^{\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}}} = \mathsf{Z}'_{-i,\mathbb{A}^{\infty,\Box}}/\mathsf{Z}'_{-i-1,\mathbb{A}^{\infty,\Box}} \overset{\sim}{\to} \mathrm{Gr}_{-i}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}} := \mathsf{Z}_{-i,\mathbb{A}^{\infty,\Box}}/\mathsf{Z}_{-i-1,\mathbb{A}^{\infty,\Box}}$ on each of the graded pieces. Then the pairs $(\varphi_{-2}, \varphi_0)$ and $(\varphi'_{-2}, \varphi'_0)$ are related by the following commutating relations:

$$\begin{tikzcd}
\mathrm{Gr}_{-2}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}} \arrow[r, "\varphi_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}", "\sim"'] & \mathrm{Hom}_{\mathbb{A}^{\infty,\Box}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1)) \arrow[d, "\wr"', "{}^t f_X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}"] \\
\mathrm{Gr}_{-2}^{\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}}} \arrow[u, "\nu(g)^{-1}\,\mathrm{Gr}_{-2}(g)", "\wr"'] \arrow[r, "\sim", "\varphi'_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}"'] & \mathrm{Hom}_{\mathbb{A}^{\infty,\Box}}(X'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1))
\end{tikzcd} \tag{5.4.3.1}$$

and

$$\begin{tikzcd}
\mathrm{Gr}_{0}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}} \arrow[r, "\varphi_{0}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}", "\sim"'] & Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \arrow[d, "\wr"', "f_Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}"] \\
\mathrm{Gr}_{-2}^{\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}}} \arrow[u, "\mathrm{Gr}_{-2}(g)", "\wr"'] \arrow[r, "\sim", "\varphi'_{0}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}"'] & Y'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}
\end{tikzcd} . \tag{5.4.3.2}$$

Lemma 5.4.3.3. *We have the approximations* $\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) = (\mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})) \cdot \mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ *and* $\mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) = (\mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})) \cdot \mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$.

Proof. Let us first note that we have the approximations $\mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) = \mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cdot \mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ and $\mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) = \mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cdot \mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ by elementary lattice theory. Then the result follows by specializing this approximation to the subgroups $\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$ and $\mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$ of $\mathcal{O}$-equivariant elements in respectively $\mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$ and $\mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$. $\Box$

Lemma 5.4.3.4. *We have the relations* $\mathrm{GL}_{\mathcal{O}}(X) = (\mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})) \cap \mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ *and* $\mathrm{GL}_{\mathcal{O}}(Y) = (\mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})) \cap \mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$.

Proof. These follows respectively from the relations $\mathrm{GL}(X) = \mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$ and $\mathrm{GL}(Y) = \mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$. $\Box$

Now we can give an alternative interpretation of f_X and f_Y. The submodule $\mathrm{Gr}^{\mathsf{Z}'}_{-2} \subset \mathrm{Gr}^{\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}}}_{-2}$ is pulled back to a submodule $(\nu^{-1}(g)\,\mathrm{Gr}_{-2}(g))(\mathrm{Gr}^{\mathsf{Z}'}_{-2}) = (\nu(g)\,\mathrm{Gr}_{-2}(g^{-1}))^{-1}(\mathrm{Gr}^{\mathsf{Z}'}_{-2}) \subset \mathrm{Gr}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}}_{-2}$. Under the isomorphism $\varphi_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} : \mathrm{Gr}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}^{\infty,\Box}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1))$, the difference between this submodule and the submodule $\mathrm{Gr}^{\mathsf{Z}}_{-2}$ of $\mathrm{Gr}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}}_{-2}$ can be given by the transpose of an element in $\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$. By Lemma 5.4.3.3, this element can be approximated by element in $\mathrm{GL}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})$ up to a difference in $\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})$. By Lemma 5.4.3.4, this element in $\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)})$ determines a *unique* choice of an $\mathcal{O}$-lattice X' in $X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, which is equivalent to the unique choice of an $\mathcal{O}$-lattice X' up to isomorphism, and an isomorphism $f_X : X'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \xrightarrow{\sim} X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$. This choice of f_X also determines an isomorphism $\varphi'_{-2} : \mathrm{Gr}^{\mathsf{Z}'}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ that makes the diagram (5.4.3.1) commutative. Similarly, the submodule $\mathrm{Gr}^{\mathsf{Z}'}_{0} \subset \mathrm{Gr}^{\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}}}_{0}$ is pulled back to a submodule $\mathrm{Gr}_0(g)(\mathrm{Gr}^{\mathsf{Z}'}_{0}) = \mathrm{Gr}_0(g^{-1})^{-1}(\mathrm{Gr}^{\mathsf{Z}'}_{0}) \subset \mathrm{Gr}^{\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}}_{0}$, and such a submodule determines a unique choice of an $\mathcal{O}$-lattice Y' in $Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$. The choice of Y' as a sublattice in $Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ is equivalent to the unique choice of an $\mathcal{O}$-lattice Y' up to isomorphism together with the unique choice of an isomorphism $f_Y : Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$. This choice of f_Y also determines an isomorphism $\varphi'_0 : \mathrm{Gr}^{\mathsf{Z}'}_{0} \xrightarrow{\sim} Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ that makes the diagram (5.4.3.2) commutative. Note that the two commuting relations (5.4.3.1) and (5.4.3.2) necessarily forces the relation $f_X\phi' f_Y = r^{-1}\phi$ above. This gives an association of $\Phi' := (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ to $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$, such that the class of the $\mathcal{H}_n$-orbit of the reduction mod n of Φ' depends only on the class of the $\mathcal{H}'_m$-orbit of the reduction mod m of Φ. In other words, this gives a well-defined association of the class of $\Phi'_{\mathcal{H}} := (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ to the class of $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$.

Let us summarize the above as the following recipe:

Proposition 5.4.3.5. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, *and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *in* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. *Then* g *defines a map from the cusp labels at level* $\mathcal{H}'$ *to cusp labels at level* $\mathcal{H}$, *as follows: Suppose we have a cusp label at level* $\mathcal{H}'$ *represented by some* $(\mathtt{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$.

1. *Let* $\hat{\alpha}$ *be any symplectic lifting of the level-*$\mathcal{H}'$ *structure* $\alpha_{\mathcal{H}'}$. *Then* $\hat{\alpha}$ *pulls back the filtration* $\mathtt{W}$ *on* $\mathrm{T}^{\Box} G_{\bar{\eta}}$ *to a fully symplectic filtration* $\mathtt{Z}$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, *together with a torus argument* $(X, Y, \phi, \varphi_{-2}, \varphi_0)$ *for* $\mathtt{Z}$. *The* $\mathcal{H}'$*-orbit of* $\mathtt{Z}$ *then determines* $\mathtt{Z}_{\mathcal{H}'}$.

2. *Let* $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ *be the filtration on* $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$ *that extends* $\mathtt{Z}$. *Then we obtain a filtration* $\mathtt{Z}' := (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, *whose* $\mathcal{H}$*-orbit is independent of the choice of* $\mathtt{Z}$ *and determines* $\mathtt{Z}'_{\mathcal{H}}$.

3. *The submodule* $\mathrm{Gr}_{-2}^{\mathtt{Z}'} \subset \mathrm{Gr}_{-2}^{\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}}$ *is pulled back to a submodule* $(\nu(g)^{-1} \mathrm{Gr}_{-2}(g))(\mathrm{Gr}_{-2}^{\mathtt{Z}'}) \subset \mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$. *Under the isomorphism* $\varphi_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} : \mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}^{\infty,\Box}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1))$, *the difference between this submodule and the submodule* $\mathrm{Gr}_{-2}^{\mathtt{Z}}$ *of* $\mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$ *can be given by the transpose of an element in* $\mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$, *which by Lemmas 5.4.3.3 and 5.4.3.4 determines the unique choice of an* $\mathcal{O}$*-lattice* X' *up to isomorphism and the unique choice of an isomorphism* $f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. *This choice of* f_X *also determines an isomorphism* φ'_{-2} *that makes the diagram (5.4.3.1) commutative.*

4. *Similarly, the submodule* $\mathrm{Gr}_{0}^{\mathtt{Z}'} \subset \mathrm{Gr}_{0}^{\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}}$ *is pulled back to a submodule* $\mathrm{Gr}_0(g)(\mathrm{Gr}_0^{\mathtt{Z}'}) \subset \mathrm{Gr}_0^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$, *and such a submodule determines a unique choice of an* $\mathcal{O}$*-lattice* Y' *in* $Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$, *which is equivalent to the unique choice of an* $\mathcal{O}$*-lattice* Y' *up to isomorphism and the unique choice of an isomorphism* $f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. *This choice of* f_Y *also determines an isomorphism* φ'_0 *that makes the diagram (5.4.3.2) commutative.*

5. *Let $r \in \mathbb{Z}^{\times}_{(\Box),>0}$ be the unique number such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}^{\Box,\times}$ in the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}^{\times}_{(\Box),>0} \cdot \hat{\mathbb{Z}}^{\Box,\times}$. Then we set $\phi' = r^{-1} f_X^{-1} \phi f_Y^{-1}$.*

The above steps determines a torus argument $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ up to equivalence, which determines a class of torus argument $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ at level $\mathcal{H}$ depending only on the class of the torus argument $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$. Since the cusp labels do not depend on the choice of the splittings $\delta_{\mathcal{H}'}$ and $\delta_{\mathcal{H}}$, we obtain a well-defined map from the cusp labels at level $\mathcal{H}'$ to cusp labels at level $\mathcal{H}$.

Definition 5.4.3.6. *Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, and suppose we have two open compact subgroups $\mathcal{H}' \subset \mathcal{H}$ such that $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. We say that a triple $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ is g-**associated** to $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$, written as $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}) \to_g (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, if there are isomorphisms $f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ and $f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ as in Proposition 5.4.3.5 that associate some lifting $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ of $\Phi'_{\mathcal{H}}$ to some lifting $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ of $\Phi_{\mathcal{H}'}$. In this case we say that there is a g-**association** $(f_X, f_Y) : (\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}) \to_g (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.*

Remark 5.4.3.7. The pair (f_X, f_Y) of the two possible maps f_X and f_Y in Definition 5.4.3.6 is unique only up to multiplication by elements in $\mathrm{GL}_{\phi'}$ (defined analogous to GL_{ϕ} as in Definition 5.4.1.7) that leaves $\Phi'_{\mathcal{H}}$ invariant. (Later such a subgroup will be called $\Gamma_{\Phi'_{\mathcal{H}}}$ in Definition 6.2.4.1.)

The above result will be applied in Section 6.4.3 to the study of Hecke actions on tower of toroidal compactifications, after we have the meaning of cusp labels as part of the parameters of the stratifications on the toroidal compactifications we construct.

It is an interesting question whether $\Phi'_{\mathcal{H}}$ can be equivalent to the $\mathcal{H}$-*orbit* of $\Phi_{\mathcal{H}'}$ in its natural sense (by Convention 5.3.1.13), in which case the cusp label at level $\mathcal{H}'$ defined by $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ is mapped to the cusp label at level $\mathcal{H}$ determined by the $\mathcal{H}$-orbit of $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$. This can be interpreted as left invariance under the action of g when we consider a tower of cusp labels. Let us make this idea precise. Take any triple $(\mathsf{Z}, \Phi = (X, Y, \phi, \varphi_{-2}, \varphi_{-1}), \delta)$, so that we have a tower of cusp labels defined by its $\mathcal{H}$-orbits as $\mathcal{H}$ varies among open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. In order for the g-association to preserve this tower, we first need $\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}$ and $\mathsf{Z}'_{\mathbb{A}^{\infty,\Box}} := g^{-1}(\mathsf{Z}_{\mathbb{A}^{\infty,\Box}})$ to be identical. This is simply the condition that g preserves $\mathsf{Z}_{\mathbb{A}^{\infty,\Box}}$.

Definition 5.4.3.8. *For any* $\hat{\mathbb{Z}}^\Box$*-algebra* R*, set*

$$\mathrm{P}_{\mathtt{Z}}(R) := \{g \in \mathrm{G}(R) : g(\mathtt{Z}_{-i} \underset{\mathbb{Z}}{\otimes} R) \subset (\mathtt{Z}_{-i} \underset{\mathbb{Z}}{\otimes} R)\}.$$

Definition 5.4.3.9. *For any* $\hat{\mathbb{Z}}^\Box$*-algebra* R*, set* $\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}}(R) = \mathrm{image}(\mathrm{P}_{\mathtt{Z}}(\hat{\mathbb{Z}}^\Box) \to \mathrm{P}(R)$*.*

Then we see that the condition that g preserves $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ means $g \in \mathrm{P}_{\mathtt{Z}}(\mathbb{A}^{\infty,\Box}) = \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}}(\mathbb{A}^{\infty,\Box})$. The next question is whether g preserves the equivalence class of Φ.

Lemma 5.4.3.10. *Let* $\mathtt{Z}$ *be an admissible filtration of* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ *that is fully symplectic with respect to* $(L, \langle \cdot\,, \cdot \rangle)$*, and let* $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ *be any torus argument as in Definition 5.4.1.3. For each* $\hat{\mathbb{Z}}^\Box$*-algebra* R*, let us define a map*

$$\begin{aligned} \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}}(R) &\to \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} R) \times \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} R) \\ g &\mapsto (g_X, g_Y) := ({}^t((\varphi_{-2}^{-1})^*(\nu(g)^{-1}\,\mathrm{Gr}_{-2}(g)), (\varphi_0^{-1})^*(\mathrm{Gr}_0(g))). \end{aligned}$$

Then the image satisfies $\phi = g_X \phi g_Y$ *and defines an element in* $\mathrm{GL}_\phi(R)$ *(defined as in Definition 5.4.1.7).*

Definition 5.4.3.11. *The above assignment with image in* $\mathrm{GL}_\phi(R)$ *for any* $\hat{\mathbb{Z}}^\Box$*-algebra* R *defines a map* $\zeta_{\mathtt{Z},\Phi} : \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}} \to \mathrm{GL}_\phi$ *between group functors over* $\hat{\mathbb{Z}}^\Box$*.*

Back to the question of whether g preserves the equivalence class of Φ. For this to be the case, we need Φ to be mapped to a tuple $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ such that X' and Y' are isomorphic to respectively X and Y as $\mathcal{O}$ lattices. According to the recipe described in Proposition 5.4.3.5, we see this is equivalent to the statement that $\zeta_{\mathtt{Z},\Phi}(g)$ lies in the image of $\mathrm{GL}_\phi(\mathbb{Z}_{(\Box)}) \to \mathrm{GL}_\phi(\mathbb{A}^{\infty,\Box}))$. In this case, there exists an isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y') : \Phi \xrightarrow{\sim} \Phi'$. This motivates the following definition:

Definition 5.4.3.12.

$$\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}} := \{g \in \mathrm{P}_{\mathtt{Z}}(\mathbb{A}^{\infty,\Box}) : \zeta_{\mathtt{Z},\Phi}(g) \in \mathrm{image}(\mathrm{GL}_\phi(\mathbb{Z}_{(\Box)}) \to \mathrm{GL}_\phi(\mathbb{A}^{\infty,\Box}))\}.$$

Then what we have said can be written as:

Proposition 5.4.3.13. *The Hecke action of* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$ *preserves the tower of cusp labels defined by the triple* $(\mathtt{Z}, \Phi, \delta)$ *if and only if* $g \in \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}}$.

Corollary 5.4.3.14. *Starting with a particular triple* $(\mathtt{Z}, \Phi, \delta)$ *as above, the cusp labels at level* $\mathcal{H}$ *that we could produce by the Hecke action of elements in* $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ *is parameterized by* $\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}}\backslash\mathrm{G}(\mathbb{A}^{\infty,\Box})/\mathcal{H}$.

It is also interesting to know what we can produce by the Hecke action of elements in $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ only, as we do not have to modify the underlying geometric object when performing such actions.

Definition 5.4.3.15. *For each* $\hat{\mathbb{Z}}^{\Box}$*-algebra* R*, set*

$$\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi}(R) := \{g \in \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}}(R) : \zeta_{\mathtt{Z},\Phi}(g) \in \mathrm{image}(\Gamma_\phi \to \mathrm{GL}_\phi(R))\}.$$

Lemma 5.4.3.16. $\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}} \cap \mathrm{G}(\hat{\mathbb{Z}}^{\Box}) = \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi}(\hat{\mathbb{Z}}^{\Box})$.

Proof. Simply because $\mathrm{GL}_\phi(\mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_\phi(\hat{\mathbb{Z}}^{\Box}) = \mathrm{GL}_\phi(\mathbb{Z}) = \Gamma_\phi$. ☐

Lemma 5.4.3.17. *The natural injection* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box}) \hookrightarrow \mathrm{G}(\mathbb{A}^{\infty,\Box})$ *induces a natural injection*

$$\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi}(\hat{\mathbb{Z}}^{\Box})\backslash\mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{H} \hookrightarrow \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}}\backslash\mathrm{G}(\mathbb{A}^{\infty,\Box})/\mathcal{H}.$$

The injection is also a surjection if the approximation $\mathrm{G}(\mathbb{A}^{\infty,\Box}) = \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}} \cdot \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *holds.*

Proof. Suppose two elements g_1 and g_2 in $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ define the same element in $\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}}\backslash\mathrm{G}(\mathbb{A}^{\infty,\Box})/\mathcal{H}$. By definition, this means there are elements $p \in \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}}$ and $u \in \mathcal{H}$ that satisfy $g_1 = pg_2u$. Then $p = g_1u^{-1}g_2^{-1} \in \mathrm{G}^{\mathrm{ess}}(\hat{\mathbb{Z}}^{\Box})$ shows that $p \in \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}} \cap \mathrm{G}(\hat{\mathbb{Z}}^{\Box}) = \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi}(\hat{\mathbb{Z}}^{\Box})$. In other words, g_1 and g_2 define the same element in $\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi}(\hat{\mathbb{Z}}^{\Box})\backslash\mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{H}$. This proves the first statement. The second statement is a tautology. ☐

Remark 5.4.3.18. We do not expect the injection to be also a surjection, because there is no reason for the approximation $\mathrm{G}(\mathbb{A}^{\infty,\Box}) = \mathrm{P}^{\mathrm{ess}}_{\mathtt{Z},\Phi,\mathbb{A}^{\infty,\Box}} \cdot \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ to hold. In general there should be many distinct isomorphism classes of Φ, in particular of X. It holds anyway in the easiest Siegel moduli case in [37].

Then a special case of Corollary 5.4.3.19 is:

Corollary 5.4.3.19. *Starting with a particular representative* $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of a cusp label at level* $\mathcal{H}$*, the cusp labels at level* $\mathcal{H}$ *that we could produce by the Hecke action of elements in* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *on* $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *is parameterized by* $\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z},\Phi}(\hat{\mathbb{Z}}^{\Box})\backslash\mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{H}$*.*

Note that this is in bijection with the set of objects $(G, \lambda, i, \alpha_{\mathcal{H}})$ of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ sharing the same underlying triple (G, λ, i). (That is, this is in bijection with the level-$\mathcal{H}$ structures that can exist on the particular underlying object.)

Chapter 6

Algebraic Constructions of Toroidal Compactifications

We will generalize the techniques in [37] and construct the toroidal compactifications of the moduli problems we have considered in Chapter 1.

The main objective of this chapter is to state and prove Theorem 6.4.1, with byproducts concerning Hecke actions given in Sections 6.4.2 and 6.4.3. Technical results worth noting are Propositions 6.2.2.5, 6.2.3.2, 6.2.3.25, and 6.2.5.14 in Section 6.2; and Lemma 6.3.1.7, Proposition 6.3.2.5, 6.3.2.13, 6.3.3.3, 6.3.3.11, and 6.3.3.18 in Section 6.3.

6.1 Review of Toroidal Embeddings

In this section, we shall discuss not only the usual toroidal embeddings defined using actions of tori, but the slightly more generalized toroidal embeddings defined using actions of groups of multiplicative type of finite type.

Remark 6.1.1. We do not know if there is any direct reference for these generalized toroidal embeddings. Nevertheless, since the groups of multiplicative type of finite type contains the torus defined by the torsion-free quotient of its character group, it is possible to define the toroidal embeddings using the action of this torus. The resulted theory is essentially the same. The theory of the usual toroidal embeddings can be found in [72]. Although we will only need the case when H is a split torus in Section 6.2.5, we do want to introduce these more generalized toroidal embeddings, because they suggest that we can construct boundary charts providing the positivity condition without having to assume the liftability condition of the tautological degeneration data. In other words, the two notions are essentially independent, even if we do not need this independence for the arithmetic compactification of PEL-type Shimura varieties.

6.1.1 Rational Polyhedral Cone Decompositions

Let H be a group of multiplicative type of finite type over S, so that its character group $\underline{X}(H) = \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$ is an étale sheaf of finitely generated abelian groups.

Definition 6.1.1.1. *The* **cocharacter group** $\underline{X}(H)^\vee$ *of* H (*over* S) *is the étale sheaf of finitely generated* (*free*) *abelian groups* $\underline{\mathrm{Hom}}_S(\mathbf{G}_{\mathrm{m},S}, H) \cong \underline{\mathrm{Hom}}_S(\underline{X}(H), \mathbb{Z})$.

Remark 6.1.1.2. Since $\underline{X}(H)^\vee$ is always *torsion-free*, the usual natural pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : \underline{X}(H) \times \underline{X}(H)^\vee \to \mathbb{Z}$$

is perfect only when H is a torus.

Assumption 6.1.1.3. *From now on, we shall assume that* H *is split.*

Let us consider the $\mathbb{R}$-vector space $\underline{X}(H)^\vee_\mathbb{R} = \underline{X}(H)^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Definition 6.1.1.4. *A subset of* $\underline{X}(H)^\vee_\mathbb{R}$ *in* $\underline{X}(H)^\vee_\mathbb{R}$ *is called a* **cone** *if it is invariant under the natural action of* $\mathbb{R}^\times_{>0}$ *on the* $\mathbb{R}$*-vector space.*

Definition 6.1.1.5. *A cone in* $\underline{X}(H)^\vee_\mathbb{R}$ *is* **nondegenerate** *if its closure does not contain any nonzero* $\mathbb{R}$*-vector subspace of* $\underline{X}(H)^\vee_\mathbb{R}$.

Definition 6.1.1.6. *A* **rational polyhedral cone** *in* $\underline{X}(H)^\vee_\mathbb{R}$ *is a cone in* $\underline{X}(H)^\vee_\mathbb{R}$ *of the form* $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ *with* $v_1, \ldots, v_n \in \underline{X}(H)^\vee_\mathbb{Q} = \underline{X}(H)^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Note that σ is an open subset in its closure $\overline{\sigma} = \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_n$, and also open in the smallest $\mathbb{R}$-vector subspace $\mathbb{R}v_1 + \ldots + \mathbb{R}v_n$ containing σ.

Definition 6.1.1.7. *A* **supporting hyperplane** H *of* σ *is a hyperplane in* $\underline{X}(H)^\vee_\mathbb{R}$ *such that* σ *lies entirely on one side of* H.

Definition 6.1.1.8. *A* **face** *of σ is a rational polyhedral cone τ such that $\overline{\tau} = \overline{\sigma} \cap H$ for some supporting hyperplane H of σ.*

Note that the natural pairing $\langle \cdot \, , \cdot \rangle : \underline{X}(H) \times \underline{X}(H)^\vee \to \mathbb{Z}$ defines by extension of scalar a natural pairing

$$\langle \cdot \, , \cdot \rangle : \underline{X}(H) \times \underline{X}(H)^\vee_{\mathbb{R}} \to \mathbb{R}$$

whose restriction to $\underline{X}(H) \times \underline{X}(H)^\vee$ gives the original pairing. (We do not tensor $\underline{X}(H)$ with $\mathbb{R}$, because the map $\underline{X}(H) \to \underline{X}(H) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is not injective when there exist nontrivial torsion elements in $\underline{X}(H)$.)

Definition 6.1.1.9. *If σ is a rational polyhedral cone in $\underline{X}(H)^\vee_{\mathbb{R}}$, then $\sigma^\vee$ is the semi-subgroup (with unit 0) of $\underline{X}(H)$ defined by*

$$\sigma^\vee := \{x \in \underline{X}(H) : \langle x, y \rangle \geq 0, \forall y \in \sigma\},$$

and $\sigma^\vee_0$ is the semi-subgroup (without unit 0) of $\underline{X}(H)$ defined by

$$\sigma^\vee_0 := \{x \in \underline{X}(H) : \langle x, y \rangle > 0, \forall y \in \sigma\}.$$

Remark 6.1.1.10. In the case that H is a torus, $\sigma^\vee$ is defined to be a cone in the Euclidean space $\underline{X}(H)_{\mathbb{R}}$, called the *dual* (*closed*) *cone* of σ.

Remark 6.1.1.11. Even if we consider the cones they span in $\underline{X}(H)_{\mathbb{R}}$, the cone $\mathbb{R}^\times_{>0} \cdot \sigma^\vee_0$ is *not* the interior of the closed cone $\mathbb{R}^\times_{>0} \cdot \sigma^\vee$ in general: Try any top-dimensional nondegenerate rational polyhedral cone σ in $\mathbb{R}^2$.

Let Γ be any group acting on $\underline{X}(H)$, which induces an action on H and hence also an action on $\underline{X}(H)^\vee$. Let C be any cone in $\underline{X}(H)^\vee_{\mathbb{R}}$.

Definition 6.1.1.12. *A* **Γ-admissible rational polyhedral cone decomposition** *of C is a collection $\Sigma = \{\sigma_j\}_{j \in J}$ with some indexing set J such that:*

1. *Every σ_j is a nondegenerate rational polyhedral cone.*
2. *C is the disjoint union of all the σ_j's in Σ. For each $j \in J$, the closure of σ_j in C is a disjoint union of σ_k's with $k \in J$. In other words, $C = \coprod\limits_{j \in J} \sigma_j$ is a stratification of C.*
3. *Σ is Γ-invariant under the action of Γ on $\underline{X}(H)^\vee_{\mathbb{R}}$, in the sense that Γ permutes the cones in Σ. Under this action, the set Σ/Γ of Γ-orbits is finite.*

Definition 6.1.1.13. *A rational polyhedral cone σ in $\underline{X}(H)^\vee_{\mathbb{R}}$ is* **smooth** *with respect to the integral structure given by $\underline{X}(H)^\vee$ if we have $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ with $v_1, \ldots, v_n$ part of a $\mathbb{Z}$-basis of $\underline{X}(H)^\vee$.*

Definition 6.1.1.14. *A* **Γ-admissible smooth rational polyhedral cone decomposition** *of C is a Γ-admissible rational polyhedral cone decomposition $\{\sigma_j\}_{j\in J}$ of C in which every σ_j is* **smooth***.*

6.1.2 Toroidal Embeddings of Torsors

Let $\mathcal{M}$ be an H-torsor over a scheme Z. Then $\mathcal{M}$ is relatively affine over Z, and the H-action on $\mathscr{O}_{\mathcal{M}}$ gives a decomposition

$$\mathscr{O}_{\mathcal{M}} = \underset{\chi\in\underline{X}(H)}{\oplus} \mathscr{O}_{\mathcal{M},\chi},$$

where $\mathscr{O}_{\mathcal{M},\chi}$ is the invertible sheaf of χ-eigenspaces under H-action, together with isomorphisms

$$\mathscr{O}_{\mathcal{M},\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M},\chi'} \xrightarrow{\sim} \mathscr{O}_{\mathcal{M},\chi+\chi'} \tag{6.1.2.1}$$

giving the structure of $\mathscr{O}_{\mathcal{M}}$ as an $\mathscr{O}_Z$-sheaf of algebras.

Remark 6.1.2.2. The specification of the isomorphisms in (6.1.2.1) is necessary because we are not assuming that Z satisfies the Assumption 3.1.2.7 needed for application of Theorem 3.1.3.7. We can not necessarily rigidified our H-torsor $\mathcal{M}$ or the invertible sheaves $\mathscr{O}_{\mathcal{M},\chi}$ in this general case.

Definition 6.1.2.3. *For any rational polyhedral cone σ in $\underline{X}(H)^\vee_{\mathbb{R}}$, the* **affine toroidal embedding** *$\mathcal{M}(\sigma)$ along σ is defined to be the relative affine scheme*

$$\mathcal{M}(\sigma) = \underline{\mathrm{Spec}}_{\mathscr{O}_Z}(\underset{\chi\in\sigma^\vee}{\oplus} \mathscr{O}_{\mathcal{M},\chi}).$$

over Z, where $\underset{\chi\in\sigma^\vee}{\oplus} \mathscr{O}_{\mathcal{M},\chi}$ has the structure of an $\mathscr{O}_Z$-sheaf of algebras given by the isomorphisms in (6.1.2.1).

Note that by construction, the H-action on $\mathcal{M}$ extends naturally to $\mathcal{M}(\sigma)$.

Lemma 6.1.2.4. *In Definition 6.1.2.3, if τ is a face of σ, then there is an H-equivariant canonical embedding $\mathcal{M}(\tau) \hookrightarrow \mathcal{M}(\sigma)$ defined by the natural inclusion of structural sheaves.*

Definition 6.1.2.5. *For any rational polyhedral cone σ in $\underline{X}(H)^\vee_{\mathbb{R}}$, define*

$$\sigma^\perp := \{x \in \underline{X}(H) : \langle x, y \rangle = 0, \forall y \in \sigma.\}.$$

This is an abelian subgroup of $\underline{X}(H)$, and defines a quotient group H_σ of multiplicative type of finite type of H over Z.

Lemma 6.1.2.6. *With the setting as above, the closed subscheme*

$$\mathcal{M}_\sigma := \mathcal{M}(\sigma) - \bigcup_{\substack{\tau \text{ is a} \\ \text{face of } \sigma \\ \tau \neq \sigma}} \mathcal{M}(\tau)$$

can be defined by the sheaf of ideals

$$\mathscr{I}_\sigma := \bigoplus_{\chi \in \sigma_0^\vee} \mathscr{O}_{\mathcal{M},\chi},$$

which can be identified with

$$\mathcal{M}_\sigma = \underline{\mathrm{Spec}}_{\mathscr{O}_Z}\big(\bigoplus_{\chi \in \sigma^\perp} \mathscr{O}_{\mathcal{M},\chi}\big).$$

This $\mathcal{M}_\sigma$ is a torsor over Z under the quotient group H_σ of H defined in Definition 6.1.2.5.

Definition 6.1.2.7. *We call the subscheme $\mathcal{M}_\sigma$ of $\mathcal{M}(\sigma)$ defined in Lemma 6.1.2.6 the* σ-**stratum** *of $\mathcal{M}(\sigma)$.*

Let C be a cone in $\underline{X}(H)^\vee_{\mathbb{R}}$, let Γ be a group acting on $\underline{X}(H)$, and let $\Sigma = \{\sigma_j\}_{j \in J}$ be a Γ-admissible rational polyhedral cone decomposition. Using Lemma 6.1.2.4, we can glue together affine toroidal embeddings $\mathcal{M}(\sigma_j)$ defined by various cones σ_j in $\Sigma = \{\sigma_j\}_{j \in J}$, which we denote by $\overline{\mathcal{M}}_\Sigma$, or simply by $\overline{\mathcal{M}}$ if Σ is clear from the context. This is the *toroidal embedding* of the H-torsor $\mathcal{M}$ defined by Σ.

Let us list the main properties of $\overline{\mathcal{M}}_\Sigma$ as follows:

Theorem 6.1.2.8. *1. $\overline{\mathcal{M}}_\Sigma$ is separated and locally of finite type over the base scheme Z, which contains $\mathcal{M}$ as an open dense subscheme. The action of H on $\mathcal{M}$ extends naturally to $\overline{\mathcal{M}}_\Sigma$, and makes $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}_\Sigma$ an H-equivariant embedding.*

2. *For each $\sigma_j \in \Sigma$, the affine toroidal embeddings $\mathcal{M}(\sigma_j)$ embeds as a relatively affine H-invariant open dense subscheme of $\overline{\mathcal{M}}_\Sigma$ over S, containing $\mathcal{M}$. We can view $\mathcal{M}$ as the union of various $\mathcal{M}(\sigma_j)$'s. Then we understand by construction that if $\overline{\sigma}_l = \overline{\sigma}_j \cap \overline{\sigma}_k$, where σ_l is the largest common face of σ_j and σ_k, then $\mathcal{M}(\sigma_l) = \mathcal{M}(\sigma_j) \cap \mathcal{M}(\sigma_k)$.*

3. *$\overline{\mathcal{M}}_\Sigma$ has a natural stratification by locally closed subschemes $\mathcal{M}_{\sigma_j}$, for $j \in J$, as defined in Lemma 6.1.2.6. Moreover, by construction, $\overline{\mathcal{M}}_{\sigma_j} \supset \overline{\mathcal{M}}_{\sigma_k}$ if and only if $\overline{\sigma}_j \subset \overline{\sigma}_k$, where $\overline{\mathcal{M}}_{\sigma_j}$ and $\overline{\mathcal{M}}_{\sigma_k}$ denote respectively the closures of $\mathcal{M}_{\sigma_j}$ and $\mathcal{M}_{\sigma_k}$ in $\overline{\mathcal{M}}_\Sigma$.*

4. *The group Γ acts on $\overline{\mathcal{M}}_\Sigma$ by sending $\mathcal{M}(\sigma_j)$ (resp. $\mathcal{M}_{\sigma_j}$) to $\mathcal{M}(\gamma\sigma_j)$ (resp. $\mathcal{M}_{\gamma\sigma_j}$), if $\gamma \in \Gamma$ sends σ_j to $\gamma\sigma_j$ in Σ. By subdividing the cone decomposition if necessary, we may assume that $(\gamma\overline{\mathcal{M}}_\sigma) \cap \overline{\mathcal{M}}_\sigma \neq \emptyset$, or equivalently if $\gamma\overline{\sigma} \cap \overline{\sigma} \neq \{0\}$, implies that γ acts trivially on the smallest (rational) linear subspace of $\underline{X}(H)^\vee_{\mathbb{R}}$.*

5. *If σ_j is smooth, then $\mathcal{M}(\sigma_j)$ is smooth over Z. If the cone decomposition $\Sigma = \{\sigma_j\}_{j \in J}$ is smooth, then $\overline{\mathcal{M}}_\Sigma$ is smooth over Z.*

6.2 Construction of Boundary Charts

6.2.1 The Setting

In this section, we will focus on the following type of general constructions:

Let B be a finite-dimensional simple algebra over $\mathbb{Q}$ with a positive involution $\star$ and center F, so that the elements in F invariant under $\star$ form a totally real extension F^+ of $\mathbb{Q}$. Let $\mathcal{O}$ be an order in B invariant under $\star$. Then $\mathcal{O}$ has an involution given by the restriction of $\star$.

Let $(L, \langle \cdot, \cdot \rangle)$ be a PEL-type $\mathcal{O}$-lattice (defined as in Definition 1.2.1.3) that satisfies Condition 1.4.3.9. (See Remark 1.4.3.8.) Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$, and let $\mathrm{I}_{\mathrm{bad}} = 2$ or 1 depending on whether or not B involves any simple factors of type D (defined as in Definitions 1.2.1.15 and 1.2.1.17). Let $n \geq 1$ be an integer that will be our level. Let $\Box$ be a set of good primes, in the sense that $\Box \nmid n\, \mathrm{I}_{\mathrm{bad}}\, \mathrm{Disc}[L^\# : L]$, so that we can define a moduli problem M_n over some base scheme $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.2. More generally, we shall consider an open compact subgroup $\mathcal{H}$ of

$\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ for some $\Box \nmid \mathrm{I}_{\mathrm{bad}} \operatorname{Disc}[L^{\#} : L]$, so that we can define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over some base scheme $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.4.

The first goal of this section is to construct formal schemes for each cusp label of $\mathsf{M}_{\mathcal{H}}$ (defined as in Definition 5.4.2.4), over which we have the so-called Mumford families playing the role of Tate curves for modular curves along the infinity. Then we approximate these Mumford families by the so-called *good algebraic models* over algebraic schemes (instead of formal schemes), and glue them together with our moduli problem $\mathsf{M}_{\mathcal{H}}$ in the étale topology (in Section 6.3) to form the arithmetic toroidal compactification.

Let us explain the setting for M_n, which will be tacitly assumed in Sections 6.2.2 and 6.2.3 below. (The setting for $\mathsf{M}_{\mathcal{H}}$ will be postponed until the beginning of Section 6.2.4.)

Let $(\mathsf{Z}_n, \Phi_n, \delta_n)$ be a representative of a cusp label at level n, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ is a torus argument at level n. Here X and Y are constant group schemes that will serve as the character groups of the torus parts, as always. More precisely, X and Y are $\mathcal{O}$-lattices of the same multi-rank, and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-linear embedding of $\mathcal{O}$-lattices.

Recall that we have defined in Definition 5.4.1.6 the subgroup $\Gamma_\phi = \Gamma_{X,Y,\phi}$ of elements in $\mathrm{GL}_{\mathcal{O}}(X)$ that leaves invariant the image of Y under ϕ, which we shall realize as the group of pairs of isomorphisms $(\gamma_X : X \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y) \in \mathrm{GL}_{\mathcal{O}}(X) \times \mathrm{GL}_{\mathcal{O}}(Y)$ such that $\phi = \gamma_X \phi \gamma_Y$.

Definition 6.2.1.1. *For any torus argument $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, the group Γ_{Φ_n} is the subgroup of elements (γ_X, γ_Y) in Γ_ϕ that satisfies $\varphi_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$ and $\varphi_{0,n} = \gamma_Y \varphi_{0,n}$. In particular, $\Gamma_{\Phi_1} = \Gamma_\phi$.*

As in Lemma 5.2.7.5, the information of Z_n alone defines a moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ over $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.2. Let $(A, \lambda_A, i_A, \varphi_{-1,n})$ be the tautological tuple over $\mathsf{M}_n^{\mathsf{Z}_n}$. Then:

1. A is a (relative) abelian scheme over $\mathsf{M}_n^{\mathsf{Z}_n}$;
2. $\lambda_A : A \xrightarrow{\sim} A^\vee$ is a prime-to-$\Box$ polarization of A.
3. $i_A : \mathcal{O} \to \operatorname{End}_{\mathsf{M}_n^{\mathsf{Z}_n}}(A)$ defines an $\mathcal{O}$-structure of (A, λ_A).
4. $\underline{\mathrm{Lie}}_{A/\mathsf{M}_n^{\mathsf{Z}_n}}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i_A satisfies the determinantal condition in Definition 1.3.4.2 given by $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$.

5. $\varphi_{-1,n} : (\mathrm{Gr}^{\mathsf{Z}}_{-1,n})_{\mathsf{M}^{\mathsf{Z}_n}_n} \xrightarrow{\sim} A[n]$ is an integral principal level-n structure for (A, λ_A, i_A) of type $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot , \cdot \rangle_{11})$ as in Definition 1.3.6.1.

Based on the above data Z_n, $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, $(A, \lambda_A, i_A, \varphi_{-1,n})$, and δ_n, we would like to construct a formal algebraic stacks $\mathfrak{X}_{\Phi_n,\delta_n}$ over which there is a tautological tuple

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c^\vee_n, \tau_n))$$

like an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$. (See Definition 5.4.1.1.) Note that we cannot really say this is an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ because our base $\mathfrak{X}_{\Phi_n,\delta_n}$ is a formal algebraic stack that do not necessarily fit into the setting of Section 5.2.1. Over each affine formal scheme that is formally étale over $\mathfrak{X}_{\Phi_n,\delta_n}$, it should induce an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$. By Theorem 5.2.7.15, an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}$, which is in particular a degenerating family of type M_n (defined as in Definition 5.3.2.1). The degenerating families over various different affine formal schemes should glue together and form a degenerating family called the *Mumford family* over $\mathfrak{X}_{\Phi_n,\delta_n}$. Therefore, stated more precisely, our goal in this section is to construct $\mathfrak{X}_{\Phi_n,\delta_n}$ and the Mumford family over it.

Note that, as in Convention 5.4.2.5, we shall not make Z_n explicit in the notations such as $\mathfrak{X}_{\Phi_n,\delta_n}$.

6.2.2 Construction without Positivity Condition and Level Structures

For simplicity, let us begin by constructions without any consideration of level-n or level-$\mathcal{H}$ structures. Note that this does not mean the construction for even the level-1 structures. (See Remark 1.3.6.2.) In fact, this section does not produce any space that we will need later.

Proposition 6.2.2.1. *Let $n = 1$. Fix choices of a cusp label $(\mathsf{Z}_1, \Phi_1, \delta_1)$ at level 1, and hence a moduli problem $\mathsf{M}^{\mathsf{Z}_1}_1$ with a tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,1})$.*

Let us consider the category fibred in groupoids over the category of locally noetherian schemes over $\mathsf{M}^{\mathsf{Z}_1}_1$ whose fiber over each locally noetherian scheme S has objects the tuples $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ describing degeneration data without positivity condition over S. Explicitly, the tuple $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ satisfy the following conditions:

1. *The tuple* $(A, \lambda_A, i_A, \varphi_{-1,1})$ *defines an object parameterized by* $\mathsf{M}_1^{\mathsf{Z}_1}$*.*
2. $c : X \to A^\vee$ *and* $c^\vee : Y \to A$ *are* $\mathcal{O}$*-equivariant group homomorphisms satisfying the compatibility relation* $\lambda_A c^\vee = c\phi$ *with the prescribed* $\phi : Y \hookrightarrow X$*.*
3. $\tau : \mathbf{1}_{Y \times X} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_A$ *is a trivialization of biextensions over* S*, which satisfy the symmetry* $\tau(y, \phi(y')) = \tau(y', \phi(y))$ *as sections of* $(c^\vee(y), c(\phi(y')))^* \mathcal{P}_A^{\otimes -1} \cong (c^\vee(y'), c(\phi(y)))^* \mathcal{P}_A^{\otimes -1}$*, and satisfy the* $\mathcal{O}$*-compatibility* $\tau(by, \chi) = \tau(y, b^\star \chi)$ *as sections of* $(c^\vee(by), c(\chi))^* \mathcal{P}_A^{\otimes -1} \cong (c^\vee(y), c(b^\star \chi))^* \mathcal{P}_A^{\otimes -1}$*. (Here it makes sense to write equalities of sections because the isomorphisms are all canonical.)*

Two tuples $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ *and* $(A', \lambda_{A'}, i_{A'}, \varphi'_{-1,1}, c', (c^\vee)', \tau')$ *are isomorphic if there are isomorphisms* $(f_X : X \xrightarrow{\sim} Y, f_Y : Y \xrightarrow{\sim} Y) \in \Gamma_{\Phi_1}$ *(defined as in Definition 6.2.1.1), and* $f_A : (A, \lambda_A, i_A, \varphi_{-1,1}) \xrightarrow{\sim} (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,1})$ *over* S*, such that:*

1. *The maps* $c : X \to A^\vee$ *and* $c' : X \to (A')^\vee$ *are related by* $cf_X = {f_A}^\vee c'$*.*
2. *The maps* $c^\vee : Y \to A$ *and* $(c^\vee)' : Y \to A'$ *are related by* $f_A c^\vee = (c^\vee)' f_Y$*.*
3. *The trivializations* $\tau : \mathbf{1}_{Y \times X} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_A$ *and* $\tau' : \mathbf{1}_{Y \times X} \xrightarrow{\sim} ((c^\vee)' \times c')^* \mathcal{P}_{A'}$ *are related by* $(\mathrm{Id}_Y \times f_X)^* \tau = (f_Y \times \mathrm{Id}_X)^* \tau'$*.*

Then there is a separated and smooth (relative) scheme $\ddot{\Xi}_{\Phi_1}$ *over* $\mathsf{M}_1^{\mathsf{Z}_1}$*, together with a tautological tuple and a natural action of* Γ_{Φ_1} *on* $\ddot{\Xi}_{\Phi_1}$*, such that the quotient* $\ddot{\Xi}_{\Phi_1} / \Gamma_{\Phi_1}$ *is isomorphic to the category described above (as categories fibred in groupoids over* $\mathsf{M}_1^{\mathsf{Z}_1}$*). Equivalently, for any scheme* S *over* $\mathsf{M}_1^{\mathsf{Z}_1}$ *together a tuple* $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ *over* S*, there is a map* $S \to \ddot{\Xi}_{\Phi_1}$*, which is unique after we fix an isomorphism* $(f_Y : Y \xrightarrow{\sim} Y, f_X : X \xrightarrow{\sim} X)$ *in* Γ_{Φ_1}*, such that the tuple over* S *is the pullback of the tautological tuple over* $\ddot{\Xi}_{\Phi_1}$ *if we identify* X *by* f_X *and* Y *by* f_Y*.*

Remark 6.2.2.2. The reason for using the clumsy notations such as $\ddot{\Xi}_{\Phi_1}$ instead of simply Ξ_{Φ_1} will be clear when we construct the level-n structures (which must satisfy the *liftability* condition in Definition 1.3.6.1).

The construction of $\ddot{\Xi}_{\Phi_1}$ with a tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ can be described as follows:

Over $\mathsf{M}_1^{\mathsf{Z}_1}$, we have the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,1})$. Therefore it only remains to construct the tautological triple $(c, c^\vee, \tau)$ that satisfies the conditions we want.

Consider the group functors $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ over $\mathsf{M}_1^{\mathsf{Z}_1}$, over which we have respectively tautological maps $c : X \to A^\vee$ and $c^\vee : Y \to A$.

Remark 6.2.2.3. When $\mathcal{O}$ is not commutative, it is not possible in general to define reasonable $\mathcal{O}$-module structures on the two group functors above.

By composition with $\phi : X \to Y$ (resp. $\lambda_A : A \to A^\vee$), we obtain a map $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$). Therefore the compatibility condition $\lambda_A c^\vee = c\phi$ can be tautologically achieved over the fiber product $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$. By Proposition 5.2.3.8, the four group functors $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$ are all representable by (relative) proper smooth schemes whose fiber-wise connected components are abelian schemes over $\mathsf{M}_1^{\mathsf{Z}_1}$. As a result, we have the following:

Corollary 6.2.2.4. *The fiber product* $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$ *above is also representable by a* (*relative*) *proper smooth group scheme* $\ddot{C}_{\Phi_1}$ *over* $\mathsf{M}_1^{\mathsf{Z}_1}$.

Here $\ddot{C}_{\Phi_1}$ may not be an abelian scheme because its geometric fibers may not be connected.

Proposition 6.2.2.5. *1. Let* $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ$ *be the* (*fiber-wise*) **identity component** *of* $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)$, *which is an abelian scheme by statement 4 of Proposition 5.2.3.8. Then the natural map*

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$$

over $\mathsf{M}_1^{\mathsf{Z}_1}$ *has kernel the finite étale group scheme*

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A))^\circ := \underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ$$

and image the identity component $\ddot{C}^\circ_{\Phi_1}$ *of* $\ddot{C}_{\Phi_1}$. (*Note that the notation* $\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A))^\circ$ *does not mean the identity component of* $\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A))$.) *Therefore,* $\ddot{C}^\circ_{\Phi_1}$ *is an abelian scheme, and the group* $\pi_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ *of fiber-wise connected components of* $\ddot{C}_{\Phi_1}$ *over* $\mathsf{M}_1^{\mathsf{Z}_1}$ *makes sense.*

2. *The rank of* $\pi_0(\dddot{\ddot{C}}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ *has no prime factors other than those of* Disc, $[X : \phi(Y)]$, *and the rank of* $\ker(\lambda_A)$.

Proof. The first claim of the lemma is clear, because the finite flat group scheme

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A)) = \underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X, \ker(\lambda_A))$$

is the kernel of

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A),$$

and $\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A))^\circ$ is just the intersection of this kernel with $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ$.

For the second claim, let $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ denote respectively the (fiber-wise) identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$, and let $\dddot{\ddot{C}}{}^{\circ\circ\circ}_{\Phi_1}$ denote the proper smooth group scheme representing the fiber product

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ.$$

By statement 4 of Proposition 5.2.3.8, the difference between the ranks of the groups $\pi_0(\dddot{\ddot{C}}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ and $\pi_0(\dddot{\ddot{C}}{}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ are given by multiplication by numbers with only prime factors of those of Disc. Therefore it suffices to show that $\pi_0(\dddot{\ddot{C}}{}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ has no prime factors other than those of $[X : \phi(Y)]$ and the rank of $\ker(\lambda_A)$.

Note that the kernel K of the natural map

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$$

is given by $\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A^\vee)^\circ \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \ker(\lambda_A))^\circ$, where

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A^\vee)^\circ := \underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A^\vee) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$$

and

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \ker(\lambda_A))^\circ := \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ.$$

Since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is an abelian scheme, the group $\pi_0(\dddot{\ddot{C}}{}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ can be identified with a quotient of K. Since the rank of K is the product of the ranks of $\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A^\vee)^\circ$ and of $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \ker(\lambda_A))^\circ$, it has no prime factors other than those of $X/\phi(Y)$ and the rank of $\ker(\lambda_A)$. Hence the result follows. $\square$

Corollary 6.2.2.6. *The (fiber-wise) connected components of $\dddot{C}_{\Phi_1}$ are torsors under the abelian scheme $\dddot{C}^{\circ}_{\Phi_1}$, each of which has a structure of an abelian scheme as soon as an identity section is chosen.*

For each section (y, χ) of $Y \times X$ over $\dddot{C}_{\Phi_1}$, we can interpret $(c^\vee(y), c(\chi))$ as a map from $\dddot{C}_{\Phi_1}$ to $A \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} A^\vee$, and consider the pullback invertible sheaf $(c^\vee(y), c(\chi))^*\mathcal{P}_A$ on $\dddot{C}_{\Phi_1}$ over $\mathsf{M}_1^{\mathsf{Z}_1}$. Note that by the biextension structure of $\mathcal{P}_A$, we have the following formal properties of this association:

1. Linear in the first variable:

$$(c^\vee(y), c(\chi))^*\mathcal{P}_A \underset{\mathscr{O}_{\dddot{C}_{\Phi_1}}}{\otimes} (c^\vee(y'), c(\chi))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y + y'), c(\chi))^*\mathcal{P}_A.$$

2. Linear in the second variable:

$$(c^\vee(y), c(\chi))^*\mathcal{P}_A \underset{\mathscr{O}_{\dddot{C}_{\Phi_1}}}{\otimes} (c^\vee(y), c(\chi'))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi + \chi'))^*\mathcal{P}_A.$$

3. Symmetric with respect to ϕ:

$$\begin{aligned}
&(c^\vee(y), c(\phi(y'))^*\mathcal{P}_A = (c^\vee(y), \lambda_A c^\vee(y'))^*\mathcal{P}_A \\
&\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c^\vee(y'))^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A \overset{\text{sym.}}{\xrightarrow{\sim}} (c^\vee(y'), c^\vee(y))^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A \\
&\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y'), \lambda_A c^\vee(y))^*\mathcal{P}_A = (c^\vee(y'), c(\phi(y))^*\mathcal{P}_A.
\end{aligned}$$

4. Hermitian:

$$\begin{aligned}
&(c^\vee(by), c(\chi))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi))^*(i_A(b) \times \mathrm{Id}_{A^\vee})^*\mathcal{P}_A \\
&\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi))^*(\mathrm{Id}_A \times i_A(b)^\vee)^*\mathcal{P}_A \\
&= (c^\vee(y), c(\chi))^*(\mathrm{Id}_A \times i_{A^\vee}(b^\star))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(b^\star\chi))^*\mathcal{P}_A.
\end{aligned}$$

Let us consider the finitely generated abelian group (i.e. $\mathbb{Z}$-module)

$$\dddot{\mathbf{S}}_{\Phi_1} := (Y \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (by) \otimes \chi - y \otimes (b^\star\chi) \end{pmatrix}_{\substack{y, y' \in Y, \\ \chi \in X, b \in \mathcal{O}}}.$$

Remark 6.2.2.7. By Proposition 1.2.2.4, the cardinality of the torsion subgroup of $\dddot{\mathbf{S}}_{\Phi_1}$ has only prime factors dividing $\mathrm{I}_{\mathrm{bad}} \operatorname{Disc}[X : \phi(Y)]$. In particular, it is prime-to-$\Box$, because $\Box \nmid \mathrm{I}_{\mathrm{bad}} \operatorname{Disc}[X : \phi(Y)]$ by assumption.

Remark 6.2.2.8. In general it is not possible to define the tensor product in the category of left $\mathcal{O}$-modules. Therefore we make our abelian group $\dddot{\mathbf{S}}_{\Phi_1}$ as a quotient of $Y \underset{\mathbb{Z}}{\otimes} X$ to make the expected relations precise.

The formal properties above shows that if we associate to each

$$\ell = \sum_i [y_i \otimes \chi_i] \in \dddot{\mathbf{S}}_{\Phi_1}$$

an invertible sheaf $\Psi_1(\ell)$ on $\ddot{C}_{\Phi_1}$ by

$$\Psi_1(\ell) := \underset{\mathscr{O}_{\ddot{C}_{\Phi_1}}, i}{\otimes} (c^\vee(y_i), c(\chi_i))^* \mathcal{P}_A,$$

then this associate is well-defined (namely independent of the expression of ℓ we choose), and there exists a canonical isomorphism

$$\Delta^*_{\ell,\ell'} : \Psi_1(\ell) \underset{\mathscr{O}_{\ddot{C}_{\Phi_1}}}{\otimes} \Psi_1(\ell') \xrightarrow{\sim} \Psi_1(\ell + \ell')$$

for any $\ell, \ell' \in \dddot{\mathbf{S}}_{\Phi_1}$. As a result, we can form an $\mathscr{O}_{\ddot{C}_{\Phi_1}}$-sheaf of algebras

$$\underset{\ell \in \dddot{\mathbf{S}}_{\Phi_1}}{\oplus} \Psi_1(\ell),$$

with algebra structure given by the isomorphisms $\Delta^*_{\ell,\ell'}$ above, and define

$$\dddot{\Xi}_{\Phi_1} := \underline{\operatorname{Spec}}_{\mathscr{O}_{\ddot{C}_{\Phi_1}}} (\underset{\ell \in \dddot{\mathbf{S}}_{\Phi_1}}{\oplus} \Psi_1(\ell)).$$

If we denote by $\dddot{E}_{\Phi_1} := \underline{\operatorname{Hom}}(\dddot{\mathbf{S}}_{\Phi_1}, \mathbf{G}_{\mathrm{m}})$ the group of multiplicative type of finite type with character group $\dddot{\mathbf{S}}_{\Phi_1}$ over $\operatorname{Spec}(\mathbb{Z})$, then we see that, after a finite étale base extension $S' \to \ddot{C}_{\Phi_1}$ that trivializes all the invertible sheaves $\Psi_1(\ell)$ (which is possible because $\dddot{\mathbf{S}}_{\Phi_1}$ is finitely generated), there is an isomorphism $\dddot{\Xi}_{\Phi_1} \underset{\ddot{C}_{\Phi_1}}{\times} S' \cong \dddot{E}_{\Phi_1} \underset{\operatorname{Spec}(\mathbb{Z})}{\times} S'$. In particular, this shows that $\dddot{\Xi}_{\Phi_1}$ is an $\dddot{E}_{\Phi_1}$-torsor.

Remark 6.2.2.9. This is essentially the same argument behind Theorem 3.1.3.7. The only issue is that the base scheme $\ddot{C}_{\Phi_1}$ does not necessarily satisfy Assumption 3.1.2.7. Therefore we have to either weaken the assumption and make the statement more clumsy in Theorem 3.1.3.7, or to include some ad hoc argument here.

For the trivialization of biextension $\tau : \mathbf{1}_{Y\times X} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_A^{\otimes -1}$ over a scheme $S \to \dddot{C}_{\Phi_1}$ to define an $\mathcal{O}$-equivariant period map $\iota : \underline{Y} \to \dddot{G}$, we need to identify $\tau(y, \phi(y'))$ with $\tau(y', \phi(y))$ under the canonical isomorphism

$$(c^\vee(y), c(\phi(y')))^*\mathcal{P}_A^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y'), c(\phi(y)))^*\mathcal{P}_A^{\otimes -1},$$

and identify $\tau(by, \chi)$ with $\tau(y, b^\star\chi)$ under the canonical isomorphism

$$(c^\vee(by), c(\chi))^*\mathcal{P}_A^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(b^\star\chi))^*\mathcal{P}_A^{\otimes -1},$$

for any $y, y' \in Y$, $\chi \in X$, and $b \in \mathcal{O}$. Therefore we can interpret τ as depending only on the class of $y \otimes \chi$ in $\dddot{\mathbf{S}}_{\Phi_1}$. The sections $\tau(y \otimes \chi)$ of $\Psi_1(y \otimes \chi)^{\otimes -1}$ by definition correspond to maps $\Psi_1(y \otimes \chi) \xrightarrow{\sim} \mathscr{O}_S$, which by linearity necessarily extend to maps $\Psi_1(\ell) \xrightarrow{\sim} \mathscr{O}_S$ for all $\ell \in \dddot{\mathbf{S}}_{\Phi_1}$. Hence they define an algebra map $\mathscr{O}_{\dddot{\Xi}_{\Phi_1}} = \bigoplus\limits_{\ell \in \dddot{\mathbf{S}}_{\Phi_1}} \Psi_1(\ell) \to \mathscr{O}_S$. In other words, the sections $\tau(y \otimes \chi)$ correspond to an S-valued point $S \to \dddot{\Xi}_{\Phi_1}$.

Conversely, since the $\Psi_1(\ell)$'s all appear in the structural sheaf of $\dddot{\Xi}_{\Phi_1}$ over $\dddot{C}_{\Phi_1}$, we have natural trivializations $\mathscr{O}_{\dddot{\Xi}_{\Phi_1}} \xrightarrow{\sim} \mathscr{O}_{\dddot{\Xi}_{\Phi_1}} \bigotimes\limits_{\mathscr{O}_{\dddot{C}_{\Phi_1}}} \Psi_1(\ell)^{\otimes -1}$ for all $\ell \in \dddot{\mathbf{S}}_{\Phi_1}$. That is, we have a tautological trivialization of biextension $\tau : \mathbf{1}_{Y\times X} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_A^{\otimes -1}$ over $\dddot{\Xi}_{\Phi_1}$. Hence any S-valued point of $\dddot{\Xi}_{\Phi_1}$ admits by pullback a tautological trivialization of biextension $\tau : \mathbf{1}_{Y\times X} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_A^{\otimes -1}$.

Note that there is an ambiguity of the identification of (X, Y, ϕ): In our definition of isomorphism classes, we always allow twisting maps involving X and Y by a pair of isomorphisms (γ_X, γ_Y) in Γ_{Φ_1}. Therefore we should consider the natural action of Γ_{Φ_1} on $\dddot{\Xi}_{\Phi_1} \to \dddot{C}_{\Phi_1} \to \mathsf{M}_1^{\mathsf{Z}_1}$, and consider the quotient $\dddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ as the universal parameter spaces, whose structure maps can be factorized as $\dddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1} \to \dddot{C}_{\Phi_1}/\Gamma_{\Phi_1} \to \mathsf{M}_1^{\mathsf{Z}_1}$. (Note that $\dddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ is not necessary an algebraic stack according to our convention, as we only allow Deligne-Mumford stacks.)

As a result, if S is a scheme over which there is a tuple $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ with the prescribed $\phi : Y \to X$ describing a degeneration datum without positivity condition, then after a choice of an isomorphism in Γ_{Φ_1} giving the identification of (X, Y, ϕ) on S and on $\dddot{\Xi}_{\Phi_1}$, there is a unique morphism $S \to \dddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ so that the tuple is the pullback of the tautological tuple on $\dddot{\Xi}_{\Phi_1}$.

This finishes the construction of $\dddot{\Xi}_{\Phi_1}$ and proves Proposition 6.2.2.1.

6.2.3 Construction with Principal Level Structures

Let us take the level-n structures (as defined in Definition 1.3.6.1) into consideration.

Let $n \geq 1$ be any positive integer such that $\Box \nmid n$, and let $(\mathsf{Z}_n, \Phi_n, \delta_n)$ be a representative of a (principal) cusp label at level n (defined as in Definition 5.4.1.9). Then Z_n alone defines a moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ with tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,n})$, as described in Section 6.2.1. We would like to construct an algebraic stack Ξ_{Φ_n,δ_n} on which there is a tautological tuple

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ (defined as in Definition 5.4.1.1) without positivity condition, so that a suitable formal completion of this algebraic stack will gives us the formal algebraic stack $\mathfrak{X}_{\Phi_n,\delta_n}$ we promised in Section 6.2.1. We shall redo our construction for $\ddot{\Xi}_{\Phi_1}$ so that we can obtain a tautological triple $(c_n, c_n^\vee, \tau_n)$ lifting $(c, c^\vee, \tau)$. Moreover, we would like to make this tautological liftable to some $(c_m, c_m^\vee, \tau_m)$ for any $n|m$, $\Box \nmid m$, as described in Definition 5.2.3.4 and Corollary 5.2.3.6.

The prescribed liftable splitting $\delta_n : \mathrm{Gr}_n^{\mathsf{Z}} \xrightarrow{\sim} L/nL$ defines two pairings

$$\langle \cdot , \cdot \rangle_{10,n} : \mathrm{Gr}_{-1,n}^{\mathsf{Z}} \times \mathrm{Gr}_{0,n}^{\mathsf{Z}} \to (\mathbb{Z}/n\mathbb{Z})(1)$$

and

$$\langle \cdot , \cdot \rangle_{00,n} : \mathrm{Gr}_{0,n}^{\mathsf{Z}} \times \mathrm{Gr}_{0,n}^{\mathsf{Z}} \to (\mathbb{Z}/n\mathbb{Z})(1).$$

The prescribed level-n structure $\varphi_{-1,n} : \mathrm{Gr}_{-1,n}^{\mathsf{Z}} \xrightarrow{\sim} A[n]_\eta$ defines a liftable map $f_{-1,n} = \varphi_{-1,n}$ by itself, and determines a necessarily unique liftable isomorphism $\nu(\hat{f}_{-1}) = \nu(\varphi_{-1,n}) : (\mathbb{Z}/n\mathbb{Z})(1) \xrightarrow{\sim} \boldsymbol{\mu}_{n,\eta}$. On the other hand, the prescribed liftable isomorphism $f_{0,n} := \varphi_{0,n} : \mathrm{Gr}_{0,n}^{\mathsf{Z}} \xrightarrow{\sim} Y/nY$ define a liftable isomorphism $\varphi_{0,n} : \mathrm{Gr}_{0,n}^{\mathsf{Z}} \xrightarrow{\sim} \frac{1}{n}Y/Y$ via the canonical isomorphism $Y/nY \xrightarrow{\sim} \frac{1}{n}Y/Y$, which we again denote by $f_{0,n}$. Therefore we have two liftable pairings

$$(f_{-1,n}^{-1} \times f_{0,n}^{-1})^*(\langle \cdot , \cdot \rangle_{10,n}) : A[n]_\eta \times \tfrac{1}{n}Y/Y \to (\mathbb{Z}/n\mathbb{Z})(1)$$

and

$$(f_{0,n}^{-1} \times f_{0,n}^{-1})^*(\langle \cdot , \cdot \rangle_{00,n}) : \tfrac{1}{n}Y \times \tfrac{1}{n}Y \to (\mathbb{Z}/n\mathbb{Z})(1).$$

Then:

Lemma 6.2.3.1. *The choice of a representative* (Φ_n, δ_n) *of a cusp label determines two particular choices of liftable maps* $b_{\Phi_n,\delta_n} : \frac{1}{n}Y \to A^\vee[n]$ *and* $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$*, by requiring*

$$(f_{-1,n}^{-1} \times f_{0,n}^{-1})^*(\langle \cdot , \cdot \rangle_{10,n})(a, \tfrac{1}{n}y) = \nu(f_{-1,n})^{-1} \circ \mathrm{e}_{A[n]}(a, b_{\Phi_n,\delta_n}(\tfrac{1}{n}y))$$

and

$$(f_{0,n}^{-1} \times f_{0,n}^{-1})^*(\langle \cdot , \cdot \rangle_{00,n})(\tfrac{1}{n}y, \tfrac{1}{n}y') = \nu(f_{-1,n})^{-1} \circ a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y')$$

for any $a \in A[n]_\eta$ *and* $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$*.*

Following Definition 5.2.7.9 and Proposition 5.2.7.10, we shall require $\lambda_A c_n^\vee - c_n\phi_n$ to agree with $b_{\Phi_n,\delta_n} : \frac{1}{n}Y \to A^\vee[n]$, and require τ_n to define a pairing that agrees with $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ in the sense that

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y, \phi(y'))^{-1} = a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y')$$

for any $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$, on the algebraic stack Ξ_{Φ_n,δ_n} we will construct. (See Lemma 6.2.3.10 for the precise definition for $a_{n,0}$ to be skew-Hermitian.)

Let us consider the map of group functors $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$) defined by restriction from $\frac{1}{n}X$ to X (resp. $\frac{1}{n}Y$ to Y). Let $\dddot{C}_{\Phi_n}$ be the (relative) proper smooth group scheme over $\mathsf{M}_n^{\mathsf{Z}_n}$ representing the fiber product

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A).$$

(See Proposition 5.2.3.8 and Corollary 6.2.2.4.) Then there is a natural map $\dddot{C}_{\Phi_n} \to \dddot{C}_{\Phi_1}$ corresponding to the restriction maps $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$ above.

The structure of $\dddot{C}_{\Phi_n}$ can be analyzed as in Proposition 6.2.2.5:

Proposition 6.2.3.2.

1. *Let* $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A)^\circ$ *be the (fiber-wise)* **identity component** *of* $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A)$*, which is an abelian scheme by statement 4 of Proposition 5.2.3.8. Then the natural map*

   $$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee)$$

   over $\mathsf{M}_n^{\mathsf{Z}_n}$ *has kernel the finite étale group scheme*

   $$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\frac{1}{n}Y), \ker(\lambda_A))^\circ \\ &\quad := \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\frac{1}{n}Y), \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^\circ\end{aligned}$$

and image the identity component $\dddot{C}^{\circ}_{\Phi_n}$ *of* $\dddot{C}_{\Phi_n}$*. Therefore,* $\dddot{C}^{\circ}_{\Phi_n}$ *is an abelian scheme, and the group* $\pi_0(\dddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ *of fiber-wise connected components of* $\dddot{C}_{\Phi_n}$ *over* $\mathsf{M}_n^{\mathsf{Z}_n}$ *makes sense.*

2. *The rank of the component group* $\pi_0(\dddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ *is has no prime factors other than those of* Disc*,* n*,* $[X : \phi(Y)]$*, and the rank of* $\ker(\lambda_A)$*. (This implies that the rank of* $\pi_0(\dddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ *does not contain prime factors other than those of* Disc*,* n *and* $[L^{\#} : L]$*.)*

Proof. The first claim of the lemma is clear, because the finite flat group scheme

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), \ker(\lambda_A)) = \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), A) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, \ker(\lambda_A))$$

is the kernel of

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^{\circ} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee}),$$

and $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), \ker(\lambda_A))^{\circ}$ is just the intersection of this kernel with $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^{\circ}$.

For the second claim, the argument is also similar to the one in the proof of Proposition 6.2.2.5. Let $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee})^{\circ}$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)^{\circ}$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})^{\circ}$ denote respectively the (fiber-wise) identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee})$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})$, and let $\dddot{C}^{\circ\circ\circ}_{\Phi_n}$ denote the proper smooth group scheme representing the fiber product

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee})^{\circ} \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})^{\circ}}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)^{\circ}.$$

By statement 4 of Proposition 5.2.3.8, the difference between the ranks of the groups $\pi_0(\dddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ and $\pi_0(\dddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ are given by multiplication by numbers with only prime factors of those of Disc. Therefore it suffices to show that $\pi_0(\dddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ has no prime factors other than those of n, $[X : \phi(Y)]$, and the rank of $\ker(\lambda_A)$.

Note that the kernel K_n of the natural map

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee})^{\circ} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})^{\circ} \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^{\vee})^{\circ}}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)^{\circ}$$

is given by $\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi(Y), A^{\vee})^{\circ} \underset{\mathsf{M}_n^{\mathsf{Z}_n}}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, \ker(n\lambda_A))^{\circ}$, where

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X/\phi(Y), A^{\vee})^{\circ} := \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X/\phi(Y), A^{\vee}) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^{\vee})^{\circ}$$

and

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, \ker(n\lambda_A))^\circ := \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, \ker(n\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)^\circ.$$

Since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is an abelian scheme, the group $\pi_0(\dddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n})$ can be identified with a quotient of K_n. Since the rank of K_n is the product of the ranks of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X/\phi(Y), A^\vee)^\circ$ and of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, \ker(n\lambda_A))^\circ$, it has no prime factors other than those of n, $X/\phi(Y)$ and the rank of $\ker(\lambda_A)$. Hence the result follows. □

Corollary 6.2.3.3. *The (fiber-wise) connected components of $\dddot{C}_{\Phi_n}$ are torsors under the abelian scheme $\dddot{C}^\circ_{\Phi_n}$, each of which has a structure of an abelian scheme as soon as an identity section is chosen.*

Let us consider the natural map

$$\partial_n^{(1)} : \dddot{C}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A^\vee)$$

defined by sending a pair $(c_n, c_n^\vee)$ to $\lambda_A c_n^\vee - c_n\phi_n \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A^\vee)$ as considered in Lemma 5.2.3.11. Note that for each pair $(c_n, c_n^\vee)$ in $\dddot{C}_{\Phi_n}$, the associated $\partial_n^{(1)}(c_n, c_n^\vee)$ actually lies in the kernel of the natural restriction map

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee),$$

or rather $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, A^\vee[n])$, because c_n and $c_n^\vee$ satisfy $\lambda_A(c_n^\vee|_Y) = \lambda_A c^\vee = c\phi = (c_n|_X)(\phi_n|_Y)$ when restricted to Y, or in other words when multiplied by n, from the definition of $\dddot{C}_{\Phi_n}$ as a fiber product. Therefore the natural map $\partial_n^{(1)}$ can be written as

$$\partial_n^{(1)} : \dddot{C}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y/Y, A^\vee[n]),$$

and it makes sense to talk about the *fiber* $\dddot{C}_{\Phi_n,b_n} := (\partial_n^{(1)})^{-1}(b_n)$ of $\partial_n^{(1)}$ over a particular section $b_n \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, A^\vee[n])$.

Over $\dddot{C}_{\Phi_n}$, we have two tautological maps $c_n : \frac{1}{n}X \to A^\vee$ and $c_n^\vee : \frac{1}{n}Y \to A$, where the restriction of c_n to X (resp. $c_n^\vee$ to Y) is the pullback of $c : X \to A^\vee$ via $\dddot{C}_{\Phi_n} \to \dddot{C}_{\Phi_1}$ (resp. pullback of $c^\vee : Y \to A$ via $\dddot{C}_{\Phi_n} \to \dddot{C}_{\Phi_1}$). Then the upshot of defining $\dddot{C}_{\Phi_n,b_n}$ is that $\lambda_A c_n^\vee - c_n\phi_n = b_n$ holds tautologically over $\dddot{C}_{\Phi_n,b_n}$.

Lemma 6.2.3.4. *The map*

$$\partial_n^{(1)} : \dddot{C}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y/Y, A^\vee[n])$$

induces a canonical morphism

$$\dddot{C}_{\Phi_n} \twoheadrightarrow \pi_0(\dddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n}) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y/Y, A^\vee[n]).$$

In particular, the fibers $\dddot{C}_{\Phi_n,b_n}$ of $\partial_n^{(1)}$ are (possibly empty) unions of fiber-wise connected components of $\dddot{C}_{\Phi_n}$ over $\mathsf{M}_n^{\mathsf{Z}_n}$.

Note that $\dddot{C}_{\Phi_n,0}$, where 0 stands for the trivial map $\frac{1}{n}Y \to A^\vee[n]$ sending everything to the identity, is simply the proper smooth group scheme representing the fiber product

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A).$$

Moreover, $\dddot{C}_{\Phi_n,b_n}$ is necessarily a translation of $\dddot{C}_{\Phi_n,0}$ by some element in $\dddot{C}_{\Phi_n}$ whose image under $\partial_n^{(1)}$ is b_n, as long as it is not empty.

Now we shall construct a scheme $\dddot{\Xi}_{\Phi_n}$ as in the case of $\dddot{\Xi}_{\Phi_1}$ in Proposition 6.2.2.1, which provides us with a tautological triple $(c_n^\vee, c_n, \tau_n)$ on top of the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,n}, X, Y, \phi, c, c^\vee, \tau)$ over $\dddot{\Xi}_{\Phi_1} \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_n^{\mathsf{Z}_n}$. Here Z_1 is the "reduction mod 1" of Z_n, in the sense that we still keep the liftability and the ranks as conditions. (See Remark 5.2.2.7.)

Let us consider the finitely generated abelian group (i.e. $\mathbb{Z}$-module)

$$\dddot{\mathbf{S}}_{\Phi_n} := ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (b\frac{1}{n}y) \otimes \chi - (\frac{1}{n}y) \otimes (b^\star \chi) \end{pmatrix}_{\substack{y,y' \in Y, \\ \chi \in X, b \in \mathcal{O}}} . \qquad (6.2.3.5)$$

Remark 6.2.3.6. By Proposition 1.2.2.4, the cardinality of the torsion subgroup of $\dddot{\mathbf{S}}_{\Phi_n}$ has only prime factors dividing $n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X : \phi(Y)]$ (cf. Remark 6.2.2.7). In particular, it is prime-to-$\Box$, because $\Box \nmid n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X : \phi(Y)]$ by assumption.

As in the construction of $\dddot{\Xi}_{\Phi_1}$, the formal properties of the pullback of the Poincaré biextension allows us to associate to each

$$\ell = \sum_i [(\tfrac{1}{n}y_i) \otimes \chi_i] \in \dddot{\mathbf{S}}_{\Phi_n}$$

a well-defined invertible sheaf

$$\Psi_n(\ell) := \underset{\mathscr{O}_{\dddot{C}_{\Phi_n}},i}{\otimes} (c_n^\vee(\tfrac{1}{n}y_i), c(\chi_i))^*\mathcal{P}_A,$$

so that there exists a canonical isomorphism

$$\Delta^*_{n,\ell,\ell'} : \Psi_n(\ell) \underset{\mathscr{O}_{\dddot{C}_{\Phi_n}}}{\otimes} \Psi_n(\ell') \xrightarrow{\sim} \Psi_n(\ell+\ell')$$

for any $\ell, \ell' \in \dddot{\mathbf{S}}_{\Phi_n}$. As a result, we can form an $\mathscr{O}_{\dddot{C}_{\Phi_n}}$-sheaf of algebras

$$\underset{\ell\in\dddot{\mathbf{S}}_{\Phi_n}}{\oplus} \Psi_n(\ell)$$

with algebra structure given by the isomorphisms $\Delta^*_{n,\ell,\ell'}$ above, and define

$$\dddot{\Xi}_{\Phi_n} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\dddot{C}_{\Phi_n}}}(\underset{\ell\in\dddot{\mathbf{S}}_{\Phi_n}}{\oplus} \Psi_n(\ell)).$$

If we denote by $\dddot{E}_{\Phi_n} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ the group of multiplicative type of finite type with character group $\dddot{\mathbf{S}}_{\Phi_n}$ over $\mathrm{Spec}(\mathbb{Z})$, then $\dddot{\Xi}_{\Phi_n}$ is an $\dddot{E}_{\Phi_n}$-torsor, with the same argument as in the case of the $\dddot{E}_{\Phi_1}$-torsor $\dddot{\Xi}_{\Phi_1}$. Moreover, we have tautological trivializations $\tau_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n^\vee \times c)^*\mathcal{P}_A^{\otimes -1}$ over $\dddot{\Xi}_{\Phi_n}$, which corresponds to a tautological map $\iota_n : \frac{1}{n}Y \to G^\natural$. Let $\tau : \mathbf{1}_{Y\times X} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_A^{\otimes -1}$ be the restriction of τ_n to $\mathbf{1}_{Y\times X}$, which corresponds to a tautological map $\iota : Y \to G^\natural$.

Let $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$ denote the torsion subgroup of $\dddot{\mathbf{S}}_{\Phi_n}$, and let $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$ denote the quotient of $\dddot{\mathbf{S}}_{\Phi_n}$ by $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$, namely the free abelian quotient group of $\dddot{\mathbf{S}}_{\Phi_n}$. Let $\dddot{E}_{\Phi_n,\mathrm{tor}} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}, \mathbf{G}_{\mathrm{m}})$ (resp. $\dddot{E}_{\Phi_n,\mathrm{free}} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}, \mathbf{G}_{\mathrm{m}})$) be the group of multiplicative type of finite type with character group $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$ (resp. $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$) over $\mathrm{Spec}(\mathbb{Z})$. Then the exact sequence

$$0 \to \dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}} \to \dddot{\mathbf{S}}_{\Phi_n} \to \dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}} \to 0$$

induces an exact sequence

$$0 \to \dddot{E}_{\Phi_n,\mathrm{free}} \to \dddot{E}_{\Phi_n} \to \dddot{E}_{\Phi_n,\mathrm{tor}} \to 0$$

in the reversed direction. Note that since $\dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$ is a finitely generated free abelian group, $\dddot{E}_{\Phi_n,\mathrm{free}}$ is a torus (by Definition 3.1.1.5).

Lemma 6.2.3.7. *Let* $\dddot{\mathbf{S}}'_{\Phi_n} \subset \dddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$ *be any torsion subgroup of* $\dddot{\mathbf{S}}_{\Phi_n}$*, and let* $\dddot{E}'_{\Phi_n} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}'_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ *be the group of multiplicative type of finite type with character group* $\dddot{\mathbf{S}}'_{\Phi_n}$ *over* $\mathrm{Spec}(\mathbb{Z})$*. Then the scheme* $\dddot{\Xi}'_{\Phi_n}$ *defined by*

$$\dddot{\Xi}'_{\Phi_n} := \mathrm{Spec}_{\mathscr{O}_{\dddot{C}_{\Phi_n}}}\big(\underset{\ell\in\dddot{\mathbf{S}}'_{\Phi_n}}{\oplus} \Psi_n(\ell)\big)$$

is canonically a trivial $\dddot{E}'_{\Phi_n}$*-torsor over* $\dddot{C}_{\Phi_n}$*. Namely, there is a canonical isomorphism* $\dddot{\Xi}'_{\Phi_n} \xrightarrow{\sim} \dddot{E}'_{\Phi_n} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} \dddot{C}_{\Phi_n}$*, which is* $\dddot{E}'_{\Phi_n}$*-equivariant. Moreover, the inclusion* $\dddot{\mathbf{S}}'_{\Phi_n} \hookrightarrow \dddot{\mathbf{S}}_{\Phi_n}$ *defines a canonical* $\dddot{E}_{\Phi_n}$*-equivariant surjection* $\dddot{\Xi}_{\Phi_n} \twoheadrightarrow \dddot{\Xi}'_{\Phi_n}$*, and defines a canonical* $\dddot{E}_{\Phi_n}$*-equivariant section*

$$\dddot{\Xi}_{\Phi_n} \to \dddot{E}'_{\Phi_n}$$

by the composition $\dddot{\Xi}_{\Phi_n} \twoheadrightarrow \dddot{\Xi}'_{\Phi_n} \twoheadrightarrow \dddot{E}'_{\Phi_n}$*.*

Proof. If $\ell \in \dddot{\mathbf{S}}_{\Phi_n}$ is a torsion element, then since $\dddot{\mathbf{S}}_{\Phi_n}$ has no $\Box$-torsion by assumption, there is an integer N prime-to-$\Box$, some $\tilde{\ell} \in (\frac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X$ representing ℓ, and elements $y_i, y'_i, y''_j \in Y$, $\chi_j \in X$, and $b_j \in \mathcal{O}$, such that

$$N\tilde{\ell} = \sum_i (y_i \otimes \phi(y'_i) - y'_i \otimes \phi(y_i) + \sum_j ((b\tfrac{1}{n}y''_j) \otimes \chi_j - (\tfrac{1}{n}y''_j) \otimes (b^\star \chi_j))$$

in $(\frac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X$. Then we have an equality

$$\begin{aligned}\tilde{\ell} = &\sum_i ((\tfrac{1}{N}y_i) \otimes \phi(y'_i) - (\tfrac{1}{N}y'_i) \otimes \phi(y_i)) \\ &+ \sum_j ((b\tfrac{1}{nN}y''_j) \otimes \chi_j - (\tfrac{1}{nN}y''_j) \otimes (b^\star \chi_j))\end{aligned}$$

in $(\frac{1}{nN}Y) \underset{\mathbb{Z}}{\otimes} X$. After a base change to an étale surjection if necessary, suppose the map $c_n^\vee : \frac{1}{n}Y \to A$ is lifted to some map $c_{nN}^\vee : \frac{1}{nN}Y \to A$ such that $c_{nN}^\vee|_{\frac{1}{n}Y} = c_n^\vee$. Then we have a canonical isomorphism between invertible sheaves:

$$\begin{aligned}\Psi_n(\ell) \overset{\text{can.}}{\xrightarrow{\sim}} &\underset{i}{\otimes} \big((c_{nN}^\vee(\tfrac{1}{N}y_i), c(\phi(y'_i)))^*\mathcal{P}_A \otimes (c_{nN}^\vee(\tfrac{1}{N}y'_i), c(\phi(y_i)))^*\mathcal{P}_A^{\otimes -1}\big) \otimes \\ &\underset{j}{\otimes} \big((c_{nN}^\vee(\tfrac{1}{nN}b_j y''_j), c(\chi_j)))^*\mathcal{P}_A \otimes (c_{nN}^\vee(\tfrac{1}{N}y''_j), c(b_j^\star \chi_j)))^*\mathcal{P}_A^{\otimes -1}\big)\end{aligned}$$

Now there are canonical symmetry isomorphisms

$$(c_{nN}^{\vee}(\tfrac{1}{N}y_i), c(\phi(y_i')))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c_{nN}^{\vee}(\tfrac{1}{N}y_i'), c(\phi(y_i)))^*\mathcal{P}_A$$

and

$$(c_{nN}^{\vee}(\tfrac{1}{nN}b_jy_j''), c(\chi_j)))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c_{nN}^{\vee}(\tfrac{1}{N}y_j''), c(b_j^{\star}\chi_j))^*\mathcal{P}_A$$

for any i and j, which therefore defines a canonical trivialization

$$\Psi_n(\ell) \xrightarrow{\sim} \mathscr{O}_{\dddot{\ddot{C}}_{\Phi_n}}$$

for any $\ell \in \dddot{\ddot{\mathbf{S}}}'_{\Phi_n}$. Since all the isomorphisms we use are canonical ones, a different lifting of $c_n^{\vee}$ does not produce a different isomorphism. In particular, the trivialization $\Psi_n(\ell) \xrightarrow{\sim} \mathscr{O}_{\dddot{\ddot{C}}_{\Phi_n}}$ must be already defined over the original base scheme by étale descent. Moreover, the trivializations for various different $\ell, \ell' \in \dddot{\ddot{\mathbf{S}}}'_{\Phi_n}$ are compatible under the canonical isomorphisms

$$\Delta_{n,\ell,\ell'}^* : \Psi_n(\ell) \otimes \Psi_n(\ell') \xrightarrow{\sim} \Psi_n(\ell + \ell'),$$

because they all involve the same canonical biextension properties of $\mathcal{P}_A$. Therefore we have an isomorphism between $\mathscr{O}_{\dddot{\ddot{C}}_{\Phi_n}}$-sheaf of algebras, which defines an isomorphism

$$\dddot{\ddot{E}}'_{\Phi_n} \underset{\operatorname{Spec}(\mathbb{Z})}{\times} \dddot{\ddot{C}}_{\Phi_n} \xrightarrow{\sim} \dddot{\ddot{\Xi}}'_{\Phi_n},$$

as desired. □

Applying Lemma 6.2.3.7 to the full torsion subgroup $\dddot{\ddot{\mathbf{S}}}_{\Phi_n,\mathrm{tor}}$, we have:

Corollary 6.2.3.8. *The scheme* $\dddot{\ddot{\Xi}}_{\Phi_n,\mathrm{tor}}$ *is canonically a trivial* $\dddot{\ddot{E}}_{\Phi_n,\mathrm{tor}}$*-torsor, and defines a canonical map*

$$\dddot{\ddot{\Xi}}_{\Phi_n} \to \dddot{\ddot{E}}_{\Phi_n,\mathrm{tor}}.$$

The preimage of each element in $\dddot{\ddot{E}}_{\Phi_n,\mathrm{tor}}$ *is a torsor under the torus* $\dddot{\ddot{E}}_{\Phi_n,\mathrm{free}}$*, which in particular is connected* $\dddot{\ddot{C}}_{\Phi_n}$*. Hence they form the connected components of* $\dddot{\ddot{\Xi}}_{\Phi_n}$ *over* $\dddot{\ddot{C}}_{\Phi_n}$*. In this case, the component group* $\pi_0(\dddot{\ddot{\Xi}}_{\Phi_n}/\dddot{\ddot{C}}_{\Phi_n})$ *makes sense and can be identified with* $\dddot{\ddot{\Xi}}_{\Phi_n,\mathrm{tor}} \xrightarrow{\sim} \dddot{\ddot{E}}_{\Phi_n,\mathrm{tor}} \underset{\operatorname{Spec}(\mathbb{Z})}{\times} \dddot{\ddot{C}}_{\Phi_n}$ *as group schemes over* $\dddot{\ddot{C}}_{\Phi_n}$

Definition 6.2.3.9. *The identity component* $\dddot{\Xi}^{\circ}_{\Phi_n}$ *of* $\dddot{\Xi}_{\Phi_n}$ *is the preimage of* 0 *of the surjection* $\dddot{\Xi}_{\Phi_n} \twoheadrightarrow \dddot{\Xi}_{\Phi_n,\mathrm{tor}}$.

Now let us consider the subgroup $\dddot{\mathbf{S}}^{(n)}_{\Phi_n}$ of $\dddot{\mathbf{S}}_{\Phi_n}$ generated by

$$[(\tfrac{1}{n}y) \otimes \phi(y')] - [(\tfrac{1}{n}y') \otimes \phi(y)],$$

where $\frac{1}{n}y$ and $\frac{1}{n}y'$ run through arbitrary elements of $\frac{1}{n}Y$. In other words, we have

$$\dddot{\mathbf{S}}^{(n)}_{\Phi_n} = ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} Y) / \begin{pmatrix} y \otimes y' - y' \otimes y \\ (b\frac{1}{n}y) \otimes y' - (\frac{1}{n}y) \otimes (b^\star y') \end{pmatrix}_{y,y' \in Y, b \in \mathcal{O}}.$$

This subgroup is torsion, because the n-th multiple of every element is part of the defining relation in $\dddot{\mathbf{S}}^{(n)}_{\Phi_n}$.

Note that we have a natural isomorphism

$$\begin{aligned} \dddot{\mathbf{S}}^{(n)}_{\Phi_n} &= ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} Y) / \begin{pmatrix} y \otimes y' - y' \otimes y \\ (b\frac{1}{n}y) \otimes y' - (\frac{1}{n}y) \otimes (b^\star y') \end{pmatrix}_{y,y' \in Y, b \in \mathcal{O}} \\ &\xrightarrow{\sim} ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} (\tfrac{1}{n}Y)) / \begin{pmatrix} n(\frac{1}{n}y \otimes \frac{1}{n}y' - \frac{1}{n}y' \otimes \frac{1}{n}y) \\ (b\frac{1}{n}y) \otimes \frac{1}{n}y' - (\frac{1}{n}y) \otimes (b^\star \frac{1}{n}y') \end{pmatrix}_{y,y' \in Y, b \in \mathcal{O}} \end{aligned}$$

Therefore we have:

Lemma 6.2.3.10. *The group* $\dddot{E}^{(n)}_{\Phi_n} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}^{(n)}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ *is isomorphic to the group of pairings*

$$a_n(\,\cdot\,,\,\cdot\,) : \tfrac{1}{n}Y \times \tfrac{1}{n}Y \to \mathbf{G}_{\mathrm{m}},$$

which are **skew-Hermitian** *in the sense that* $a_n(\frac{1}{n}y, \frac{1}{n}y') = -a_n(\frac{1}{n}y', \frac{1}{n}y)$ *and* $a_n(b\frac{1}{n}y, \frac{1}{n}y') = a_n(\frac{1}{n}y, b^\star\frac{1}{n}y')$ *for any* $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ *and any* $b \in \mathcal{O}$.

We shall henceforth identify $\dddot{E}^{(n)}_{\Phi_n} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}^{(n)}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ with the group of such skew-Hermitian pairings, and write $a_n \in \dddot{E}^{(n)}_{\Phi_n}$ in this case.

Let us write Lemma 6.2.3.7 in the case of $\dddot{\mathbf{S}}^{(n)}_{\Phi_n}$ as:

Corollary 6.2.3.11. *The scheme* $\dddot{\Xi}^{(n)}_{\Phi_n}$ *defined by*

$$\dddot{\Xi}^{(n)}_{\Phi_n} := \mathrm{Spec}_{\mathscr{O}_{\dddot{C}_{\Phi_n}}}\Big(\underset{\ell \in \dddot{\mathbf{S}}^{(n)}_{\Phi_n}}{\oplus} \Psi_n(\ell) \Big)$$

is a canonically trivial $\dddot{E}^{(n)}_{\Phi_n}$*-torsor over* $\dddot{C}_{\Phi_n}$*. Namely, there is a canonical isomorphism* $\dddot{\Xi}^{(n)}_{\Phi_n} \xrightarrow{\sim} \dddot{E}^{(n)}_{\Phi_n} \underset{\operatorname{Spec}(\mathbb{Z})}{\times} \dddot{C}_{\Phi_n}$*, which is* $\dddot{E}_{\Phi_n}$*-equivariant and defines a canonical* $\dddot{E}_{\Phi_n}$*-equivariant section*

$$\partial^{(0)}_n : \dddot{\Xi}_{\Phi_n} \to \dddot{E}^{(n)}_{\Phi_n}$$

by the composition $\dddot{\Xi}_{\Phi_n} \twoheadrightarrow \dddot{\Xi}^{(n)}_{\Phi_n} \twoheadrightarrow \dddot{E}^{(n)}_{\Phi_n}$*. In particular,* $\partial^{(0)}_n$ *is surjective.*

Following the proof of Lemma 6.2.3.7, we see that the upshot of Corollary 6.2.3.11 is that we have the trivializations $\Psi_n(\ell) \xrightarrow{\sim} \mathscr{O}_{\dddot{C}_{\Phi_n}}$, when ℓ is of the form

$$[(\tfrac{1}{n}y) \otimes \phi(y')] - [(\tfrac{1}{n}y') \otimes \phi(y)],$$

given by the canonical symmetry isomorphism

$$(c^\vee_n(\tfrac{1}{n}y), c(\phi(y')))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee_n(\tfrac{1}{n}y'), c(\phi(y)))^*\mathcal{P}_A.$$

On the other hand, for each $\frac{1}{n}y$ and $\frac{1}{n}y'$, we can define a section $\dddot{\Xi}_{\Phi_n} \to \mathbf{G}_{\mathrm{m}}$ by comparing the difference

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} \in \mathbf{G}_{\mathrm{m}}(\dddot{\Xi}_{\Phi_n})$$

between the two tautological sections $\tau_n(\frac{1}{n}y, \phi(y'))$ and $\tau_n(\frac{1}{n}y', \phi(y))$ under the canonical symmetry isomorphism

$$(c^\vee_n(\tfrac{1}{n}y), c(\phi(y')))^*\mathcal{P}_A^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee_n(\tfrac{1}{n}y'), c(\phi(y)))^*\mathcal{P}_A^{\otimes -1}$$

over $\dddot{\Xi}_{\Phi_n}$, which is the same as the one we have used in Lemma 6.2.3.7 (which leads to Corollaries 6.2.3.8 and 6.2.3.11).

Combining the observations, we have:

Corollary 6.2.3.12. *The tautological section* $\dddot{\Xi}_{\Phi_n} \to \mathbf{G}_{\mathrm{m}}$ *defined by each* $\frac{1}{n}y$ *and* $\frac{1}{n}y'$ *by*

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} \in \mathbf{G}_{\mathrm{m}}(\dddot{\Xi}_{\Phi_n})$$

agrees with the evaluation of the canonical section $\partial^{(0)}_n : \dddot{\Xi}_{\Phi_n} \to \dddot{E}^{(n)}_{\Phi_n}$ *in Corollary 6.2.3.11 at*

$$[(\tfrac{1}{n}y) \otimes \phi(y')] - [(\tfrac{1}{n}y') \otimes \phi(y)] \in \dddot{\mathbf{S}}^{(n)}_{\Phi_n}.$$

Then it makes sense to talk about the fiber $\dddot{\Xi}_{\Phi_n,a_n} := (\partial_n^{(0)})^{-1}(a_n)$ of $\partial_n^{(0)}$ over a particular section $a_n \in \dddot{E}_{\Phi_n}^{(n)}$, as in the case of $\dddot{C}_{\Phi_n,b_n}$. The upshot of defining $\dddot{\Xi}_{\Phi_n,a_n}$ is that we have a tautological relation

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_n(\tfrac{1}{n}y, \tfrac{1}{n}y')$$

for any $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ over $\dddot{\Xi}_{\Phi_n,a_n}$. If we define

$$\dddot{\Xi}_{\Phi_n,(b_n,a_n)} := \dddot{\Xi}_{\Phi_n,a_n}|_{\dddot{C}_{\Phi_n,b_n}} = \dddot{\Xi}_{\Phi_n,a_n} \underset{\dddot{C}_{\Phi_n}}{\times} \dddot{C}_{\Phi_n,b_n},$$

then we have both the tautological relations

$$\lambda_A c_n^\vee - c_n\phi_n = b_n$$

and

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_n(\tfrac{1}{n}y, \tfrac{1}{n}y')$$

for any $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ over $\dddot{\Xi}_{\Phi_n,(b_n,a_n)}$.

As a result, we have obtained a tautological triple $(c_n, c_n^\vee, \tau_n)$ over $\dddot{\Xi}_{\Phi_n,(b_n,a_n)}$, on top of the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,n}, X, Y, \phi, c, c^\vee, \tau)$ over $\dddot{\Xi}_\phi \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_n^{\mathsf{Z}_n}$, which satisfies the additional condition that $\lambda_A c_n^\vee - c_n\phi_n = b_n$ and the relation between τ_n and a_n. What remains is the liftability of such a triple to some system $(\hat{c}, \hat{c}^\vee, \hat{\tau}) := \{(c_m, c_m^\vee, \tau_m)\}_{n|m,\Box\nmid m}$ as in Definition 5.2.3.4, which are compatible with some system $(\hat{b}, \hat{a}) := \{(b_m, a_m)\}_{n|m,\Box\nmid m}$ lifting (b_n, a_n) in the natural sense.

To realize the liftability condition for $(c_n, c_n^\vee, \tau_n)$, we shall consider the common schematic image $\dddot{\Xi}^{\text{com.}}_{\Phi_n,(b_n,a_n)}$ of the natural maps $\dddot{\Xi}_{\Phi_m,(b_m,a_m)} \to \dddot{\Xi}_{\Phi_n,(b_n,a_n)}$, for m such that $n|m$ and $\Box \nmid m$. This common image will cover the common schematic image $\dddot{C}^{\text{com.}}_{\Phi_n,b_n}$ of $\dddot{C}_{\Phi_m,b_m} \to \dddot{C}_{\Phi_n,b_n}$.

Remark 6.2.3.13. It may seem unpleasant that we need to fix a choice of a system $(\hat{b}, \hat{a}) = \{(b_m, a_m)\}_{n|m,\Box\nmid m}$ lifting (b_n, a_n) when defining $\dddot{\Xi}^{\text{com.}}_{\Phi_n,(b_n,a_n)}$ and $\dddot{C}^{\text{com.}}_{\Phi_n,b_n}$. It will be clear later how our construction should be interpreted as independent of the choice of liftings.

Lemma 6.2.3.14. *Under the assumption that we have chosen a system $\hat{b} = \{b_m\}_{n|m,\Box\nmid m}$ lifting b_n, we have a recipe to produce a uniquely determined system of elements $\{\tilde{b}_m\}_{n|m,\Box\nmid m}$ such that $\tilde{b}_m \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X/\phi(Y), A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \{e\} \subset \dddot{C}_{\Phi_m}$, such that $\partial_m^{(1)}(\tilde{b}_m) = b_m$,* *and such that $\tilde{b}_l$ is mapped to $\tilde{b}_m$ under the natural surjection $\dddot{C}_{\Phi_l} \twoheadrightarrow \dddot{C}_{\Phi_m}$ for any $m|l$ in the system. Moreover, the choice of each $\tilde{b}_m$ is independent of those b_l with $l > m$.*

As a result, $\tilde{b}_m$ translates $\dddot{C}_{\Phi_m,0}$ to $\dddot{C}_{\Phi_m,b_m}$ for any $m \geq 1$, $n|m$, $\Box \nmid m$, and the translations are compatible between different m and l.

Proof. Let $m \geq 1$ be an integer such that $n|m$ and $\Box \nmid m$, and let $l \geq 1$ be another integer such that $n|m$, $\Box \nmid m$, and $\frac{1}{m}X \subset \frac{1}{l}Y$. Certainly, what is implicit in the choices of $\hat{b}$ is that there is a choice of a filtration Z lifting Z_n, which in particular induces two filtrations Z_m and Z_l by reductions. By making an étale localization to $\mathsf{M}_l^{\mathsf{Z}_l}$, we may restrict the map $b_l : \frac{1}{l}Y \to A^\vee[l]$ to $\frac{1}{m}X$, and obtain a map $\frac{1}{m}X \to A^\vee$ over $\mathsf{M}_l^{\mathsf{Z}_l}$, whose restriction to $\phi(Y)$ is necessarily trivial. That is, we obtain an element in $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{m}X/\phi(Y), A^\vee)$. Let us define $\tilde{b}_m := (-b_l|_{\frac{1}{m}X}, 0)$. (The reason for putting the negative sign will be clear later.) Since the group $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{m}X/\phi(Y), A^\vee)$ is already defined over $\mathsf{M}_m^{\mathsf{Z}_m}$, this map $\tilde{b}_m$ is defined over $\mathsf{M}_m^{\mathsf{Z}_m}$ by étale descent.

By Definition, $\partial_m^{(1)}(\tilde{b}_m) = \partial_m^{(1)}(-b_l|_{\frac{1}{m}X}, 0)$ is given by the restriction of $b_l|_{\frac{1}{m}X}$ to $\frac{1}{m}Y$, which is necessarily b_m. The same argument also shows that $\tilde{b}_m$ is compatible between different m and l.

Finally, the fact that $\tilde{b}_m$ is defined without a base change to $\mathsf{M}_l^{\mathsf{Z}_l}$ with $l > m$ shows that $\tilde{b}_m$ must be independent of those b_l with $l > m$. □

Remark 6.2.3.15. The statement that $\tilde{b}_n$ translates $\dddot{C}_{\Phi_n,0}$ to $\dddot{C}_{\Phi_n,b_n}$ shows that the underlying geometric spaces of $\dddot{C}_{\Phi_n,0}$ and $\dddot{C}_{\Phi_n,b_n}$ are isomorphic. The essential difference is that we see potentially different tautological maps 0 and b_n on them, which might lead to different tautological tuples in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}^{\text{fil.-spl.}}$ (defined as in Definition 5.4.1.1) in the end of the construction, and might produce different degenerating families that should not be glued to the same part of the same Shimura variety. (It is possible that they should be glued to different Shimura varieties.)

Lemma 6.2.3.16. *The common schematic image $\dddot{C}_{\Phi_n,0}^{\text{com.}}$ of the maps $\dddot{C}_{\Phi_m,0} \to \dddot{C}_{\Phi_n,0}$ is the identity component $\dddot{C}_{\Phi_n}^{\circ}$ of $\dddot{C}_{\Phi_n}$.*

Proof. Note that we have a canonical isomorphism

$$\dddot{C}_{\Phi_n,0} \underset{\mathsf{M}_n^{\mathsf{Z}_n}}{\times} \mathsf{M}_m^{\mathsf{Z}_m} \xrightarrow{\sim} \dddot{C}_{\Phi_1} \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_m^{\mathsf{Z}_m} \xrightarrow{\sim} \dddot{C}_{\Phi_m,0}$$

given by the canonical isomorphisms $\frac{1}{n}Y \xrightarrow{\sim} Y \xrightarrow{\sim} \frac{1}{m}Y$ and $\frac{1}{n}X \xrightarrow{\sim} X \xrightarrow{\sim} \frac{1}{m}X$ for any m we consider, because the $\dddot{C}_{\Phi_m,0}$'s are of the form

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee). \tag{6.2.3.17}$$

Therefore, the natural restriction map $\dddot{C}_{\Phi_m,0} \to \dddot{C}_{\Phi_n,0}$ defined by the inclusions $\frac{1}{n}Y \hookrightarrow \frac{1}{m}Y$ and $\frac{1}{n}X \hookrightarrow \frac{1}{n}X$ has the same image of the multiplication by $\frac{m}{n}$ on $\dddot{C}_{\Phi_n,0}$ itself. Since we know from Proposition 6.2.2.5 that the rank of the component group $\pi_0(\dddot{C}_{\Phi_n,0}/\mathsf{M}_n^{\mathsf{Z}_n})$ is prime-to-$\Box$, because it is the same as pullback of the component group $\pi_0(\dddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$, we see that there is an integer $m \geq 1$ such that $n|m$ and $\Box \nmid m$, and such that the multiplication by $\frac{m}{n}$ kills the component group $\pi_0(\dddot{C}_{\Phi_m,0}/\mathsf{M}_m^{\mathsf{Z}_m})$. In particular, the schematic image of $\dddot{C}_{\Phi_m,0} \to \dddot{C}_{\Phi_n,0}$ for this m is contained in $\dddot{C}^{\circ}_{\Phi_n}$. This forces the common schematic image $\dddot{C}^{\mathrm{com.}}_{\Phi_n,\{b_m\}}$ to agree with $\dddot{C}^{\circ}_{\Phi_n}$, because $\dddot{C}^{\circ}_{\Phi_n}$ is an abelian schemes by Proposition 6.2.3.2 and hence multiplications on it are surjective. $\Box$

Combining Lemmas 6.2.3.14 and 6.2.3.16, we obtain:

Corollary 6.2.3.18. *The common schematic image* $\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$ *is the translation of the identity component* $\dddot{C}_{\Phi_n,0}$ *of* $\dddot{C}_{\Phi_n}$ *by* $\tilde{b}_n$*, as defined in Lemma 6.2.3.14. In particular, it is an abelian scheme. Moreover, it does not depend on the choice of the liftings* $\{b_m\}_{n|m,\Box\nmid m}$ *of* b_n*.*

To find the common schematic image $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ of the natural maps $\dddot{\Xi}_{\Phi_m,(b_m,a_m)} \to \dddot{\Xi}_{\Phi_n,(b_n,a_n)}$, for m such that $n|m$ and $\Box \nmid m$, it suffices to find the common schematic image of the natural maps $\dddot{\Xi}_{\Phi_m,(b_m,a_m)}|_{\dddot{C}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,(b_n,a_n)}|_{\dddot{C}_{\Phi_n,b_n}}$, or rather $\dddot{\Xi}_{\Phi_m,a_m}|_{\dddot{C}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,a_n}|_{\dddot{C}_{\Phi_n,b_n}}$.

Lemma 6.2.3.19. *Under the assumption that we have chosen a system* $\hat{a} = \{a_m\}_{n|m,\Box\nmid m}$ *lifting* a_n*, we have a recipe (depending on whether* $2 \in \Box$ *or not) to produce a system of elements* $\{\tilde{a}_m\}_{n|m,\Box\nmid m}$ *such that* $\tilde{a}_m \in \dddot{E}_{\Phi_m}$*, such that* $\tilde{a}_m$ *is mapped to* a_m *under the surjection* $\dddot{E}_{\Phi_m} \twoheadrightarrow \dddot{E}^{(m)}_{\Phi_m}$*, and such that* $\tilde{a}_{m'}$ *is mapped to* $\tilde{a}_m$ *under the natural map* $\dddot{E}_{\Phi_l} \twoheadrightarrow \dddot{E}_{\Phi_m}$ *for any* $m|l$ *in the system. (This time the* $\tilde{a}_m$ *may depend on those* a_l *with* $l > m$*.)*

In particular, $\tilde{a}_m$ *translates* $\dddot{\Xi}_{\Phi_m,0}$ *to* $\dddot{\Xi}_{\Phi_m,a_m}$ *for any* $m \geq 1$*,* $n|m$*,* $\Box \nmid m$*, and the translations are compatible between different* m *and* l*.*

Proof. As in Lemma 6.2.3.10, the datum of a system of elements $\hat{a} = \{a_m\}_{n|m, \Box\nmid m}$ is equivalent to the datum of a system of pairings

$$\tfrac{1}{m}Y \times Y \to \mathbf{G}_{\mathrm{m}},$$

which we denote again by a_m, satisfying the two properties

$$a(\tfrac{1}{m}y, y') = -a(\tfrac{1}{m}y', y)$$

and

$$a(b\tfrac{1}{m}y, y') = a(\tfrac{1}{m}y, b^\star y')$$

for any $y, y' \in Y$ and any $b \in \mathcal{O}$. Here m runs over the integers $m \geq 1$ such that $n|m$ and $\Box \nmid m$.

On the other hand, each of the element $\tilde{a}_m$ to be produced in $\ddot{E}_{\Phi_m}$ corresponds to a linear map

$$(\tfrac{1}{m}Y) \underset{\mathbb{Z}}{\otimes} X \to \mathbf{G}_{\mathrm{m}},$$

which we denote again by $\tilde{a}_m$, such that

$$\tilde{a}_m(y, \phi(y')) = -\tilde{a}_m(y', \phi(y))$$

and

$$\tilde{a}_m(b\tfrac{1}{m}y, \chi) = \tilde{a}(\tfrac{1}{m}y, b^\star \chi)$$

for any $y, y' \in Y$, $\chi \in X$ and $b \in \mathcal{O}$, so that $\tilde{a}_m$ will be mapped to

$$\tilde{a}_m(\tfrac{1}{m}y, \phi(y')) - \tilde{a}_m(\tfrac{1}{m}y', \phi(y)),$$

which agrees with $a_m(\frac{1}{m}y, y')$. (The fact that this makes sense is actually the content of Lemma 6.2.3.10.)

Let us take an integer $l \geq 1$ such that $lX \subset \phi(Y)$. We may and we do take this integer l to be prime-to-$\Box$, because $\Box \nmid [X : \phi(Y)]$ by the original choice of $\Box$.

If $2 \notin \Box$, then a_{2lm} is defined for each $m \geq 1$ such that $n|m$ and $\Box \nmid m$. Let us define $\tilde{a}_m$ by

$$\tilde{a}_m(\tfrac{1}{m}y, \chi) := a_{2lm}(\tfrac{1}{2lm}y, \phi^{-1}(l\chi)).$$

Then we have

$$\begin{aligned}
&\tilde{a}_m(\tfrac{1}{m}y, \phi(y')) - \tilde{a}_m(\tfrac{1}{m}y', \phi(y)) \\
&= a_{2lm}(\tfrac{1}{2lm}y, \phi^{-1}(l\phi(y'))) - a_{2lm}(\tfrac{1}{2lm}y', \phi^{-1}(l\phi(y))) \\
&= a_{2lm}(\tfrac{1}{2lm}y, ly') - a_{2lm}(\tfrac{1}{2lm}y', ly) = a_{2m}(\tfrac{1}{2m}y, y') - a_{2m}(\tfrac{1}{2m}y', y) \\
&= a_{2m}(\tfrac{1}{2m}y, y') + a_{2m}(\tfrac{1}{2m}y, y') = 2a_{2m}(\tfrac{1}{2m}y, y') = a_m(\tfrac{1}{m}y, y')
\end{aligned}$$

The upshot in this argument is $\frac{1}{2}+\frac{1}{2}=1$.

If $2 \in \Box$, then all the integer $m \geq 1$ such that $\Box \nmid m$ has to be odd, and we can pick a system of elements $e_m \in \mathbb{Z}/m\mathbb{Z}$, such that $2e_m = 1$ in $\mathbb{Z}/m\mathbb{Z}$ for any odd integer $m \geq 1$. It is obvious that $\{e_m\}$ form a compatible system and define an element of $\hat{\mathbb{Z}}^{\Box}$. Then we can define $\tilde{a}_m$ by

$$\tilde{a}_m(\tfrac{1}{m}y,\chi) := e_{lm}a_{lm}(\tfrac{1}{lm}y,\phi^{-1}(l\chi)).$$

Then we have

$$\begin{aligned}
&\tilde{a}_m(\tfrac{1}{m}y,\phi(y')) - \tilde{a}_m(\tfrac{1}{m}y',\phi(y)) \\
&= e_{lm}a_{lm}(\tfrac{1}{lm}y,\phi^{-1}(l\phi(y'))) - e_{lm}a_{lm}(\tfrac{1}{lm}y',\phi^{-1}(l\phi(y))) \\
&= e_{lm}a_{lm}(\tfrac{1}{lm}y,ly') - e_{lm}a_{lm}(\tfrac{1}{lm}y',ly) = e_m a_m(\tfrac{1}{m}y,y') - e_m a_m(\tfrac{1}{m}y',y) \\
&= e_m a_m(\tfrac{1}{m}y,y') + e_m a_m(\tfrac{1}{m}y,e^{\star}y') = 2e_m a_m(\tfrac{1}{m}y,y') = a_m(\tfrac{1}{m}y,y')
\end{aligned}$$

The upshot in this argument is simply $2e_m = 1$.

This gives our recipes for producing a system $\{\tilde{a}_m\}_{n|m,\Box\nmid m}$ explicitly for each system $\hat{a} = \{a_m\}_{n|m,\Box\nmid m}$ lifting a_n. $\Box$

Remark 6.2.3.20. Due to the presence of a_{lm} or a_{2lm} in the above construction, it is not always possible to make $\tilde{a}_m$ independent of the choice of a_l for $l > m$. This may seem unpleasant for our construction. But if we consider for example the case $n = 1$, when most of the data are trivially given, then this should not be not surprising. As in Remark 6.2.3.15, the statement that $\tilde{a}_m$ translates $\dddot{\Xi}_{\Phi_m,0}$ to $\dddot{\Xi}_{\Phi_m,a_m}$ shows that there is no practical different between the two underlying geometric spaces of $\dddot{\Xi}_{\Phi_m,0}$ and $\dddot{\Xi}_{\Phi_m,a_m}$. The essential difference is that we will see different tautological tuples in the end. We will see more complete explanations to this in Remark 6.2.3.23.

Lemma 6.2.3.21. *For any $\hat{b}$ lifting b_n, the common schematic image $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,0)}$ of the maps $\dddot{\Xi}_{\Phi_m,0}|_{\ddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,0}|_{\ddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ is the identity component $\dddot{\Xi}^{\circ}_{\Phi_n}|_{\ddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ of $\dddot{\Xi}_{\Phi_n}|_{\ddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ (defined as in Definition 6.2.3.9).*

Proof. Note that, for any $m \geq 1$ such that $n|m$ and $\Box \nmid m$, the fiber $\dddot{\Xi}_{\Phi_m,0} = (\partial_m^{(1)})^{-1}(0)$ can be constructed explicitly as follows:

Let $\dddot{\mathbf{S}}_{\Phi_m,0}$ be the finitely-generated abelian group defined by

$$\dddot{\mathbf{S}}_{\Phi_m,0} := ((\tfrac{1}{m}Y)\underset{\mathbb{Z}}{\otimes} X)/\begin{pmatrix} (\frac{1}{m}y)\otimes\phi(y') - (\frac{1}{m}y')\otimes\phi(y) \\ (b\frac{1}{m}y)\otimes\chi - (\frac{1}{m}y)\otimes(b^{\star}\chi) \end{pmatrix}_{\substack{y,y'\in Y,\\ \chi\in X, b\in\mathcal{O}}}$$

As in the construction of $\dddot{\Xi}_{\Phi_n}$, the formal properties of the pullback of the Poincaré biextension allows us to associate to each

$$\ell = \sum_i [(\tfrac{1}{m} y_i) \otimes \chi_i] \in \dddot{\mathbf{S}}_{\Phi_m,0}$$

a well-defined invertible sheaf

$$\Psi_{m,0}(\ell) := \bigotimes_{\dddot{C}_{\Phi_m},i} (c_m^\vee(\tfrac{1}{m} y_i), c(\chi_i))^* \mathcal{P}_A,$$

so that there exists a canonical isomorphism

$$\Delta^*_{m,0,\ell,\ell'} : \Psi_{m,0}(\ell) \bigotimes_{\mathscr{O}_{\dddot{C}_{\Phi_m}}} \Psi_{m,0}(\ell') \xrightarrow{\sim} \Psi_m(\ell + \ell')$$

for any $\ell, \ell' \in \dddot{\mathbf{S}}_{\Phi_m,0}$. Note that here we are using a stronger relation than those used for $\dddot{\mathbf{S}}_{\Phi_m}$, namely we are using the canonical symmetry isomorphisms

$$(c_m^\vee(\tfrac{1}{m} y), c(\phi(y')))^* \mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c_m^\vee(\tfrac{1}{m} y'), c(\phi(y)))^* \mathcal{P}_A$$

for all $y, y' \in Y$, instead of simply

$$(c^\vee(y), c(\phi(y')))^* \mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y'), c(\phi(y)))^* \mathcal{P}_A.$$

As a result, we can form an $\mathscr{O}_{\dddot{C}_{\Phi_m}}$-sheaf of algebras

$$\bigoplus_{\ell \in \dddot{\mathbf{S}}_{\Phi_m,0}} \Psi_{m,0}(\ell)$$

with algebra structure given by the isomorphisms $\Delta^*_{m,0,\ell,\ell'}$ above, and define

$$\dddot{\Xi}_{\Phi_m,0} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\dddot{C}_{\Phi_m}}} (\bigoplus_{\ell \in \dddot{\mathbf{S}}_{\Phi_m,0}} \Psi_{m,0}(\ell)).$$

This definition produces the same $\dddot{\Xi}_{\Phi_m,0}$ defined as the fiber $(\partial_m^{(1)})^{-1}(0)$. Moreover, if we consider $\dddot{E}_{\Phi_m,0} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_m,0}, \mathbf{G}_{\mathrm{m}})$ to be the group of multiplicative type of finite type over $\mathrm{Spec}(\mathbb{Z})$ with character group $\dddot{\mathbf{S}}_{\Phi_m,0}$, then $\dddot{\Xi}_{\Phi_m,0}$ is naturally a $\dddot{E}_{\Phi_m,0}$-torsor.

Note that we have

$$\dddot{\mathbf{S}}_{\Phi_m,0} \xrightarrow{\sim} \dddot{\mathbf{S}}_{\Phi_1} = (Y \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (by) \otimes \chi - y \otimes (b^\star \chi) \end{pmatrix}_{\substack{y,y' \in Y, \\ \chi \in X, b \in \mathcal{O}}},$$

for any m we consider. That is, they are all isomorphic. Therefore, the natural map

$$\dddot{\mathbf{S}}_{\Phi_n,0} \to \dddot{\mathbf{S}}_{\Phi_m,0}$$

induced by $(\frac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X \hookrightarrow (\frac{1}{m}Y) \underset{\mathbb{Z}}{\otimes} X$ for each m can be viewed as multiplication maps

$$\dddot{\mathbf{S}}_{\Phi_n,0} \overset{[\frac{m}{n}]}{\to} \dddot{\mathbf{S}}_{\Phi_n,0}.$$

Note that the cardinality of the torsion subgroup has only prime factors dividing $\mathrm{I}_{\mathrm{bad}} \operatorname{Disc}[X : \phi(Y)]$, which is prime-to-$\Box$ (cf. Remarks 6.2.2.7 and 6.2.3.6). Consequently, any sufficiently divisible choice of $m \geq 1$ such that $n|m$ and $\Box \nmid m$ kills the torsion subgroup $\dddot{\mathbf{S}}_{\Phi_n,0,\mathrm{tor}}$ of $\dddot{\mathbf{S}}_{\Phi_n,0}$.

According to Lemma 6.2.3.7, the torsion subgroup $\dddot{\mathbf{S}}_{\Phi_n,0,\mathrm{tor}}$ defines a canonical map

$$\dddot{\Xi}_{\Phi_n,0} \to \dddot{E}_{\Phi_n,0,\mathrm{tor}} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_n,0,\mathrm{tor}}, \mathbf{G}_{\mathrm{m}}),$$

which makes the diagram commute

$$\begin{tikzcd}
\dddot{\Xi}_{\Phi_n,0} \arrow[r] \arrow[d] & \dddot{\Xi}_{\Phi_n} \arrow[d] \\
\dddot{E}_{\Phi_n,0,\mathrm{tor}} \arrow[r] & \dddot{E}_{\Phi_n,\mathrm{tor}}
\end{tikzcd},$$

so that we can identify the pullback of $\dddot{E}_{\Phi_n,0,\mathrm{tor}}$ to $\dddot{C}_{\Phi_n}$ as the component group $\pi_0(\dddot{\Xi}_{\Phi_n,0}/\dddot{C}_{\Phi_n})$ of $\dddot{\Xi}_{\Phi_n,0}$ over $\dddot{C}_{\Phi_n}$. Since the groups $\dddot{E}_{\Phi_n,0,\mathrm{tor}}$ are all isomorphic to $\dddot{E}_{\Phi_1,\mathrm{tor}} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_1,\mathrm{tor}}, \mathbf{G}_{\mathrm{m},\mathsf{M}_n^{\mathsf{z}_n}})$, the bottom map of the $\dddot{E}_{\Phi_m,0}$-equivariant commutative diagram

$$\begin{tikzcd}
\dddot{\Xi}_{\Phi_m,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \arrow[r] \arrow[d] & \dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}} \arrow[d] \\
\dddot{E}_{\Phi_m,0,\mathrm{tor}} \arrow[r] & \dddot{E}_{\Phi_n,0,\mathrm{tor}}
\end{tikzcd}$$

induced by the inclusions

$$\dddot{\mathbf{S}}_{\Phi_n,0} \hookrightarrow \dddot{\mathbf{S}}_{\Phi_m,0}$$

can be identified with the multiplication by $\frac{m}{n}$ on $\dddot{E}_{\Phi_m,0,\mathrm{tor}}$, which kills the whole group as soon as $\frac{m}{n}$ is divisible by the rank of $\dddot{E}_{\Phi_m,0,\mathrm{tor}}$. As a result, we see that the common image of $\dddot{\Xi}_{\Phi_m,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ must lie inside the identity component $\dddot{\Xi}^{\circ}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ of $\dddot{\Xi}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$. This proves our result, because $\dddot{\Xi}^{\circ}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ is a torsor under the torus $\dddot{E}_{\Phi_n,\mathrm{free}}$ by Corollary 6.2.3.8, and maps between torus-torsors that are equivariant under surjective maps on the tori must be surjective as well. $\square$

Combining Lemmas 6.2.3.19 and 6.2.3.21, we obtain:

Corollary 6.2.3.22. *The common schematic image* $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ *is the translation of the* **identity component** $\dddot{\Xi}^{\circ}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ *of* $\dddot{\Xi}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ *by* $\tilde{a}_n$*, as defined in Lemma 6.2.3.19. In particular, it is a torsor under the torus* $\dddot{E}_{\Phi_n,\mathrm{free}}$ *over the abelian scheme* $\dddot{C}_{\Phi_n,b_n}$.

Remark 6.2.3.23. As a continuation of Remark 6.2.3.20, we can give a more satisfactory explanation now to the ambiguity in the statements of Lemma 6.2.3.19 that $\tilde{a}_m$ may depend on a_l for $l > m$: Indeed, there is an ambiguity in choosing the $\tilde{a}_n$ with only the information of a_n. However, no matter what choice of $\tilde{a}_m$ we take, which is based on some lifting $\hat{a} = \{a_m\}_{n|m,\Box\nmid m}$, we obtain the same tautological a_m on $\dddot{\Xi}_{\Phi_m,a_m}$. Therefore, we obtain the same tautological tuple in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ in the end. (See Definition 5.4.1.1.) Different choices of $\tilde{a}_n$ that produce the same a_n does not produce any difference for our purpose. They lead to isomorphic boundary charts and isomorphic degenerating families over them. Therefore, there is no essential need to distinguish them.

Remark 6.2.3.24. A more conceptual way to interpret Remark 6.2.3.23 is to realize that there is actually no ambiguity at all: The liftability condition is imprecise unless we can specify where to lift to. Once we make the context of liftability precise, the system $\hat{a} = \{a_m\}_{n|m,\Box\nmid m}$ is then also made precise, and there is no room for a different $\tilde{a}_n$. This should be a much better approach, and we hope we could come back and say more about this later.

Proposition 6.2.3.25. *Let* $n \geq 1$ *be an integer prime-to-*$\Box$ (*in the setting of Section 6.2.1*)*. Fix a choice of a representative* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *of a cusp label at level* n*, where* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*, and hence a moduli problem* $\mathsf{M}_n^{\mathsf{Z}_n}$ *with a tautological tuple* $(A, \lambda_A, i_A, \varphi_{-1,n})$*. Then such a choice of cusp label determines two maps* b_{Φ_n,δ_n} *and* a_{Φ_n,δ_n} *as in Lemma 6.2.3.1.*

Let us consider the category fibred in groupoids over the category of locally noetherian schemes over $\mathsf{M}_n^{\mathsf{Z}_n}$ *whose fiber over each locally noetherian scheme* S *has objects the tuples*

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^{\vee}, \tau_n))$$

describing degeneration data without positivity condition over S. Explicitly, the tuple as above satisfies the following conditions:

1. *The tuple $(A, \lambda_A, i_A, \varphi_{-1,n})$ defines an object parameterized by $\mathsf{M}_n^{\mathsf{Z}_n}$.*

2. *$c_n : \frac{1}{n}X \to A^\vee$ and $c_n^\vee : \frac{1}{n}Y \to A$ are $\mathcal{O}$-equivariant group homomorphisms satisfying the compatibility relation $\lambda_A c_n^\vee - c_n \phi_n = b_{\Phi_n,\delta_n}$ with $\phi_n : \frac{1}{n}Y \hookrightarrow \frac{1}{n}X$ induced by the prescribed $\phi : Y \hookrightarrow X$.*

3. *$\tau_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n^\vee \times c)^* \mathcal{P}_A$ is a trivialization of biextensions over S which satisfy the relation*

   $$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y') \in \boldsymbol{\mu}_n(S)$$

   for any $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ under the canonical symmetry isomorphism

   $$(c_n^\vee(\tfrac{1}{n}y), c(\phi(y')))^* \mathcal{P}_A \xrightarrow{\sim} (c_n^\vee(\tfrac{1}{n}y') c(\phi(y)))^* \mathcal{P}_A,$$

   and satisfy the $\mathcal{O}$-compatibility

   $$\tau(b\tfrac{1}{n}y, \chi) = \tau(\tfrac{1}{n}y, b^\star \chi)$$

   for any $\frac{1}{n}y \in Y$ and $\chi \in X$ under the canonical isomorphism

   $$(c_n^\vee(b\tfrac{1}{n}y), c(\chi))^* \mathcal{P}_A \cong (c_n^\vee(\tfrac{1}{n}y), c(b^\star\chi))^* \mathcal{P}_A.$$

   (*Here it makes sense to write equalities of sections because the isomorphisms are all canonical.*)

4. *The triple $(c_n, c_n^\vee, \tau_n)$ is* **liftable** *in the following sense: For any $m \geq 1$ such that $n|m$ and $\Box \nmid m$, suppose we have lifted all the other data to some liftable tuple*

   $$(\mathsf{Z}_m, (X, Y, \phi, \varphi_{-2,m}, \varphi_{0,m}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_m)$$

   at level m. Then the triple $(c_n, c_n^\vee, \tau_n)$ is also liftable to some $(c_m, c_m^\vee, \tau_m)$ that has the same kind of compatibility as $(c_n, c_n^\vee, \tau_n)$ with other data.

Two tuples

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$

and

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,n}), \delta_n, (c'_n, (c_n^\vee)', \tau'_n))$$

are isomorphic if there are isomorphisms $(f_X : X \xrightarrow{\sim} Y, f_Y : Y \xrightarrow{\sim} Y) \in \Gamma_{\Phi_n}$ (*defined as in Definition 6.2.1.1*), *and* $f_A : (A, \lambda_A, i_A, \varphi_{-1,n}) \xrightarrow{\sim} (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,n})$ *over* S, *such that:*

1. *The maps* $c_n : \frac{1}{n}X \to A^\vee$ *and* $c'_n : \frac{1}{n}X \to (A')^\vee$ *are related by* $cf_X = f_A{}^\vee c'$. (*Here* f_X *also stands for the isomorphism* $\frac{1}{n}X \xrightarrow{\sim} \frac{1}{n}X$ *canonically induced by* f_X.)
2. *The maps* $c_n^\vee : \frac{1}{n}Y \to A$ *and* $(c_n^\vee)' : \frac{1}{n}Y \to A'$ *are related by* $f_A c_n^\vee = (c_n^\vee)' f_Y$. (*Here* f_Y *also stands for the isomorphism* $\frac{1}{n}Y \xrightarrow{\sim} \frac{1}{n}Y$ *canonically induced by* f_X.)
3. *The trivializations* $\tau_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n^\vee \times c)^*\mathcal{P}_A$ *and* $\tau'_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} ((c_n^\vee)' \times c')^*\mathcal{P}_{A'}$ *are related by* $(\mathrm{Id}_{\frac{1}{n}Y} \times f_X)^*\tau = (f_Y \times \mathrm{Id}_X)^*\tau'$.

Then there is a separated and smooth (*relative*) *scheme* Ξ_{Φ_n,δ_n} *over* $\mathsf{M}_n^{\mathsf{Z}_n}$, *together with a tautological tuple and a natural action of* Γ_{Φ_n} *on* Ξ_{Φ_n,δ_n}, *such that the quotient* $\Xi_{\Phi_n,\delta_n}/\Gamma_{\Phi_n}$ *is isomorphic to the category described above* (*as categories fibred in groupoids over* $\mathsf{M}_n^{\mathsf{Z}_n}$). *Equivalently, for every scheme* S *over* $\mathsf{M}_n^{\mathsf{Z}_n}$, *together with a tuple*

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$

over S, *there is a map* $S \to \Xi_{\Phi_n,\delta_n}$, *which is unique after we fix an isomorphism* $(f_Y : Y \xrightarrow{\sim} Y, f_X : X \xrightarrow{\sim} X)$ *in* Γ_{Φ_n}, *such that the tuple over* S *is the pullback of the tautological tuple over* Ξ_{Φ_n,δ_n} *if we identify* X *by* f_X *and* Y *by* f_Y.

Proof. Simply take $\Xi_{\Phi_n,\delta_n} := \overset{\dots}{\ddot{\Xi}}{}^{\mathrm{com.}}_{\Phi_n,\delta_n,(b_{\Phi_n,\delta_n},a_{\Phi_n,\delta_n})}$ as in Corollary 6.2.3.22. Then the expected universal properties including the relations given by b_{Φ_n,δ_n} and a_{Φ_n,δ_n} and the liftability follow from the construction. Note that the ambiguity of identification of $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ forces us to allow an action of Γ_{Φ_n}, as in the case of $\overset{\dots}{\ddot{\Xi}}_{\Phi_1}$. This explains why we need the quotient by Φ_n. □

However, to really invoke Theorem 5.2.7.15, we need a tautological object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}$, or equivalently (by Lemma 5.4.2.10) an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ if we fix a choice of a representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$ of a cusp label, both of which do not literally make sense if our base scheme is not the spectrum $\mathrm{Spec}(R)$ of some I-adic complete local ring as in Section 5.2.1. Moreover, we need the positivity condition with respect to the ideal of definition I of R. These two issues will be addressed in Section 6.2.5 with the introduction of toroidal embeddings.

For simplicity, we shall set up the following convention:

Convention 6.2.3.26. *We shall set up the following simplification of notations:*

1. $\mathbf{S}_{\Phi_n} := \dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$.
2. $E_{\Phi_n} := \dddot{E}_{\Phi_n,\mathrm{free}}$.
3. $C_{\Phi_n,\delta_n} := \dddot{C}^{\mathrm{com.}}_{\Phi_n,b_{\Phi_n,\delta_n}}$.
4. $\Xi_{\Phi_n,\delta_n} := \dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,\delta_n,(b_{\Phi_n,\delta_n},a_{\Phi_n,\delta_n})}$ *(already set up in Proposition 6.2.3.25).*

Moreover, as a E_{Φ_n}-torsor, we shall describe Ξ_{Φ_n,δ_n} using a weight space decomposition

$$\mathscr{O}_{\Xi_{\Phi_n,\delta_n}} = \bigoplus_{\ell \in \mathbf{S}_{\Phi_n}} \Psi_{\Phi_n,\delta_n}(\ell).$$

Here the notation Ψ_{Φ_n,δ_n} means that, when we set up the equivalences between, for example, $(c_n^\vee(\frac{1}{n}y), c\phi(y'))^*\mathcal{P}_A$ and $(c_n^\vee(\frac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$, the isomorphisms

$$(c_n^\vee(\tfrac{1}{n}y), c\phi(y'))^*\mathcal{P}_A \xrightarrow{\sim} (c_n^\vee(\tfrac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$$

should differ from the canonical symmetry isomorphism

$$(c_n^\vee(\tfrac{1}{n}y), c\phi(y'))^*\mathcal{P}_A \overset{\mathrm{can.}}{\xrightarrow{\sim}} (c_n^\vee(\tfrac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$$

by the $a_{\Phi_n,\delta_n}(\frac{1}{n}y, \frac{1}{n}y')$ prescribed by Lemma 6.2.3.1.

6.2.4 Construction with General Level Structures

Let us take general level-$\mathcal{H}$ structures (as defined in Definition 1.3.7.8) into consideration.

With the setting as in Section 6.2.1, let $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ be an open compact subgroup, and let $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a (principal) cusp label at level $\mathcal{H}$ (defined as in Definition 5.4.2.4), where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ is a torus argument at level $\mathcal{H}$. Here X and Y are constant group schemes that will serve as the character group of the torus parts, as always.

Definition 6.2.4.1. *For any torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, the group $\Gamma_{\Phi_{\mathcal{H}}}$ is the subgroup of elements (γ_X, γ_Y) in Γ_ϕ that satisfies $\varphi_{-2,\mathcal{H}} = {}^t\gamma_X \varphi_{-2,\mathcal{H}}$ and $\varphi_{0,\mathcal{H}} = \gamma_Y \varphi_{0,\mathcal{H}}$ if we view them as collections of orbits.*

That is, the compositions with respectively ${}^t\gamma_X$ and γ_Y do not change the orbits. (Certainly, this does not mean the elements in the orbits are unchanged.)

As explained in Definition 5.4.2.6, the information of $\mathsf{Z}_{\mathcal{H}}$ alone defines a moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.2. Let $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ be the tautological tuple over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Then:

1. A is a (relative) abelian scheme over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$;

2. $\lambda_A : A \xrightarrow{\sim} A^\vee$ is a prime-to-$\Box$ polarization of A.

3. $i_A : \mathcal{O} \to \mathrm{End}_{\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}}(A)$ defines an $\mathcal{O}$-structure of (A, λ_A).

4. $\underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i_A satisfies the determinantal condition in Definition 1.3.4.2 given by $(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$.

5. $\varphi_{-1,\mathcal{H}}$ is an integral level-$\mathcal{H}$ structure for (A, λ_A, i_A) of type $(\mathrm{Gr}_{-1}^{\mathsf{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ as in Definition 1.3.7.8.

Based on the above data $\mathsf{Z}_{\mathcal{H}}$, $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$, and $\delta_{\mathcal{H}}$, we would like to construct a formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over which there is a tautological tuple

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c_{\mathcal{H}}^\vee, \tau_{\mathcal{H}})),$$

like an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}^{\text{fil.-spl.}}$. (See Definition 5.4.2.8.) Note that we cannot really say this is an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}^{\text{fil.-spl.}}$ because our base is a formal algebraic stack that does not necessarily fit into the setting of Section 5.2.1. Over each affine formal scheme that is formally étale over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, it should induce an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$. By Theorem 5.3.1.17, an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$, which is in particular a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ (as in Definition 5.3.2.1). The degenerating families over various different affine formal schemes should glue together and form a degenerating family called the *Mumford family* over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Therefore, stated more precisely, our goal in this section is to construct $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and the Mumford family over it.

Note that, as in Convention 5.4.2.5, we shall not make $\mathsf{Z}_{\mathcal{H}}$ explicit in the notations such as $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

For each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, set $\mathcal{H}_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$ as always. Then we can interpret $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ as a collection $\{(\mathsf{Z}_{\mathcal{H}_n}, \Phi_{\mathcal{H}_n}, \delta_{\mathcal{H}_n})\}$, each $(\mathsf{Z}_{\mathcal{H}_n}, \Phi_{\mathcal{H}_n}, \delta_{\mathcal{H}_n})$ being an $\mathcal{H}_n$-orbit of some representative of cusp label $(\mathsf{Z}_n, \Phi_n, \delta_n)$ at level n. For each such representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$, we have constructed in Section 6.2.3 the algebraic stacks

$$\Xi_{\Phi_n,\delta_n} \twoheadrightarrow C_{\Phi_n,\delta_n} \twoheadrightarrow \mathsf{M}_n^{\mathsf{Z}_n},$$

such that the first morphism is a torsor under some torus E_{Φ_n} with character group $\mathbf{S}_{\Phi_n}$, such that the second morphism is an abelian scheme, and such that the quotient functor $\Xi_{\Phi_n,\delta_n}/\Gamma_{\Phi_n}$ is universal for tuples of the form

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^{\vee}, \tau_n)).$$

We claim that we can construct algebraic stacks

$$\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \twoheadrightarrow C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}, \tag{6.2.4.2}$$

such that the first morphism in (6.2.4.2) is a torsor under some torus $E_{\Phi_{\mathcal{H}}}$ with some character group $\mathbf{S}_{\Phi_{\mathcal{H}}}$, such that the second morphism in (6.2.4.2) is a (relative) abelian scheme, and such that the quotient functor $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ is universal for tuples of the form

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c_{\mathcal{H}}^{\vee}, \tau_{\mathcal{H}})).$$

If we take any $(\mathsf{Z}_n, \Phi_n, \delta_n)$ in the $\mathcal{H}_n$-orbit $(\mathsf{Z}_{\mathcal{H}_n}, \Phi_{\mathcal{H}_n}, \delta_{\mathcal{H}_n})$, then we have the natural inclusions

$$\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset \mathcal{H}_n$$

as in Definition 5.3.1.9. Note that the quotient $\mathcal{H}_n/\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ describes other elements in the orbit $\mathsf{Z}_{\mathcal{H}_n}$, and the quotient $\mathcal{H}_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}/\mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ describes the elements in the orbit $\Phi_{\mathcal{H}_n}$ that are torus arguments for Z_n. Once we have fixed a choice of $(\mathsf{Z}_n, \Phi_n, \delta_n)$, the action of $\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}$ on the remaining objects can be described by the following commutative diagram, in which the squares are all Cartesian by definition:

$$\begin{tikzcd}
\Xi_{\Phi_n,\delta_n} \arrow[r, two heads] \arrow[dr, two heads] & \Xi_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d, two heads] & \Xi_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d, two heads] & \Xi_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \arrow[d, two heads] & . \\
& C_{\Phi_n,\delta_n} \arrow[r, two heads] \arrow[dr, two heads] & C_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d, two heads] & C_{\Phi_n,\delta_n}/\mathcal{H}_{n,(\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n})} \arrow[d, two heads] & \\
& & \mathsf{M}_n^{\mathsf{Z}_n} \arrow[r, two heads] \arrow[dr, two heads] & \mathsf{M}_n^{\mathsf{Z}_n}/\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \arrow[d, two heads] & \\
& & & \mathsf{S}_0 &
\end{tikzcd}$$

(6.2.4.3)

Here, by Definition 5.3.1.9, $\mathcal{H}_{n,(\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n})} := \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$, $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} := \mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ and $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} := \mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$, and we have used the facts that $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ and $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ act trivially on respectively $\mathsf{M}_n^{\mathsf{Z}_n}$ and C_{Φ_n,δ_n} by their interpretation as universal spaces.

Lemma 6.2.4.4. *The finite group $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ can be canonically identified as a finite étale subgroup of the n-torsion elements in the torus E_{Φ_n}, and the action of $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ on the E_{Φ_n}-torsor Ξ_{Φ_n,δ_n} over C_{Φ_n,δ_n} can be canonically identified with the torsor-action of this subgroup of E_{Φ_n}, so that $\Xi_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ is the torsor under the quotient torus $E_{\Phi_{\mathcal{H}_n}} := E_{\Phi_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$. We shall denote its character group by $\mathbf{S}_{\Phi_{\mathcal{H}_n}}$. The torus $E_{\Phi_{\mathcal{H}_n}}$ and its character group $\mathbf{S}_{\Phi_{\mathcal{H}_n}}$ are independent of the choice of n, and hence define a torus $E_{\Phi_{\mathcal{H}}}$ with character group $\mathbf{S}_{\Phi_{\mathcal{H}}}$.*

Lemma 6.2.4.5. *The finite group $\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$ can be canonically identified with a finite étale subgroup of C_{Φ_n,δ_n}, and its action on C_{Φ_n,δ_n} can be given canonically by the action of this subgroup. As a result, the quotient $C_{\Phi_{\mathcal{H}_n},\delta_{\mathcal{H}_n}} := C_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$ is an abelian scheme over M_n.*

Lemma 6.2.4.6. *The finite group $\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ acts on $\mathsf{M}_n^{\mathsf{Z}_n}$ by twisting the level-n structures, and hence defines a map $\mathsf{M}_n^{\mathsf{Z}_n}/\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, which is an isomorphism by its moduli interpretation.*

Since the quotients in (6.2.4.3) all exist as algebraic stacks, the above lemmas show that the claim would follow if we identify (6.2.4.2) with the vertical arrows $\Xi_{\Phi_n,\delta_n}/\mathcal{H}_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \twoheadrightarrow C_{\Phi_n,\delta_n}/\mathcal{H}_{n,(\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n})} \twoheadrightarrow \mathsf{M}_n^{\mathsf{Z}_n}/\mathcal{H}_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ in (6.2.4.3). Note that the construction is independent of the n we choose, because the construction using some m (such that $n|m$ and $\Box \nmid m$) will factor through the above quotient diagram (6.2.4.3). This enables us to state the following analogue of Proposition 6.2.3.25 for $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$:

Proposition 6.2.4.7. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as above. Fix a choice of a representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of a cusp label at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, and hence a moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ with a tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$.*

Let us consider the category fibred in groupoids over the category of locally noetherian schemes over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ whose fiber over each locally noetherian scheme S has objects the tuples

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

describing degeneration data without positivity condition over S, which is a collection of $\mathcal{H}_n$-orbits of objects defined as in Proposition 6.2.3.25 for each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$.

Then there is a separated and smooth (relative) scheme $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, together with a tautological tuple and a natural action of $\Gamma_{\Phi_{\mathcal{H}}}$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, such that the quotient $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ is isomorphic to the category described above (as categories fibred in groupoids over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$). Equivalently, for every scheme S over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, together with a tuple

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

over S, there is a map $S \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, which is unique after we fix an isomorphism $(f_Y : Y \xrightarrow{\sim} Y, f_X : X \xrightarrow{\sim} X)$ in $\Gamma_{\Phi_{\mathcal{H}}}$, such that the tuple over S is the pullback of the tautological tuple over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ if we identify X by f_X and Y by f_Y.

Note then it makes sense to consider $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ as an invertible sheaf on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, for any $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, as in Convention 6.2.3.26.

6.2.5 Construction with Positivity Condition

We now construct toroidal embeddings $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over which the action of $\Gamma_{\Phi_{\mathcal{H}}}$ extends naturally, so that we can interpret the quotient $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ as some partial toroidal compactification of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$, and so that the period map could vanish along a suitable sub-algebraic stack of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} - \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. (Note that it is not true in general that the period map should vanish along the whole *boundary* of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.)

Let $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbb{Z})$ be the $\mathbb{Z}$-dual of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, and let $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}} := \mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbb{R})$ be the $\mathbb{R}$-dual of $\mathbf{S}_{\Phi_{\mathcal{H}}}$. By definition of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ (in Lemma 6.2.4.4), the $\mathbb{R}$-vector space $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ is isomorphic to the space of Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$, by sending a Hermitian pairing $(\!|\cdot,\cdot|\!)$ to the function $y \otimes \phi(y') \mapsto \mathrm{Tr}_{B/\mathbb{Q}}(\!|y,y'|\!)$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbb{R})$. (See Lemma 1.1.4.6 and Remark 1.1.4.7.)

Remark 6.2.5.1. Up to isomorphisms between $\mathbb{R}$-vector spaces, the space $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ is independent of $\Phi_{\mathcal{H}}$. Even so, the identification of the space depends on the identification of $\Phi_{\mathcal{H}}$. Therefore, things we do for this space will make sense only if they are canonical up to the action of $\Gamma_{\Phi_{\mathcal{H}}}$. This is why we will have to make our cone decompositions admissible under $\Gamma_{\Phi_{\mathcal{H}}}$ later.

Definition 6.2.5.2. *An element b in $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is* **symmetric** *if $b^{\star} = b$.*

Definition 6.2.5.3. *An element b in $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is* **positive** (*resp.* **semi-positive**) *if it is symmetric and*

$$(B \underset{\mathbb{Q}}{\otimes} \mathbb{R}) \times (B \underset{\mathbb{Q}}{\otimes} \mathbb{R}) \to \mathbb{R} : (x,y) \mapsto \mathrm{Tr}_{B/\mathbb{Q}}(ybx^{\star})$$

defines a **positive definite** (*resp.* **positive semi-definite**) *symmetric $\mathbb{Z}$-bilinear pairing. We denote this by $b > 0$ (resp. $b \geq 0$).*

Definition 6.2.5.4. *A Hermitian pairing $(\!|\cdot,\cdot|\!) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is* **positive definite** (*resp.* **positive semi-definite**) *if $(\!|y,y|\!) > 0$ (resp. $(\!|y,y|\!) \geq 0$) for any nonzero $y \in Y$.*

Let $\mathbf{P}_{\Phi_{\mathcal{H}}}$ be the subset of $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ corresponding to positive semi-definite Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$, with radical (namely the annihilator of the whole space) given by the $\mathbb{R}$-span of some *admissible submodule* Y' of Y. We say that the radical is *admissible.*

Remark 6.2.5.5. An admissible radical is automatically *rational* in the sense that it is spanned by elements in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. When $\mathcal{O}$ is maximal, the two notions are identical.

This subset $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is a cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$. Let $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be the subset of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ corresponding to positive definite Hermitian pairings. Then $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ is also a cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$.

Lemma 6.2.5.6. *If $h \in \mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $h \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$), then $h(y, \phi(y)) \geq 0$ (resp. $h(y, \phi(y)) > 0$) for any nonzero $y \in Y$.*

Proof. By definition, for any $y, y' \in Y$, $h(y, \phi(y')) = \mathrm{Tr}_{B/\mathbb{Q}}(\!| y, y' |\!)$ for some positive definite (resp. positive semi-definite) Hermitian form $(\!| \cdot, \cdot |\!)$. Hence $(\!| y, y |\!) > 0$ (resp. $(\!| y, y |\!) \geq 0$) for any nonzero $y \in Y$, which means we have $h(y, \phi(y)) > 0$ (resp. $h(y, \phi(y)) \geq 0$) for any nonzero $y \in Y$ after taking trace. □

By the arguments in [79, §2], there is no loss of generality in identifying $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ with products of one of those standard examples $\mathrm{M}_r(\mathbb{R})$, $\mathrm{M}_r(\mathbb{C})$, and $\mathrm{M}_r(\mathbb{H})$, with the cone $\mathbf{P}_{\Phi_{\mathcal{H}}}$ identified with the positive semi-definite matrices with admissible radicals, a condition no stronger than the condition with rational radicals (as explained in Remark 6.2.5.5). Hence, by [15, Ch. II], with minor error corrected by Looijenga as remarked in [37, Ch. IV, §2], it is known that there exist $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decompositions of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ with respect to the integral structure given by $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$.

Let $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j \in J}$ be any such cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$. Let $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ be the toroidal embedding of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ defined by $\Sigma_{\Phi_{\mathcal{H}}}$ as in Definition 6.1.2.3. Note that the choice of $\delta_{\mathcal{H}}$ is unrelated to the choice of $\Sigma_{\Phi_{\mathcal{H}}}$. (see also Definition 6.2.6.2 below. This is why we use the notation $\Sigma_{\Phi_{\mathcal{H}}}$ rather than $\Sigma_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.) By construction, we know that $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ has the properties described in Theorem 6.1.2.8, with the following additional properties:

Proposition 6.2.5.7. *1. There are constructible $\Gamma_{\Phi_{\mathcal{H}}}$-equivariant étale constructible sheaves $\underline{X}$ and $\underline{Y}$ on $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, together with an embedding $\underline{\phi} : \underline{Y} \hookrightarrow \underline{X}$, which are defined as follows:*

Any admissible surjection $X \twoheadrightarrow X'$ of $\mathcal{O}$-lattices (defined as in Definition 1.2.6.7) determines a surjection from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to some representative $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of a cusp label at level $\mathcal{H}$ by Lemma 5.4.2.11, where $\mathsf{Z}'_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are uniquely determined by the construction. In particular, it makes sense to define $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ and an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ for any admissible surjection $X \twoheadrightarrow X'$.

Over the locally closed stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_j}$, $\underline{X}$ is the constant quotient sheaf X_{σ_j} of X, with quotient map $X \twoheadrightarrow X_{\sigma_j}$ an admissible surjection defining a pair $(\mathsf{Z}_{\mathcal{H},\sigma_j}, \Phi_{\mathcal{H},\sigma_j} = (X_{\sigma_j}, Y_{\sigma_j}, \phi_{\sigma_j}, \varphi_{-2,\mathcal{H},\sigma_j}, \varphi_{0,\mathcal{H},\sigma_j}))$, such that σ_j is contained in the image of the embedding $\mathbf{P}^{+}_{\phi_{\sigma_j}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$. We would like to interpret this as having a sheaf version of $\Phi_{\mathcal{H}}$, written as $\underline{\Phi}_{\mathcal{H}} = (\underline{X}, \underline{Y}, \underline{\phi}, \underline{\varphi}_{-2,\mathcal{H}}, \underline{\varphi}_{0,\mathcal{H}})$.

2. *The formation of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ from $\Phi_{\mathcal{H}}$ applies to the above context of $\underline{\Phi}_{\mathcal{H}}$, and defines a sheaf $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}}$.*

3. *There is a tautological linear map of constructible sheaves $\underline{B} : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}} \to \underline{\mathrm{Inv}}(\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})$ (see Definition 4.2.4.1), which sends the class of $\ell \in \mathbf{S}_{\Phi_{\mathcal{H},\sigma_j}}$ to the sheaf of ideals $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_j)} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_j)$, so that:*

   (a) *This map $\underline{B}$ is $\Gamma_{\Phi_{\mathcal{H}}}$-equivariant (because the map is compatible with twists of identification of $\Phi_{\mathcal{H}}$) and $E_{\Phi_{\mathcal{H}}}$-invariant (because $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ corresponds to a weight subspace of the $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-sheaf of algebras $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ under $E_{\Phi_{\mathcal{H}}}$-action), and is trivial on the open subscheme $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.*

   (b) *For any local section y of $\underline{Y}$, the support of $\underline{B}(y \otimes \phi(y))$ is effective, and is the same as the support of y. This is because $\sigma(y, \phi(y)) \geq 0$ for any $y \in Y$ and $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}$, and $\sigma(y, \phi(y)) > 0$ when $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H},\sigma}}$ and $y \in Y_\sigma$.*

For any nondegenerate rational polyhedral cone σ in $\mathbf{P}_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$, we can define the affine toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$, which can be interpreted as the *moduli space* for certain degeneration data without positivity condition, as follows:

Let R be a noetherian normal local integral domain with quotient field K, and suppose we have a map $t_R : \mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ that is liftable over $\mathrm{Spec}(K)$ to a map $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

The map $t_R : \mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ defines by the universal property of $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ a tuple

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, (A_R, \lambda_{A_R}, i_{A_R}, \varphi_{-1,\mathcal{H},R}), \delta_{\mathcal{H}}, (c_{\mathcal{H},R}, c^{\vee}_{\mathcal{H},R}))$$

describing a degeneration datum without the trivialization $\tau_{\mathcal{H}}$ (and hence without the positivity condition for τ) over $\mathrm{Spec}(R)$. Let us denote their restrictions to $\mathrm{Spec}(K)$ by the same notations with the subscript R replaced by K. The lifting $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ defines the additional datum $\tau_{\mathcal{H}}$ over $\mathrm{Spec}(K)$, which in particular determines an additional datum $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}^{\otimes -1}_{A,K}$. This defines sections $\tau(y,\chi) : (c_K^\vee(y), c_K(\chi))^*\mathcal{P}_{A_K} \overset{\sim}{\to} K$ for any $y \in Y$ and any $\chi \in X$, so that the natural restriction

$$(c_R^\vee(y), c_R(\chi))^*\mathcal{P}_{A_R} \overset{\text{res.}}{\to} (c_K^\vee(y), c_K(\chi))^*\mathcal{P}_{A_K} \overset{\tau(y,\chi)}{\overset{\sim}{\to}} K$$

defines an embedding of the invertible sheaf $(c_R^\vee(y), c_R(\chi))^*\mathcal{P}_{A_R}$ as an invertible R-submodule $I_{y,\chi}$ of K. By using $\tau_{\mathcal{H}}$, and by looking only at the images, this defines a map

$$B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R) : \ell \mapsto I_\ell$$

(see Definition 4.2.4.1). For any discrete valuation $\upsilon : K^\times \to \mathbb{Z}$ of K, since I_ℓ is locally principal for any ℓ, it makes sense to consider the composition

$$\upsilon \circ B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z} : \ell \mapsto \upsilon(I_\ell),$$

which is an element in $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee$.

Proposition 6.2.5.8. *Let R be a noetherian normal local integral domain with quotient field K, and suppose we have a map $t_R : \mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ that is liftable over $\mathrm{Spec}(K)$ to a map $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Then we have a unique map $B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R)$ by the above construction.*

The universal property of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is as follows: The map $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a map $\tilde{t}_R : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ if and only if for every discrete valuation $\upsilon : K^\times \to \mathbb{Z}$ of K the corresponding linear map $\upsilon \circ B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z}$ lies in the closure $\overline{\sigma}$ of σ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$.

Remark 6.2.5.9. The linear map $B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R)$ cannot tell us precisely how the invertible module $(c_R^\vee(y), c_R(\chi))^*\mathcal{P}_{A_R}$ should be embedded as an R-submodule $I_{y,\chi}$ of K. Nevertheless, since the embedding always exists and is canonically determined by the trivialization $\tau_{\mathcal{H}}$ (given by $\tilde{t}_K$), the only information we need is whether $I_{y,\chi}$ lies in R or not.

Remark 6.2.5.10. Recall that the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is defined (as in Lemma 6.1.2.6 and Definition 6.1.2.7) by the sheaf of ideals $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} =$

$\underset{\ell\in\sigma_0^\vee}{\oplus}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} = \underset{\ell\in\sigma^\vee}{\oplus}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. (See Convention 6.2.3.26.) Since $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}$ is positive semi-definite, we have $\sigma(\ell) \geq 0$ for any ℓ of the form $[y \otimes \phi(y)]$. As a result, the trivialization

$$\tau(y,\phi(y)) = \tau_{\mathcal{H}}(y,\phi(y)) : \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(y\otimes\phi(y)) \xrightarrow{\sim} \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a section

$$\tau(y,\phi(y)) : \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(y\otimes\phi(y)) \to \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \tag{6.2.5.11}$$

over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$. If moreover $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, then by Lemma 6.2.5.6, we have $\sigma(y\otimes\phi(y)) > 0$ for any $y \neq 0$. In this case, the section $\tau(y,\phi(y))$ over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ as in (6.2.5.11) has image contained in $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. This is almost exactly the positivity condition, except that the base scheme is not completed along $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$.

Since we have the tautological presence of $G^\natural$ and τ (defined by the tautological tuple $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$) over the smooth algebraic stack $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, we can define (by étale descent if necessary) as in Section 4.6.2 a Kodaira-Spencer map

$$\mathrm{KS}_{(G^\natural,\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{G^\natural/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]. \tag{6.2.5.12}$$

Let $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ be the morphism defined by the tautological data $\lambda_A : A \to A^\vee$ and $\phi : Y \to X$. Then $\lambda^\natural$ induces an $\mathcal{O}$-linear morphism $(\lambda^\natural)^* : \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \underline{\mathrm{Lie}}^\vee_{G^\natural/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$. Let $i^\natural : \mathcal{O} \to \mathrm{End}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(G^\natural)$ denote the tautological $\mathcal{O}$-action map on $G^\natural$. Then we define the sheaf $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G^\natural,\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ as the quotient of

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

by the submodule of relations

$$\begin{pmatrix} (\lambda^\natural)^*(y)\otimes z - (\lambda^\natural)^*(z)\otimes y \\ ((i^\natural(b))^*x)\otimes y - x\otimes((i^\natural(b)^\vee)^*y) \end{pmatrix}_{\substack{x\in\underline{\mathrm{Lie}}^\vee_{G^\natural/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}},\\ y,z\in\underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}},\\ b\in\mathcal{O}}}$$

(cf. Definitions 2.3.4.1 and 6.3.1).

Remark 6.2.5.13. When $\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$ is not invertible in the base scheme, the formation of $\underline{\mathrm{KS}}$ as a quotient may produce nontrivial torsion elements. But this is not the case here by Proposition 1.2.2.4 and the assumptions on the Lie algebra conditions.

Proposition 6.2.5.14. *The Kodaira-Spencer map* (*6.2.5.12*) *factors through the sheaf* $\underline{\mathrm{KS}}$ *defined above, and induces an isomorphism*

$$\underline{\mathrm{KS}} \xrightarrow{\sim} \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty].$$

Proof. Let us analyze the structure map $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{S}_0$ as a composition of *smooth* morphisms:

$$\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \xrightarrow{\pi_0} C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \xrightarrow{\pi_1} \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \xrightarrow{\pi_2} \mathsf{S}_0.$$

For simplicity, let us denote the composition $\pi_1 \circ \pi_0$ by π_{10}. Then $\Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ has an increasing filtration

$$0 \subset \pi^*_{01}\,\Omega^1_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0} \subset \pi^*_0\,\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} \subset \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0},$$

with graded pieces given by $\pi^*_{01}\,\Omega^1_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$, $\pi^*_0\,\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$ and $\Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$.

On the other hand, the sheaf $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G^\natural,\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ has an increasing filtration given by $\pi^*_{01}\,\underline{\mathrm{KS}}_{(A,\lambda)/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$, the pullback (under π_0) of the quotient $\underline{\mathrm{KS}}_{(A,c,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ of

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

by the relations as in the definition of $\underline{\mathrm{KS}}$, and the whole sheaf $\underline{\mathrm{KS}}$. Hence it suffices to show that the map (6.2.5.12) respects the filtrations and matches isomorphically the graded pieces.

By Proposition 2.3.4.2, the Kodaira-Spencer map $\mathrm{KS}_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$ for (A,λ) over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ induces an isomorphism $\underline{\mathrm{KS}}_{(A,\lambda)/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$, and hence the same remains true under pullback by π_{01}. Since the Kodaira-Spencer map $\mathrm{KS}_{A/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} = \pi^*_{01}\,\mathrm{KS}_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$ for (A,λ) over $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the restriction of the Kodaira-Spencer map KS in (6.2.5.12), we see that the first filtered pieces are respected.

By the deformation theoretic interpretation of the Kodaira-Spencer maps $\mathrm{KS}_{(A,c)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ and $\mathrm{KS}_{(A^\vee,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ in Section 4.6.1 (see in particular Definition 4.6.1.2), we see that both of the restrictions of them to

$$\underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

agree with $\mathrm{KS}_{(A,\lambda)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$, which is a surjection onto $\pi_1^*\ \Omega^1_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$. Hence they define an map from

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

to $\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$, which induces a map from

$$\underline{\mathrm{Lie}}^\vee_{T/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

to $\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$. The realization of $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a finite étale quotient of some C_{Φ_n,δ_n} with the condition that $\lambda_A c_n^\vee - c_n\phi_n = b_{\Phi_n,\delta_n}$ for a tautological map $b_{\Phi_n,\delta_n} : \frac{1}{n}Y/Y \to A^\vee[n]$ defined over $\mathsf{M}_n^{\mathsf{Z}_n}$ implies that the above morphism factors through the (the same) quotient image of either $\underline{\mathrm{Lie}}^\vee_{T/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ or $\underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$, which can be identified with the quotient of the above $\underline{\mathrm{KS}}_{(A,c,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ by $\underline{\mathrm{KS}}_{(A,\lambda)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} = \pi_1^*\ \underline{\mathrm{KS}}_{(A,\lambda)/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$. By the fact that $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the universal space for $(A, c_{\mathcal{H}}, c_{\mathcal{H}}^\vee)$, it defines an isomorphism

$$\underline{\mathrm{KS}}_{(A,c,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}/\underline{\mathrm{KS}}_{(A,\lambda)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}},$$

and hence an isomorphism

$$\underline{\mathrm{KS}}_{(A,c,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}.$$

Since the pullback of the map (under π_0) to $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the restriction of the Kodaira-Spencer map KS in (6.2.5.12), we see that the second filtered pieces are also respected, with an induced isomorphism between the second graded pieces.

Finally, we arrive at the top filtered pieces, and the only question is whether the induced morphism

$$\underline{\mathrm{Lie}}^{\vee}_{T/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^{1}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty] \tag{6.2.5.15}$$

is an isomorphism. For notational simplicity, let us denote by $\underline{\mathrm{KS}}_{T/\mathsf{S}_0}$ the quotient of $\underline{\mathrm{Lie}}^{\vee}_{T/\mathsf{S}_0} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/\mathsf{S}_0}$ by same relations as in the definition of $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G^{\natural},\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ as above, and by $\underline{\mathrm{KS}}_{T/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ and $\underline{\mathrm{KS}}_{T/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ their pullbacks to respectively $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Let us first consider the map

$$\underline{\mathrm{Lie}}^{\vee}_{T/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^{1}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \tag{6.2.5.16}$$

that (6.2.5.15) induces on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, which is induced by the Kodaira-Spencer map $\mathsf{KS}_{(G^{\natural},\iota)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ defined deformation-theoretically as in Definition 4.6.2.7. The realization of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a finite étale quotient of some Ξ_{Φ_n,δ_n} with the condition that $\iota_n(\frac{1}{n}y,\phi(y'))\iota_n(\frac{1}{n}y',\phi(y))^{-1} = a_{\Phi_n,\delta_n}(\frac{1}{n}y,\frac{1}{n}y')$ for a tautological map $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ over C_{Φ_n,δ_n} implies that (6.2.5.16) factors through the quotient $\underline{\mathrm{KS}}_{T/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ of its domain. By the fact that $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the universal space for ι over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, we see that the induced morphism $\underline{\mathrm{KS}}_{T/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^{1}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ is an isomorphism.

If we work over $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, then the map (6.2.5.15) is induced by the extended Kodaira-Spencer map $\mathsf{KS}_{(G^{\natural},\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ defined as in Definition 4.6.2.10. In other words, it is the restriction of (6.2.5.16) to $\underline{\mathrm{Lie}}^{\vee}_{T/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$. Since its image in $\Omega^{1}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty]$ contains $d\log(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$ for all $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, which are exactly the generators, we see that (6.2.5.15) induces an isomorphism $\underline{\mathrm{KS}}_{T/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^{1}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty]$ between the top filtered pieces, as desired. $\square$

Back to the context that we have the toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \hookrightarrow \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ defined by a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$. Let $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ be the formal completion of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ along the union of the σ_j-strata $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_j}$ for $\sigma_j \in \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, and if $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, let $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$

be the formal completion of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ along the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Then, using the language of relative schemes over formal algebraic stacks (see [61]), there are tautological tuples of the form

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

over the formal algebraic stacks $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, the one on the latter being the pullback of the one on the former under the natural map $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Moreover, this tautological tuple over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ satisfies the positive condition in the following sense: We have a functorial assignment that to every affine formal scheme U with a formally étale map $U \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ assigns a tuple

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

with positivity condition over the (smooth) scheme $\mathrm{Spec}(\Gamma(U, \mathscr{O}_U))$. By Theorem 5.3.1.17 (see also Definition 5.4.2.8 and Remark 5.4.2.9), Mumford's construction defines an object $(\,{}^{\heartsuit}G,\,{}^{\heartsuit}\lambda,\,{}^{\heartsuit}i,\,{}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(\Gamma(U, \mathscr{O}_U))$ in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$, which we call a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ as in Definition 5.3.2.1. Moreover, any fiber of ${}^{\heartsuit}G$ over the support of U has torus part a split torus with character group X. If we have a formally étale map $\mathrm{Spf}(R_1) \to \mathrm{Spf}(R_2)$ and if the degeneration datum over $\mathrm{Spec}(R_2)$ pulls back to the datum over $\mathrm{Spec}(R_1)$, then the family constructed by Mumford's construction over $\mathrm{Spec}(R_1)$ pulls back to a family over $\mathrm{Spec}(R_2)$ with the same degeneration datum as the datum over $\mathrm{Spec}(R_2)$. The functoriality in Theorem 4.4.18 over $\mathrm{Spec}(R_2)$ then assures that this pullback family must agree with the family constructed from the datum over $\mathrm{Spec}(R_2)$. In particular, we see that the assignment of $(\,{}^{\heartsuit}G,\,{}^{\heartsuit}\lambda,\,{}^{\heartsuit}i,\,{}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(\Gamma(U, \mathscr{O}_U))$ to U is functorial. Hence the assignment defines a (relative) degenerating family $(\,{}^{\heartsuit}G,\,{}^{\heartsuit}\lambda,\,{}^{\heartsuit}i,\,{}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Since the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ is $\Gamma_{\Phi_{\mathcal{H}}}$-admissible, the group $\Gamma_{\Phi_{\mathcal{H}}}$ acts naturally on all the objects involved in the degeneration data, and hence by functoriality on the degenerating family $(\,{}^{\heartsuit}G,\,{}^{\heartsuit}\lambda,\,{}^{\heartsuit}i,\,{}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Definition 6.2.5.17. *Let σ be any nondegenerate rational polyhedral cone in $\mathbf{P}_{\Phi_{\mathcal{H}}}$. The group $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is defined as the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ consisting of elements that leaves σ invariant under the natural action of $\Gamma_{\Phi_{\mathcal{H}}}$ on $\mathbf{P}_{\Phi_{\mathcal{H}}}$.*

A similar story describes the degenerating family $(\,{}^{\heartsuit}G,\,{}^{\heartsuit}\lambda,\,{}^{\heartsuit}i,\,{}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, together with an equivariant action of $\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Condition 6.2.5.18. *The cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is chosen so that* $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j \neq \{0\}$ *for some* $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$ *implies that* γ *acts as the identity on the smallest* (*rational*) *linear subspace of* $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ *containing* σ_j.

Remark 6.2.5.19. By Definition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$, this smallest linear subspace of $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ containing σ_j is necessarily of the image of $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}} \hookrightarrow (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ for some surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, and the condition that γ acts as the identity on $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ shows that the eigenvalues of the actions of γ on X and on Y are roots of unity. This forces the actions to be trivial if $\mathcal{H}$ is neat (defined as in Definition 1.4.1.8).

Lemma 6.2.5.20. *Suppose that Condition 6.2.5.18 is satisfied. If* $\mathcal{H}$ *is neat* (*defined as in Definition 1.4.1.8*), *then* $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *acts trivially on* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ *and* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. *Hence* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} = \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *is an algebraic space and* $\mathcal{H}$ *is neat. As a consequence,* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *is a* (*Deligne-Mumford*) *algebraic stack for general open compact subgroups* $\mathcal{H}$ *in* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$.

Proof. Suppose Condition 6.2.5.18 is satisfied, and suppose $\mathcal{H}$ is neat, then $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is forced to be trivial as explained in Remark 6.2.5.19. The general case then follows from the existence of a surjection from a similar parameter space (with the same remaining data) defined by some neat open compact subgroup $\mathcal{H}'$ of $\mathcal{H}$. □

Let us assume from now that the following the cone decompositions $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ are chosen so that Condition 6.2.5.18 is satisfied. Then the quotients $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ are (Deligne-Mumford) algebraic stacks, and the equivariant action of $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ on $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ implies that we have a descended family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

The various degenerating families $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}})$ constructed above are called *Mumford families*.

Remark 6.2.5.21. By abuse of notation, we will not change the notation for the Mumford families $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}})$ even if they are over different bases.

Remark 6.2.5.22. If $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ is in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ defining $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, then $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ under the natural map $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Remark 6.2.5.23. If $\sigma, \sigma' \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ are two smooth rational polyhedral cones such that $\sigma \subset \sigma'$, then $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma'}$ under the natural map $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma'}$.

However, the map $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma')$ between affine toroidal embeddings may blow-down certain strata to smaller dimensional subschemes, for example when a face τ of σ is not contained in any face τ' of σ' of the same dimension as τ. Hence we cannot expect the map to be flat in general.

6.2.6 Identifications Between Parameter Spaces

Let us continue with the assumptions and notations as in the previous section.

Definition 6.2.6.1. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, let $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, and let $\sigma' \subset (\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}}$. We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are* **equivalent** *if there exists an isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ such that:*

1. *The two representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ as in Definition 5.4.2.4. In other words, $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}'_{\mathcal{H}}$ are identical, and $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are equivalent under $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ as in Definition 5.4.2.2.*

2. *The isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ induces an isomorphism $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ that sends σ' to σ.*

In this case we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are equivalent under the isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.

Definition 6.2.6.2. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are* **equivalent** *if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, and if for one (and hence all) such isomorphism the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this case we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are equivalent under the isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.*

Definition 6.2.6.3. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of*

$\mathbf{P}_{\Phi_{\mathcal{H}}}$ (*resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). *We say that the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *is a* **refinement** *of the* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *if* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent under some isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, *and if for one* (*and hence all*) *such isomorphism the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is a refinement of the cone decomposition* $\Sigma_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. *In this case we say that the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *is a refinement of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *under the isomorphism* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.

Definition 6.2.6.4. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels at level* $\mathcal{H}$, *and let* $\Sigma_{\Phi_{\mathcal{H}}}$ (*resp.* $\Sigma_{\Phi'_{\mathcal{H}}}$) *be a* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible* (*resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible*) *smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (*resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). *A* **surjection** $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *is given by a surjection* $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (*defined as in Definition 5.4.2.12*) *that induces an embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ *such that the restriction* $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ *of the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *to* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *is the cone decomposition* $\Sigma_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$.

Then we have the following formal observations:

Proposition 6.2.6.5. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels at level* $\mathcal{H}$, *let* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, *and let* $\sigma' \subset \mathbf{P}^+_{\Phi'_{\mathcal{H}}}$. *Then the Mumford families* $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *and* $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma'}/\Gamma_{\Phi'_{\mathcal{H}},\sigma'}$ *are* (*uniquely*) *isomorphic over* $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ *if and only if the triples* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *are equivalent as in Definition 6.2.6.1.*

That is, the construction of the Mumford family $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ from the equivalence class of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ is a well-defined process.

Proposition 6.2.6.6. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels, and let* $\Sigma_{\Phi_{\mathcal{H}}}$ (*resp.* $\Sigma_{\Phi'_{\mathcal{H}}}$) *be a* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible* (*resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible*) *smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (*resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). *Then the corresponding Mumford families* $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ *and* $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ *are isomorphic over* $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (*up to an isomorphism that is unique up to actions of* $\Gamma_{\Phi_{\mathcal{H}}}$ *and* $\Gamma_{\Phi'_{\mathcal{H}}}$) *if and only if the triples* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *are equivalent as in Definition 6.2.6.2.*

Proposition 6.2.6.7. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels, and let* $\Sigma_{\Phi_{\mathcal{H}}}$ *(resp.* $\Sigma_{\Phi'_{\mathcal{H}}}$*) be a* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible (resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). Suppose the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *is a refinement of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *as in Definition 6.2.6.3. Then the corresponding Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ *is the pullback of the Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ *over* $\mathsf{M}'_{\mathcal{H}}$ *via a surjection* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \twoheadrightarrow \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ *(which is unique up to actions of* $\Gamma_{\Phi_{\mathcal{H}}}$ *and* $\Gamma_{\Phi'_{\mathcal{H}}}$*).*

6.3 Approximation and Gluing

Let us continue with the setting of Section 6.2.1 in this section. Assume moreover that Condition 6.2.5.18 is satisfied for all the cone decompositions we choose.

For the ease of expositions we shall make the following definition:

Definition 6.3.1. *Let* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *be a degenerating family of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *(as defined in Definition 5.3.2.1) over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*. Let* $\underline{\mathrm{Lie}}^{\vee}_{G/S} := e^*_G\Omega^1_{G/S}$ *be the dual of* $\underline{\mathrm{Lie}}_{G/S}$*, and let* $\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} = e^*_G\Omega^1_{G^{\vee}/S}$ *be the dual of* $\underline{\mathrm{Lie}}_{G^{\vee}/S}$*. Note that* $\lambda : G \to G^{\vee}$ *induces an* $\mathcal{O}$*-linear morphism* $\lambda^* : \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} \to \underline{\mathrm{Lie}}^{\vee}_{G/S}$*. Then we define the sheaf* $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G,\lambda)/S} = \underline{\mathrm{KS}}_{(G,\lambda,i,\alpha_{\mathcal{H}})/S}$ *by*

$$\underline{\mathrm{KS}} := (\underline{\mathrm{Lie}}^{\vee}_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S}) / \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ (i(b)^*(x)) \otimes y - x \otimes ((i(b)^{\vee})^*(y)) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^{\vee}_{G/S}, \\ y,z \in \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S}, \\ b \in \mathcal{O}}}$$

(cf. Definition 2.3.4.1).

6.3.1 Good Formal Models

Construction 6.3.1.1. Suppose that we are given a torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ at level $\mathcal{H}$ for some $\mathsf{Z}_{\mathcal{H}}$. Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be any degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over an integral normal noetherian local base scheme S. Suppose that the character group of the torus part of the special fiber of G (resp. $G^{\vee}$) is constant and identified with X (resp. Y), and suppose that $\lambda : G \to G^{\vee}$ induces an $\mathcal{O}$-linear embedding from Y to X. Then by Theorem 3.3.1.11, the étale sheaf $\underline{X}(G)$ is a quotient sheaf of the constant sheaf X such that for any $s \in S$ there is a quotient map $s_X : X \twoheadrightarrow X_s$ from X to the restriction X_s of $\underline{X}(G)$ to s. By the argument in Lemma 5.4.2.11 and Proposition 6.2.5.7, this defines the sheaves

$$\underline{\Phi}_{\mathcal{H}}(G) = (\underline{X}(G), \underline{Y}(G), \underline{\phi}(G), \underline{\varphi}_{-2,\mathcal{H}}(G), \underline{\varphi}_{0,\mathcal{H}}(G))$$

and $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}$ on S. Moreover, for any discrete valuation ring V with fraction field $\mathrm{Frac}(V)$, the pullback $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha^\dagger_{\mathcal{H}})$ of $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathrm{Spec}(V)$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$, and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$. The pullbacks of $\underline{\Phi}_{\mathcal{H}}(G)$ and $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}$ determine the objects $\Phi^\dagger_{\mathcal{H}}$ and $\mathbf{S}_{\Phi^\dagger_{\mathcal{H}}}$ on $\mathrm{Spec}(V)$, and hence a choice of an object

$$(\mathsf{Z}^\dagger_{\mathcal{H}}, (X^\dagger, Y^\dagger, \phi^\dagger, \varphi^\dagger_{-2,\mathcal{H}}, \varphi^\dagger_{0,\mathcal{H}}), (A^\dagger, \lambda^\dagger_A, i^\dagger_A, \varphi^\dagger_{-1,\mathcal{H}}), \delta^\dagger_{\mathcal{H}}, (c^\dagger_{\mathcal{H}}, (c^\vee_{\mathcal{H}})^\dagger, \tau^\dagger_{\mathcal{H}}))$$

over $\mathrm{Spec}(V)$ in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$, which (by Lemma 5.4.2.10) is unique up to identification of $\Phi^\dagger_{\mathcal{H}}$ by an element $\Gamma_{\Phi^\dagger_{\mathcal{H}}}$. In particular, for any $y \in Y^\dagger$ and $\chi \in X^\dagger$, we can define invertible V-submodules $I^\dagger_{y,\chi}$ of $\mathrm{Frac}(V)$ to be the image of

$$((c^\vee)^\dagger(y), c^\dagger(\chi))^*\mathcal{P}_{A^\dagger} \overset{\text{res.}}{\to} ((c^\vee)^\dagger_{\mathrm{Frac}(V)}(y), c^\dagger_{\mathrm{Frac}(V)}(\chi))^*\mathcal{P}_{A^\dagger_{\mathrm{Frac}(V)}} \overset{\tau^\dagger(y,\chi)}{\overset{\sim}{\to}} \mathrm{Frac}(V),$$

By using $\tau_{\mathcal{H}}$, this defines a map

$$B^\dagger : \mathbf{S}_{\Phi^\dagger_{\mathcal{H}}} \to \mathrm{Inv}(V).$$

Note that by composition with the discrete valuation $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ of V, we obtain a linear map

$$\upsilon \circ B^\dagger : \mathbf{S}_{\Phi^\dagger_{\mathcal{H}}} \to \mathbb{Z},$$

in $\mathrm{Hom}(\mathbf{S}_{\Phi^\dagger_{\mathcal{H}}}, \mathbb{Z}) = \mathbf{S}^\vee_{\Phi^\dagger_{\mathcal{H}}}$. Since the above construction is functorial and commute with base change, we obtain a sheaf map

$$\underline{B}(G) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \underline{\mathrm{Inv}}(S)$$

as well (see Definition 4.2.4.1).

Let us summarize the properties of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}, \sigma) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ as follows:

Proposition 6.3.1.2. *Let* $S = \mathrm{Spf}(R, I)$ *be an affine formal scheme, with a formally étale map* $S \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, and let* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$ *be the pullback of* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *to* S*. Then* R *is an* I*-adically complete excellent ring, which is formally smooth over the abelian scheme* $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*, and hence also formally smooth over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*. Let* K *be the quotient field of* R*.*

1. *The stratification of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *determines a stratification of* $\mathrm{Spec}(R)$ *parameterized by* $\{\text{faces } \tau \text{ of } \sigma\}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

2. *The formal completion of* ${}^{\diamondsuit}G$ *along the inverse image of* $\mathrm{Spec}(R/I)$ *is canonically isomorphic to the pullback of* $G^{\natural}$ *under* $S \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(as a formal algebraic stack, instead of a relative scheme).*

3. *The étale sheaf* $\underline{X}({}^{\diamondsuit}G)$ *is the quotient sheaf of the constant sheaf* X *such that on the* $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})$*-stratum* $\underline{X}({}^{\diamondsuit}G)$ *is a constant quotient* $X_{(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ *of* X*, with an admissible surjection* $X \twoheadrightarrow X_{(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ *inducing a torus argument* $\Phi_{\mathcal{H},(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ *from* $\Phi_{\mathcal{H}}$ *as in Lemma 5.4.2.11, so that* τ *is contained in the* $\Gamma_{\Phi_{\mathcal{H}}}$*-orbit of the image of the induced embedding* $\mathbf{P}^{+}_{\Phi_{\mathcal{H},(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$*. (We known the surjection is admissible because of the existence of level-*$\mathcal{H}$ *structures.) This produces a sheaf version* $\underline{\Phi}_{\mathcal{H}}({}^{\diamondsuit}G)$ *of* $\Phi_{\mathcal{H}}$ *over* $\mathrm{Spec}(R)$*.*

   The formation of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ *from* $\Phi_{\mathcal{H}}$ *applies to the above context of* $\underline{\Phi}_{\mathcal{H}}({}^{\diamondsuit}G)$ *and defines a sheaf* $\underline{\mathbf{S}}(\underline{\Phi}_{\mathcal{H}})$*. (See Proposition 6.2.5.7.)*

   Then $\underline{\Phi}_{\mathcal{H}}({}^{\diamondsuit}G)$ *is equivalent to the pullback of the tautological* $\underline{\Phi}_{\mathcal{H}}$ *on* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(described in Proposition 6.2.5.7) via* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*. (See Definition 5.4.2.2.)*

4. *Under the equivalence between* $\underline{\Phi}_{\mathcal{H}}({}^{\diamondsuit}G)$ *and the pullback of* $\underline{\Phi}_{\mathcal{H}}$ *above, the pullback* $\underline{B} : \underline{\mathbf{S}}(\underline{\Phi}_{\mathcal{H}})({}^{\diamondsuit}G) \to \underline{\mathrm{Inv}}(\mathrm{Spec}(R))$ *of the tautological map* $\underline{B}$ *on* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(described in Proposition 6.2.5.7) via* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *agrees with the linear map* $\underline{B}({}^{\diamondsuit}G)$ *associated to* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$ *defined as in Construction 6.3.1.1.*

5. *Let* $\underline{\mathrm{KS}}_{{}^{\diamondsuit}G/\mathrm{Spec}(R)}$ *be the sheaf defined by* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$ *as in Definition 6.3.1. Let* $\Omega^1_{\mathrm{Spec}(R)/\mathsf{S}_0}[d\log\infty]$ *be the sheaf of modules of logarithmic 1-differentials on* $\mathrm{Spec}(R)$ *with respect to the divisor*

$\operatorname{Spec}(R/I)$ *with normal crossings. Then the extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *defines an isomorphism*

$$\underline{\mathrm{KS}}_{\diamondsuit G/\operatorname{Spec}(R)} \xrightarrow{\sim} \Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty].$$

6. *The map* $S = \operatorname{Spf}(R,I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, or rather the map* $\operatorname{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, is tautological with respect to its universal property, in the following sense:*

   The setting is as follows: The base scheme R*, with its ideal of definition* I*, satisfies the setting of Section 6.2.1. Let* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$ *be any degenerating family over* $\operatorname{Spec}(R)$ *of type* $\mathsf{M}_{\mathcal{H}}$ (*defined as in Definition 5.3.2.1*) *over* $\operatorname{Spec}(R)$ *that defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ (*as in Definition 5.3.1.15*)*. Then the family determines an object of* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *by Theorem 5.3.1.17, which determines in particular a cusp label. Suppose that the cusp label is represented by* $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$*. By Lemma 5.4.2.10, there exists a tuple*

$$(A,\lambda_A,i_A,X,Y,\phi,c,c^\vee,\tau,[\alpha^\natural_{\mathcal{H}}])$$

   that defines the above object of $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$*, together with a representative*

$$\alpha^\natural_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}},\varphi_{-2,\mathcal{H}},\varphi_{-1,\mathcal{H}},\varphi_{0,\mathcal{H}},\delta_{\mathcal{H}},c_{\mathcal{H}},c^\vee_{\mathcal{H}},\tau_{\mathcal{H}})$$

   of $[\alpha^\natural_{\mathcal{H}}]$*. By Proposition 6.2.4.7, this tuple without its positivity condition defines a map* $\operatorname{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *that is unique up to an action of* $\Gamma_{\Phi_{\mathcal{H}}}$ *on the identification of* $\Phi_{\mathcal{H}}$*. Let* $\underline{B}(G) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \underline{\mathrm{Inv}}(\operatorname{Spec}(R))$ *be the map defined as in Construction 6.3.1.1.*

   Then the universal property is as follows: Suppose there exists an identification of $\Phi_{\mathcal{H}}$ *such that, for any discrete valuation* $\upsilon : (\operatorname{Frac}(\operatorname{Spec}(R)))^\times \to \mathbb{Z}$ *defined by a prime of* R *of hight one, the composition* $\upsilon \circ \underline{B}(G) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \mathbb{Z}$ *defines an element in the closure* $\overline{\sigma}$ *of* σ *in* $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$*. Such an identification of* $\Phi_{\mathcal{H}}$ *is unique up to an element in* $\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, and those maps* $\operatorname{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *appearing above that do respect such identifications of* $\Phi_{\mathcal{H}}$ *descend to the same unique map* $\operatorname{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*. Then this map extends to a* (*necessarily unique*) *map* $\operatorname{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, sending the subscheme* $\operatorname{Spec}(R/I)$ *to the* σ*-stratum of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *and hence inducing a map* $S = \operatorname{Spf}(R,I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *between formal schemes, which makes* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$ *isomorphic to the pullback family* $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$*.*

Proof. The only nontrivial part that we have not discussed is the calculation of the Kodaira-Spencer map. The case without PEL-structures has been discussed in Section 4.6, especially Theorems 4.6.3.13 and 4.6.3.30, in which we have identified a map

$$\mathrm{KS}_{{}^\diamondsuit G/\operatorname{Spec}(R)/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G/\operatorname{Spec}(R)} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^\vee/\operatorname{Spec}(R)} \to \Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty] \tag{6.3.1.3}$$

(between locally free sheaves) canonical with the map

$$\mathrm{KS}_{({}^\diamondsuit G^\natural,\,{}^\diamondsuit\iota)/\operatorname{Spec}(R)/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^\natural/\operatorname{Spec}(R)} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^{\vee,\natural}/\operatorname{Spec}(R)} \to \Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty]$$

defined by the pair $({}^\diamondsuit G^\natural, {}^\diamondsuit\iota)$ in DD underlying the degeneration datum associated to $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$. Now since $S = \operatorname{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is formally étale, and since $({}^\diamondsuit G^\natural, {}^\diamondsuit\iota)$ is the pullback of the tautological $(G^\natural, \iota)$ over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ under the underlying map $\operatorname{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ of the map between the formal schemes, the result follows simply from Proposition 6.2.5.14. ☐

As a byproduct of our usage of Proposition 6.2.5.14 in the proof:

Corollary 6.3.1.4. *A strata-preserving morphism* $S = \operatorname{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *is* **formally étale** *if and only if the structural map* $\operatorname{Spf}(R, I) \to \mathsf{S}_0$ *is formally smooth, the underlying map* $\operatorname{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *is flat, and the extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *induces a surjection*

$$\underline{\mathrm{KS}}_{{}^\diamondsuit G/\operatorname{Spec}(R)} \twoheadrightarrow \Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty], \tag{6.3.1.5}$$

where $\underline{\mathrm{KS}}_{{}^\diamondsuit G/\operatorname{Spec}(R)}$ *is the sheaf defined as in Definition 6.3.1 by the pullback* $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$ *of* $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *to* S.

Proof. Let us denote the map $S \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ by $\hat{f}$, and the induced map $\operatorname{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ by f. Then the question is to show that $\hat{f}$ is formally étale if and only if the conditions in the corollary are satisfied. It is clear that the conditions are necessary for $\hat{f}$ to be formally étale. Conversely, suppose we know that $\operatorname{Spf}(R, I) \to \mathsf{S}_0$ is formally smooth, f is flat, and that (6.3.1.5) is surjective. We would like to show that $\hat{f}$ is formally unramified. By the fact that the formation of $\underline{\mathrm{KS}}$ commutes with base extensions, we may pullback the isomorphism in Proposition 6.2.5.14 to an isomorphism $\underline{\mathrm{KS}}_{(\,\diamond G,\iota)/\operatorname{Spec}(R)} \xrightarrow{\sim} f^*\Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\mathsf{S}_0}[d\log\infty]$ over $\operatorname{Spec}(R)$. Then the surjectivity of (6.3.1.5) implies the surjectivity of the canonical map

$$f^*\Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\mathsf{S}_0}[d\log\infty] \to \Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty]. \tag{6.3.1.6}$$

On the other hand, under the assumption that f is strata-preserving, we can identify $\Omega^1_{\operatorname{Spec}(R)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)}$ as a submodule of the cokernel of (6.3.1.6), because the generators of the logarithmic differentials in $\Omega^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty]$ are all in the image of (6.3.1.5). Hence $\Omega^1_{\operatorname{Spec}(R)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} = 0$, which implies that $\hat{f}$ is formally unramified, as desired. □

To check the flatness condition in Corollary 6.3.1.4, let us reproduce the following useful criterion from [70, pp. 507–508, in Notes Added in Proof]:

Lemma 6.3.1.7. *Let $f : Z_1 \to Z_2$ be morphism between locally noetherian schemes equidimensional of the same dimension. Suppose f is quasi-finite, Z_1 is Cohen-Macaulay, and Z_2 is regular. Then f is automatically flat.*

Here we follow [70] and define a scheme to be *equidimensional* if all its open subschemes have the same dimension. Note that this is stronger than the definition in [51, IV, §13] because it has a different purpose.

Proof of Lemma 6.3.1.7. Since Z_2 is regular and locally noetherian, it is the disjoint union of its irreducible component. Then we may assume that Z_2 is irreducible. Let Z_1' be any irreducible component of Z_1. By [50, IV, 5.4.1(i)], the assumption on dimensions implies that $f|_{Z_1'} : Z_1' \to Z_2$ is dominant. By [50, IV, 5.6.4 and 5.6.5.3], the quasi-finiteness of $f|_{Z_1'}$ implies that $\dim \mathscr{O}_{Z_1',z} = \dim \mathscr{O}_{Z_2,f(z)}$ at any point z of Z_1'. Since Z_1 is Cohen-Macaulay, [49, 0_{IV}, 16.5.4] implies that $\dim \mathscr{O}_{Z_1',z} = \dim \mathscr{O}_{Z_1,z}$ at any point z of Z_1'. Since Z_1' is arbitrary, we have equivalently $\dim \mathscr{O}_{Z_1,z} = \dim \mathscr{O}_{Z_2,f(z)}$ at any point z of Z_1. Then the result follows from [51, IV, 15.4.2 e')⇒b)]. □

Remark 6.3.1.8. If Z_1 and Z_2 are local and regular, then a similar criterion can be found in [2, Ch. V, Cor. 3.6]. This is no longer mentioned in [70, Notes Added in Proof], but used for example in the main text of the proof of [70, Prop. 5.2.2].

Corollary 6.3.1.9. *Let* $f : Z_1 \to Z_2$ *be a morphism between locally noetherian schemes equidimensional of the same dimension. Suppose* f *is unramified,* Z_1 *is Cohen-Macaulay, and* Z_2 *is regular. Then* f *is automatically étale.*

Proof. By [52, IV, 17.4.1 a)⇒d')], f is quasi-finite because it is unramified. Then the result follows from Lemma 6.3.1.7. $\square$

Corollary 6.3.1.10. *Corollary 6.3.1.4 remains true if we replace the flatness condition on the underlying map* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *by the condition that* R *is equidimensional and has the same dimension as* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Now we are ready to define the so-called *good formal models*, as in [37, Ch. IV, §3].

Definition 6.3.1.11. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *be a representative of a cusp label at level* $\mathcal{H}$*, and let* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ *be a (nondegenerate) smooth rational polyhedral cone. A* **good formal** $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$**-model** *is a degenerating family* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R)$*, where:*

1. R *is a strict local ring that is complete with respect to an ideal* $I = \mathrm{rad}(I)$*, together with a stratification of* $\mathrm{Spec}(R)$ *with strata parameterized by faces of* σ *modulo* $\Gamma_{\Phi_{\mathcal{H}},\sigma}$.
2. *There exists a strata-preserving map* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *that makes* R *isomorphic to the completion of a strict local ring of* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *along its* σ*-stratum. Under this map, the tuple* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R)$ *is the pullback of the Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Remark 6.3.1.12. As in Proposition 6.3.1.2, the morphism $\mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in Definition 6.3.1.11 (which makes the degenerating family $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}})$ over $\mathrm{Spec}(R)$ the pullback of the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$) is necessarily unique.

Remark 6.3.1.13. The existence of the morphism $\mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, and its strata-preserving property, is encoded in the sheaf map $\underline{B}({}^{\diamondsuit}G) : \underline{\mathbf{S}}(\underline{\Phi}_{\mathcal{H}}) \to \underline{\mathrm{Inv}}(\mathrm{Spec}(R))$. (See Proposition 6.2.5.8.)

Then we can restate the combination of Corollaries 6.3.1.4 and 6.3.1.10 as:

Corollary 6.3.1.14. *Suppose* $\mathrm{Spf}(R, I)$ *is formally smooth over* S_0*. The statement that the strata-preserving morphism* $\mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *makes* R *isomorphic to the completion of a strict local ring of* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *along its* σ*-stratum can be checked by the condition that* $\mathrm{Spec}(R)$ *is equidimensional and has the same dimension as* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, and by the calculation that the extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *defines an isomorphism*

$$\underline{\mathrm{KS}}_{{}^\diamondsuit G/\,\mathrm{Spec}(R)} \xrightarrow{\sim} \Omega^1_{\mathrm{Spec}(R)/\mathsf{S}_0}[d\log\infty],$$

when the degenerating family $({}^\diamondsuit G,\ {}^\diamondsuit\lambda,\ {}^\diamondsuit i,\ {}^\diamondsuit\alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R)$ *is the pullback of the Mumford family* $({}^\heartsuit G,\ {}^\heartsuit\lambda,\ {}^\heartsuit i,\ {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

Remark 6.3.1.15. The various maps from $\mathrm{Spec}(R/I)$ to the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}}\sigma}$, for the various good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-models, *cover* the whole σ-stratum.

Remark 6.3.1.16. A good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-model $({}^\diamondsuit G,\ {}^\diamondsuit\lambda,\ {}^\diamondsuit i,\ {}^\diamondsuit\alpha_{\mathcal{H}})$ over $\mathrm{Spec}(R)$ is a good formal $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$-model if and only if $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ is equivalent to $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ (defined as in Definition 6.2.6.1).

Remark 6.3.1.17. For two smooth rational polyhedral cones $\sigma, \sigma' \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \sigma'$, a good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-model is not necessarily a good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma')$-model. (See Remark 6.2.5.23.)

6.3.2 Good Algebraic Models

For the purpose of gluing, the formal models are not enough. We need to construct algebraic models, namely families over schemes (instead of formal schemes), which are approximate enough to the formal models so that a gluing process for the purpose of compactification can still be performed.

Proposition 6.3.2.1. *Let* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *be a representative of a cusp label at level* $\mathcal{H}$*, and let* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ *be a* (*nondegenerate*) *smooth rational polyhedral cone.*

Let R *be the strict local ring of a geometric point* $\bar{x}$ *of the* σ*-stratum of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *for some* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$*, and let* $R^\wedge$ *be the completion of* R *along the ideal* I *defining the* σ*-stratum. Suppose* $({}^\diamondsuit G,\ {}^\diamondsuit\lambda,\ {}^\diamondsuit i,\ {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R^\wedge)$ *defines a good formal* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$*-model over* $\mathrm{Spec}(R^\wedge)$*. Then we can find* (*non-canonically*) *a degenerating family* $(G,\lambda,i,\alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R)$ *as in Definition 5.3.2.1, which approximates* $({}^\diamondsuit G,\ {}^\diamondsuit\lambda,\ {}^\diamondsuit i,\ {}^\diamondsuit\alpha_{\mathcal{H}})$ *in the following sense:*

1. *Over* $\mathrm{Spec}(R/I)$, $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i) \cong (G, \lambda, i)$. (*We do not compare* ${}^{\diamondsuit}\alpha_{\mathcal{H}}$ *and* $\alpha_{\mathcal{H}}$ *because they are not defined over* $\mathrm{Spec}(R/I)$.)

2. *The objects* $\underline{\Phi}_{\mathcal{H}}(G)$, $\mathbf{S}_{\underline{\Phi}(G)}$, *and* $\underline{B}(G)$ *as defined in Construction 6.3.1.1 induce respectively objects* $\Phi_{\mathcal{H}}({}^{\diamondsuit}G)$, $\mathbf{S}_{\underline{\Phi}({}^{\diamondsuit}G)}$, *and* $\underline{B}({}^{\diamondsuit}G)$ *as defined in Proposition 6.3.1.2* (*via* $R \hookrightarrow R^{\wedge}$, *since the objects are sheaves defined on the base*).

3. *The pullback* $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^{\wedge} \rightarrow \mathrm{Spec}(R^{\wedge})$ *of* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *via the canonical map* $R \rightarrow R^{\wedge}$ *defines a good formal* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*-model, and can be realized as the pullback of the Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \rightarrow \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *via a canonically defined morphism* $\mathrm{Spf}(R^{\wedge}, I) \rightarrow \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. *Comparing this isomorphism with the original map* $\mathrm{Spf}(R^{\wedge}, I) \rightarrow \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *that makes the formal good* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*-model* $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}}) \rightarrow \mathrm{Spec}(R^{\wedge})$ *a pullback of the Mumford family, we see that they are approximate in the sense that the induced maps from* $\mathrm{Spec}(R/I)$ *to the* σ*-stratum of* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (*i.e. between the supports of the formal schemes*) **coincide**.

4. *The extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *defines an isomorphism*

$$\underline{\mathrm{KS}}_{G/\mathrm{Spec}(R)} \xrightarrow{\sim} \Omega^1_{\mathrm{Spec}(R)/\mathsf{S}_0}[d\log\infty].$$

The proof of this requires the approximation techniques of Artin:

Proposition 6.3.2.2 ([8, Thm. 1.10]). *Let* R_0 *be a field or an excellent discrete valuation ring, and let* R *be the Henselization of an* R_0*-algebra of finite type at a prime ideal. Let* I *be a proper ideal of* R*, and let* $R^{\wedge}$ *be the* I*-adic completion of* R*. Suppose* R_1 *is a subalgebra of* $R^{\wedge}$ *of finite type over* R*. Then the natural inclusion* $R \hookrightarrow R_1$ *has sections* $R_1 \rightarrow R$ *such that the compositions* $R_1 \rightarrow R \hookrightarrow R^{\wedge}$ *are arbitrarily close to the natural inclusion* $R_1 \hookrightarrow R^{\wedge}$ *in the* I*-adic topology.*

Proof. By writing R_1 as a quotient of a polynomial ring over R by finitely many relations, the natural inclusion $R_1 \hookrightarrow R^{\wedge}$ can be interpreted as giving a solution in $R^{\wedge}$ to a system of polynomial equations with coefficients in R. Then the approximation result of [8, Thm. 1.10] tells us that we can find approximate solutions in R that are arbitrarily close to the given solution in $R^\wedge$ in the I-adically topology. In other words, we have maps $R_1 \to R$ that define sections to the natural inclusion $R \hookrightarrow R_1$ with the desired properties. □

Proof of Proposition 6.3.2.1. Let R and $R^\wedge$ be as in Proposition 6.3.2.1, both of which are excellent normal by assumption. Note that $R^\wedge$ is the filtering direct union of its normal subalgebras R_1 of finite type over R. By Proposition 6.3.2.2, for any such algebra R_1, the natural inclusion $R \hookrightarrow R_1$ has sections $R_1 \to R$ such that the compositions $R_1 \to R \hookrightarrow R^\wedge$ can be made arbitrarily close to the natural inclusion $R_1 \hookrightarrow R^\wedge$ in the I-adic topology.

Since ${}^\heartsuit G \to \operatorname{Spec}(R^\wedge)$ is of finite presentation, by [51, IV, 8.8.2], we may take some R_1 as above such that $({}^\diamondsuit G, {}^\diamondsuit \lambda, {}^\diamondsuit i, {}^\diamondsuit \alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^\wedge)$ is already defined over $\operatorname{Spec}(R_1)$. By enlarging R_1 if necessary, we may assume that the equivalence between $(\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G), \underline{B}({}^\diamondsuit G))$ and the pullback of $(\underline{\Phi}_{\mathcal{H}}, \underline{B})$ (described in Proposition 6.3.1.2) is already defined over R_1. (Then the various character torus parts are trivialized over R_1.) By pullback with a section $R_1 \to R$ of $R \hookrightarrow R_1$ such that $R_1 \to R \hookrightarrow R^\wedge$ is close to the natural inclusion $R_1 \hookrightarrow R^\wedge$, we obtain a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$, together with an equivalence between $(\underline{\Phi}_{\mathcal{H}}(G), \underline{B}(G))$ and the pullback of $(\underline{\Phi}_{\mathcal{H}}, \underline{B})$.

Note that $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \operatorname{Spec}(R^\wedge)$ is not identical to $({}^\diamondsuit G, {}^\diamondsuit \lambda, {}^\diamondsuit i, {}^\diamondsuit \alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^\wedge)$, but they can be made close in the I-adic topology. This already shows that (G, λ, i) and $({}^\diamondsuit G, {}^\diamondsuit \lambda, {}^\diamondsuit i)$ are isomorphic over $\operatorname{Spec}(R/I)$. We claim that the degeneration datum

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c_{\mathcal{H}}^\vee, \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ associated to $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \operatorname{Spec}(R^\wedge)$ can be made I-adically close to the degeneration datum

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, ({}^\diamondsuit A, {}^\diamondsuit \lambda_A, {}^\diamondsuit i_A, {}^\diamondsuit \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, ({}^\diamondsuit c_{\mathcal{H}}, {}^\diamondsuit c_{\mathcal{H}}^\vee, {}^\diamondsuit \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ associated to $({}^\diamondsuit G, {}^\diamondsuit \lambda, {}^\diamondsuit i, {}^\diamondsuit \alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^\wedge)$. Since $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is prescribed, since choices of i_A and $\varphi_{-1,\mathcal{H}}$ are discrete in nature, and since the determination of $(c_{\mathcal{H}}, c_{\mathcal{H}}^\vee, \tau_{\mathcal{H}})$ from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $\alpha_{\mathcal{H}}$ is discrete in nature (by Proposition 5.2.7.10 and Theorem 5.2.7.15), this is essentially a statement on (A, λ_A) and $(c, c^\vee, \tau)$.

The statement that (A, λ_A) and $(c, c^\vee)$ can be made I-adically close to $({}^\diamondsuit A, {}^\diamondsuit \lambda_A)$ and $({}^\diamondsuit c, {}^\diamondsuit c^\vee)$ means, for any prescribed integer $a > 0$, we can take a section $R_1 \to R$ such that (A, λ_A) and $(c, c^\vee)$ agree with respectively $({}^\diamondsuit A, {}^\diamondsuit \lambda_A)$ and $({}^\diamondsuit c, {}^\diamondsuit c^\vee)$ over $\mathrm{Spec}(R/I^a)$. This is the case because $({}^\diamondsuit A, {}^\diamondsuit \lambda_A)$ and $({}^\diamondsuit c, {}^\diamondsuit c^\vee)$ is determined by the I-adic completion $({}^\diamondsuit G_{\mathrm{for}}, {}^\diamondsuit \lambda_{\mathrm{for}})$.

The corresponding statement for τ is more tricky, because τ as a trivialization of biextensions is not defined over $\mathrm{Spec}(R/I^a)$ for any a. However, for each $y \in Y$ and $\chi \in X$, we can interpret $\tau(y, \chi)$ as an $R^\wedge$-module isomorphism from $(c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1}$ to an $R^\wedge$-invertible submodule $I_{y,\chi}$ of $K = \mathrm{Frac}(R^\wedge)$ (as in the proof of Lemma 4.2.1.6 in Section 4.2.4). Then the statement that τ can be made I-adically close to ${}^\diamondsuit \tau$ means, for each prescribed integer $a > 0$, the two $R^\wedge$-module isomorphisms $\tau(y, \chi) : (c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1} \overset{\sim}{\to} I_{y,\chi}$ and ${}^\diamondsuit \tau(y, \chi) : ({}^\diamondsuit c^\vee(y), {}^\diamondsuit c(\chi))^* \mathcal{P}_{{}^\diamondsuit A}^{\otimes -1} \overset{\sim}{\to} {}^\diamondsuit I_{y,\chi}$ are identical modulo I^a (under the identification of $(c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1}$ and $({}^\diamondsuit c^\vee(y), {}^\diamondsuit c(\chi))^* \mathcal{P}_{{}^\diamondsuit A}^{\otimes -1}$ over $\mathrm{Spec}(R/I^a)$). Since $(c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1}$ and $({}^\diamondsuit c^\vee(y), {}^\diamondsuit c(\chi))^* \mathcal{P}_{{}^\diamondsuit A}^{\otimes -1}$ agree over R/I, they are isomorphic as R/I^a-modules for any integer $a > 0$ (cf. the proof of Lemma 2.1.1.1). Therefore their difference over $\mathrm{Spec}(R/I^a)$ is measured a priori by an invertible element in R/I^a, and we only need to show that this invertible element is the identity. To justify our claim above that the degeneration data can be made close, it only remains to show that we can make τ close to ${}^\diamondsuit \tau$ in this sense.

Let us take ${}^\diamondsuit \mathcal{L}_\eta := (\mathrm{Id}_{{}^\diamondsuit G_\eta}, {}^\diamondsuit \lambda_\eta)^* \mathcal{P}_{{}^\diamondsuit G_\eta}$. Note that ${}^\diamondsuit \mathcal{L}_\eta$ is symmetric. Let ${}^\diamondsuit \mathcal{L}$ be the unique cubical extension of ${}^\diamondsuit \mathcal{L}_\eta$ (given by Theorem 3.3.2.3), and let ${}^\diamondsuit \mathcal{M} := (\mathrm{Id}_{{}^\diamondsuit A}, {}^\diamondsuit \lambda_{{}^\diamondsuit A})^* \mathcal{P}_{{}^\diamondsuit A}$. Then ${}^\diamondsuit \mathcal{L}^\natural = {}^\diamondsuit \pi^* \, {}^\diamondsuit \mathcal{M}$, where ${}^\diamondsuit \pi : {}^\diamondsuit G^\natural \to {}^\diamondsuit A$ is the structural morphism. We know that ${}^\diamondsuit \mathcal{L}$ induces $2\, {}^\diamondsuit \lambda$, and so ${}^\diamondsuit \tau$ is part of the degeneration datum associated to $({}^\diamondsuit G, {}^\diamondsuit \mathcal{L})$ by $\mathrm{F}_{\mathrm{ample}}$ (by Theorem 4.2.1.8 and the statement in Remark 4.2.1.10 that ${}^\diamondsuit \lambda$ and $2\, {}^\diamondsuit \lambda$ have the same associated ${}^\diamondsuit \tau$). More precisely, there is an ${}^\diamondsuit \psi$ such that for any section ${}^\diamondsuit s$ in $\Gamma({}^\diamondsuit G, {}^\diamondsuit \mathcal{L}) \underset{R^\wedge}{\otimes} \mathrm{Frac}(R^\wedge)$, we have

$$ {}^\diamondsuit \psi(y) \, {}^\diamondsuit \tau(y, \chi) \, T^*_{{}^\diamondsuit c^\vee(y)} \circ {}^\diamondsuit \sigma_\chi({}^\diamondsuit s) = {}^\diamondsuit \sigma_{\chi + \phi(y)}({}^\diamondsuit s) \tag{6.3.2.3} $$

for any $y \in Y$ and any $\chi \in X$, where ${}^\diamondsuit \sigma : \Gamma({}^\diamondsuit G, {}^\diamondsuit \mathcal{L}) \to \Gamma({}^\diamondsuit A, {}^\diamondsuit \mathcal{M}_\chi)$ is defined as in Section 4.3.1. As in the case for τ, for each $y \in Y$ and $\chi \in X$, we can interpret $\psi(y)$ as an $R^\wedge$-module isomorphism from $c^\vee(y)^* \mathcal{M}^{\otimes -1}$ to an $R^\wedge$-invertible submodule I_y of $K = \mathrm{Frac}(R^\wedge)$. (See the proof of Lemma

4.2.1.6 in Section 4.2.4.) Then the statement that ψ can be made I-adically close to ${}^\diamondsuit\psi$ means, for each prescribed integer $a > 0$, the two $R^\wedge$-module isomorphisms $\psi(y) : c^\vee(y)^*\mathcal{M}^{\otimes -1} \xrightarrow{\sim} I_y$ and ${}^\diamondsuit\psi(y) : {}^\diamondsuit c^\vee(y)^*\mathcal{M}^{\otimes -1} \xrightarrow{\sim} {}^\diamondsuit I_y$ are identical modulo I^a (under the identification of $c^\vee(y)^*\mathcal{M}^{\otimes -1}$ and ${}^\diamondsuit c^\vee(y)^* \,{}^\diamondsuit\mathcal{M}^{\otimes -1}$ over $\mathrm{Spec}(R/I^a)$). Note that situation for ψ is easier than for τ because all the I_y are actually invertible submodules of $R^\wedge$ by positivity of ψ (defined as in Definition 4.2.1.5).

Since $({}^\diamondsuit G, {}^\diamondsuit\lambda)$ is defined over R_1, we may take all the above objects (including a basis of the sections ${}^\diamondsuit s$) to be defined over R_1. If we take $\mathcal{L} := (R_1 \to R)^*({}^\diamondsuit\mathcal{L})$, $\mathcal{M} := (R_1 \to R)^*({}^\diamondsuit\mathcal{M})$, and $s := (R_1 \to R)^*({}^\diamondsuit s)$, then we see that the same relations is true as above for the objects defined by $(G, \mathcal{L})$, and for any prescribed integer $a > 0$, we can take a section $R_1 \to R$ so that the maps σ_χ for $(G, \mathcal{L})$ and the maps ${}^\diamondsuit\sigma_\chi$ for $({}^\diamondsuit G, {}^\diamondsuit\mathcal{L})$ coincide over $\mathrm{Spec}(R/I^a)$. Take a finite number of basis elements of sections ${}^\diamondsuit s$ of $\Gamma({}^\diamondsuit G, {}^\diamondsuit\mathcal{L}) \underset{R^\wedge}{\otimes} \mathrm{Frac}(R^\wedge)$ such that each of the elements lies in some $\bar{\chi}$-weight space for some $\bar{\chi} \in X/\phi(Y)$ as in Section 4.3.2. Since ${}^\diamondsuit\psi$ is determined by comparisons between (translations of) images of these basis elements (by taking for example $\chi = 0$ in (6.3.2.3)), we see that ψ can be made close to ${}^\diamondsuit\psi$.

Let us repeat [37, Ch. IV, Lem. 4.2] as follows:

Lemma 6.3.2.4. *Suppose R is a noetherian ring, I an ideal of R, such that $\mathrm{Gr}_I(R)$ is an integral domain, and such that the I-adic topology on R is separated. Suppose moreover that $f, g \neq 0$ are elements of R such that f/g lies in R. If $f_i, g_i \neq 0$ are sequences of elements of R converging I-adically to respectively f and g, and if all the quotients f_i/g_i lie in R. Then the quotients f_i/g_i converges I-adically to f/g.*

Proof. The separateness assumption shows that there is an injection from R to $\mathrm{Gr}_I(R)$. By the I-adic order $\mathrm{ord}_I(x)$ of an element x in R, we mean the degree of the first nonzero entry of its image in $\mathrm{Gr}_I(R)$. Then any sequences of elements $x_i \in R$ satisfies $x_i \to \infty$ (as $i \to \infty$) in the I-adic topology if and only if $\mathrm{ord}_I(x_i) \to \infty$. Since $\mathrm{Gr}_I(R)$ is an integral domain, the I-adic order of a product is the sum of the orders of its terms.

Now note that $g \cdot g_i \cdot (f/g - f_i/g_i) = f \cdot g_i - f_i \cdot g$ converges to 0. The I-adic order of $g \cdot g_i$ is constant for sufficiently large i, which is simply twice of the I-adic order of g. Then

$$\mathrm{ord}_I(f/g - f_i/g_i) = \mathrm{ord}_I(f \cdot g_i - f_i \cdot g) - \mathrm{ord}_I(g \cdot g_i) \to \infty$$

as $i \to \infty$. This proves the lemma. $\square$

Back to the proof of Proposition 6.3.2.1. Now that we know ψ can be made close to ${}^{\diamondsuit}\psi$, the relation

$$ {}^{\diamondsuit}\tau(y_1, y_2) = {}^{\diamondsuit}\psi(y_1 + y_2)\, {}^{\diamondsuit}\psi(y_1)^{-1}\, {}^{\diamondsuit}\psi(y_2)^{-1} $$

and Lemma 6.3.2.4 show that τ can be made close to ${}^{\diamondsuit}\tau$ over the subgroup $Y \times \phi(Y)$ of $Y \times X$. By Remark 4.3.1.7, and by using the same technique as in the proof of Lemma 4.3.4.1, this shows that τ can be made close to ${}^{\diamondsuit}\tau$ over the whole group $Y \times X$. This concludes the proof of the claim that the degeneration datum associated to $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^{\wedge} \to \operatorname{Spec}(R^{\wedge})$ can be made close to the degeneration datum associated to $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}})$.

The proposition now follows from Remarks 6.3.1.12 and 6.3.1.13, and Corollaries 6.3.1.4 and 6.3.1.14. $\square$

Note that, in the statement of Proposition 6.3.2.1, the strict local ring R is the inductive limit of the coordinate rings of all affine étale neighborhoods of $\bar{x}$. By taking R_{alg} to be such an affine étale neighborhood over which all the objects, sheaves, and isomorphisms in Proposition 6.3.2.1 and its proof are defined, we may replace Proposition 6.3.2.1 by the following:

Proposition 6.3.2.5. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ be a (nondegenerate) smooth rational polyhedral cone.*

Let R be the strict local ring of a geometric point $\bar{x}$ of the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for some $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, and let $R^{\wedge}$ be the completion of R along the σ-stratum. Then there exists (non-canonically) an étale neighborhood $\operatorname{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *of $\bar{x}$, with the* **natural inclusion** $\imath^{\mathrm{nat}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$*, and a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over* $\operatorname{Spec}(R_{\mathrm{alg}})$ *as in Definition 5.3.2.1, with an embedding $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$, such that the embedding $\imath^{\mathrm{alg}}$ is* **close to the natural inclusion** $\imath^{\mathrm{nat}}$ *in the following sense:*

1. *The stratification of* $\operatorname{Spec}(R_{\mathrm{alg}})$ *is parameterized by faces of σ mod $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ and agrees with the stratification induced from $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ via* $\operatorname{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

2. *There are isomorphisms between the objects $\underline{\Phi}_{\mathcal{H}}(G)$, $\mathbf{S}_{\underline{\Phi}(G)}$, and $\underline{B}(G)$ as defined in Construction 6.3.1.1, and the pullbacks of the tautological objects $\underline{\Phi}_{\mathcal{H}}$, $\underline{\mathbf{S}}$, and $\underline{B}$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (described in Proposition 6.2.5.7) via* $\operatorname{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

3. *The following two maps* $\mathrm{Spf}(R^\wedge, I) \to \mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma}/\Gamma_{\Phi_\mathcal{H},\sigma}$ **coincide** *on the* σ*-stratum:*

   (a) *The pullback* $(G, \lambda, i, \alpha_\mathcal{H}) \underset{R_{\mathrm{alg}}}{\otimes} R^\wedge \to \mathrm{Spec}(R^\wedge)$ *under the natural inclusion* $\imath^{\mathrm{nat}} : R_{\mathrm{alg}} \hookrightarrow R^\wedge$ *defines a formal good* $(\Phi_\mathcal{H}, \delta_\mathcal{H})$*-model by the existence of the isomorphisms in 2, and hence defines a canonical map* $\mathrm{Spf}(R^\wedge, I) \to \mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma}/\Gamma_{\Phi_\mathcal{H},\sigma}$.

   (b) *The embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^\wedge$ *defines a* (*strata-preserving*) *composition*

   $$\mathrm{Spec}(R^\wedge) \overset{\mathrm{Spec}(\imath^{\mathrm{alg}})}{\to} \mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma)/\Gamma_{\Phi_\mathcal{H},\sigma},$$

   which induces a map $\mathrm{Spf}(R^\wedge, I) \to \mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma}/\Gamma_{\Phi_\mathcal{H},\sigma}$.

4. *The extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *defines an isomorphism*

   $$\underline{\mathrm{KS}}_{G/\,\mathrm{Spec}(R_{\mathrm{alg}})} \xrightarrow{\sim} \Omega^1_{\mathrm{Spec}(R_{\mathrm{alg}})/\mathsf{S}_0}[d\log\infty].$$

Definition 6.3.2.6. *We call such an étale neighborhood* $\mathrm{Spec}(R_{\mathrm{alg}})$*, together with the degenerating family* $(G, \lambda, i, \alpha_\mathcal{H})$ *and the existence of an embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R$ *as above, a* **good algebraic** $(\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma)$**-model**.

Or, in a more self-contained manner:

Definition 6.3.2.7. *Let* $(\Phi_\mathcal{H}, \delta_\mathcal{H})$ *be a representative of a cusp label at level* $\mathcal{H}$*, and let* $\sigma \subset \mathbf{P}^+_{\Phi_\mathcal{H}}$ *be a* (*nondegenerate*) *smooth rational polyhedral cone. A* **good algebraic** $(\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma)$**-model** *consists of the following data:*

1. *An affine scheme* $\mathrm{Spec}(R_{\mathrm{alg}})$*, together with a stratification of* $\mathrm{Spec}(R_{\mathrm{alg}})$ *with strata parameterized by faces of* σ *modulo* $\Gamma_{\Phi_\mathcal{H},\sigma}$.

2. *A strata-preserving map* $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma)/\Gamma_{\Phi_\mathcal{H},\sigma}$ *making* $\mathrm{Spec}(R_{\mathrm{alg}})$ *an étale neighborhood of some geometric point* $\bar{x}$ *of* $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma)/\Gamma_{\Phi_\mathcal{H},\sigma}$ *at the* σ*-stratum.*

   Let $R^\wedge$ *be the completion of the strict local ring of* $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma)/\Gamma_{\Phi_\mathcal{H},\sigma}$ *at* $\bar{x}$ *along the* σ*-stratum. Then there is a "natural inclusion"* $\imath^{\mathrm{nat}} : R_{\mathrm{alg}} \hookrightarrow R^\wedge$.

3. *A degenerating family* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R_{\mathrm{alg}})$ *as in Definition 5.3.2.1 together with the existence of an embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$*, such that:*
   (a) *There are isomorphisms between the objects* $\underline{\Phi}_{\mathcal{H}}(G)$*,* $\underline{\mathbf{S}}_{\underline{\Phi}(G)}$*, and* $\underline{B}(G)$ *as defined in Construction 6.3.1.1, and the pullbacks of the tautological objects* $\underline{\Phi}_{\mathcal{H}}$*,* $\underline{\mathbf{S}}$*, and* $\underline{B}$ *on* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(described in Proposition 6.2.5.7) via* $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

   (b) *The embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ *is* **close to the natural inclusion** $\imath^{\mathrm{nat}}$ *in the sense that the following two maps* **coincide** *on the* σ*-stratum:*
      i. *The pullback* $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge} \to \mathrm{Spec}(R^{\wedge})$ *under the natural inclusion* $\imath^{\mathrm{nat}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ *defines a formal good* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*-model by the existence of the isomorphisms in 3a, and hence defines a canonical map* $\mathrm{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

      ii. *The embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ *defines a (strata-preserving) composition*

$$\mathrm{Spec}(R^{\wedge}) \overset{\mathrm{Spec}(\imath^{\mathrm{alg}})}{\to} \mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma},$$

      which induces a map $\mathrm{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

   (c) *The extended Kodaira-Spencer map (defined as in Definition 4.6.3.32) defines an isomorphism*

$$\underline{\mathrm{KS}}_{G/\mathrm{Spec}(R_{\mathrm{alg}})} \overset{\sim}{\to} \Omega^1_{\mathrm{Spec}(R_{\mathrm{alg}})/\mathsf{S}_0}[d\log\infty].$$

Proposition 6.3.2.8 (reformulation of Proposition 6.3.2.5)**.** *Good algebraic* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*-models* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$ *exist, and the maps from the various* $\mathrm{Spec}(R_{\mathrm{alg}})$ *to* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ **cover** *the* σ*-stratum of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

Remark 6.3.2.9. What is implicit behind Proposition 6.3.2.8 is that, although we need to approximate the (possibly infinitely many) good formal models at all geometric points of the σ-stratum, we only need *finitely many* good algebraic models to cover it, by quasi-compactness.

Remark 6.3.2.10. Suppose a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \operatorname{Spec}(R_{\mathrm{alg}})$ together with an embedding $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ defines a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model, then its pullback via the embedding defines a good formal $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^{\wedge})$. It is the pullback of the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ via the map $\operatorname{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ induced by the composition $\operatorname{Spec}(R^{\wedge}) \xrightarrow{\operatorname{Spec}(\imath^{\mathrm{alg}})} \operatorname{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ defined by the embedding $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$, which makes $(R^{\wedge}, I)$ isomorphic to the completion of a strict local ring of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. This property essentially determines the embedding $\imath^{\mathrm{alg}}$, and hence the notation $\imath^{\mathrm{alg}}$ can be suppressed when we talk about good algebraic models.

Remark 6.3.2.11. A good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $({}^{\diamondsuit}G, {}^{\diamondsuit}\lambda, {}^{\diamondsuit}i, {}^{\diamondsuit}\alpha_{\mathcal{H}})$ over $\operatorname{Spec}(R)$ is also a good algebraic $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$-model if and only if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ is equivalent to $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$. (See Remark 6.3.1.16.)

Remark 6.3.2.12. Even if $\sigma, \sigma' \in \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ are two smooth rational polyhedral cones such that $\sigma \subset \sigma'$, a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model is not necessarily a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$-model. (See Remarks 6.2.5.23 and 6.3.1.17.)

Proposition 6.3.2.13 (*openness of versality*)**.** *Suppose $\bar{x}$ is any geometric point in the $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\mathcal{H},\sigma})$-stratum of a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model, where τ is a face of σ. By pulling back to the completion $R^{\wedge}_{\bar{x}}$ of the strict local ring $R_{\bar{x}}$ at $\bar{x}$, we obtain a good formal $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')$-model, where:*

1. *$\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ is the pullback of $\underline{\Phi}_{\mathcal{H}}$ to $\bar{x}$, which comes with a surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ (as in Definition 5.4.2.12) from the definition of $\Phi_{\mathcal{H}}$.*

2. *$\delta'_{\mathcal{H}}$ is any splitting that makes $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ a representative of a cusp label. Then there is a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$. (The actual choice of $\delta'_{\mathcal{H}}$ does not matter.)*

3. *$\tau' \subset \mathbf{P}^{+}_{\Phi'_{\mathcal{H}}}$ is any (nondegenerate) smooth rational polyhedral cone whose image under the embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ induced by the surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of τ.*

Proof. With the choice of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, the pullback of G to $R_{\bar{x}}$ determines (by Lemma 5.4.2.10) an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ over $R^{\wedge}_{\bar{x}}$. As we already know the pullback of the linear map $\underline{B}(G)$, we obtain (by Proposition 6.2.5.8) a canonical (strata-preserving) morphism from $\operatorname{Spf}(R^{\wedge}_{\bar{x}})$ to $\mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau'}/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$.

Since $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\bar{x}}^{\wedge})$ is the pullback of the Mumford family over $\mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau'}/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$, we know that the pullback of the extended Kodaira-Spencer map (defined as in Definition 4.6.3.32) is necessarily the extended Kodaira-Spencer map for $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\bar{x}}^{\wedge})$. Moreover, as explained in Remark 4.6.3.31, Proposition 4.5.3.10 implies that the logarithmic 1-differentials we need for the extensions are the *same*. As a result, Corollary 6.3.1.14 implies that the above canonical morphism identifies $R_{\bar{x}}^{\wedge}$ with the completion of a strict local ring of $\Xi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\tau')/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$ under this map. $\square$

Inspired by Proposition 6.3.2.13:

Definition 6.3.2.14. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ be a (nondegenerate) smooth rational polyhedral cone. We say that a triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a* **face** *of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$, if:*

1. *$(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ is the representative of some cusp label at level $\mathcal{H}$, such that there exists a surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ as in Definition 5.4.2.12.*
2. *$\sigma' \subset \mathbf{P}^{+}_{\Phi'_{\mathcal{H}}}$ is a (nondegenerate) smooth rational polyhedral cone, such that for one (and hence all) surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ as above, the image of σ' under the induced embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of a face of σ.*

Note that this definition is insensitive to the choices of representatives in the classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$. This justifies the following:

Definition 6.3.2.15. *We say that the equivalence class $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a* **face** *of the equivalence class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ if (any triple that is equivalent to) $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a face of (any triple that is equivalent to) $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.*

Remark 6.3.2.16. Suppose $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a *face* of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$, so that σ' is identified with some face τ of σ under some surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$. Then there always exists some good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model that has a nonempty $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})$-stratum on the base scheme.

6.3.3 Étale Presentation and Gluing

To construct the arithmetic toroidal compactification $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ as an algebraic stack, it suffices to give an étale presentation $\mathsf{U}_{\mathcal{H}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, such that $\mathsf{R}_{\mathcal{H}} := \mathsf{U}_{\mathcal{H}} \underset{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}{\times} \mathsf{U}_{\mathcal{H}}$ is étale over $\mathsf{U}_{\mathcal{H}}$ via the two projections. Equivalently, it suffices to construct the $\mathsf{U}_{\mathcal{H}}$ and $\mathsf{R}_{\mathcal{H}}$ that satisfy the required groupoid relations, which then realizes $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ as the quotient of $\mathsf{U}_{\mathcal{H}}$ by $\mathsf{R}_{\mathcal{H}}$. Let us first explain our choices of $\mathsf{U}_{\mathcal{H}}$ and $\mathsf{R}_{\mathcal{H}}$, then show that they have the desired properties.

Take a finite number of good algebraic models, which cover our potential compactification, as follows:

1. Choose a complete set of representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$ as in Definition 5.4.2.4.

   This is a finite set because there can only be finitely many $\mathsf{Z}_{\mathcal{H}}$, $\Phi_{\mathcal{H}}$, and $\delta_{\mathcal{H}}$ at level $\mathcal{H}$. (We do not need to know anything about the exact parameterization of them.)

2. Make compatible choices of a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ for each $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ chosen above. We assume that each $\Sigma_{\Phi_{\mathcal{H}}}$ satisfies Condition 6.2.5.18.

   The compatibility means:

   Condition 6.3.3.1. *For every surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels, we require the chosen cone decompositions $\Sigma_{\Phi_{\mathcal{H}}}$ and $\Sigma_{\Phi'_{\mathcal{H}}}$ to define a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ as in Definition 6.2.6.4.*

   Let us proceed by assuming that such a compatible choice is possible. (See Proposition 6.3.3.3 below.)

3. For each chosen $\Sigma_{\Phi_{\mathcal{H}}}$ above, choose a complete set of representatives σ in $\Sigma_{\Phi_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$. This gives a complete set of representatives of equivalence classes of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (defined as in Definition 6.2.6.1).

   This is a finite set by the $\Gamma_{\Phi_{\mathcal{H}}}$-admissibility (defined as in Definition 6.1.1.12) of each $\Sigma_{\Phi_{\mathcal{H}}}$. (So the question is rather about the existence of $\Sigma_{\Phi_{\mathcal{H}}}$ in the previous step.)

4. For each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ above that satisfies moreover $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, choose *finitely many* good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-models $\mathrm{Spec}(R_{\mathrm{alg}})$ (defined as in Definition 6.3.2.7) such that the corresponding étale maps from the various $\mathrm{Spec}(R_{\mathrm{alg}}/I)$ to the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, where I denotes the ideal of R_{alg} defining the σ-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$, *cover* the whole σ-stratum. (See Proposition 6.3.2.8 and Remark 6.3.2.9.)

   This is possible by the quasi-compactness of the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, which follows from the realization of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ as a toroidal embedding of a torus-torsor over an abelian scheme over the moduli problem $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. (See Section 6.2 and Theorem 1.4.1.12.)

Note that the compatible choices of $\Sigma_{\Phi_{\mathcal{H}}}$ for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ extends by the equivalence defined in Definition 6.2.6.2 to compatible choices for all possible pairs $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ representing cusp labels. In particular, there is only one choice of cone decomposition necessary for each individual cusp label.

Definition 6.3.3.2. *A* **compatible choice of admissible smooth rational polyhedral cone decomposition data** *for $\mathsf{M}_{\mathcal{H}}$ is a complete set $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of* **compatible choices** *of $\Sigma_{\Phi_{\mathcal{H}}}$ (satisfying Condition 6.2.5.18) in the sense of Condition 6.3.3.1, for $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ running through all possible pairs representing cusp labels.*

Proposition 6.3.3.3. *A compatible choice Σ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$* **exists***.*

Remark 6.3.3.4. This is a combinatorial question unrelated to the question of integral models at all. It is already needed in the existing works on complex analytic or rational models of toroidal compactifications, such as in [107] or related works such as [63]. Since the argument is not complicated in our case, and since the translation of different notations and settings will take too much effort, we would like to give a direct treatment here.

For ease of exposition, let us introduce the following notions:

Definition 6.3.3.5. *Let $r = (r_{[\tau]})$ be the multi-rank of an $\mathcal{O}$-lattice as in Definition 1.2.1.20. The* **magnitude** *$|r|$ of r is defined to be $|r| := \sum\limits_{[\tau]} r_{[\tau]}$.*

Definition 6.3.3.6. *Let $r = (r_{[\tau]})$ and $r' = (r'_{[\tau]})$ be the multi-rank of an $\mathcal{O}$-lattice as in Definition 1.2.1.20. We say r is* **greater** *than r', denoted $r > r'$, if $|r| > |r'|$ and $r_{[\tau]} \geq r'_{[\tau]}$ for any $[\tau]$. We say r is* **smaller** *than r', denoted $r < r'$, if $r > r'$. We say r is equal to r' if $r = r'$. These relations define a* **partial order** *on the set of all possible multi-ranks.*

Proof of Proposition 6.3.3.3. It is clear that it suffices to take a complete set $\{(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})\}$ of (mutually inequivalent) representatives of cusp labels of $\mathsf{M}_{\mathcal{H}}$ at level $\mathcal{H}$, and then to construct $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (satisfying Condition 6.2.5.18) for each $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ in the collection so that Condition 6.3.3.1 is satisfied. Then there is no compatibility to satisfy between cone decompositions of representatives of cusp labels of the same multi-rank.

If we begin with only the representatives of cusp labels $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of multi-rank r of magnitude one (defined as in Definition 6.3.3.5), then there is no compatibility condition to satisfy, and hence we can take any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (satisfying Condition 6.2.5.18) for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$.

Let us consider any multi-rank r with magnitude greater than one. Suppose we have constructed a compatible collection of cone decompositions as above for each representative of cusp labels of multi-rank (defined as in Definition 6.3.3.6) smaller than r. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be any representative of cusp label of multi-rank r (if it exists). The *admissible boundary* $\mathbf{P}_{\Phi_{\mathcal{H}}} - \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is the cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ formed by the union of the images of $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ of surjections $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ from $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to a representative $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of a cusp label of multi-rank smaller than r. The choices of cone decompositions we have made for $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ determines a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth polyhedral cone decomposition (satisfying Condition 6.2.5.18) for the cone $\mathbf{P}_{\Phi_{\mathcal{H}}} - \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, which is independent of the choice of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ by the definition of admissibility. (See Definitions 6.1.1.12 and 6.1.1.14.) Then we can take any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth polyhedral cone decomposition (satisfying Condition 6.2.5.18) for the cone $\mathbf{P}_{\Phi_{\mathcal{H}}}$ that extends the above cone decomposition, which is clearly possible by construction. This enables us to construct a compatible collection of cone decompositions as above for each representative of cusp labels of multi-rank r.

Now the result follows by repeating the above process until we have exhausted all representatives of cusp labels. ☐

When the multi-rank of a cusp label is zero, there is only one possible class $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) = (0, 0)$ representing this cusp label, and also only one possible $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma) = (0, 0, 0)$. In this case, the Mumford family is just the tautological tuple over the moduli problem $\mathsf{M}_{\mathcal{H}}$ we wanted to compactify. Then the good algebraic models $\mathrm{Spec}(R_{\mathrm{alg}})$ are simply affine schemes with étale maps $\mathrm{Spec}(R_{\mathrm{alg}}) \to \mathsf{M}_{\mathcal{H}}$, which altogether cover $\mathsf{M}_{\mathcal{H}}$.

Let us form the smooth scheme

$$\mathsf{U}_{\mathcal{H}} = \begin{pmatrix} \text{disjoint union of the (finitely many)} \\ \text{good algebraic } (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)\text{-models} \\ \mathrm{Spec}(R_{\mathrm{alg}}) \text{ chosen above} \end{pmatrix},$$

which comes naturally with a stratification labeled as follows: On a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $\mathrm{Spec}(R_{\mathrm{alg}})$ used in the construction of $\mathsf{U}_{\mathcal{H}}$ above, the σ-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$ is certainly defined to belong to the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-*stratum* of $\mathsf{U}_{\mathcal{H}}$, where $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ is the class of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (defined as in Definition 6.2.6.1). For a general face τ of σ, the $(\tau \bmod \Gamma_{\Phi_{\mathcal{H}},\sigma})$-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$ is defined to belong to the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$-*stratum* of $\mathsf{U}_{\mathcal{H}}$ if $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$ is the class of any triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')$ determined as in Proposition 6.3.2.13. (In this case, $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$ is a *face* of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, as defined in Definition 6.3.2.15.) By the compatibility of the choice of Σ (given by Condition 6.3.3.1) in Definition 6.3.3.2, we know τ' is a cone in the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ that we have chosen in Σ. Since τ' is unique up to the translation by an element in $\Gamma_{\Phi'_{\mathcal{H}}}$, we see that the equivalent class $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$ of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')$ is well-defined. Hence we may label all the strata by the equivalence classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ of triples that we have taken in the construction of $\mathsf{U}_{\mathcal{H}}$. For simplicity, we call the $[(0, 0, 0)]$-stratum the $[0]$-stratum of $\mathsf{U}_{\mathcal{H}}$, which we denote by $\mathsf{U}_{\mathcal{H}}^{[0]}$.

The good algebraic models $(G, \lambda, i, \alpha_{\mathcal{H}})$ over the various $\mathrm{Spec}(R_{\mathrm{alg}})$'s define a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{U}_{\mathcal{H}}$, whose restriction to the $[0]$-stratum $\mathsf{U}_{\mathcal{H}}^{[0]}$ is a tuple $(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}})$ parameterized by the moduli problem $\mathsf{M}_{\mathcal{H}}$. This determines a canonical morphism $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$. This morphism is étale, because $\mathsf{U}_{\mathcal{H}}^{[0]}$ is locally of finite presentation, and the map $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$ is formally étale at every geometric point of $\mathsf{U}_{\mathcal{H}}^{[0]}$ by the calculation of Kodaira-Spencer maps (using condition 3c of Definition 6.3.2.7, Theorem 4.6.3.13, and Proposition 2.3.4.2). As a result, the morphism $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$ (surjective by definition) defines an étale presentation of $\mathsf{M}_{\mathcal{H}}$. This identifies $\mathsf{M}_{\mathcal{H}}$ as the quotient of $\mathsf{U}_{\mathcal{H}}^{[0]}$ by the étale groupoid $\mathsf{R}_{\mathcal{H}}^{[0]}$ over $\mathsf{U}_{\mathcal{H}}^{[0]}$, defined by the representable functor

$$\mathsf{R}_{\mathcal{H}}^{[0]} = \underline{\mathrm{Isom}}_{\mathsf{U}_{\mathcal{H}}^{[0]} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}^{[0]}}(\mathrm{pr}_1^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}), \mathrm{pr}_2^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}})),$$

where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathsf{U}_{\mathcal{H}}^{[0]} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{U}_{\mathcal{H}}^{[0]}$ denote respectively the projections to the first and the second components.

Proposition 6.3.3.7. *Suppose R is a normal complete local ring with quotient field K and algebraically closed residue field k. Assume that we have a degenerating family $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha_{\mathcal{H}}^\dagger)$ of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(R)$ as in Definition 5.3.2.1. Then the following conditions are equivalent:*

1. *There exists a morphism $\mathrm{Spec}(R) \to \mathsf{U}_{\mathcal{H}}$ sending the generic point $\mathrm{Spec}(K)$ to the $[0]$-stratum, such that $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha_{\mathcal{H}}^\dagger) \to \mathrm{Spec}(R)$ is the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{U}_{\mathcal{H}}$.*

2. *The degenerating family $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha_{\mathcal{H}}^\dagger) \to \mathrm{Spec}(R)$ is the pullback of a Mumford family $({}^\heartsuit G, {}^\heartsuit \lambda, {}^\heartsuit i, {}^\heartsuit \alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}/\Gamma_{\Phi_{\mathcal{H}}, \sigma}$ via a map $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}/\Gamma_{\Phi_{\mathcal{H}}, \sigma}$, or equivalently a map $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}}, \sigma}$, for some $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (which can be assumed to be a triple used in the construction of $\mathsf{U}_{\mathcal{H}}$).*

3. *The degenerating family $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha_{\mathcal{H}}^\dagger)$ over $\mathrm{Spec}(R)$ defines an object of $\mathrm{DEG}_{\mathrm{PEL}, \mathsf{M}_{\mathcal{H}}}$ over $\mathrm{Spec}(R)$, which corresponds to a tuple*

   $$(A^\dagger, \lambda_A^\dagger, i_A^\dagger, \underline{X}^\dagger, \underline{Y}^\dagger, \phi^\dagger, c^\dagger, (c^\vee)^\dagger, \tau^\dagger, [\alpha_{\mathcal{H}}^{\natural}{}^\dagger])$$

   in $\mathrm{DD}_{\mathrm{PEL}, \mathsf{M}_{\mathcal{H}}}$ under Theorem 5.3.1.17. Then we have a fully symplectic-liftable admissible filtration $\mathsf{Z}_{\mathcal{H}}^\dagger$ determined by $[\alpha_{\mathcal{H}}^{\natural}{}^\dagger]$. Moreover, the étale sheaves $\underline{X}^\dagger$ and $\underline{Y}^\dagger$ are necessarily constant, because the base scheme R is strictly Henselian. Hence it makes sense to say we also have a uniquely determined torus argument $\Phi_{\mathcal{H}}^\dagger$ at level $\mathcal{H}$ for $\mathsf{Z}_{\mathcal{H}}^\dagger$.

   On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^\dagger)$, $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}}(G^\dagger)$, and $\underline{B}(G^\dagger)$, which defines objects $\Phi_{\mathcal{H}}^\dagger$, $\mathbf{S}_{\Phi_{\mathcal{H}}^\dagger}$ and in particular $B^\dagger : \mathbf{S}_{\Phi_{\mathcal{H}}^\dagger} \to \mathrm{Inv}(R)$ over the special fiber.

   If $\upsilon : K^\times \to \mathbb{Z}$ is any discrete valuation defined by a prime of R of height one, then $\upsilon \circ B^\dagger : \mathbf{S}_{\Phi_{\mathcal{H}}^\dagger} \to \mathbb{Z}$ makes sense and defines an element *of* $\mathbf{S}^\vee_{\Phi^\dagger_{\mathcal{H}}}$*. Then the condition is that, for any choice of* $\delta^\dagger_{\mathcal{H}}$ *(which does not matter), there is a cone* $\sigma^\dagger$ *in the chosen cone decomposition* $\Sigma_{\Phi^\dagger_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^\dagger_{\mathcal{H}}}$ *(given by the choice of* Σ*; cf. Definition 6.3.3.2) such that* $\overline{\sigma}^\dagger$ *contains all* $\upsilon \circ B^\dagger$ *obtained in this way.*

Proof. The implication from 1 to 2 is clear, as the map from $\mathrm{Spec}(R)$ to $\mathsf{U}_{\mathcal{H}}$ necessarily factors through the completion of a strict local ring of $\mathsf{U}_{\mathcal{H}}$.

The implication from 2 to 3 is analogous to Proposition 6.3.1.2.

For the implication from 3 to 1, choose $\sigma^\dagger$ as in the statement of 3 such that $\overline{\sigma}^\dagger$ contains all the $\upsilon \circ B^\dagger$. Then, by positivity of τ, some positive linear combination of the $\upsilon \circ B^\dagger$'s lies in $\sigma^\dagger$, the interior of $\overline{\sigma}^\dagger$, and we may identify the character group $X^\dagger$ of the torus part of the special fiber with the tautological X_σ on $\mathsf{U}_{\mathcal{H}}$. This shows that we may identify $\Phi^\dagger_{\mathcal{H}}$ with $\Phi_{\mathcal{H}}$ as well. Let us take any $\delta_{\mathcal{H}}$ so that $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ represents a cusp label at level $\mathcal{H}$ as in Definition 5.4.2.4. By modifying the identification of $\Phi^\dagger_{\mathcal{H}}$ with $\Phi_{\mathcal{H}}$ and the choice of $\delta_{\mathcal{H}}$ if necessary, we may and we do assume that $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ is a representative chosen in the construction $\mathsf{U}_{\mathcal{H}}$.

By Proposition 6.2.4.7, the degeneration datum without positivity condition associated to $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha^\dagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ determines a map $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By Proposition 6.2.5.8 and the assumption on $\upsilon \circ B^\dagger$, the map $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a map $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$, which identifies $\underline{B}(G^\dagger)$ with the pullback of $\underline{B}$ under an identification of $\underline{\Phi}_{\mathcal{H}}(G^\dagger)$ with the pullback of $\underline{\Phi}_{\mathcal{H}}$. The ambiguity of the identifications can be removed (or rather intrinsically incorporated) if we form the quotient $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Hence we have a uniquely determined strata preserving map $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, which is independent of the identification of $\Phi^\dagger_{\mathcal{H}}$ with $\Phi_{\mathcal{H}}$ we have chosen. This determines a morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ as in 2.

Let us denote the image of the closed point of $\mathrm{Spec}(R)$ by x, which necessarily lies in the σ-stratum. By construction, there is some good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$ used in construction of $\mathsf{U}_{\mathcal{H}}$ such that the image of the structural morphism $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ covers x. Let ${R_{\mathrm{alg}}}^\wedge$ be the completion of R_{alg} along the ideal defining the σ-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$. Then the étale morphism $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ extends to a formally étale morphism $\mathrm{Spf}({R_{\mathrm{alg}}}^\wedge) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. By formal étaleness, the morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ can be lifted uniquely to a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R_{\mathrm{alg}}{}^{\wedge})$. The corresponding morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_{\mathrm{alg}}{}^{\wedge})$ identifies the degeneration datum associated to $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to \mathrm{Spec}(R)$ with the degeneration datum associated to the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R_{\mathrm{alg}}{}^{\wedge} \to \mathrm{Spec}(R_{\mathrm{alg}}{}^{\wedge})$. Hence the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_{\mathrm{alg}}{}^{\wedge}) \to \mathrm{Spec}(R_{\mathrm{alg}})$ identifies $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to \mathrm{Spec}(R)$ with the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$, as desired. □

Remark 6.3.3.8. For the representative $(\Phi^{\dagger}_{\mathcal{H}}, \delta^{\dagger}_{\mathcal{H}})$ in condition 3 of Proposition 6.3.3.7, there is a unique representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of a cusp label chosen in the construction of $\mathsf{U}_{\mathcal{H}}$ such that $(\Phi_{\mathcal{H}}, \delta)$ is equivalent to $(\Phi^{\dagger}_{\mathcal{H}}, \delta^{\dagger}_{\mathcal{H}})$ for any choice of $\delta_{\mathcal{H}}$, under some isomorphism $(\gamma_X : X^{\dagger} \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y^{\dagger})$, which is unique up to composition with elements in $\Gamma_{\Phi_{\mathcal{H}}}$. Hence the cone $\sigma^{\dagger}$ in the chosen cone decomposition $\Sigma_{\Phi^{\dagger}_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is carried to some σ in the chosen decomposition $\Sigma_{\Phi^{\dagger}_{\mathcal{H}}}$ of $\mathbf{P}^{\dagger}_{\Phi_{\mathcal{H}}}$. By modifying the identification $(Y^{\dagger} \xrightarrow{\sim} Y, X \xrightarrow{\sim} X^{\dagger})$ by composition with an element in $\Gamma_{\Phi_{\mathcal{H}}}$, and by assuming the cone σ to be the smallest such cone, we may assume that the cone σ is a representative of $\Sigma_{\Phi_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ chosen in the construction of $\mathsf{U}_{\mathcal{H}}$.

Remark 6.3.3.9. Proposition 6.3.3.7 actually characterizes the compatible choice Σ of admissible smooth rational polyhedral cone decomposition data (defined as in Definition 6.3.3.2) chosen in the construction of $\mathsf{U}_{\mathcal{H}}$.

Remark 6.3.3.10. The condition 3 in Proposition 6.3.3.7 is *always fulfilled* if R is a complete discrete valuation ring. This is because there is only one choice of the discrete valuation υ, and hence only one linear map $\upsilon \circ B^{\dagger}$ to be considered. Since the union of the cones in $\Sigma_{\Phi^{\dagger}_{\mathcal{H}}}$ cover $\mathbf{P}_{\Phi^{\dagger}_{\mathcal{H}}}$ by definition of *admissibility* (in Definition 6.1.1.12), $\upsilon \circ B^{\dagger}$ must lie in one of the cones. Here the completeness of R can be dropped by usual limit arguments. We shall see in the proof of Proposition 6.3.3.18 how this observation will be applied in showing the *compactness* (or rather *properness*) of the toroidal compactification that we will construct.

Now let $\mathsf{R}_{\mathcal{H}}$ be the *normalization* of $\mathsf{R}^{[0]}_{\mathcal{H}} \to \mathsf{U}^{[0]}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}^{[0]}_{\mathcal{H}}$. By definition, $\mathsf{R}_{\mathcal{H}}$ is noetherian normal. By Proposition 3.3.1.7, the tautological isomorphism

$$h^{[0]} : (\mathrm{pr}_1^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}))_{\mathsf{R}^{[0]}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}))_{\mathsf{R}^{[0]}_{\mathcal{H}}}$$

over

$$\mathsf{R}^{[0]}_{\mathcal{H}} = \underline{\mathrm{Isom}}_{\mathsf{U}^{[0]}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}^{[0]}_{\mathcal{H}}}(\mathrm{pr}_1^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}), \mathrm{pr}_2^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}))$$

extends uniquely to an isomorphism

$$h : (\mathrm{pr}_1^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}}$$

over $\mathsf{R}_{\mathcal{H}}$.

The key to the gluing process is the following proposition:

Proposition 6.3.3.11. *The two projections from $\mathsf{R}_{\mathcal{H}}$ to $\mathsf{U}_{\mathcal{H}}$ are* **étale***.*

Proof. The plan is as follows: For any geometric point z of $\mathsf{R}_{\mathcal{H}}$, let $x = \mathrm{pr}_1(z)$ and $y = \mathrm{pr}_2(z)$. Let R_{12} (resp. R_1, resp. R_2) be the completion of the strict local ring of $\mathsf{R}_{\mathcal{H}}$ (resp. $\mathsf{U}_{\mathcal{H}}$, resp. $\mathsf{U}_{\mathcal{H}}$) at the geometric point z (resp. x, resp. y). As any irreducible component of $\mathsf{R}_{\mathcal{H}}$ dominates a component of $\mathsf{U}_{\mathcal{H}}$ via any projection, (say, by restriction to $\mathsf{U}_{\mathcal{H}}^{[0]}$,) R_1 (resp. R_2) can be naturally embedded into R_{12} via the local homomorphism $\mathrm{pr}_1^* : R_1 \to R_{12}$ (resp. $\mathrm{pr}_2^* : R_2 \to R_{12}$). Our goal is to prove $R_{12} = \mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$.

Step 1. There are triples $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)$ and $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)$, with $\sigma_1 \subset \mathbf{P}_{\phi_1}^+$ and $\sigma_2 \subset \mathbf{P}_{\phi_2}^+$, such that x (resp. y) lies on the $[(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)]$-stratum (resp. $[(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)]$-stratum) of $\mathsf{U}_{\mathcal{H}}$. By Proposition 6.3.2.13 that we have the *openness of versality*, there exists a unique isomorphism from R_1 (resp. R_2) to the completion of the strict local ring of $\mathfrak{X}_{\Phi_{\mathcal{H},1},\delta_{\mathcal{H},1},\sigma_1}/\Gamma_{\Phi_{\mathcal{H},1},\sigma_1}$ (resp. $\mathfrak{X}_{\Phi_{\mathcal{H},2},\delta_{\mathcal{H},2},\sigma_2}/\Gamma_{\Phi_{\mathcal{H},2},\sigma_2}$) at a (unique) geometric point in the σ_1-stratum (resp. σ_2-stratum), so that the pullback $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ (resp. $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$) of $(G, \lambda, i, \alpha_{\mathcal{H}})$ via $\mathrm{Spec}(R_1) \to \mathsf{U}_{\mathcal{H}}$ (resp. $\mathrm{Spec}(R_2) \to \mathsf{U}_{\mathcal{H}}$) defines a good formal $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)$-model (resp. good formal $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)$-model) via this isomorphism.

Step 2. By the theory of degeneration data, we have an object

$$(A_1, \lambda_{A,1}, i_{A,1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1, [\alpha_{\mathcal{H},1}^\natural])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ over $\mathrm{Spec}(R_1)$, where $[\alpha_{\mathcal{H},1}^\natural]$ is represented by some

$$\alpha_{\mathcal{H},1}^\natural = (\mathsf{Z}_{\mathcal{H},1}, \varphi_{-2,\mathcal{H},1}, \varphi_{-1,\mathcal{H},1}, \varphi_{0,\mathcal{H},1}, \delta_{\mathcal{H},1}, c_{\mathcal{H},1}, c_{\mathcal{H},1}^\vee, \tau_{\mathcal{H},1})$$

that corresponds to the family $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ under the functor $\mathrm{M}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ in Theorem 5.3.1.17. Here we may and we do take the datum $(\mathsf{Z}_{\mathcal{H},1}, (X_1, Y_1, \phi_1, \varphi_{-2,\mathcal{H},1}, \varphi_{0,\mathcal{H},1}), \delta_{\mathcal{H},1})$ to be the one given by

$(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1})$, because of the above-mentioned isomorphism from $\mathrm{Spec}(R_1)$ to $\mathfrak{X}_{\Phi_{\mathcal{H},1},\delta_{\mathcal{H},1},\sigma_1}/\Gamma_{\Phi_{\mathcal{H},1},\sigma_1}$. Similarly, we have an object

$$(A_2, \lambda_{A,2}, i_{A,2}, X_2, Y_2, \phi_2, c_2, c_2^\vee, \tau_2, [\alpha^\natural_{\mathcal{H},2}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}$ over $\mathrm{Spec}(R_2)$, where $[\alpha^\natural_{\mathcal{H},2}]$ is represented by some

$$\alpha^\natural_{\mathcal{H},2} = (\mathsf{Z}_{\mathcal{H},2}, \varphi_{-2,\mathcal{H},2}, \varphi_{-1,\mathcal{H},2}, \varphi_{0,\mathcal{H},2}, \delta_{\mathcal{H},2}, c_{\mathcal{H},2}, c^\vee_{\mathcal{H},2}, \tau_{\mathcal{H},2})$$

that corresponds to the family $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$. Again, we may and we do take the datum $(\mathsf{Z}_{\mathcal{H},2}, (X_2, Y_2, \phi_2, \varphi_{-2,\mathcal{H},2}, \varphi_{0,\mathcal{H},2}), \delta_{\mathcal{H},2})$ to be the one given by $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$.

As $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ and $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$ becomes isomorphic over $\mathrm{Spec}(R_{12})$ via a tautological isomorphism

$$h : (\mathrm{pr}_1^*(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}))_{R_{12}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}))_{R_{12}},$$

we have a corresponding isomorphism between the degeneration data over $\mathrm{Spec}(R_{12})$. In particular, h must match $[\alpha^\natural_{\mathcal{H},1}]$ with $[\alpha^\natural_{\mathcal{H},2}]$. Namely, $\alpha^\natural_{\mathcal{H},1}$ must be equivalent to $\alpha^\natural_{\mathcal{H},2}$. (See Definition 5.3.1.14.) In particular, $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1})$ and $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$ must be equivalent as representatives of cusp labels. (See Definition 5.4.2.4.) Since we have taken only one representative in each cusp label, they must be identical. We shall henceforth assume that $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}) = (\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}) = (\Phi_\mathcal{H}, \delta_\mathcal{H})$, where $(\Phi_\mathcal{H} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), \delta_\mathcal{H})$ is some representative that we have chosen. In particular, the equivalence induced by h between $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}) = (\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$ is now given by an element $(h_X : X \xrightarrow{\sim} Y, h_Y : Y \xrightarrow{\sim} Y)$ in $\Gamma_{\Phi_\mathcal{H}}$.

Step 3. We claim that the isomorphism $(h_X : X \xrightarrow{\sim} X, h_Y : Y \xrightarrow{\sim} Y)$ identifies the cone $\sigma_1 \subset \mathbf{P}^+_{\Phi_\mathcal{H}}$ with the cone $\sigma_2 \subset \mathbf{P}^+_{\Phi_\mathcal{H}}$ up to $\Gamma_{\Phi_\mathcal{H}}$.

This follows because the linear map $B_1 : \mathbf{S}_{\Phi_\mathcal{H}} \to \mathrm{Inv}(R_1)$ (resp. $B_2 : \mathbf{S}_{\Phi_\mathcal{H}} \to \mathrm{Inv}(R_2)$) determines the cone σ_1 (resp. σ_2) up to $\Gamma_{\Phi_\mathcal{H}}$, and B_1 and B_2 becomes identified under the element $(h_X : X \xrightarrow{\sim} X, h_Y : Y \xrightarrow{\sim} Y)$ in $\Gamma_{\Phi_\mathcal{H}}$ when extended to $\mathrm{Inv}(R_{12})$. In other words $(\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma_1)$ and $(\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma_2)$ are equivalent (under a twist of $(h_X : X \xrightarrow{\sim} X, h_Y : Y \xrightarrow{\sim} Y)$ by possibly another element of $\Gamma_{\Phi_\mathcal{H}}$).

Thus σ_1 and σ_2 are necessarily the same cone σ, because we are using only one representative in each equivalence class of $(\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma)$ in our construction of $\mathsf{U}_\mathcal{H}$. By Proposition 6.3.2.13, we may compare R_1 and R_2 by viewing them as the completions of strict local rings (over the σ-strata) of the same algebraic stack $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Step 4. Now we have two maps

$$\mathrm{Spec}(R_{12}) \overset{\mathrm{pr}_1}{\to} \mathrm{Spec}(R_1) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$$

and

$$\mathrm{Spec}(R_{12}) \overset{\mathrm{pr}_2}{\to} \mathrm{Spec}(R_2) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$$

defined by the two degeneration data, which actually *coincide*. As both $\mathrm{pr}_1^*(R_1)$ and $\mathrm{pr}_2^*(R_2)$ are given by the image of the completion of the strict local ring at the *same* image of the closed point of $\mathrm{Spec}(R_{12})$, they must *coincide* as subrings of R_{12}. Therefore, the identification $\mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$ transforms one degeneration datum to the other. By the functoriality in the theory of degeneration data, the isomorphism between $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1})_{R_{12}}$ and $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2})_{R_{12}}$ should be *already defined* over $\mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$.

By definition of $\mathsf{R}_{\mathcal{H}}^{[0]}$, we get a map from the *generic point* of $\mathrm{Spec}(R_1)$ to $\mathsf{R}_{\mathcal{H}}^{[0]}$, which covers the map to $\mathsf{U}_{\mathcal{H}}^{[0]} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}^{[0]}$ with components given by

$$(\mathrm{Id}_{\mathrm{Spec}(R_1)}, \mathrm{Spec}(R_1) \overset{\sim}{\to} \mathrm{Spec}(R_2) \text{ given by } \mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2) \text{ in } R_{12}).$$

By definition of $\mathsf{R}_{\mathcal{H}}$ as a normalization, this extends to a *regular* map

$$\mathrm{Spec}(R_1) \to \mathsf{R}_{\mathcal{H}},$$

which sends the closed point x of $\mathrm{Spec}(R_1)$ to the closed point z of R_{12}. This gives a homomorphism $R_{12} \to R_1$, which is a left inverse to $\mathrm{pr}_1^* : R_1 \hookrightarrow R_{12}$. Hence $R_{12} = \mathrm{pr}_1^*(R_1)$. Consequently, $R_{12} = \mathrm{pr}_2^*(R_2)$ as well. ☐

Remark 6.3.3.12. The key point in the proof of separateness is the tautological fact that our equivalence classes of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *do not overlap.*

Remark 6.3.3.13. Certainly, one do not really have to take representatives that are all in different equivalence classes as in our proof. Those in the same equivalence class will necessarily be identified by the gluing.

Corollary 6.3.3.14. *The scheme* $\mathsf{R}_{\mathcal{H}}$ *over* $\mathsf{U}_{\mathcal{H}}$ *defines an étale groupoid, which extends the étale groupoid* $\mathsf{R}_{\mathcal{H}}^{[0]}$ *over* $\mathsf{U}_{\mathcal{H}}^{[0]}$. *The scheme* $\mathsf{R}_{\mathcal{H}}$ *is* **finite** *over* $\mathsf{U}_{\mathcal{H}} \underset{S}{\times} \mathsf{U}_{\mathcal{H}}$, *and hence* $\mathsf{U}_{\mathcal{H}}/\mathsf{R}_{\mathcal{H}}$ *defines a* **separated** *algebraic stack over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.

Proof. Fiber products of $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}}$ are étale over $\mathsf{U}_{\mathcal{H}}$ as well. Hence they are normal and equal to respectively the normalizations of fiber products of $\mathsf{R}_{\mathcal{H}}^{[0]}$. The necessary maps between the fiber products defining the groupoid relation of $\mathsf{R}_{\mathcal{H}}^{[0]}$ over $\mathsf{U}_{\mathcal{H}}^{[0]}$ extend to the normalizations. Hence $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}}$ is a groupoid as well.

Since $\mathsf{R}_{\mathcal{H}}^{[0]}$ is *finite* over $\mathsf{U}_{\mathcal{H}}^{[0]} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}^{[0]}$ by the property of the $\underline{\mathrm{Isom}}$ functor of abelian schemes, so is the normalization $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}$. □

Definition 6.3.3.15. *The separated algebraic stack constructed as* $\mathsf{U}_{\mathcal{H}}/\mathsf{R}_{\mathcal{H}}$ *above will be denoted by* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.

Remark 6.3.3.16. The algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ depends on the compatible choice Σ of cone decompositions in the construction, but not on the actual choice of good algebraic models used to form the étale presentation $\mathsf{U}_{\mathcal{H}}$. To signify the choice we have made, we shall also denote $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ by $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ or $\mathsf{M}_{\mathcal{H},\{\Sigma_{\Phi_{\mathcal{H}}}\}}^{\mathrm{tor}}$, where $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ is the compatible choice of admissible smooth rational polyhedral cone decomposition data used in the construction.

Corollary 6.3.3.17. *The degenerating family* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *and the stratification on* $\mathsf{U}_{\mathcal{H}}$ *descends to a degenerating family* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$*, which we again denote by the same notations. This realizes* $\mathsf{M}_{\mathcal{H}}$ *as the* $[0]$*-stratum in the stratification, and identifies the restriction of* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *to* $\mathsf{M}_{\mathcal{H}}$ *with the tautological tuple over* $\mathsf{M}_{\mathcal{H}}$.

Proof. The degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{U}_{\mathcal{H}}$ has a descent datum over $\mathsf{R}_{\mathcal{H}}$ defined by $h : (\mathrm{pr}_1^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}}$. Hence the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ descends to a degenerating family over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. The proof of Proposition 6.3.3.11 shows that the two étale projections from $\mathsf{R}_{\mathcal{H}}$ to $\mathsf{U}_{\mathcal{H}}$ respect the stratification on $\mathsf{U}_{\mathcal{H}}$. Hence the stratification on $\mathsf{U}_{\mathcal{H}}$ descends to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ as well. The remaining claim is clear from definitions. □

Proposition 6.3.3.18. $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *is proper over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.

Proof. The proof is based on the valuative criterion for algebraic stacks of finite type: Let V be a discrete valuation ring with an algebraically closed residue field k and fraction field K. Let $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ be a morphism. Since the separateness of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is known by Proposition 6.3.3.11 and Corollary 6.3.3.14, the point is to establish the existence of an extension $\mathrm{Spec}(V) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.

Since $\mathsf{M}_{\mathcal{H}}$ is open and dense in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, it suffices (by Remark A.6.1.10) to treat the special case when the map $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ has image in $\mathsf{M}_{\mathcal{H}}$. The morphism $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}$ (the latter being an algebraic stack) gives a tuple $(G_K, \lambda_K, i_K, \alpha_{\mathcal{H},K})$ over K, where G_K is an abelian scheme, where $\lambda_K : G_K \xrightarrow{\sim} G_K^\vee$ is a principal polarization, where $i_K : \mathcal{O} \to \mathrm{End}_K(G_K)$ is an $\mathcal{O}$-endomorphism structure of (G_K, λ_K), and where $\alpha_{\mathcal{H},K}$ is a level-$\mathcal{H}$ structure of G_K. By Theorem 3.3.2.4 and our assumption on V, we may assume that G_K extends to a semi-abelian scheme $G_V \to \mathrm{Spec}(V)$. The principal polarization λ_K and the $\mathcal{O}$-endomorphism structure of (G_K, λ_K) then extends to an isomorphism $\lambda_V : G_V \xrightarrow{\sim} G_V^\vee$ and a map $i_V : \mathcal{O} \to \mathrm{End}_V(G_V)$. Since the base scheme V is a discrete valuation ring with algebraically closed residue field k, by Remark 6.3.3.10 and hence by Proposition 6.3.3.7, there always exists a map $\mathrm{Spec}(V) \to \mathsf{U}_{\mathcal{H}}$ such that $(G_V, \lambda_V, i_V, \alpha_{\mathcal{H},K}) \to \mathrm{Spec}(V)$ is the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{U}_{\mathcal{H}}$ via this map. This establishes the existence of a map $\mathrm{Spec}(V) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ extending the given $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}$, as desired. □

6.4 Arithmetic Toroidal Compactifications

With the same setting as in Definition 1.4.1.2, assume moreover that L is chosen so that it satisfies 1.4.3.9. (See Remark 1.4.3.8.)

6.4.1 Main Results

Theorem 6.4.1.1 (arithmetic toroidal compactifications)**.** *To each compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data as in Definition 6.3.3.2, there is associated a proper smooth algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ containing $\mathsf{M}_{\mathcal{H}}$ as an open dense sub-algebraic stack, which is an algebraic space when $\mathcal{H}$ is neat* (*defined as in Definition 1.4.1.8*)*, together with a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$* (*defined as in Definition 5.3.2.1*) *such that:*

1. *The restriction $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}})$ of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ is the universal tuple over $\mathsf{M}_{\mathcal{H}}$.*

2. *$\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ has a stratification by locally closed sub-algebraic stacks*

$$\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]},$$

with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *running through a complete set of equivalence classes of* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (*as in Definition 6.2.6.1*). (*Here* $\mathsf{Z}_{\mathcal{H}}$ *is suppressed in the notations by Convention 5.4.2.5.*)

In this stratification, the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]}$ *lies in the closure of the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *if and only if* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *is a face of* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ *as in Definition 6.3.2.15.* (*See also Remark 6.3.2.16.*)

The $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *is isomorphic to the support of the formal algebraic stack* $\mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}/\Gamma_{\Phi_{\mathcal{H}}, \sigma}$ *for any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, *which admits a canonically defined structure of a torus-torsor over an abelian scheme over the smooth algebraic stack* $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *representing a smaller dimensional moduli problem defined by some PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_{\mathcal{H}}})$ *associated to the cusp label represented by* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *as described in Definition 5.4.2.6.* (*Note that* $\mathsf{Z}_{\mathcal{H}}$ *depends only on the class* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. *The actual representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *we take does not matter.*)

In particular, $\mathsf{M}_{\mathcal{H}}$ *is an open stratum in this stratification.*

3. *The complement of* $\mathsf{M}_{\mathcal{H}}$ *in* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ (*with its reduced structure*) *is a relative Cartier divisor* $\mathsf{D}_{\infty,\mathcal{H}}$ *with normal crossings, such that each stratum of* $\mathsf{D}_{\infty,\mathcal{H}}$ *is open dense in an intersection of components of* $\mathsf{D}_{\infty,\mathcal{H}}$ (*including possible self-intersections*).

4. *The extended Kodaira-Spencer map* (*defined as in Definition 4.6.3.32*) *for* $G \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *induces an isomorphism*

$$\underline{\mathrm{KS}}_{G/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}[d\log \mathsf{D}_{\infty,\mathcal{H}}].$$

Here the formation of $\underline{\mathrm{KS}}_{G/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ *over* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *is given étale locally over each locally noetherian scheme as in Definition 6.3.1. The sheaf* $\Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}[d\log \mathsf{D}_{\infty,\mathcal{H}}]$ *is the sheaf of modules of logarithmic 1-differentials on* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *over* S_0, *with respect to the Cartier divisor* $\mathsf{D}_{\infty,\mathcal{H}}$ *with normal crossings.*

5. *The formal completion* $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}}$ *of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *along the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *is* **isomorphic** *to the formal algebraic stack* $\mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}/\Gamma_{\Phi_{\mathcal{H}}, \sigma}$ *for any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$.

This isomorphism respects stratifications in the sense that if R *is a complete strict local ring,* $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *identifies* $\mathrm{Spf}(R)$ *with a completion of a strict local ring of* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, and gives a good formal* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$*-model over* $\mathrm{Spec}(R)$*, then the stratification of* $\mathrm{Spec}(R)$ *makes the induced map* $\mathrm{Spec}(R) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *a strata-preserving map.*

The pullback of the degenerating family $(G,\lambda,i,\alpha_{\mathcal{H}})$ *over* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *to* $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *is the Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}})$ *over* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *after we identify the bases using the isomorphism.*

6. *Let* S *be an irreducible noetherian normal scheme over* S_0*, and suppose that we have a degenerating family* $(G^{\dagger},\lambda^{\dagger},i^{\dagger},\alpha_{\mathcal{H}}^{\dagger})$ *of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *as in Definition 5.3.2.1. Then* $(G^{\dagger},\lambda^{\dagger},i^{\dagger},\alpha_{\mathcal{H}}^{\dagger}) \to S$ *is the pullback of* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *via a (necessarily unique) map* $S \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *if and only if the following condition is satisfied:*

   For every geometric point $\bar{s}$ *of* S*, and for any morphism* $\mathrm{Spec}(V) \to S$ *centered at* $\bar{s}$*, where* V *is a complete discrete valuation ring with quotient field* K*, algebraically closed residue field* k*, and discrete valuation* υ*. Let* $(G^{\ddagger},\lambda^{\ddagger},i^{\ddagger},\alpha_{\mathcal{H}}^{\ddagger}) \to \mathrm{Spec}(V)$ *be the pullback of* $(G^{\dagger},\lambda^{\dagger},i^{\dagger},\alpha_{\mathcal{H}}^{\dagger}) \to S$*. This pullback family defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *over* $\mathrm{Spec}(V)$*, which corresponds to a tuple*

   $$(A^{\ddagger},\lambda_A^{\ddagger},i_A^{\ddagger},\underline{X}^{\ddagger},\underline{Y}^{\ddagger},\phi^{\ddagger},c^{\ddagger},(c^{\vee})^{\ddagger},\tau^{\ddagger},[{\alpha_{\mathcal{H}}^{\natural}}^{\ddagger}])$$

   in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *under Theorem 5.3.1.17. Then we have a fully symplectic-liftable admissible filtration* $\mathsf{Z}_{\mathcal{H}}^{\ddagger}$ *determined by* $[{\alpha_{\mathcal{H}}^{\natural}}^{\ddagger}]$*. Moreover, the étale sheaves* $\underline{X}^{\ddagger}$ *and* $\underline{Y}^{\ddagger}$ *are necessarily constant, because the base scheme* R *is strictly Henselian. Hence it makes sense to say we also have a uniquely determined torus argument* $\Phi_{\mathcal{H}}^{\ddagger}$ *at level* $\mathcal{H}$ *for* $\mathsf{Z}_{\mathcal{H}}^{\ddagger}$*.*

   On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^{\ddagger})$*,* $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}}(G^{\ddagger})$*, and* $\underline{B}(G^{\ddagger})$*, which defines objects* $\Phi_{\mathcal{H}}^{\ddagger}$*,* $\mathbf{S}_{\Phi_{\mathcal{H}}^{\ddagger}}$ *and in particular* $B^{\ddagger} : \mathbf{S}_{\Phi_{\mathcal{H}}^{\ddagger}} \to \mathrm{Inv}(V)$ *over the special fiber.*

   If $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ *is the discrete valuation of* V*, then* $\upsilon \circ B^{\ddagger} : \mathbf{S}_{\Phi_{\mathcal{H}}^{\ddagger}} \to \mathbb{Z}$ *makes sense and defines an element of* $\mathbf{S}_{\Phi_{\mathcal{H}}^{\ddagger}}^{\vee}$*. Then the condition is that, for any choice of* $\delta_{\mathcal{H}}^{\ddagger}$ *(which does not matter), there is a cone* $\sigma^{\ddagger}$ *in the* *chosen cone decomposition* $\Sigma_{\Phi^{\ddagger}_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^{\ddagger}_{\mathcal{H}}}$ *(given by the choice of* Σ*; cf. Definition 6.3.3.2) such that* $\overline{\sigma}^{\ddagger}$ *contains all* $v \circ B^{\ddagger}$ *obtained in this way.*

This is essentially a restatement of Proposition 6.3.3.7, which characterizes $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *uniquely for each choice of* Σ*.*

Proof. By Lemma 6.2.5.20, we know that $/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ acts trivially on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ and $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, because we have assumed in Definition 6.3.3.2 that each cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ satisfies Condition 6.2.5.18. Hence the claim that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is an algebraic space when $\mathcal{H}$ is neat follows.

Statements 1 and 2 follows from Corollaries 6.3.3.14 and 6.3.3.17 and Proposition 6.3.3.18. Statements 3 and 4 are étale local in nature, and hence are inherited from $\mathsf{U}_{\mathcal{H}}$ as $\mathsf{U}_{\mathcal{H}}$ is finite étale over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ by construction.

Let us prove statement 6 by explaining why it is essentially a restatement of Proposition 6.3.3.7. Suppose we have a degenerating family $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$ as in the statement. Then there is an open dense subscheme S_1 of S such that the restriction of the family defines an object of $\mathsf{M}_{\mathcal{H}}$. Hence we have a morphism $S_1 \to \mathsf{M}_{\mathcal{H}}$ by the universal property of $\mathsf{M}_{\mathcal{H}}$. The question is whether this morphism extends to a morphism $S \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, so that $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$ is the pullback of the universal tuple $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along this morphism. If this is the case, then the condition in the statement certainly holds. On the other hand, assume that the condition holds. Since extendability is a local question (because $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is separated), we may (work with $\mathsf{U}_{\mathcal{H}}$) and apply Proposition 6.3.3.7. Note that in the language of algebraic stacks this actually means there is an étale surjection $S' \to S$ such that the pullback of the map $S_1 \to \mathsf{M}_{\mathcal{H}}$ extends to a map $S' \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, and such that the pullback of $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$ along $S' \to S$ is isomorphic to the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along $S' \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, together with a descent datum of this isomorphism over $S' \underset{S}{\times} S'$. (We shall not try to make precise all the necessary technical relations here.)

Finally, let us prove statement 5. By statement 6 that we have just proved, we know that there is a unique morphism from $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (More precisely, this morphism is patched from the unique morphisms from irreducible noetherian normal schemes giving an formally étale covering of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.) This induces a unique surjective formally étale morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. For an inverse morphism, note that by construction there is a canonical morphism from the formal completion of $\mathsf{U}_{\mathcal{H}}$ along its $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-stratum to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Since this canonical morphism is determined by the degeneration data associated to the pullback of the tautological tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ to the completion, and since the two pullbacks of the tautological tuple to $\mathsf{R}_{\mathcal{H}}$ are tautologically isomorphic by definition of $\mathsf{R}_{\mathcal{H}}$, we see that the morphism from the completion of $\mathsf{U}_{\mathcal{H}}$ along its $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ descends to a map $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. It is clear from the construction that these two canonical morphisms are inverses to each other. $\square$

Remark 6.4.1.2. By our choice of $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ such that each $\Sigma_{\Phi_{\mathcal{H}}}$ satisfies Condition 6.2.5.18, we know by Lemma 6.2.5.20 that $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is a (Deligne-Mumford) algebraic stack, and is an algebraic space when $\mathcal{H}$ is neat. If we are willing to accept the Corollary 7.2.3.10 below for the (smaller) moduli problems $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, then we even know that $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is a formal scheme when $\mathcal{H}$ is neat. This removes the unpleasant situation that we have to deal with algebraic stacks even when talking about local structures.

Following [37, Ch. IV, 5.10], we may deduce from Theorem 6.4.1.1 the following formal consequence:

Corollary 6.4.1.3. *All geometric fibers of* $\mathsf{M}_{\mathcal{H}} \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ *have the same number of connected components.*

Proof. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is proper and smooth over S_0, the analogue of Zariski's connectedness theorem as formulated in [28, Thm. 4.17] implies that all geometric fibers of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{S}_0$ have the same number of connected components. (The precise assumption we need is that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{S}_0$ is proper flat and has geometrically normal fibers.) Then the results follows from the fact that $\mathsf{M}_{\mathcal{H}}$ is *fiber-wise* dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. $\square$

Remark 6.4.1.4. As an application of Corollary 6.4.1.3, the connected components of the geometric fiber over a finite field can be matched with the connected components of the complex fiber, the latter of which can be understood using the complex uniformization by unions of Hermitian symmetric spaces.

6.4.2 Towers of Toroidal Compactifications

Definition 6.4.2.1. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels. Let* σ *(resp.* σ'*) be any nondegenerate rational polyhedral cone in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ **dominates** *the triple*

$(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$, if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent as in Definition 5.4.2.4, and if for one (and hence all) isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ that identifies $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ with $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, the cone σ is contained in a $\Gamma_{\Phi'_{\mathcal{H}}}$-translate of the cone σ' under the identification between $\mathbf{P}_{\Phi_{\mathcal{H}}}$ and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ defined by (f_X, f_Y). In this case, we say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ dominates the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ under the isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.

Definition 6.4.2.2. *Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ and $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}$ be two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$. We say that Σ* **dominates** *Σ' if the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma'_{\Phi_{\mathcal{H}}})$, as in Definition 6.2.6.3, for $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ running through all possible pairs representing cusp labels.*

Proposition 6.4.2.3. *Suppose $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ and $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}$ are two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ such that Σ dominates Σ' as in Definition 6.4.2.2. Then the family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is the pullback of the family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ via a (unique) surjection $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. This surjection is proper, and is an isomorphism over $\mathsf{M}_{\mathcal{H}}$. (This is a global algebraized version of Proposition 6.2.6.7.)*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ if and only if there are representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ of respectively $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ such that $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ dominates $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$ as in Definition 6.4.2.1.

Proof. The first statement follows from statement 6 in Theorem 6.4.1.1: The restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ maps uniquely to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. These maps patches uniquely, and hence descends. Therefore there exists a unique map from $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, in the sense of relative schemes. The restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}_{\mathcal{H}}$ is the universal tuple over $\mathsf{M}_{\mathcal{H}}$, which is mapped isomorphically to the restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ to $\mathsf{M}_{\mathcal{H}}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the map is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.7. □

Remark 6.4.2.4. Proposition 6.4.2.3 shows that there is a *tower* of toroidal compactifications labeled by the compatible choices Σ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$. The partial order of refinements on the set of all possible Σ is translated into the partial order on the toroidal compactifications given by surjections that are proper and are isomorphisms over $\mathsf{M}_{\mathcal{H}}$. When properly interpreted, this tower can be viewed as a canonical compactification of $\mathsf{M}_{\mathcal{H}}$, which does not depend on the choices of Σ.

Let us now also consider the varying of the level $\mathcal{H}$:

Definition 6.4.2.5. *Suppose $\mathcal{H}' \subset \mathcal{H}$ are two open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$. Suppose $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are representatives of cusp labels at levels respectively $\mathcal{H}'$ and $\mathcal{H}$, where $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ and $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. We say that $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$* **dominates** *$(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ if the $\mathcal{H}$-orbit determined by $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ in its natural sense (by Convention 5.3.1.13) is equivalent to $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ as in Definition 5.4.2.4. In other words, the $\mathcal{H}$-orbit determined by $\mathsf{Z}_{\mathcal{H}'}$ is identical to $\mathsf{Z}'_{\mathcal{H}}$, and there exists an isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ that satisfies $\phi = \gamma_X \phi' \gamma_Y$ and identifies $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ with the $\mathcal{H}$-orbit determined by $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$. In this case we say that the triple $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ dominates the triple $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ under the isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$. For simplicity, we shall suppress $\mathsf{Z}_{\mathcal{H}'}$ and $\mathsf{Z}'_{\mathcal{H}}$ from the notations as in Convention 5.4.2.5.*

Definition 6.4.2.6. *Suppose $\mathcal{H}' \subset \mathcal{H}$ are two open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$. Suppose $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are representatives of cusp labels at levels respectively $\mathcal{H}'$ and $\mathcal{H}$, where $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ and $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. Let σ (resp. σ') be any nondegenerate rational polyhedral cone in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the triple $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$* **dominates** *the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$, if $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ dominates $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ as in Definition 6.4.2.5, and if for one (and hence all) isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ that identifies $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ with the $\mathcal{H}$-orbit determined by $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$, the cone σ is contained in a $\Gamma_{\Phi'_{\mathcal{H}}}$-translate of the cone σ' under the identification between $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ defined by (f_X, f_Y). In this case, we say that the triple $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ dominates the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ under the isomorphism $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.*

Definition 6.4.2.7. *Suppose $\mathcal{H}' \subset \mathcal{H}$ are two open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$. Suppose $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are representatives of cusp labels at levels respectively $\mathcal{H}'$ and $\mathcal{H}$, where $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ and $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. Suppose $\Sigma_{\Phi_{\mathcal{H}'}}$ (resp. $\Sigma'_{\Phi'_{\mathcal{H}}}$) is a $\Gamma_{\Phi_{\mathcal{H}'}}$-admissible* *(resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ **dominates** *the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$*, if* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *dominates* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *as in Definition 6.4.2.5, and if for one (and hence all) isomorphism* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *that identifies* $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ *with the* $\mathcal{H}$*-orbit determined by* $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$*, the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}'}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *is a refinement of the cone decomposition* $\Sigma'_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*. In this case, we say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ *dominates the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ *under the isomorphism* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.

Definition 6.4.2.8. *Suppose* $\mathcal{H}' \subset \mathcal{H}$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$*. Let* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ *be compatible choices of admissible smooth rational polyhedral cone decomposition data for respectively* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$*. We say that* Σ **dominates** Σ' *if, for any* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *dominating* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$*, both running through all possible pairs representing cusp labels at levels respectively* $\mathcal{H}'$ *and* $\mathcal{H}$*, the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ *dominates* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ *as in Definition 6.4.2.7.*

Proposition 6.4.2.9. *Suppose* $\mathcal{H}' \subset \mathcal{H}$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$*. Suppose* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ *are two compatible choices of admissible smooth rational polyhedral cone decomposition data for respectively* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$ *such that* Σ *dominates* Σ' *as in Definition 6.4.2.8. Then the family* $(G, \lambda, i, \alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *is the pullback of the family* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *via a (unique) surjection* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$*. This surjection is proper, and is the natural surjection* $\mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ *over* $\mathsf{M}_{\mathcal{H}}$ *determined by the reduction mod* $\mathcal{H}$ *of the level-*$\mathcal{H}'$ *structure* $\alpha_{\mathcal{H}'}$*.*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *to the* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *of respectively* $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ *such that* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *dominates* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *as in Definition 6.4.2.6.*

Proof. The first statement again follows from statement 6 in Theorem 6.4.1.1: Consider the restriction of $(G, \lambda, i, \alpha_{\mathcal{H}'} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma})$ to $\mathsf{M}_{\mathcal{H}'}$. Then there is a natural surjection from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$ identifying the $\mathcal{H}$-orbit of $\alpha_{\mathcal{H}'}$ with the pullback of $\alpha_{\mathcal{H}}$, as in the statement of the corollary. By abuse of notations, let us denote the reduction mod $\mathcal{H}$ of $\alpha_{\mathcal{H}'}$ also by $\alpha_{\mathcal{H}}$. By statement 6 in Theorem 6.4.1.1, the restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ maps uniquely to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. These maps patches uniquely, and hence descends to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}'}$. Therefore there exists a unique map from $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, in the sense of relative schemes. By construction, the restriction of the map is the natural map from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$ determined by the $\mathcal{H}$-orbit of $\alpha_{\mathcal{H}'}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the map is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.7. □

Remark 6.4.2.10. Proposition 6.4.2.3 is now a special case of Proposition 6.4.2.9.

Remark 6.4.2.11. Proposition 6.4.2.9 shows that there is also a *double tower* of toroidal compactifications labeled by both the levels $\mathcal{H}$ given by open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, and the compatible choices Σ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$. The partial order of dominance on the set of all possible Σ is translated into the partial order on the toroidal compactifications given by surjections as described in Proposition 6.4.2.9. When properly interpreted, this double tower can be viewed as a canonical compactification of the tower $\mathsf{M}^{\Box}$ as defined in Remark 1.4.3.10, which does not depend on the choices of Σ. This is essentially a level-varying version of Remark 6.4.2.4, but the upshot is that it is now possible to extend the Hecke action of $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ on $\mathsf{M}^{\Box}$ to this double tower. We shall explain this notion in Section 6.4.3.

6.4.3 Hecke Actions

Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, and suppose we have two open compact subgroups $\mathcal{H}' \subset \mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. Then the *Hecke action* defined by g induces a natural surjection from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$. More precisely, it is determined as follows: Let us consider the universal tuple $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},\alpha_{\mathcal{H}'})$ over $\mathsf{M}_{\mathcal{H}'}$, and consider the rational version $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},[\hat{\alpha}]_{\mathcal{H}'})$ over $\mathsf{M}_{\mathcal{H}'}$ (defined as in Construction 1.3.7.10 and Definition 1.3.7.15). Let us denote by $(G'_{\mathsf{M}_{\mathcal{H}'}},\lambda'_{\mathsf{M}_{\mathcal{H}'}},i'_{\mathsf{M}_{\mathcal{H}'}},\alpha'_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}'}$ the tuple associated to $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},[\hat{\alpha}\circ g]_{\mathcal{H}})$ under Proposition 1.4.3.3, and call $(G'_{\mathsf{M}_{\mathcal{H}'}},\lambda'_{\mathsf{M}_{\mathcal{H}'}},i'_{\mathsf{M}_{\mathcal{H}'}},\alpha'_{\mathcal{H}})$ the *Hecke twist* of $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},\alpha_{\mathcal{H}'})$ by g over $\mathsf{M}_{\mathcal{H}'}$. This determines a natural morphism $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$. This morphism $[g]$ is surjective, because it is étale and surjective over all of its geometric points. We say that this is the natural surjection defined by the *Hecke action* of g on $\mathsf{M}^{\Box}$.

Note that the argument in the beginning of Section 5.4.3 shows that $G'_{\mathsf{M}_{\mathcal{H}'}}$ can be realized as a quotient of $G_{\mathsf{M}_{\mathcal{H}'}}$ by a finite flat group scheme over $\mathsf{M}_{\mathcal{H}'}$, and this quotient extends to the whole degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}'})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ for any smooth toroidal compactification of $\mathsf{M}_{\mathcal{H}'}$ as in Theorem 6.4.1.1. We call the resulted quotient $(G', \lambda', i', \alpha'_{\mathcal{H}})$ the *Hecke twist* of $(G, \lambda, i, \alpha_{\mathcal{H}'})$ by g over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Therefore the tuple $(G'_{\mathsf{M}_{\mathcal{H}'}}, \lambda'_{\mathsf{M}_{\mathcal{H}'}}, i'_{\mathsf{M}_{\mathcal{H}'}}, \alpha'_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}'}$ extends to a degenerating family $(G', \lambda', i', \alpha'_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$, and the question is whether the natural surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ extends to some natural surjection $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ when Σ and Σ' satisfy some appropriate relation.

Definition 6.4.3.1. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, *and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. *Suppose* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels respectively* $\mathcal{H}'$ *and* $\mathcal{H}$, *where* $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. *Let* σ *(resp.* σ'*) be any nondegenerate rational polyhedral cone in* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ g**-dominates** *the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$, *if there is a pair* $(f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}, f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)})$ *of isomorphisms that defines a g-association* $\Phi_{\mathcal{H}'} \to_g \Phi'_{\mathcal{H}}$ *as in Definition 5.4.3.6, and if for one such isomorphism* (f_X, f_Y), *the cone* σ *is contained in a* $\Phi'_{\mathcal{H}}$*-translate of the cone* σ' *under the identification between* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *and* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *defined by* (f_X, f_Y). *In this case, we say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *g-dominates the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *under the isomorphism* (f_X, f_Y).

Definition 6.4.3.2. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, *and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$. *Suppose* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels respectively* $\mathcal{H}'$ *and* $\mathcal{H}$, *where* $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$. *Suppose* $\Sigma_{\Phi_{\mathcal{H}'}}$ *(resp.* $\Sigma'_{\Phi'_{\mathcal{H}}}$*) is a* $\Gamma_{\Phi_{\mathcal{H}'}}$*-admissible (resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ g**-dominates** *the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$, *if there is a pair* $(f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}, f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)})$ *of isomorphisms that defines a g-association* $\Phi_{\mathcal{H}'} \to_g \Phi'_{\mathcal{H}}$ *as in Definition 5.4.3.6, and if for one (and hence all, as explained in Remark 5.4.3.7) such isomorphism* (f_X, f_Y), *the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}'}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *is a refinement of the cone decomposition* $\Sigma'_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *under the identification between* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *and* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *defined by* (f_X, f_Y). *In this case, we say that the triple*

$(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ *g-dominates the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ *under the isomorphism* (f_X, f_Y).

Definition 6.4.3.3. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$*, and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$*. Let* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ *be compatible choices of admissible smooth rational polyhedral cone decomposition data for respectively* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$*. We say that* Σ ***g-dominates*** Σ' *if, for any* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *that dominates* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$*, both running through all possible pairs representing cusp labels at levels respectively* $\mathcal{H}'$ *and* $\mathcal{H}$*, the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ *g-dominates* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ *as in Definition 6.4.3.2.*

Proposition 6.4.3.4. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$*, and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$*. Suppose* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ *are two compatible choices of admissible smooth rational polyhedral cone decomposition data for respectively* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$ *such that* Σ *g-dominates* Σ' *as in Definition 6.4.3.3. Then the Hecke twist of the family* $(G, \lambda, i, \alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *by* g *is the pullback of the family* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *via a (unique) surjection* $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$*. This surjection is proper, and its restriction to* $\mathsf{M}_{\mathcal{H}}$ *is the natural surjection* $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ *over* $\mathsf{M}_{\mathcal{H}}$ *defined by the* **Hecke action** *of* g *on* $\mathsf{M}^{\Box}$*.*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *to the* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *of respectively* $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ *such that* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *g-dominates* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$ *as in Definition 6.4.3.1.*

Proof. The first statement follows from a combination of Proposition 5.4.3.5 and statement 6 in Theorem 6.4.1.1: Let $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ be the Hecke twist of $(G, \lambda, i, \alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ by g. Then the restriction of Let $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to $\mathsf{M}_{\mathcal{H}'}$ determines the natural surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ as in the statement of the corollary. By statement 6 in Theorem 6.4.1.1, the restriction of $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ maps uniquely to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, by our assumption of g-dominance from Σ' to Σ. (Slightly more precisely, we need to show that the cones containing pairings of the form $\upsilon \circ B' : Y' \times X' \to \mathbb{Z}$ are carried to cones containing pairings of the form $\upsilon \circ B : Y \times X \to \mathbb{Z}$ under the identification between $\mathbf{P}_{\Phi_{\mathcal{H}'}}$

and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ defined by $(f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}, f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)})$, when we have the objects as in the context of Definition 6.4.3.3.) These maps patches uniquely, and hence descends to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}'}$. Therefore there exists a unique map $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ extending $[g]$, which pulls $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ back to $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the map $[g]^{\mathrm{tor}}$ is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.7. $\Box$

Remark 6.4.3.5. Propositions 6.4.2.3 and 6.4.2.9 are now special cases of Proposition 6.4.3.4.

Remark 6.4.3.6. Proposition 6.4.3.4 suggests that it is not always possible to realize the Hecke action on a tower of toroidal compactifications with only a single admissible smooth rational polyhedral cone decomposition data $\Sigma_{\mathcal{H}} = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ as in Definition 6.3.3.2 at each level $\mathcal{H}$. (This is nevertheless possible when each of the $\mathbb{Q}$-simple factors of the algebraic group $\mathbf{G}^{\mathrm{ad}}$ has both $\mathbb{R}$-rank no greater than one, because then there is a unique choice of $\Sigma_{\mathcal{H}}$ for each $\mathcal{H}$.) This is why we have suggested the use of a double tower as in Remark 6.4.2.11. However, the introduction of a double tower will make some of the representation-theoretic objects, such as the étale cohomology associated to the direct limit of the double tower, become non-admissible modules. More precisely, when we fix any level $\mathcal{H}$, the direct limit along the tower of all possible Σ is infinite-dimensional in general. This is not the case for some other cohomology theories, such as the ones defined using Mumford's canonical extensions of equivariant coherent modules. Since this could be a rather case-by-case discussion, we shall leave this to the readers.

Chapter 7

Algebraic Constructions of Minimal Compactifications

In this chapter we collect several useful byproducts of the construction of arithmetic toroidal compactifications. All of them have their analytic analogues over the complex numbers. The complex analytic constructions involve sections of certain algebraically defined invertible sheaves, which also make sense over an integral base scheme. However, there is no reason that the theory should carry over automatically. We need the theory of theta constants to establish the positivity of the above-mentioned invertible sheaves. This should be considered as the main technical input of this chapter.

The main objective is to state and proof Theorem 7.2.4.1, with Proposition 7.1.2.15, Corollary 7.2.4.9, Proposition 7.2.5.1, and Theorem 7.3.3.4 as important byproducts. Technical results worth noting are Propositions 7.2.1.1, 7.2.1.2, 7.2.2.3, 7.2.3.2, 7.2.3.7, 7.2.3.11, and 7.2.4.5 in Section 7.2; and Proposition 7.3.1.5 in Section 7.3.

Throughout this chapter, we shall assume the same setting as in Section 6.4.

7.1 Automorphic Forms and Fourier-Jacobi Expansions

7.1.1 Automorphic Forms

Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ be a compatible choice of admissible smooth rational polyhedral cone decomposition data as in Definition 6.3.3.2, and let $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over $\mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$ be the proper smooth algebraic stack associated to Σ as in Theorem 6.4.1.1. Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be the degenerating family described in Theorem 6.4.1.1. For ease of later exposition, we shall change our notations and denote them by $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha_{\mathcal{H}}^{\mathrm{tor}})$. Let $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} = \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ be the *Hodge invertible sheaf*.

Definition 7.1.1.1. *Let M be a module over $\mathcal{O}_{F_0,(\Box)}$, and let $k \geq 0$ be an integer. An* (**arithmetic**) **automorphic form** *over $\mathsf{M}_{\mathcal{H}}$ of parallel weight k and with coefficients in M is an element of $\mathrm{AF}(k, M) := \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)$. For simplicity, when the context is clear, we shall call simply call such an element an automorphic form of weight k.*

Remark 7.1.1.2. For most applications, it suffices to consider those M that are algebras over $\mathcal{O}_{F_0,(\Box)}$. However, the theory becomes most systematic if we allow M to be arbitrary modules: As a functor in M (as modules over $\mathcal{O}_{F_0,(\Box)}$), $\mathrm{AF}(k, -)$ is left exact and commute with filtering direct limits for any $k \geq 0$.

To justify our definition:

Lemma 7.1.1.3. *Definition 7.1.1.1 is independent of the choice of Σ we made in the construction of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.*

Proof. By taking common refinements if necessary, it suffices to show that if Σ' is another compatible choices of admissible smooth rational polyhedral cone decomposition data such that Σ' dominates Σ as in Definition 6.4.2.2. According to Proposition 6.4.2.3, there is a surjective morphism $p : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, which is proper and is the identity morphism on $\mathsf{M}_{\mathcal{H}}$. Moreover, according to the construction of the arithmetic toroidal compactifications, this morphism p can be étale locally identified with a proper morphism between toric varieties, which is equivariant under the same torus action. By the arguments in [72, Ch. I, §3], this shows that $R^i p_* \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}} = 0$ for any $i > 0$, and that $p_* \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}} = \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$. As a result, we obtain:

Lemma 7.1.1.4. *Let $\mathcal{E}$ be a quasi-coherent sheaf on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Then the canonical morphism $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \mathcal{E}) \to H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}, p^*\mathcal{E})$ is an isomorphism for any $i \geq 0$.*

Back to the proof of Lemma 7.1.1.3, let $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma'} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}}$ and $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ denote the Hodge invertible sheaves on respectively $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. By construction, $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma'} = p^*\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Applying Lemma 7.1.1.4 to $\mathcal{E} := (\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M$, we see that Definition 7.1.1.1 is indeed independent of the choice of Σ. $\Box$

7.1.2 Fourier-Jacobi Expansions

Let us take any *nonempty* stratum of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ labeled by some $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, with some choice of a representative $(\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), \delta_{\mathcal{H}}, \sigma)$. According to statement 5 of Theorem 6.4.1.1, the formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is isomorphic to the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, and the pullback of $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}})$ over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Moreover, the pullback of $\omega^{\mathrm{tor}} = \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is ${}^{\heartsuit}\omega := \omega_{{}^{\heartsuit}G/(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma})} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G/(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma})}$. Note that there is a map from the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to an abelian scheme $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over a moduli scheme $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ defined by some $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}})$. Let $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ denote the universal tuple over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. Let $\omega_A := \omega_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$ denote the Hodge invertible sheaf over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. We shall often use the same notations for the pullbacks of ω_A over other bases. Let T denote the torus over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ with character group X. Then we have $T \cong \underline{\mathrm{Hom}}_{\mathsf{S}_0}(X, \mathbf{G}_{\mathrm{m},\mathsf{S}_0})$, $\mathrm{Lie}_{T/\mathsf{S}_0} \cong \underline{\mathrm{Hom}}_{\mathsf{S}_0}(X, \mathscr{O}_{\mathsf{S}_0})$, $\underline{\mathrm{Lie}}^{\vee}_{T/\mathsf{S}_0} \cong X \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{S}_0}$, and $\omega_T := \omega_{T/\mathsf{S}_0} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{T/\mathsf{S}_0} \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{S}_0}$. Similar to the case of ω_A, we shall often use the same notations for the pullbacks of ω_T over other bases.

Lemma 7.1.2.1. *There is a canonical isomorphism ${}^{\heartsuit}\omega \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ as a relative object over the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.*

Proof. This is essentially an étale local statement. Therefore it suffices to verify this statements over any formally étale morphism $S = \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, so that R and I satisfies the setting as in Section 5.2.1. Let us also denote the pullback of various objects by the same notations. The upshot is that then it makes sense to talk about the Raynaud extension ${}^{\heartsuit}G^{\natural}$ associated to ${}^{\heartsuit}G$, so that we have an isomorphism ${}^{\heartsuit}G^{\natural}_{\mathrm{for}} \cong {}^{\heartsuit}G_{\mathrm{for}}$ along the I-adic completion S_{for} of S. Then there is a canonical isomorphism $\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G/S} \cong \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}/S}$, because they have the same formal completion $\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G_{\mathrm{for}}/S_{\mathrm{for}}} = \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}}$ over S_{for}. (See Corollary 2.3.1.3.) As a result, there are canonical isomorphisms

$$\begin{aligned}\omega_{{}^{\heartsuit}G/S} &\cong \omega_{{}^{\heartsuit}G^{\natural}/S} := \wedge^{\mathrm{top}}\, \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}/S} \cong \wedge^{\mathrm{top}}\, \underline{\mathrm{Lie}}^{\vee}_{T/S} \underset{\mathscr{O}_S}{\otimes} \wedge^{\mathrm{top}}\, \underline{\mathrm{Lie}}^{\vee}_{A/S} \\ &\cong \omega_T \underset{\mathscr{O}_S}{\otimes} \omega_A \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}}\, X) \underset{\mathbb{Z}}{\otimes} \omega_A,\end{aligned}$$

as desired. □

Recall that $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is by construction the formal completion of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ along its σ-stratum, and that $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is by definition

$$\underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}\big(\bigoplus_{\ell\in\sigma^{\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\big),$$

where $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ is defined linearly in ℓ so that $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}([y \otimes \chi]) \cong (c^{\vee}(y), c(\chi))^{*}\mathcal{P}_A$. As described in Remark 6.2.5.10, the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is defined by the sheaf of ideals

$$\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} = \bigoplus_{\ell\in\sigma_0^{\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$$

in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} = \bigoplus\limits_{\ell\in\sigma^{\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. Therefore we may write symbolically

$$\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} = \hat{\bigoplus_{\ell\in\sigma^{\vee}}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell),$$

and

$$\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}} = \big(\hat{\bigoplus_{\ell\in\sigma^{\vee}}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\big)^{\Gamma_{\Phi_{\mathcal{H}},\sigma}}.$$

Let us denote the structural map $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ by $\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Note that this is the structural map of an abelian scheme, which is in particular proper and smooth. For the ease of notations:

Definition 7.1.2.2. $\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} := (\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$.

Now it makes sense to consider the following composition of natural maps:

$$\begin{aligned}
&\mathrm{AF}(k,M) = \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \\
&\to \Gamma((\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \cong \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}, {}^{\heartsuit}\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \\
&\to \left[\prod_{\ell\in\sigma^\vee} \Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\right]^{\Gamma_{\Phi_{\mathcal{H}},\sigma}} \\
&\cong \left[\prod_{\ell\in\sigma^\vee} \Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}, (\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \underset{\mathscr{O}_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\right]^{\Gamma_{\Phi_{\mathcal{H}},\sigma}} \\
&= \left[\prod_{\ell\in\sigma^\vee} \Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}, \underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\right]^{\Gamma_{\Phi_{\mathcal{H}},\sigma}}
\end{aligned} \tag{7.1.2.3}$$

Definition 7.1.2.4. *The above composition* (*7.1.2.3*) *is called the* **Fourier-Jacobi map** *along* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$, *which we denote by* $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. *The image of an element* $f \in \mathrm{AF}(k,M)$ *has a natural expansion*

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f) = \sum_{\ell\in\sigma^\vee} \mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f),$$

where the sum can be infinite and where each $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *lies in*

$$\mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M) := \Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}, \underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M).$$

The expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *is called the* **Fourier-Jacobi expansion** *of* f *along* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$, *with* **Fourier-Jacobi coefficients** $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *of each degree* $\ell \in \sigma^\vee$.

Remark 7.1.2.5. These are generalizations of the q-expansions and Fourier-Jacobi expansions for modular, Hilbert modular, or Siegel modular forms, with which the readers might be familiar.

Remark 7.1.2.6. The name Fourier appears naturally because we are forming an infinite sum of sections according to the weights under the torus action of $E_{\Phi_{\mathcal{H}}}$. The name Jacobi appears because, in the case of Siegel modular forms, the sections of $\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ are closely related to the classical Jacobi theta functions. This remains to be true in more generality, as for example in the work of Shimura and other people.

Suppose that we have two cones σ_1 and σ_2 in $\Sigma_{\Phi_{\mathcal{H}}}$ such that $\sigma_1 \subset \overline{\sigma}_2$ and such that they are both in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$. In this case we have $\sigma_2^\vee \subset \sigma_1^\vee$, and therefore an open embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_2) \hookrightarrow \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_1)$, equivariant with respect to the torus action of $E_{\Phi_{\mathcal{H}}}$. This induces a canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2} \hookrightarrow \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1}$ such that the Mumford family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}})$ over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1}$ is pulled back to the Mumford family over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2}$. This shows that $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}$ is mapped to $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}$ under the canonical map

$$\Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1}, {}^\heartsuit\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2}, {}^\heartsuit\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M),$$

which maps $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}$ to $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}$ as long as both of them are defined. In particular, we must have $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1} = 0$ for those $\ell \in \sigma_1^\vee - \sigma_2^\vee$. That is, we only need the Fourier-Jacobi coefficients of degrees lying in the smaller of $\sigma_1^\vee$ and $\sigma_2^\vee$. As any two cones in $\Sigma_{\Phi_{\mathcal{H}}}$ that are both in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ can be related by a sequence of inclusions (of closures) of cones, we see that any of the Fourier-Jacobi maps has degrees of nonzero coefficients supported on

$$\Sigma^\vee_{\Phi_{\mathcal{H}}} := \bigcap_{\sigma \in \Sigma_{\Phi_{\mathcal{H}}}} \sigma^\vee.$$

Here it is harmless to take also those $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ that might not be in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, because they are necessarily faces of some cones lying in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$. Since $\Sigma_{\Phi_{\mathcal{H}}}$ is a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$, which means in particular that the union of the cones in $\Sigma_{\Phi_{\mathcal{H}}}$ is $\mathbf{P}_{\Phi_{\mathcal{H}}}$, we see that $\Sigma^\vee_{\Phi_{\mathcal{H}}}$ is simply $\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}$, which is independent of the choice of $\Sigma_{\Phi_{\mathcal{H}}}$. Thus we see that we may define a canonical map

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} : \mathrm{AF}(k, M) \to \mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k, M) := \prod_{\ell \in \mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} \mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k, M),,$$

which is invariant under any of the groups $\Gamma_{\Phi_{\mathcal{H}},\sigma}$. It is also invariant under the full group $\Gamma_{\Phi_{\mathcal{H}}}$, because the Mumford families over each $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of the Mumford family over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and because the latter is invariant under the action of $\Gamma_{\Phi_{\mathcal{H}}}$. Since the pullback objects are naturally invariant under $\Gamma_{\Phi_{\mathcal{H}}}$, we may redefine $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as:

$$\begin{aligned}\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}:\ \mathrm{AF}(k,M) &\to \mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)^{\Gamma_{\Phi_{\mathcal{H}}}} \\ &:= \left[\prod_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} \mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)\right]^{\Gamma_{\Phi_{\mathcal{H}}}}.\end{aligned} \tag{7.1.2.7}$$

Definition 7.1.2.8. *The above map (7.1.2.7) is called the* **Fourier-Jacobi map** *along $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, which we denote by $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. The image of an element $f\in\mathrm{AF}(k,M)$ has a natural expansion*

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f) = \sum_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} \mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f),$$

where each $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ lies in $\mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)$. The expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ is called the **Fourier-Jacobi expansion** *of f along $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, with* **Fourier-Jacobi coefficients** *$\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ of each degree $\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}$.*

Let us record the above argument as follows:

Proposition 7.1.2.9. *The Fourier-Jacobi map $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ can be computed by any $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ as in Definition 7.1.2.4. The definition is independent of the σ we use.*

Moreover:

Proposition 7.1.2.10. *The Fourier-Jacobi map $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is independent of the $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ we use.*

Proof. This is a consequence of Lemma 7.1.1.3, Proposition 6.4.2.3, and the construction of $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ using Mumford families. □

Definition 7.1.2.11. *The* **constant term** *of a Fourier-Jacobi expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ of an element $f\in\mathrm{AF}(k,M)$ is the Fourier-Jacobi coefficient*

$$\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)\in\mathrm{FJC}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)=\Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}},((\wedge^{\mathrm{top}}_{\mathbb{Z}}X)\underset{\mathbb{Z}}{\otimes}\omega_A)^{\otimes k}\underset{\mathcal{O}_{F_0,(\Box)}}{\otimes}M)$$

in degree zero.

Lemma 7.1.2.12. *For every element ℓ of the semi-subgroup $\mathbf{P}_{\Phi_{\mathcal{H}}}^{\vee}$ of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, there exists an integer $N \geq 1$ such that $N\ell$ is a finite sum $\sum\limits_{i} [y_i \otimes \phi(y_i)]$ for some elements $y_i \in Y$.*

Proof. It suffices to check that $(\mathbf{P}_{\Phi_{\mathcal{H}}}^{\vee}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is contained in the $\mathbb{Q}_{>0}$-span of elements of the form $[y \otimes \phi(y)]$ for some $y \in Y$. But this is rather equivalent to the definition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ as the cone of positive semi-definite Hermitian pairings in $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ whose radicals are admissible (and hence rational) subspaces. $\square$

Corollary 7.1.2.13. *The set $\mathbf{P}_{\Phi_{\mathcal{H}}}^{\vee} - \{0\}$ is the semi-subgroup of elements in $\mathbf{S}_{\Phi_{\mathcal{H}}}$ that pairs positively with some element in $\mathbf{P}_{\Phi_{\mathcal{H}}}$, or equivalently the semi-subgroup of elements in $\mathbf{S}_{\Phi_{\mathcal{H}}}$ that pairs positively with any element in $\mathbf{P}_{\Phi_{\mathcal{H}}}^{+}$.*

Proposition 7.1.2.14. *The value of an element $f \in \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M)$ along the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is determined by its constant term $\mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}^{(0)}(f)$. In particular, it is* **constant along the fibers** *of the structural map $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. We say in this case that it depends only on the* **abelian part** *of $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha_{\mathcal{H}}^{\mathrm{tor}})$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$.*

Proof. As described in Remark 6.2.5.10, the σ-stratum $\Xi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ of $\Xi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)$ is defined by the sheaf of ideals $\mathscr{I}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma} = \underset{\ell \in \sigma_0^{\vee}}{\oplus} \Psi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\ell)$ in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)} = \underset{\ell \in \sigma^{\vee}}{\oplus} \Psi_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\ell)$. In particular, the Fourier-Jacobi coefficient $\mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}^{(\ell)}(f)$ vanishes along $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ if ℓ lies in $\ell \in \sigma_0^{\vee}$. However, by Lemma 7.1.2.12, any of the possible nonzero degree ℓ in the Fourier-Jacobi expansion is up to a rational multiple a nontrivial finite sum of elements of the form $[y \otimes \phi(y)]$, which necessarily lies in $\sigma_0^{\vee}$ because $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}^{+}$ consists of positive definite Hermitian pairings. This shows that $\mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(f) = \mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}^{(0)}(f)$ along $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$. Since the value of f along $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ is determined in particular by its pullback to the formal completion $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}}$, it is also determined by $\mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(f)$. In other words, it is determined by its constant term $\mathrm{FJ}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}^{(0)}(f)$, as desired. $\square$

Proposition 7.1.2.15. *Let $\{\mathsf{Z}_{[(\Phi_{\mathcal{H}}^{(i)}, \delta_{\mathcal{H}}^{(i)}, \sigma^{(i)})]}\}_{i \in I}$ be a finite collection of strata of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ such that the union of members in the collection intersects all irreducible components of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. For each $i \in I$, let $(\Phi_{\mathcal{H}}^{(i)}, \delta_{\mathcal{H}}^{(i)}, \sigma^{(i)})$ be some* *representative of* $[(\Phi_{\mathcal{H}}^{(i)}, \delta_{\mathcal{H}}^{(i)}, \sigma^{(i)})]$*. Let* M *be an* $\mathcal{O}_{F_0,(\Box)}$*-module and let* $k \geq 0$ *be an integer. Let* f *be an automorphic form over* $\mathsf{M}_{\mathcal{H}}$*, of parallel weight* k*, with coefficients in* M*, and regular at infinity. Then the following are true:*

1. *If* $\mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}(f) = \sum\limits_{\ell \in \mathbf{P}_{\Phi_{\mathcal{H}}}^{\vee}} \mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}^{(\ell)}(f) = 0$ *for all* $i \in I$*, then* $f = 0$*.*

2. (**Fourier-Jacobi expansion principle**) *Suppose* M_1 *is an* $\mathcal{O}_{F_0,(\Box)}$*-submodule of* M*, and suppose the Fourier-Jacobi expansions* $\mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}(f) \in \mathrm{FJE}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}(k, M)$ *lies in* $\mathrm{FJE}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}(k, M_1)$ *for all* $i \in I$*. Then* f *is an element in* $\mathrm{AF}(k, M_1)$*.*

Proof. Note that the association of $\mathrm{AF}(k,-)$, $\mathrm{FJC}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}^{(\ell)}(k,-)$ and $\mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,-)$ are left exact because they are defined by taking global sections of sheaves.

Let us prove the first statement. Note that M is a direct limit of finitely-generated $\mathcal{O}_{F_0,(\Box)}$-modules. Since taking cohomology and tensor products commute with direct limits, we may assume that M is a finitely-generated $\mathcal{O}_{F_0,(\Box)}$-module. Since $\mathcal{O}_{F_0,(\Box)}$ is a Dedekind domain, the torsion-free quotient of M is automatically projective. Hence we may split M as a sum of a torsion submodule and a torsion-free submodule. The torsion submodule can be written as a sum of modules of the form $\mathcal{O}_{F_0,(\Box)}/\mathfrak{n}$, where $\mathfrak{n}$ is a nonzero ideal, because it is supported over a finite number of unramified primes. On the other hand, the torsion-free submodule is automatically flat over $\mathcal{O}_{F_0,(\Box)}$, and hence is a limit of its free submodules. By the same fact that taking cohomology and tensor products commute with direct limits, we may assume that it is free. In any case, we are reduced to the case that M is the sum of modules of the form $\mathcal{O}_{F_0,(\Box)}$ or $\mathcal{O}_{F_0,(\Box)}/\mathfrak{n}$, and hence (by additivity of $\mathrm{AF}(k,-)$) to the case that M is a ring, and work over the base change from $\mathcal{O}_{F_0,(\Box)}$ to M. Since $\{\mathsf{Z}_{[(\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)},\sigma^{(i)})]}\}_{i \in I}$ intersects all irreducible components of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, the (finite) direct product of Fourier-Jacobi maps $\mathrm{FJ}_I := \prod\limits_{i \in I} \mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}} = \prod\limits_{i \in I} \mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)},\sigma^{(i)}}$ is injective because each Fourier-Jacobi expansion map $\mathrm{FJ}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)},\sigma^{(i)}}$ is defined by pullback of a global section of $(\omega^{\mathrm{tor}})^{\otimes k}$ to the completion along $\mathsf{Z}_{[(\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)},\sigma^{(i)})]}$, over which ω^{tor} is trivialized.

Let us prove the second statement. For simplicity of notations, let us define $\mathrm{FJE}_I(k,-) := \prod\limits_{i \in I} \mathrm{FJE}_{\Phi_{\mathcal{H}}^{(i)},\delta_{\mathcal{H}}^{(i)}}(k,-)$. Consider the following commutative diagram, in which the two rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{AF}(k, M_1) \arrow[r] \arrow[d, hook, "\mathrm{FJ}_I"'] & \mathrm{AF}(k, M) \arrow[r] \arrow[d, hook, "\mathrm{FJ}_I"'] & \mathrm{AF}(k, M/M_1) \arrow[d, hook, "\mathrm{FJ}_I"'] & . \\
0 \arrow[r] & \mathrm{FJE}_I(k, M_1) \arrow[r] & \mathrm{FJE}_I(k, M) \arrow[r] & \mathrm{FJE}_I(k, M/M_1) &
\end{tikzcd}$$

If $f \in \mathrm{AF}(k, M)$ and $\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}}, \delta^{(i)}_{\mathcal{H}}}(f) \in \mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}}, \delta^{(i)}_{\mathcal{H}}}(k, M_1)$ for all $i \in I$, then $\mathrm{FJ}_I(f) \in \mathrm{FJE}_I(k, M)$ is sent to zero in $\mathrm{FJE}_I(k, M/M_1)$. By injectivity of the map $\mathrm{FJ}_I : \mathrm{AF}(k, M/M_1) \to \mathrm{FJE}_I(k, M/M_1)$ and the commutativity of the diagram, this means f is sent to zero in $\mathrm{AF}(k, M/M_1)$. Then we must have $f \in \mathrm{AF}(k, M_1)$, as desired. ☐

7.2 Arithmetic Minimal Compactifications

7.2.1 Positivity of Hodge Invertible Sheaves

Proposition 7.2.1.1 ([37, Ch. V, Prop. 2.1])**.** *Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ be any (smooth) arithmetic toroidal compactification of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1, with a degenerating family $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the universal tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}$. Consider the invertible sheaf $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} = \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, and consider its restriction ω to $\mathsf{M}_{\mathcal{H}}$. Then there exists an integer $N_0 \geq 1$ such that $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is generated by its relative global sections over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.*

Proof. This is a special case of [96, Ch. IX, Thm. 2.1] if we replace $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ by a normal excellent scheme. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is of finite type over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, its étale charts are excellent and normal. Hence the result follows. ☐

Proposition 7.2.1.2 ([37, Ch. V, Prop. 2.2])**.** *Let C be a proper smooth irreducible curve over an algebraically closed field k. Let $f : G \to C$ be a semi-abelian scheme over C, and let $\omega_{G/C} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G/C} = \wedge^{\mathrm{top}} e^*_G \Omega^1_{G/C}$. Suppose that $\deg(\omega_{G/C}) \leq 0$. Then:*

1. *$\omega_{G/C}$ is a torsion bundle. That is, some positive power of it is trivial. Hence $\deg(\omega_{G/C}) = 0$.*

2. *G is an extension of an isotrivial abelian scheme by a torus over C.*

Here an abelian scheme $A \to C$ is *isotrivial* if it becomes constant over a *finite étale* surjection $\tilde{C}$ of C.

Proof. First let us note that if $A \underset{C}{\times} C'$ is constant for some proper smooth curve C' over C, then A is also isotrivial. To see this, take any $n \geq 3$ that is prime to $\mathrm{char}(k)$, and take $\tilde{C} := A[n]$, which is finite étale over C. Then $A \underset{C}{\times} \tilde{C} \to \tilde{C}$ is constant because $A \underset{C}{\times} \tilde{C} \underset{C}{\times} C' \to S \underset{C}{\times} C'$ is. Hence we may assume that C is projective.

By the theory of torus parts in Section 3.3.1, in particular Proposition 3.3.1.9 and Theorem 3.3.1.11, we may write $G \to C$ as an extension of a semi-abelian scheme G' by a torus H over C, so that G'_η is an abelian scheme over the generic point η of C. By replacing G by G', it suffices to treat case when G_η is an abelian scheme.

By assumption $\deg(\omega_{G/C}^{\otimes n}) \leq 0$ for any $n \geq 0$. In other words, all global sections of $\omega_{G/C}^{\otimes n}$ are constant, in the sense that they are either zero or nowhere zero. Combining with Proposition 7.2.1.1, this shows that $\deg(\omega_{G/C}^{\otimes n}) = 0$ for any $n \geq 0$.

By [96, Ch. XI, Thm. 4.5(b), (v bis)$\Longrightarrow$(iv)] and [96, Ch. X, Prop. 4.4, (i)$\Longleftrightarrow$(iii)], which uses implicitly the fact that *theta constants determine the moduli*, we see that G_η has potentially good reduction everywhere, and that G is isotrivial over S. ☐

Remark 7.2.1.3. The proofs of both Propositions 7.2.1.1 and 7.2.1.2 involves the theory of *theta constants*, which is not entirely surprising as the theta functions can be used to describe the moduli spaces. More precisely, the upshot is to compare the Hodge invertible sheaf ω (defined over various bases) with the relative sections of a relatively ample invertible sheaf on the abelian scheme.

7.2.2 Stein Factorizations and Finite Generation

In this section, we include several standard results that we will need for our main construction in Section 7.2.3 below.

Fix a noetherian base ring R and let $S = \mathrm{Spec}(R)$. Suppose Y is a proper algebraic stack over S. Suppose $\mathcal{L}$ is an invertible sheaf such that there is an integer $N_0 \geq 1$ such that $\mathcal{L}^{\otimes N_0}$ is generated by its global sections. Then the global sections of $\mathcal{L}^{\otimes N_0}$ defines a morphism

$$f : Y \to \mathbb{P}_S^{r_0}$$

for some integer $r_0 \geq 1$. This is a proper map from an algebraic stack to a scheme, both of which are noetherian. The push-forward $f_*\mathscr{O}_Y$ is a finite $\mathscr{O}_{\mathbb{P}_S^{r_0}}$-algebra, and it determines the *Stein factorization* (see [47, III, 4.3.3])

$$f^{\mathrm{st}} : Y \to Y^{\mathrm{st}} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\mathbb{P}_S^{r_0}}}(f_*\mathscr{O}_Y)$$

of f, such that the natural map $\mathscr{O}_{Y^{\mathrm{st}}} \to (f^{\mathrm{st}})_*\mathscr{O}_Y$ is an isomorphism. In this case, the pullback of $\mathcal{O}(1)$ of $\mathbb{P}_S^{r_0}$ to f^{st} is an ample invertible sheaf, which we also denote by $\mathcal{O}(1)$ if there is no confusion, and its further pullback to Y is the original $\mathcal{L}^{\otimes N_0}$.

Lemma 7.2.2.1. *For any locally free sheaf $\mathcal{E}$ of finite rank on f^{st}, we have a canonical isomorphism $f_*^{\mathrm{st}}(f^{\mathrm{st}})^*\mathcal{E} \cong \mathcal{E}$. As a byproduct, we have $\Gamma(Y, (f^{\mathrm{st}})^*\mathcal{E}) \cong \Gamma(Y^{\mathrm{st}}, \mathcal{E})$.*

Proof. By the projection formula (see for example [64, Ch. III, Exer. 8.3]), we have $f_*^{\mathrm{st}}(f^{\mathrm{st}})^*\mathcal{E} \cong (f_*^{\mathrm{st}}\mathscr{O}_Y) \underset{\mathscr{O}_{Y^{\mathrm{st}}}}{\otimes} \mathcal{E} \cong \mathcal{E}$. Hence the result follows. $\square$

By [48, II, 4.6.3], we have an isomorphism

$$Y^{\mathrm{st}} \cong \mathrm{Proj}(\underset{k\geq 0}{\oplus} \Gamma(Y^{\mathrm{st}}, \mathcal{O}(1))).$$

By Lemma 7.2.2.1, this implies that we have

$$Y^{\mathrm{st}} \cong \mathrm{Proj}(\underset{k\geq 0}{\oplus} \Gamma(Y, \mathcal{L}^{\otimes N_0 k})).$$

It is desirable to explain the finite generation of algebras such as $\underset{k\geq 0}{\oplus} \Gamma(Y, \mathcal{L}^{\otimes N_0 k})$ over R (as they appear in the above construction of Proj). The fundamental tool is *Serre's vanishing theorem*:

Theorem 7.2.2.2 ([64, Ch. III, Thm. 5.2 and Prop. 5.3])**.** *Let Z be a proper scheme over $S = \mathrm{Spec}(R)$ where R is a noetherian ring. Then an invertible sheaf $\mathcal{M}$ is ample on Y if and only if, for any coherent sheaf $\mathcal{E}$ on Y, there is an integer k_0 (depending on $\mathcal{E}$) such that $H^i(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes k}) = 0$ for any $i > 0$ and $k \geq k_0$.*

Proposition 7.2.2.3 ([86, Exer. 1.2.22])**.** *Let Z be a projective scheme over $S = \operatorname{Spec}(R)$ where R is a noetherian ring, and let $\mathcal{M}$ be an ample invertible sheaf on Z. For any coherent sheaves $\mathcal{E}$ and $\mathcal{F}$, there is an integer k_0 such that*

$$\Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes a}) \otimes \Gamma(Z, \mathcal{F} \otimes \mathcal{M}^{\otimes b}) \to \Gamma(Z, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{M}^{\otimes (a+b)})$$

is **surjective** *for any $a, b \geq k_0$.*

Proof. Let $S := \operatorname{Spec}(R)$. Consider the diagonal embedding $\Delta : Z \to Z \underset{S}{\times} Z$ and denote its image by $\Delta(Z)$. Let $\mathscr{I}$ be the sheaf of ideals that fits into the exact sequence

$$0 \to \mathscr{I} \to \operatorname{pr}_1^* \mathcal{E} \otimes \operatorname{pr}_2^* \mathcal{F} \to \mathscr{O}_{\Delta(Z)} \otimes \operatorname{pr}_1^* \mathcal{E} \otimes \operatorname{pr}_2^* \mathcal{F} \to 0$$

over $Z \underset{S}{\times} Z$. If we tensor the whole sequence with $\operatorname{pr}_1^*(\mathcal{M}^{\otimes a}) \otimes \operatorname{pr}_2^*(\mathcal{M}^{\otimes b})$, and take cohomology over $Z \underset{S}{\times} Z$, then we obtain (by *Künneth formula*) the exact sequence

$$\begin{aligned} \Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes a}) \otimes \Gamma(Z, \mathcal{F} \otimes \mathcal{M}^{\otimes b}) &\to \Gamma(Z, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{M}^{\otimes (a+b)}) \\ &\to H^1(Z \underset{S}{\times} Z, \mathscr{I} \otimes \operatorname{pr}_1^*(\mathcal{M}^{\otimes a}) \otimes \operatorname{pr}_2^*(\mathcal{M}^{\otimes b})). \end{aligned}$$

Now the point is the vanishing of the last term. Let us denote the ample line bundle $\operatorname{pr}_1^* \mathcal{M} \otimes \operatorname{pr}_2^* \mathcal{M}$ over $Z \underset{S}{\times} Z$ by $\mathcal{O}(1)$, and its m-th tensor power by $\mathcal{O}(m)$. By [64, Ch. II, Cor. 5.18], there is a resolution

$$\ldots \to \mathcal{O}(-m_i)^{\oplus d_i} \to \ldots \to \mathcal{O}(-m_0)^{\oplus d_0} \to \mathscr{I} \otimes \operatorname{pr}_1^*(\mathcal{M}^{\otimes a}) \otimes \operatorname{pr}_2^*(\mathcal{M}^{\otimes b}) \to 0$$

on $Z \underset{S}{\times} Z$. Therefore the question is to show that there is an k_0 such that

$$H^i(Z \underset{S}{\times} Z, \operatorname{pr}_1^*(\mathcal{M}^{\otimes (a-m_{i-1})}) \otimes \operatorname{pr}_2^*(\mathcal{M}^{\otimes (b-m_{i-1})})) = 0$$

for any $i > 0$ and $a, b \geq k_0$. It suffices to verify this for $0 < i \leq \dim(Z \underset{S}{\times} Z)$, which involves only finitely many terms. Using Künneth formula again, this reduces the question of the vanishing of the various

$$H^j(Z, \mathcal{M}^{\otimes a} \otimes \mathcal{M}^{\otimes -m_{i-1}}) \otimes H^{i-j}(Z, \mathcal{M}^{\otimes b} \otimes \mathcal{M}^{\otimes -m_{i-1}})$$

for $a, b \geq M_0$. Hence the result follows from Theorem 7.2.2.2. □

Corollary 7.2.2.4 ([86, Exer. 2.1.30])**.** *Let Z be a projective scheme over $S = \mathrm{Spec}(R)$ where R is a noetherian ring, and let $\mathcal{M}$ be an ample invertible sheaf on Z. Then the algebra $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{M}^{\otimes k})$ is finitely generated over R.*

Proof. Apply Proposition 7.2.2.3 with $\mathcal{E} = \mathcal{F} = \mathscr{O}_Z$. ☐

Corollary 7.2.2.5. *Let Z be a projective scheme over $S = \mathrm{Spec}(R)$ where R is a noetherian ring, and let $\mathcal{E}$ be a coherent sheaf on Z. Then the module $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes k})$ is finitely generated over the algebra $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{M}^{\otimes k})$.*

Proof. Apply Proposition 7.2.2.3 with $\mathcal{F} = \mathscr{O}_Z$. ☐

Corollary 7.2.2.6. *Back to the context of Y and $\mathcal{L}$ above. The algebra $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{L}^{\otimes N_1 k})$ is finitely generated over R for any integer N_1.*

Proof. For the purpose of proving this corollary, we may replace N_0 by its multiple (and accordingly $f : Z \to \mathbb{P}^{r_0}_S$, f^{st}, Z^{st}, etc) and assume that it is a multiple of N_1.

Let $\mathcal{E} := f^{\mathrm{st}}_* \left(\overset{(N_0/N_1)-1}{\underset{k=0}{\oplus}} \mathcal{L}^{\otimes N_1 k} \right)$, which is a coherent sheaf on Z^{st}. Apply Corollary 7.2.2.5 to $Z := Y^{\mathrm{st}}$, $\mathcal{M} := \mathcal{O}(1)$, and $\mathcal{E}$ just defined above, we see that

$$\underset{k\geq 0}{\oplus} \Gamma(Y, \mathcal{L}^{\otimes k}) \cong \underset{k\geq 0}{\oplus} \Gamma(Y^{\mathrm{st}}, \mathcal{E} \otimes \mathcal{O}(1)^{\otimes k})$$

is a finitely generated module over

$$\underset{k\geq 0}{\oplus} \Gamma(Y, \mathcal{L}^{\otimes N_0 k}) \cong \underset{k\geq 0}{\oplus} \Gamma(Y^{\mathrm{st}}, \mathcal{O}(1)^{\otimes k}),$$

the latter being finitely generated by Corollary 7.2.2.4. Hence the result follows. ☐

7.2.3 Main Construction

With the same setting as in Section 7.1, let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ be any (smooth) arithmetic toroidal compactification of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1, with a degenerating family $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the universal tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}$. Let $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} = \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ be the invertible sheaf on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ that extends ω naturally. According to Proposition 7.2.1.1, there is an integer $N_0 \geq 1$ such that $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is generated by its global sections. Let us fix a choice of such an integer N_0.

As in Section 7.2.2, the global sections of $(\omega^{\mathrm{tor}})^{\otimes N_0}$ defines a morphism $\int_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathbb{P}_{\mathsf{S}_0}^{r_0}$ to some projective r_0-space $\mathbb{P}_{\mathsf{S}_0}^{r_0}$ for some integer $r_0 \geq 1$, together with a *Stein factorization*

$$\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\mathbb{P}_{\mathsf{S}_0}^{r_0}}}(\textstyle\int_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$$

of $\int_{\mathcal{H}}$, such that the natural map $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}} \to \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism. The induced morphism $\bar{\int}_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{min}} \to \mathbb{P}_{\mathsf{S}_0}^{r_0}$ is finite, and $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is projective and of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. In this case, we have an isomorphism

$$\mathsf{M}_{\mathcal{H}}^{\mathrm{min}} \cong \mathrm{Proj}(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes N_0 k})),$$

Note that this is independent of the choice of $N_0 \geq 1$ because we have canonical isomorphisms (by [48, II, 2.4.7])

$$\mathrm{Proj}(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes N_0 k})) \cong \mathrm{Proj}(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k}))$$

for any $N_0 \geq 1$, and because the right-hand side is independent of the choice of the cone decomposition Σ by Lemma 7.1.1.3. For psychological comfort we would like to confirm the finite generation of the algebra $\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})$ over $\mathcal{O}_{F_0,(\Box)}$. This is indeed the case by applying Corollary 7.2.2.6 to our context.

According to [47, III, 4.3.1, 4.3.3, 4.3.4], with its natural generalization to the context of algebraic stacks, we see that the first factor $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ has nonempty connected fibers over any geometric point of the target. Since $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is the pullback of $\mathcal{O}(1)$ (of $\mathbb{P}_{\mathsf{S}_0}^{r_0}$) to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, we see that the restriction of $(\omega^{\mathrm{tor}})^{\otimes N_0}$ to any fiber of $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is trivial. Therefore we may apply Proposition 7.2.1.2 to morphisms of complete curves to these fibers. Note that the geometric fibers are all connected as we have seen above. Hence it follows that the isomorphism class of the abelian part of G^{tor} is constant on each of the fibers. In particular, if a fiber meets $\mathsf{M}_{\mathcal{H}}$, then it has only one geometric point.

Lemma 7.2.3.1. *Let $f : Z_1 \to Z_2$ be a quasi-compact morphism from an algebraic stack to a scheme such that $\mathscr{O}_{Z_2} \to f_* \mathscr{O}_{Z_1}$ is an isomorphism. Suppose Z_1 is normal. Then Z_2 is also normal.*

Proof. Since $\mathscr{O}_{Z_2} \xrightarrow{\sim} f_*\mathscr{O}_{Z_1}$, the local rings of Z_2 are domains. Take $\tilde{Z}_2$ to be the normalization of Z_2. By universal property of $\tilde{Z}_2$ and the normality of Z_1, the map $f : Z_1 \to Z_2$ factors as a composition of morphisms $Z_1 \xrightarrow{\tilde{f}} \tilde{Z}_2 \xrightarrow{\bar{f}} Z_2$, corresponding to a composition of canonical injections $\mathscr{O}_{Z_2} \to \bar{f}_*\mathscr{O}_{\tilde{Z}_2} \to \bar{f}_*\tilde{f}_*\mathscr{O}_{Z_1} \cong f_*\mathscr{O}_{Z_1}$, the latter composition being an isomorphism by assumption. This forces $\mathscr{O}_{Z_2} \xrightarrow{\sim} \bar{f}_*\mathscr{O}_{\tilde{Z}_2}$, or rather $\tilde{Z}_2 \xrightarrow{\sim} Z_2$, which implies that Z_2 is normal. $\Box$

Proposition 7.2.3.2. $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *is normal.*

Proof. Note that $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is normal because it is smooth over the normal base scheme $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. (See [52, IV, 17.5.7].) Moreover, the natural map $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}} \to \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism by construction. Hence the result follows from Lemma 7.2.3.1. $\Box$

Recall (see Section A.6.4) that each algebraic stack Z has an associated *coarse moduli space* $[Z]$, which is an algebraic space together with a canonical morphism $Z \to [Z]$ such that any morphism $f : Z \to Z'$ from Z to an algebraic space Z' factors uniquely as a composition $Z \to [Z] \xrightarrow{[f]} Z'$. The formation of $[Z]$ commutes with flat base change. In particular, taking étale neighborhoods and forming completions commute with such a process. If Z is representable by an algebraic space, then its coarse moduli space is just itself. In particular, if $\mathcal{H}$ is *neat* (defined as in Definition 1.4.1.8), then the canonical morphism $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}]$ is an isomorphism.

Let us quote the following version of Zariski's main theorem:

Proposition 7.2.3.3 (Zariski's main theorem)**.** *A proper morphism of locally noetherian algebraic spaces is finite over the set of points over which the morphism has discrete fibers. Moreover, such a set of points is open with open inverse image.*

Proof. The statement for schemes can be found in [47, III, 4.4.3, 4.4.11]. A weaker statement for algebraic spaces can be found in [76, V, 4.2], whose proof also explains how to translate stronger statements for schemes into statements for algebraic spaces. $\Box$

As mentioned above, the restriction of $\oint_{\mathcal{H}}$ to $\mathsf{M}_{\mathcal{H}}$ is a morphism $\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}} : \mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ from an algebraic stack to a scheme, each of whose geometric fibers has only one single point. Since $\mathsf{M}_{\mathcal{H}}$ is open in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, and since the formation of coarse moduli spaces commutes with flat base change, we see that $[\mathsf{M}_{\mathcal{H}}]$ is an open sub-algebraic space of $[\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}]$. The morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ factors as $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \overset{[\oint_{\mathcal{H}}]}{\to} \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$, whose restriction to $\mathsf{M}_{\mathcal{H}}$ is the factorization $\mathsf{M}_{\mathcal{H}} \to [\mathsf{M}_{\mathcal{H}}] \overset{[\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}}]}{\to} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$. Applying Zariski's main theorem (Proposition 7.2.3.3) to $[\oint_{\mathcal{H}}]$, and taking into account the fact (Proposition 7.2.3.2) that $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is normal, we see that $[\oint_{\mathcal{H}}]$ is an isomorphism over an open subscheme of $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ containing the image of $[\mathsf{M}_{\mathcal{H}}]$. (We will see below that the image of $[\mathsf{M}_{\mathcal{H}}]$ is actually open, with complements given by closed subschemes, and hence $[\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}}]$ is an open immersion.)

More generally, suppose a fiber of $\oint_{\mathcal{H}}$ meets the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ be any representative of the class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. By Proposition 7.1.2.14, the restriction of any element $f \in \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})$ to $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ depends only on its constant term $\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$, which is constant along each fiber of the structural morphism $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, where $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is the moduli problem defined by the cusp label represented by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$. Applying this to those $k \geq 0$ divisible by N_0, we see that $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ factors through $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. This induces a morphism $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ from an algebraic stack to a scheme, each of whose geometric fibers has only one single point. Since the restriction $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is proper over its image, the induced map $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is necessarily also proper over its image. Combining with Zariski's main theorem (Proposition 7.2.3.3) as above and what we have, we obtain:

Lemma 7.2.3.4. *The map* $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *induced by* $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *is a finite morphism over its image, which induces a bijection on geometric points.*

Remark 7.2.3.5. The question of whether the induced map $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is an isomorphism over its image will be addressed by Corollary 7.2.3.16 below.

Remark 7.2.3.6. Do not confuse the issue mentioned in Remark 7.2.3.5 with the issues of having a finite Galois cover or finite quotient of Shimura varieties as in [107, §4.11, §6.3]. (see also the remark in [62, §1.7].) The integral models of Shimura varieties defined using adelic data as in [79] are naturally unions of several Shimura varieties (as explained in Remark 1.2.1.9), which are not necessarily spaces for variations of Hodge structures. Therefore our theory is insensitive to the above-mentioned considerations in [107].

The argument used in proving Proposition 7.1.2.9 shows that any two restrictions $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\min}$ and $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}} : \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]} \to \mathsf{M}_{\mathcal{H}}^{\min}$ have the same image when there exist representatives $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ of respectively $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ and $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ such that $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ are equivalent as in Definition 5.4.2.4 and represent the same cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})] = [(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$. Let us denote this common image by $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$. We claim that the converse is also true:

Proposition 7.2.3.7. *If* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} := \mathrm{image}(\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}})$ *and* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]} := \mathrm{image}(\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}})$ *have a nonempty intersection, then the two cusp labels* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *and* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$ *are the same.* (*In this case, we have seen above that* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$.)

Proof. Suppose there is any geometric point $\bar{x}$ in the intersection of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ and $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$. Let C be any proper smooth curve that is mapped to the fiber of $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ over $\bar{x}$. By the same argument using Proposition 7.2.1.2 as before, we see that the pullback of G^{tor} to C is globally an extension of an isotrivial abelian scheme by a torus. If we take any geometric point $\bar{z}$ of C, and take the pullback of $(G^{\mathrm{tor}},\lambda^{\mathrm{tor}},i^{\mathrm{tor}},\alpha_{\mathcal{H}}^{\mathrm{tor}})$ to the strict local ring of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ at $\bar{z}$ completed along the curve C, then we obtain a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over a base ring R_z that fits into the setting of Section 5.2.1. (The key point here is that the pullback of G^{tor} to C is globally an extension of an abelian scheme by a *split* torus.) Therefore, it makes sense to consider the degeneration datum associated to such a family, and in particular the equivalence class of the discrete data $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ associated to it. In other words, there is a locally constant association of a cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ over any such proper smooth curve C. Since the fiber of $\oint_{\mathcal{H}}$ over $\bar{x}$ is connected, we see that the associated cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ must be globally constant over the whole fiber. This forces $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})] = [(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$, as desired. ☐

Corollary 7.2.3.8. *The subschemes* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *form a stratification*

$$\mathsf{M}_{\mathcal{H}}^{\min} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \tag{7.2.3.9}$$

of $\mathsf{M}_{\mathcal{H}}^{\min}$ *by locally closed subscheme, with* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *running through a complete set of cusp labels, such that the* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *lies in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *if and only if there is a surjection* *from the cusp label* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$ *to the cusp label* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *as in Definition 5.4.2.13.*

Proof. According to statement 2 of Theorem 6.4.1.1, the closure of the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is the union of the $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$-strata $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ such that $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ is a face of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ as in Definition 6.3.2.15. Since the map $\oint_{\mathcal{H}}:\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}\to\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ is proper, we see that the closure of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is mapped to the closure of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, which is by definition the union of those $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ such that there is a surjection from $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ to $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$. By Proposition 7.2.3.7, this union is disjoint. Hence we may conclude (by induction on the incidence relations in the stratification of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$) that (7.2.3.9) is indeed a stratification of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$. □

As a byproduct, we have shown the following complement to Theorem 1.4.1.12, promised in Remark 1.4.1.14:

Corollary 7.2.3.10. *The coarse moduli space* $[\mathsf{M}_{\mathcal{H}}]$ *of* $\mathsf{M}_{\mathcal{H}}$ *is a quasi-projective scheme over* $\mathsf{S}_0=\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. *In particular,* $\mathsf{M}_{\mathcal{H}}$ *is a quasi-projective scheme when* $\mathcal{H}$ *is neat.*

Proof. The stratification shows that $[\mathsf{M}_{\mathcal{H}}]$ is an open sub-algebraic space in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$. Then the result follows from the fact that a sub-algebraic space of a projective scheme is a scheme. (See [76, II, 3.8].) □

Proposition 7.2.3.11. *The codimension of the complement of* $[\mathsf{M}_{\mathcal{H}}]$ *in* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *is at least two, unless the canonical surjection* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}\twoheadrightarrow\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *induces an isomorphism* $[\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}]\xrightarrow{\sim}\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *(for one and hence any choice of* Σ *in the construction of* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$*).*

Proof. Let us first exclude the trivial case that $\mathsf{M}_{\mathcal{H}}$ is proper over S_0. Let us take any top-dimensional stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ that is not the whole of $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$. Since there is a finite morphism from $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}]$ to $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (by Lemma 7.2.3.4), the codimension of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ is the difference between the dimensions of $\mathsf{M}_{\mathcal{H}}$ and of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, which is never greater than the codimension of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ for any $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Therefore, by statements 2, 3, and 5 of Theorem 6.4.1.1, it suffices to show that, unless we are in the case when $[\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}]\xrightarrow{\sim}\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, there exists some stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ such that the smooth structural morphism $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\twoheadrightarrow\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ has relative dimension greater than zero.

Let us suppose that all the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ has relative dimension zero over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Since any of their structural morphisms factors through $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, this forces $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ to be of relative dimension zero over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Since $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ has multi-rank of magnitude one (defined as in Definition 6.3.3.5), this is possible only when $L^{\mathsf{Z}_{\mathcal{H}}} = \{0\}$. This already forces $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ to be zero-dimensional over S_0. The next essential question is the dimension of $E_{\Phi_{\mathcal{H}}}$. Since the boundary of toroidal embeddings are relative Cartier divisors, the relative dimension of the relative Cartier divisors is zero if and only if the torus $E_{\Phi_{\mathcal{H}}}$ itself is dimension one, or equivalently if the multi-rank of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ has magnitude one. In this case, there is a unique cone decomposition for $\mathbf{P}_{\Phi_{\mathcal{H}}}^{+}$, and hence there is a unique stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. As a result, the proper morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is finite over any stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Since $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is normal, and since $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}} \xrightarrow{\sim} \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$, Zariski's main theorem (Proposition 7.2.3.3) implies that the induced morphism $[\oint_{\mathcal{H}}] : [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is an isomorphism, as desired. □

Corollary 7.2.3.12 (of the proof of Proposition 7.2.3.11)**.** *The canonical surjection* $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *can induce an isomorphism* $[\oint_{\mathcal{H}}] : [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *(for one and hence any choice of* Σ *in the construction of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$*) if and only if the Hermitian symmetric space associated to each* $\mathbb{Q}$*-simple factor of* $\mathrm{G}^{\mathrm{ad}}(\mathbb{R})$ *is one-dimensional. Explicitly, this can be achieved when each of the associated Hermitian symmetric space is isomorphic to* $\mathrm{SU}_{1,1}(\mathbb{R})/\,\mathrm{S}(\mathrm{U}_1(\mathbb{R}) \times \mathrm{U}_1(\mathbb{R})) = \mathrm{Sp}_2(\mathbb{R})/\,\mathrm{U}_1(\mathbb{R}) = \mathrm{SO}_4^*(\mathbb{R})/\,\mathrm{U}_2(\mathbb{R})$. *(The group classified as* $\mathrm{Sp}_{2g}(\mathbb{R})$ *here stands for the symplectic group of rank* g*.)*

Proof. A quick way is to take a look at the classification of irreducible Riemannian global symmetric spaces as in [66, Ch. X, §6, Table V in 2, the statements in 3, and the coincidence stated in 4(xi)]. The only cases to consider are the noncompact types A_{III}, C_I, and D_{III}. (See also [79, §1–5].) Checking the real dimension, we have $2pq = n_1(n_1+1) = n_2(n_2-1) = 2 = \dim_{\mathbb{R}} \mathbb{C}$ only when $p = q = 1 = n_1 = 1$ and $n_2 = 2$. By coincidence, the associated Hermitian symmetric spaces are all the same. □

Remark 7.2.3.13. The only other possible (noncompact) Hermitian cases are the types BD_I $(q = 2)$, E_{III}, and E_{VII}, which can have real dimension 2 only in case BD_I with $(p,q) = (1,2)$. This corresponds to the space $\mathrm{SO}_{1,2}(\mathbb{R})/(\mathrm{SO}_1(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R}))$, which is actually the same as $\mathrm{SU}_{1,1}(\mathbb{R})/\,\mathrm{S}(\mathrm{U}_1(\mathbb{R}) \times \mathrm{U}_1(\mathbb{R})) = \mathrm{Sp}_2(\mathbb{R})/\,\mathrm{U}_1(\mathbb{R}) = \mathrm{SO}_4^*(\mathbb{R})/\,\mathrm{U}_2(\mathbb{R})$. (More precisely, the exceptional cases are respectively $(\mathfrak{e}_{6(-14)}/(\mathfrak{so}_{10} + \mathbb{R}))$ and

$(\mathfrak{e}_{7(-25)}/(\mathfrak{e}_6+\mathbb{R}))$, which have real dimensions respectively 32 and 54, with ranks respectively 2 and 3. They do not fit into our consideration even if we had worked out the theory for non-PEL-type Shimura varieties.)

Proposition 7.2.3.14. *Let* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *be a cusp label, and let* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *be a representative of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$. *Let* $\bar{x}$ *be a geometric point of* $\mathsf{M}^{\min}_{\mathcal{H}}$ *over the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$*, which by abuse of notation we also identify as a geometric point of* $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}]$ *by Lemma 7.2.3.4. Let* $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$ *denote the completion of the strict localization of* $\mathsf{M}^{\min}_{\mathcal{H}}$ *at* $\bar{x}$. *Let* $([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^\wedge_{\bar{x}}$ *denote the completion of the strict localization of* $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}]$ *at* $\bar{x}$ *(as a geometric point of* $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$*), and let* $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ *denote the pullback of* $\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *under the canonical morphism* $(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. *For convenience, let us also use the notations of the various sheaves supported on* $\bar{x}$ *to denote their underlying algebras or modules. Then we have a canonical isomorphism*

$$\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}} \cong \left[\prod_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\right]^{\mathrm{Aut}(\bar{x})\times\Gamma_{\Phi_{\mathcal{H}}}} \tag{7.2.3.15}$$

of rings, which is adic if we interpret the product at the right-hand side as the completion of the elements that are finite sums with respect to ideal generated by the elements without constant terms, i.e. with trivial projection to $(\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$. *Let us denote by* $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^\wedge_{\bar{x}}$ *the completion of the strict localization of* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *at* $\bar{x}$. *Then (7.2.3.15) induces a* **structural morphism** *from* $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$ *to* $([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^\wedge_{\bar{x}}$*, whose pre-composition with the canonical morphism* $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^\wedge_{\bar{x}} \to (\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$ *defines a canonical isomorphism* $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^\wedge_{\bar{x}} \xrightarrow{\sim} ([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^\wedge_{\bar{x}}$.

Proof. By [47, III, 4.1.5], with its natural generalization to the context of algebraic stacks, the ring $\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}}$ is isomorphic to the $\mathrm{Aut}(\bar{x})$-invariants in the ring of regular functions over the completion of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the fiber of $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\min}_{\mathcal{H}}$ at $\bar{x}$. By Proposition 7.2.3.7, the inverse image $\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} := \oint_{\mathcal{H}}^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})$ of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ under $\oint_{\mathcal{H}}$ is the union

$$\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \bigcup_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$$

of those strata $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. According to statement 5 of Theorem 6.4.1.1 and Lemma 6.2.5.20, there are canonical isomorphisms

$(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, for any representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$. Therefore, the ring of regular functions over the completion of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ along the fiber of $\oint_{\mathcal{H}}$ at $\bar{x}$ is isomorphic to the *common intersection* of the rings of regular functions over the various completions of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ along the fibers of the structural morphisms $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. In other words, it is isomorphic to the common intersection of the $\Gamma_{\Phi_{\mathcal{H}},\sigma}$-invariants in the completions of $\hat{\bigoplus}_{\ell\in\sigma^\vee} \underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ along $\bar{x}$. Note that the identifications $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma'}$ for equivalent triples $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ involve the canonical actions of $\Gamma_{\Phi_{\mathcal{H}}}$ on the structure sheaves. Hence the process of taking common intersection involves also the process of taking $\Gamma_{\Phi_{\mathcal{H}}}$-invariants. This shows the existence of (7.2.3.15).

The claim that (7.2.3.15) is adic and that the composition $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \to (\mathsf{M}_{\mathcal{H}}^{\min})^{\wedge}_{\bar{x}} \to ([\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}])^{\wedge}_{\bar{x}}$ is an isomorphism follows from the fact that the support $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of each formal completion $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is defined by the vanishing of the ideal generated by terms in $\hat{\bigoplus}_{\ell\in\sigma^\vee} \underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ with trivial constant term, i.e. with trivial projection to $\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Then the result follows by taking $\mathrm{Aut}(\bar{x})$-invariants and by noting that $(\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\mathrm{Aut}(\bar{x})} \cong (\mathscr{O}_{(\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}})^{\wedge}_{\bar{x}}})^{\mathrm{Aut}(\bar{x})} \cong \mathscr{O}_{[(\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}})^{\wedge}_{\bar{x}}]} \cong \mathscr{O}_{([\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}])^{\wedge}_{\bar{x}}}$. ☐

Corollary 7.2.3.16. *The canonical finite surjection* $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *defined by* $\oint_{\mathcal{H}}$ *is an isomorphism.*

Proof. The proof of Proposition 7.2.3.14 shows that the composition of the completion $([\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}])^{\wedge}_{\bar{x}} \to (\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}}$ of the finite surjection $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ defined by $\oint_{\mathcal{H}}$ (described in Lemma 7.2.3.4) with the canonical *structural isomorphism* $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \xrightarrow{\sim} ([\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}])^{\wedge}_{\bar{x}}$ is the identity map. This forces $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to be an isomorphism as the property of being an isomorphism satisfies fpqc descent. ☐

7.2.4 Main Results

Theorem 7.2.4.1 (arithmetic minimal compactification)**.** *There exists a normal projective scheme* $\mathsf{M}_{\mathcal{H}}^{\min}$ *over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$*, such that:*

1. $\mathsf{M}_{\mathcal{H}}^{\min}$ *contains the* **coarse moduli space** $[\mathsf{M}_{\mathcal{H}}]$ *of* $\mathsf{M}_{\mathcal{H}}$ *as an open dense* **subscheme**.

2. *Let* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *be the tautological tuple over* $\mathsf{M}_{\mathcal{H}}$. *Let us define the invertible sheaf* $\omega := \omega_{G/\mathsf{M}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}_{\mathcal{H}}} = \wedge^{\mathrm{top}} e_G^* \Omega^1_{G/\mathsf{M}_{\mathcal{H}}}$ *on* $\mathsf{M}_{\mathcal{H}}$. *Then there is a smallest integer* $N_0 \geq 1$ *such that* $\omega^{\otimes N_0}$ *is the pullback of an ample invertible sheaf* $\mathcal{O}(1)$ *on* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$.

   If $\mathcal{H}$ *is neat* (*defined as in Definition 1.4.1.8*), *then* $\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} [\mathsf{M}_{\mathcal{H}}]$ *is an open subscheme of* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, *and the invertible sheaf* ω *extends to an ample invertible sheaf on* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, *which we denote as* ω^{min}. *In this case, we may take the* $\mathcal{O}(1)$ *and* N_0 *above to be respectively* ω^{min} *and* 1.

   By abuse of notation, we shall denote $\mathcal{O}(1)^{\otimes k/N_0}$ *as* $(\omega^{\mathrm{min}})^{\otimes k}$ *even if* ω^{min} *is not defined.*

3. *For any* (*smooth*) *arithmetic toroidal compactification* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *of* $\mathsf{M}_{\mathcal{H}}$ *as in Theorem 6.4.1.1, with a degenerating family* $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ *over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *extending the universal tuple* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* $\mathsf{M}_{\mathcal{H}}$, *let* $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} = \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *be the invertible sheaf on* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *that extends* ω *naturally. Then the graded algebra* $\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, (\omega^{\mathrm{tor}})^{\otimes k})$, *with its natural algebra structure induced by tensor products, is finitely generated over* $\mathcal{O}_{F_0,(\Box)}$, *and is independent of the choice of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.

   The normal projective scheme $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *is canonically isomorphic to* $\mathrm{Proj}(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, (\omega^{\mathrm{tor}})^{\otimes k}))$, *and there is a canonical morphism* $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, *determined by* ω^{min} *and the universal property of* Proj *such that* $\oint^*_{\mathcal{H}} \mathcal{O}(1) \cong (\omega^{\mathrm{tor}})^{\otimes N_0}$ *on* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, *such that* $\mathscr{O}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}} \cong \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$. *Moreover, the morphisms* $\oint_{\mathcal{H}}$ *are compatible with the natural maps between different arithmetic toroidal compactifications as in Proposition 6.4.2.3.*

   When $\mathcal{H}$ *is neat, we have* $\oint^*_{\mathcal{H}} \omega^{\mathrm{min}} \cong \omega^{\mathrm{tor}}$ *and* $\oint_{\mathcal{H},*} \omega^{\mathrm{tor}} \cong \omega^{\mathrm{min}}$.

4. $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *has a natural stratification by locally closed subschemes*

$$\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]},$$

   with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *running through a complete set of cusp labels as in Definition 5.4.2.4, such that the* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *lies in the* *closure of the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *if and only if there is a surjection from the cusp label* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$ *to the cusp label* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *as in Definition 5.4.2.13.*

Each $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *is canonically isomorphic to the coarse moduli space* $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}]$ (*which is a scheme*) *of the corresponding smooth algebraic stack* $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *representing the moduli problem defined by the PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot\,,\cdot \rangle^{\mathsf{Z}_{\mathcal{H}}})$ *associated to the cusp label* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *as described in Definition 5.4.2.6.*

Let us define the multi-rank of a stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *to be the multi-rank of the cusp label represented by* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ (*defined as in Definition 5.4.2.7*)*. The only stratum with multi-rank zero is the open stratum* $\mathsf{Z}_{[(0,0)]} \cong [\mathsf{M}_{\mathcal{H}}]$*, and those strata* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *with nonzero multi-ranks are called* **cusps**. (*This explains the name of the cusp labels.*)

5. *The restriction of* $\oint_{\mathcal{H}}$ *to the stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *is a proper smooth surjection to the stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$.

   Under the identification $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \xrightarrow{\sim} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *on the target as mentioned above, this surjection can be viewed as a quotient of a torsor under a torus* $E_{\Phi_{\mathcal{H}},\sigma}$ *over an abelian scheme* $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ (*as in the construction*) *over the algebraic stack* $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *over the coarse moduli space* $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}]$ (*which is a scheme*)*. More precisely, this torus* $E_{\Phi_{\mathcal{H}},\sigma}$ *is the quotient of the torus* $E_{\Phi_{\mathcal{H}}} := \underline{\mathrm{Hom}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbf{G}_{\mathrm{m},S})$ *corresponding to the subgroup* $\mathbf{S}_{\Phi_{\mathcal{H}},\sigma} := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle x, y\rangle = 0, \forall y \in \sigma\}$ *of* $\mathbf{S}_{\Phi_{\mathcal{H}}}$. (*See the definition of* $E_{\Phi_{\mathcal{H}}}$ *in Lemma 6.2.4.4, and definition of* σ*-stratum in Definition 6.1.2.5 and Lemma 6.1.2.6.*)

Proof. Let us take $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ to be the normal projective scheme constructed in Section 7.2.3. The first concern is whether its properties as described by the theorem depend on the toroidal compactifications we choose. It is clear from the construction that statements 1, 4, and 5 are satisfied regardless of the choices. Let us verify that this is also the case for statements 2 and 3.

Suppose Σ' dominates Σ as in Definition 6.4.2.2, and suppose the morphism $p : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ and the invertible sheaves $\omega_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ and $\omega_{\mathcal{H},\Sigma'}^{\mathrm{tor}}$ are defined as in the proof of Lemma 7.1.1.3. Let $\oint_{\mathcal{H},\Sigma} : \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ and $\oint_{\mathcal{H},\Sigma'} : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$. Then $\oint_{\mathcal{H},\Sigma'} = \oint_{\mathcal{H},\Sigma} \circ p$ and $p_*\mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}} \cong \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ implies that $\oint_{\mathcal{H},\Sigma}^* \mathcal{O}(1) \cong (\omega_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\otimes N_0}$ if and only if $\oint_{\mathcal{H},\Sigma'}^* \mathcal{O}(1) \cong (\omega_{\mathcal{H},\Sigma'}^{\mathrm{tor}})^{\otimes N_0}$. In other words, we can move freely between different choices of Σ by taking pullbacks or push-forwards. This shows that there is a choice of $\mathcal{O}(1)$ with smallest value of $N_0 \geq 1$ that works for any particular choice of Σ.

From now on, let us fix a choice of Σ and suppress it from the notations. We would like to show that ω extends to an ample invertible sheaf on $\mathsf{M}_{\mathcal{H}}^{\min}$ when $\mathcal{H}$ is *neat*. For this purpose, we need to distinguish two cases.

Suppose the natural map $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ is an isomorphism. Then we may simply take

$$\omega^{\min} := (\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}}^{\min})_* \omega^{\mathrm{tor}},$$

where ω^{tor} is defined as in statement 3.

Suppose $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ is not an isomorphism (for one and hence all $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$). Then Proposition 7.2.3.11 implies that the complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$ has codimension at least two. Since $\mathsf{M}_{\mathcal{H}}^{\min}$ is noetherian normal, this codimension statement implies that it suffices to show that

$$\omega^{\min} := (\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}_{\mathcal{H}}^{\min})_* \omega$$

is an invertible sheaf. By fpqc descent, it suffices to verify this statement over completions along the geometric points of $\mathsf{M}_{\mathcal{H}}^{\min}$.

Let $\bar{x}$ be a geometric point over some $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$, and consider any $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ that maps surjectively to $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Let $\tilde{x}_\sigma$ denote the preimage of $\bar{x}$ in $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ under the morphism $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ be any representative of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. Note that, under our assumption on $\mathcal{H}$, our choice of Σ (defined as in Definition 6.3.3.2) forces $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to act trivially on $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (by Lemma 6.2.5.20). Therefore we have $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (by statement 5 of Theorem 6.4.1.1), and we may also identify $\tilde{x}_\sigma$ with its image in $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ under this isomorphism. According to Proposition 7.2.3.14, there is a *structural morphism* $(\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}} \to (\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}})^\wedge_{\bar{x}}$ such that the natural composition

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\tilde{x}_\sigma} \cong (\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^\wedge_{\tilde{x}_\sigma} \to (\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}} \to (\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}})^\wedge_{\bar{x}}$$

agrees with the morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\tilde{x}_\sigma} \to (\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}})^\wedge_{\bar{x}}$ induced by the structural morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Over $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}$, the pullback ${}^{\heartsuit}\omega$ of ω^{tor} from $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is isomorphic to $(\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ by Lemma 7.1.2.1, which does descend to $(\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}}$, because the pullback of ω_A from $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ makes sense also there. Since the complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$ has codimension at least two, pullback of $\omega^{\min}$ has to agree with the pullback of $(\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ to $(\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}}$. In particular, it is invertible, as desired.

Since $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ satisfies $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}} \xrightarrow{\sim} \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ by construction as a Stein factorization, we see that two locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ of finite rank on $\mathsf{M}_{\mathcal{H}}^{\min}$ are isomorphic if and only if $\oint_{\mathcal{H}}^{*} \mathcal{E} \cong \oint_{\mathcal{H}}^{*} \mathcal{F}$. Indeed, for the nontrivial implication we just need $\mathcal{E} \cong \oint_{\mathcal{H},*} \oint_{\mathcal{H}}^{*} \mathcal{E} \cong \oint_{\mathcal{H},*} \oint_{\mathcal{H}}^{*} \mathcal{F} \cong \mathcal{F}$ (by Lemma 7.2.2.1). Since $\oint_{\mathcal{H}}^{*} \omega^{\min} \cong \omega^{\mathrm{tor}}$, we have $\oint_{\mathcal{H},*} \omega^{\mathrm{tor}} \cong \omega^{\min}$, and the $\mathcal{O}(1)$ above such that $\oint_{\mathcal{H}}^{*} \mathcal{O}(1) \cong (\omega^{\mathrm{tor}})^{\otimes N_0}$ has to satisfy $\mathcal{O}(1) \cong (\omega^{\min})^{\otimes N_0}$. This shows that $\omega^{\min}$ is ample and verifies statements 2 and 3 fully. □

Remark 7.2.4.2. $\mathsf{M}_{\mathcal{H}}^{\min}$ is in general not smooth over S_0.

Corollary 7.2.4.3. *Let M be a module over $\mathcal{O}_{F_0,(\square)}$, let $k \geq 0$ be an integer divisible by the smallest value of $N_0 \geq 1$ as in statement 2 of Theorem 7.2.4.1, and let us denote by $(\omega^{\min})^{\otimes k}$ the invertible sheaf $\mathcal{O}(1)^{\otimes k/N_0}$. Then natural map*

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M) \to \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M)$$

induced by the relation $\oint_{\mathcal{H}}^{*}(\omega^{\min})^{\otimes k} \cong (\omega^{\mathrm{tor}})^{\otimes k}$ in statement 3 of Theorem 7.2.4.1 is an isomorphism.

Proof. This follows immediately from another natural relation $\oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \cong \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}}$ in statement 3 of Theorem 7.2.4.1. □

Now we may pretend that automorphic forms can be defined intrinsically:

Definition 7.2.4.4 (fake reformulation of Definition 7.1.1.1)**.** *Let M be a module over $\mathcal{O}_{F_0,(\square)}$, let $k \geq 0$ be an integer divisible by the smallest value of $N_0 \geq 1$ as in statement 2 of Theorem 7.2.4.1, and let us denote by $(\omega^{\min})^{\otimes k}$ the invertible sheaf $\mathcal{O}(1)^{\otimes k/N_0}$. An* (**arithmetic**) **automorphic form** *over $\mathsf{M}_{\mathcal{H}}$, of parallel weight k, with coefficients in M, and regular at infinity, is an element of $\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M)$. For simplicity, when the context is clear, we shall simply call such an element an* **automorphic form** *of weight k.*

Proposition 7.2.4.5. *If we repeat the construction of $\mathsf{M}_{\mathcal{H}}^{\min}$ with $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$ replaced by any affine noetherian* **normal** *scheme $S \to \mathsf{S}_0$, then we obtain a normal projective scheme $\mathsf{M}_{\mathcal{H},S}^{\min}$ over S, with analogous characterizing properties as described in Theorem 7.2.4.1, together with a canonical* **finite** *morphism*

$$\mathsf{M}_{\mathcal{H},S}^{\min} \to \mathsf{M}_{\mathcal{H}}^{\min} \underset{\mathsf{S}_0}{\times} S. \tag{7.2.4.6}$$

If $S \to \mathsf{S}_0$ is flat, then it is indeed an isomorphism.

Note that, by Zariski's main theorem (Proposition 7.2.3.3), (7.2.4.6) is an isomorphism if $\mathsf{M}_{\mathcal{H}}^{\min} \underset{\mathsf{S}_0}{\times} S$ is noetherian normal.

Proof. Let us take any $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ as in the construction so that we have the canonical surjection

$$\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\min} \cong \mathrm{Proj}(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})).$$

If we repeat the construction of $\mathsf{M}_{\mathcal{H}}^{\min}$ over S, then we obtain a canonical surjection

$$\oint_{\mathcal{H},S} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \underset{\mathsf{S}_0}{\times} S \twoheadrightarrow \mathsf{M}_{\mathcal{H},S}^{\min} \cong \mathrm{Proj}(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \underset{\mathsf{S}_0}{\times} S, (\omega^{\mathrm{tor}} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \mathscr{O}_S)^{\otimes k})).$$

By the description of the projective spectra, we obtain a canonical proper morphism as in (7.2.4.6), and we know it is an isomorphism when $S \to \mathsf{S}_0$ is flat.

For each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}_{\mathcal{H}}^{\min}$, any surjection $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ that defines it factors through a canonical isomorphism $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Consider the analogous construction over S, we may decompose the above morphism as a composition

$$[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \underset{\mathsf{S}_0}{\times} S] \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],S} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{\mathsf{S}_0}{\times} S, \tag{7.2.4.7}$$

which forces the second morphism in (7.2.4.7) to be quasi-finite. Since the second morphism is necessarily the restriction of (7.2.4.6) to $\mathsf{Z}_{[\Phi_{\mathcal{H}},\delta_{\mathcal{H}}],S}$, it forces (7.2.4.6) to be a finite morphism by Zariski's main theorem (Proposition 7.2.3.3). $\square$

Remark 7.2.4.8. The proof of Proposition 7.2.4.5 suggests that the normality of $\mathsf{M}_{\mathcal{H}}^{\min} \underset{\mathsf{S}_0}{\times} S$ can be related to the question of whether arithmetic automorphic forms (of sufficiently divisible weights) defined over S are generated by the arithmetic automorphic forms defined over some affine normal scheme that is flat over S_0. In particular, when S is the spectrum of a finite field, this is asking whether arithmetic automorphic forms defined over this finite field is *liftable* to characteristic zero. It is fortunate that the question of liftability from finite fields does have a positive answer.

Corollary 7.2.4.9 (*Koecher's principle*)**.** *Let M be a module over $\mathcal{O}_{F_0,(\Box)}$, let $k \geq 0$ be an integer divisible by the smallest value of $N_0 \geq 1$ as in statement 2 of Theorem 7.2.4.1, and let us denote by $(\omega^{\min})^{\otimes k}$ the invertible sheaf $\mathcal{O}(1)^{\otimes k/N_0}$. Suppose that $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \neq \mathsf{M}_{\mathcal{H}}^{\min}$ for some* (*and hence all*) (*smooth*) *toroidal compactification $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1. Suppose there is a noetherian normal $\mathcal{O}_{F_0,(\Box)}$-algebra M_0 over which M is flat. Then the natural restriction maps*

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \to \Gamma(\mathsf{M}_{\mathcal{H}}, \omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \tag{7.2.4.10}$$

and

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \to \Gamma(\mathsf{M}_{\mathcal{H}}, \omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) \tag{7.2.4.11}$$

are bijections.

In other words, *automorphic forms of parallel weight k with coefficients in M are automatically regular at infinity* under the assumption.

Proof. Let $S := \mathrm{Spec}(M_0)$. By Corollary 7.2.4.3, the two maps (7.2.4.10) and (7.2.4.11) are essentially the same map. Moreover, we may identify the right-hand side of (7.2.4.11) with $\Gamma([\mathsf{M}_{\mathcal{H}}], (\omega^{\min}|_{[\mathsf{M}_{\mathcal{H}}]})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)$. By statement 1 of Theorem 7.2.4.1, $[\mathsf{M}_{\mathcal{H},S}]$ is embedded as a subscheme of $\mathsf{M}_{\mathcal{H},S}^{\min}$, while the (rather linear algebraic) assumption that $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \neq \mathsf{M}_{n}^{\min}$ (for one and hence all $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$) shows that the complement of $[\mathsf{M}_{\mathcal{H},S}]$ in $\mathsf{M}_{\mathcal{H},S}^{\min}$ has codimension at least two. Therefore, the noetherian normality of $\mathsf{M}_{\mathcal{H},S}^{\min}$ as asserted by Proposition 7.2.4.5 forces the bijectivity of the analogue of (7.2.4.11) with $\mathsf{M}_{\mathcal{H}}^{\min}$ replaced by $\mathsf{M}_{\mathcal{H},S}^{\min}$ when $M = M_0$, and hence when M is flat over M_0. $\Box$

Remark 7.2.4.12. It is not necessary to know if (7.2.4.6) is an isomorphism for $S = \mathrm{Spec}(M_0)$ in this proof.

7.2.5 Hecke Actions

Let us state the following analogue of Proposition 6.4.3.4 for arithmetic minimal compactifications. Since the minimal compactifications are canonical for each level, we do not need a double tower for defining the Hecke actions.

Proposition 7.2.5.1. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$*, and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$*. Then there is a canonical morphism* $[g]^{\min} : \mathsf{M}^{\min}_{\mathcal{H}'} \to \mathsf{M}^{\min}_{\mathcal{H}}$ *extending the canonical morphism* $[[g]] : [\mathsf{M}_{\mathcal{H}'}] \to [\mathsf{M}_{\mathcal{H}}]$ *induced by the canonical morphism* $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ *defined by the Hecke action of* g*, such that* $(\omega^{\min})^{\otimes k}$ *on* $\mathsf{M}^{\min}_{\mathcal{H}}$ *is pulled back to* $(\omega^{\min})^{\otimes k}$ *on* $\mathsf{M}^{\min}_{\mathcal{H}'}$ *whenever the former is defined.*

Moreover, the surjection $[g]^{\min}$ *maps the* $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ *of* $\mathsf{M}^{\min}_{\mathcal{H}'}$ *to the* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *of* $\mathsf{M}^{\min}_{\mathcal{H},\Sigma'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *of respectively* $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]$ *and* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$ *such that* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$ *is* g*-associated to* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *as in Definition 5.4.3.6.*

For any two compatible choices of admissible smooth rational polyhedral cone decomposition data $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ *for respectively* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$ *such that* Σ g*-dominates* Σ' *as in Definition 6.4.3.3, the canonical surjection* $\mathsf{M}^{\min}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ *is compatible with the surjection* $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *given by Proposition 6.4.3.4.*

Proof. Let us take any two compatible choices of admissible smooth rational polyhedral cone decomposition data $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ and $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}$ for respectively $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$ such that Σ g-dominates Σ' as in Definition 6.4.3.3. Let $\oint_{\mathcal{H}'} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}'}$ and $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ be the surjections given by statement 3 of Theorem 7.2.4.1. Let $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ be the canonical surjection given by Proposition 6.4.3.4 extending the canonical morphism $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ defined by the Hecke action of g. The composition of $[g]^{\mathrm{tor}}$ with $\oint_{\mathcal{H}}$ gives a morphism $\oint_{\mathcal{H}} \circ [g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \to \mathsf{M}^{\min}_{\mathcal{H}}$, which pulls $(\omega^{\min})^{\otimes k}$ (whenever it is defined) on $\mathsf{M}^{\min}_{\mathcal{H}}$ back to $(\omega^{\mathrm{tor}})^{\otimes k}$ on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. By the universal property stated in statement 3 of Theorem 7.2.4.1, this composition morphism factors through $\oint_{\mathcal{H}'}$, and induces a morphism $[g]^{\min} : \mathsf{M}^{\min}_{\mathcal{H}'} \to \mathsf{M}^{\min}_{\mathcal{H}}$. By the fact that the restriction of $\oint_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}'}$ is the canonical morphism $\mathsf{M}_{\mathcal{H}'} \to [\mathsf{M}_{\mathcal{H}'}]$, we see that the restriction of $[g]^{\min}$ to $[\mathsf{M}_{\mathcal{H}'}]$ is the canonical surjection $[[g]] : [\mathsf{M}_{\mathcal{H}'}] \to [\mathsf{M}_{\mathcal{H}}]$ induced by the canonical surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ defined by the Hecke action of g. Since $\mathsf{M}^{\min}_{\mathcal{H}'}$ is proper over S_0, and since $[\mathsf{M}_{\mathcal{H}}]$ is dense in $\mathsf{M}^{\min}_{\mathcal{H}}$, we see that $[g]^{\min}$ is a surjection pulling $(\omega^{\min})^{\otimes k}$ on $\mathsf{M}^{\min}_{\mathcal{H}}$ back to $(\omega^{\min})^{\otimes k}$ on $\mathsf{M}^{\min}_{\mathcal{H}'}$.

The statements about the images of the strata of $\mathsf{M}^{\min}_{\mathcal{H}'}$ under $[g]^{\min}$ follow from the corresponding statements about the images of the strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ under $[g]^{\mathrm{tor}}$. □

Corollary 7.2.5.2. *Suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$*. Then the Hecke actions of those elements in the group* $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ *on* $\mathsf{M}^\Box$ *that map* $\mathsf{M}_{\mathcal{H}'}$ *to* $\mathsf{M}_{\mathcal{H}}$ *induce maps from* $[\mathsf{M}_{\mathcal{H}'}]$ *to* $[\mathsf{M}_{\mathcal{H}}]$*, which extend naturally to maps from* $\mathsf{M}_{\mathcal{H}'}^{\mathrm{min}}$ *to* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *by the canonical morphisms defined in Proposition 7.2.5.1. The open compact subgroup* $\mathcal{H}'$ *of* $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ *acts trivially on* $\mathsf{M}_{\mathcal{H}'}^{\mathrm{min}}$*, and hence induces an action of the finite group* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)/\mathcal{H}'$ *on* $\mathsf{M}_{\mathcal{H}'}^{\mathrm{min}}$*. The canonical surjection* $[1]^{\mathrm{min}} : \mathsf{M}_{\mathcal{H}'}^{\mathrm{min}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *defined by Proposition 7.2.5.1 can be identified with the quotient of* $\mathsf{M}_{\mathcal{H}'}^{\mathrm{min}}$ *by the finite group* $\mathcal{H}/\mathcal{H}'$*.*

Proof. The existence of such an action is clear. The claim that the induced morphism $\mathsf{M}_{\mathcal{H}'}^{\mathrm{min}}/(\mathcal{H}/\mathcal{H}') \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ (with noetherian normal target) is an isomorphism follows from Zariski's main theorem (Proposition 7.2.3.3). ☐

7.3 Projectivity of Toroidal Compactifications

Assume that $\mathcal{H}$ is neat (defined as in Definition 1.4.1.8). By Corollary 7.2.3.10 (and the fact that $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ is projective), the algebraic space $\mathsf{M}_{\mathcal{H}}$ is a quasi-projective scheme. However, the arithmetic toroidal compactifications $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, which depend on choices of the admissible smooth rational polyhedral cone decomposition data $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ we take in Theorem 6.4.1.1, are not schemes in general. It is a natural question whether there exists any nice condition on Σ that guarantees the projectivity of $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$. In the complex analytic case, this question is solved by Tai in [15, Ch. IV]. With suitable reinterpretations, the same technique has a purely algebraic analogue: Assuming that Σ satisfies certain convexity condition (to be defined in Section 7.3.1), then the toroidal compactification $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ can be realized as the normalization of a blow-up of the minimal compactification $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ along certain sheaf of ideal that vanishes to some sufficiently high power along the boundary of $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$. This algebraic analogue is first proved in [22, Ch. IV] for Siegel moduli schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, then in [37, Ch. V, §5] for Siegel moduli schemes over $\mathrm{Spec}(\mathbb{Z})$. Our goal in this section is to make the statements precise and prove it in general.

7.3.1 Convexity Conditions on Cone Decompositions

The following definition follows Tai's original one [15, Ch. IV, §2] very closely:

Definition 7.3.1.1. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *be a representative of a cusp label at level* $\mathcal{H}$*, and let* $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j \in J}$ *be any* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *with respect to the integral structural given by* $\mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee}$ *in* $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$*. An* (**invariant**) **polarization function** *on* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *for the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is a* $\Gamma_{\Phi_{\mathcal{H}}}$*-invariant continuous piecewise linear function* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *such that:*

1. $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear* (*i.e. coincides with a linear function*) *on each cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}}$*.* (*In particular,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(tx) = t\mathsf{pol}_{\Phi_{\mathcal{H}}}(x)$ *for any* $x \in \mathbf{P}_{\Phi_{\mathcal{H}}}$ *and* $t \in \mathbb{R}_{\geq 0}$*.*)

2. $\mathsf{pol}_{\Phi_{\mathcal{H}}}((\mathbf{P}_{\Phi_{\mathcal{H}}} \cap \mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee}) - \{0\}) \subset \mathbb{Z}_{>0}$*.* (*In particular,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x) > 0$ *for any nonzero* x *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$*.*)

3. *Any rational polyhedral cone* σ *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *on which* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear* (*in the above sense*) *is contained in some cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}$*.*

4. *For any* $x, y \in \mathbf{S}_{\Phi_{\mathcal{H}}}$*, we have* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x+y) \geq \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) + \mathsf{pol}_{\Phi_{\mathcal{H}}}(y)$*. This is called the* **convexity** *of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$*.* (*These functions are rather called* **concave** *in the context of calculus or elementary mathematics.*)

If such a polarization function exists, then we say that the $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is* **projective**.

Following [15, Ch. II], with minor error pointed out by Looijenga, as in [37, Ch. V, §5] (cf. similar remark in Section 6.2.5), we may summarize the information we need as follows:

Proposition 7.3.1.2.

1. *Given any* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$*, there exists refinements* $\Sigma'_{\Phi_{\mathcal{H}}}$ *of* $\Sigma_{\Phi_{\mathcal{H}}}$ *that are either projective, smooth, or both projective and smooth.*

2. *Let* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *be a polarization function of a* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$*. Let us denote by* $\overline{\mathbf{P}}_{\Phi_{\mathcal{H}}}$ *the closure of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *in* $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$*. Let*

$$K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} := \{x \in \mathbf{P}_{\Phi_{\mathcal{H}}} : \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) \geq 1\}.$$

This is a convex subset of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *not containing* $\{0\}$ *such that* $\mathbb{R}_{\geq 1} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *and* $\mathbb{R}_{\geq 0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \supset \overline{\mathbf{P}}_{\Phi_{\mathcal{H}}}$*, whose closure* $\overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *in* $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ *is a* **cocore** *in the context of [15, Ch. II, §5].*

For simplicity, we shall also call $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *a cocore in the following text.*

3. *The* **dual** *of* $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *is defined as*

$$\begin{aligned} K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} &:= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1, \forall y \in K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\} \\ &= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1, \forall y \in \overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}. \end{aligned}$$

   This is a convex subset in $((\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R})^{\circ}$, *the interior of* $(\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, *such that* $\mathbb{R}_{\geq 1} \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *and* $\mathbb{R}_{>0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = ((\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R})^{\circ}$, *which is a* **core** *in the context of [15, Ch. II, §5].*

4. *The top-dimensional cones* σ*'s in the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *in statement 2 correspond bijectively to the vertices* ℓ *of the core* $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, *which are linear forms whose restrictions to each* σ *coincide with the restriction of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *to* σ.

5. *Suppose we have a surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *as in Definition 6.2.6.4, and suppose* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *is a polarization function for* $\Sigma_{\Phi_{\mathcal{H}}}$. *By definition of a surjection, there is a surjection* $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (*defined as in Definition 5.4.2.12*) *that induces an embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ *such that the restriction* $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ *of the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *to* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *is the cone decomposition* $\Sigma_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. *Then the restriction of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *to* $\Sigma_{\Phi'_{\mathcal{H}}}$ (*via any choice of* (s_X, S_Y)) *is an* (*invariant*) *polarization function for* $\Sigma_{\Phi'_{\mathcal{H}}}$.

Definition 7.3.1.3. *We say that a compatible choice* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ *of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$ (*defined as in Definition 6.3.3.2*) *is* **projective** *if it satisfies the following condition: There is a collection* $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}\}$ *of polarization functions labeled by representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of cusp labels, each* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *being a polarization function of the cone decompositions* $\Sigma_{\Phi_{\mathcal{H}}}$ *in* Σ (*defined as in Definition 7.3.1.1*), *which are* **compatible** *in the following sense: For any surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *of representatives of cusp labels* (*defined as in Definition 5.4.2.12*) *inducing an embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$, *we have the identification* $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$.

Remark 7.3.1.4. For any two representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, the condition $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$ in Definition 7.3.1.3 is independent of the surjection

$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ we take (as long as it exists), because pol and $\mathsf{pol}_{\Phi'_{\mathcal{H}}}$ are invariant under respectively $\Gamma_{\Phi_{\mathcal{H}}}$ and $\Gamma_{\Phi'_{\mathcal{H}}}$.

Proposition 7.3.1.5. *There exists a compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ (defined as in Definition 6.3.3.2) that is* **projective** *in the sense of Definition 7.3.1.3.*

Remark 7.3.1.6. As in Remark 6.3.3.4, this is a combinatorial question unrelated to the question of compactifying integral models at all. It is already needed in the existing works on complex analytic or rational models of toroidal compactifications.

Proof of Proposition 7.3.1.5. Following exactly same steps in the proof of Proposition 6.3.3.3, we simply have to impose projectivity on all the cone decompositions that we construct. ☐

Let us now fix a choice of a polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ for some projective cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$. Let us quote the following useful combinatorial results from [37, Ch. V, §5]:

Lemma 7.3.1.7 (cf. [37, Ch. V, Lem. 5.3])**.** *For any open compact subgroup $\mathcal{H}$ of $\mathcal{U}^{\square}(n)$, there is an open compact subgroup $\mathcal{H}' \subset \mathcal{H}$ (which can be taken to be normal) such that for any vertex ℓ_0 of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional cone σ_0, we have*

$$\langle \ell_0, x\rangle < \langle \gamma \cdot \ell_0, x\rangle$$

for any $x \in \overline{\sigma}_0 \cap \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ and any $\gamma \in \Gamma_{\Phi_{\mathcal{H}'}}$, where $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ is any lifting of $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ to level $\mathcal{H}'$ in its natural sense.

Lemma 7.3.1.8 (cf. [37, Ch. V, Lem. 5.5])**.** *Let σ_0 be a (not necessarily top-dimensional) cone in $\Sigma_{\Phi_{\mathcal{H}}}$, let $\sigma_1, \ldots, \sigma_k$ be the top-dimensional cones in $\Sigma_{\Phi_{\mathcal{H}}}$ containing σ_0, and let $\sigma_{k+1}, \ldots, \sigma_l$ be the top-dimensional cones in $\Sigma_{\Phi_{\mathcal{H}}}$ other than $\sigma_1, \ldots, \sigma_k$ that have a common positive-dimensional face with any of $\sigma_1, \ldots, \sigma_k$. Let $\ell_0, \ell_1, \ldots, \ell_m$ be the vertices of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to $\sigma_0, \sigma_1, \ldots, \sigma_m$. Then the $\mathbb{R}_{\geq 0}$-span of the dual cone $\sigma_0^{\vee}$ of σ_0 can be described by*

$$\mathbb{R}_{\geq 0} \cdot \sigma_0^{\vee} = \sum_{1 \leq i,j \leq k} \mathbb{R}(\ell_j - \ell_i) + \sum_{1 \leq i \leq k < j \leq l} \mathbb{R}_{\geq 0}(\ell_j - \ell_i).$$

Remark 7.3.1.9. The integral version of Lemma 7.3.1.8 is not true in general. We cannot replace $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$) by $\mathbb{Z}$ (resp. $\mathbb{Z}_{\geq 0}$) in its statements. This difference is immaterial because we are taking normalizations later in the proof of projectivity of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. (Indeed, it is the main reason that we have to take normalizations.)

7.3.2 Generalities on Normalizations of Blow-Ups

Definition 7.3.2.1. *Let W be any noetherian scheme, and let $\mathcal{I}$ be any coherent sheaf of ideals on W. Then we denote by $\mathrm{Bl}_{\mathcal{I}}(W)$ the* **blow-up** *of W along $\mathcal{I}$, and we denote by $\mathrm{NBl}_{\mathcal{I}}(W)$ the* **normalization** *of $\mathrm{Bl}_{\mathcal{I}}(W)$.*

Definition 7.3.2.2. *Let W be any noetherian scheme, let $\mathcal{I}$ be any coherent sheaf of ideals on W, and let $f : \tilde{W} \to W$ be any morphism from a noetherian scheme $\tilde{W}$ such that $f^*\mathcal{I}$ is an invertible sheaf. Then we denote by*

$$\mathrm{Bl}_{\mathcal{I}}(f) : \tilde{W} \to \mathrm{Bl}_{\mathcal{I}}(W)$$

the canonical morphism induced by the universal property of $\mathrm{Bl}_{\mathcal{I}}(W)$, so that f is the composition of $\mathrm{Bl}_{\mathcal{I}}(f)$ with the structural morphism $\mathrm{Bl}_{\mathcal{I}}(W) \to W$.

If moreover $\tilde{W}$ is a normal scheme, then we denote by

$$\mathrm{NBl}_{\mathcal{I}}(f) : \tilde{W} \to \mathrm{NBl}_{\mathcal{I}}(W)$$

the canonical morphism induced by the universal property of $\mathrm{NBl}_{\mathcal{I}}(W)$, so that f is the composition of $\mathrm{NBl}_{\mathcal{I}}(f)$ with the structural morphism $\mathrm{NBl}_{\mathcal{I}}(W) \to W$.

Let us quote the following useful result concerning blow-ups from [37, Ch. V, §5]:

Proposition 7.3.2.3 ([37, Ch. V, Prop. 5.13])**.** *Suppose we have a commutative diagram*

$$\begin{tikzcd}
\tilde{W}_1 \arrow[r, "\tilde{g}"] \arrow[d, "f_1"'] & \tilde{W}_2 \arrow[d, "f_2"] \\
W_1 \arrow[r, "g"'] & W_2
\end{tikzcd}$$

of noetherian normal integral schemes such that f_1 and f_2 are proper, and such that the canonical morphisms $W_1 \to f_{1,*}\tilde{W}_1$ and $W_2 \to f_{2,*}\tilde{W}_2$ are *isomorphisms. Suppose that there is a finite group* H *acting on* $\tilde{W}_1$ *and* W_1, *which is equivariant with respect to* f_1, *and suppose that* $\tilde{g}$ *and* g *can be identified with the quotient maps by* H. *Suppose that* $\imath_1$ *and* $\imath_2$ *are invertible sheaves of ideals on respectively* $\tilde{W}_1$ *and* $\tilde{W}_2$ *such that* $\imath_1 := \tilde{g}^*\imath_2$. *For any integer* $d \geq 1$, *set* $\mathcal{I}_1 := f_{1,*}\imath_1$ *and* $\mathcal{I}_2^{(d)} := f_{2,*}\imath^{\otimes d}$. *Then* $\mathcal{I}_1$ *and* $\mathcal{I}_2^{(d)}$ *are coherent sheaf of ideals on respectively* W_1 *and* W_2.

Suppose $\imath_2^{\otimes d} \xrightarrow{\sim} (\tilde{g}_*\tilde{g}^*\imath_2^{\otimes d})^H$ *for any integer* $d \geq 1$ (*which is automatic when* $\tilde{g}$ *is flat*). *Suppose that* $\imath_1 \xrightarrow{\sim} f_1^*\mathcal{I}_1$, *and that the induced canonical morphism* $\mathrm{NBl}_{\mathcal{I}_1}(f_1) : \tilde{W}_1 \to \mathrm{NBl}_{\mathcal{I}_1}(W_1)$ (*defined as in Definition 7.3.2.2*) *is an isomorphism. Then, for some integer* $d_0 \geq 1$, *we have* $\imath_2^{\otimes k} \xrightarrow{\sim} f_2^*\mathcal{I}_2^{(d_0)}$, *and the induced canonical morphism* $\mathrm{NBl}_{\mathcal{I}_2^{(d_0)}}(f_2) : \tilde{W}_2 \to \mathrm{NBl}_{\mathcal{I}_2^{(d_0)}}(W_2)$ *is an isomorphism.*

7.3.3 Main Result

Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ be any *projective* compatible choice of smooth rational polyhedral cone decomposition data, with a compatible collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}$ of polarization functions as in Definition 7.3.1.3.

Definition 7.3.3.1. *Let us maintain the setting of* Σ, $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}$, *and* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ *as above. According to statement 3 of Theorem 6.4.1.1, the complement* $\mathsf{D}_{\infty,\mathcal{H}}$ *of* $\mathsf{M}_{\mathcal{H}}$ *in* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ (*with its reduced structure*) *is a relative Cartier divisor with normal crossings, which has irreducible components of the form* $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, *each being the closure of some strata* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *labeled by the equivalence class* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *of some triple* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *with* σ *a one-dimensional cone in the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$. *Let* $\jmath_{\mathcal{H},\mathsf{pol}}$ *be the invertible sheaf of ideals on* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *supported on* $\mathsf{D}_{\infty,\mathcal{H}}$ *defined so that the order of* $\jmath_{\mathcal{H},\mathsf{pol}}$ *along any component* $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *of* $\mathsf{D}_{\infty,\mathcal{H}}$ *is the value of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *at the* $\mathbb{Z}_{\geq 0}$*-generator of* $\sigma \cap \mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee}$ *for any choice of representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$. *This is well-defined because of the compatibility condition on* $\{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}$ *as in Definition 7.3.1.3.*

Definition 7.3.3.2. *For any* $d \geq 1$, *let* $\mathcal{J}_{\mathcal{H},\mathsf{pol}}^{(d)} := \oint_{\mathcal{H},*}(\jmath_{\mathcal{H},\mathsf{pol}}^{\otimes d})$, *where* $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *is the canonical morphism* (*as described in statement 3 of Theorem 7.2.4.1*). (*Note that* $\mathcal{J}_{\mathcal{H},\mathsf{pol}}^{(d)}$ *is a coherent sheaf of ideals on* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *because the canonical morphism* $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}} \to \oint_{\mathcal{H},*}\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ *is an isomorphism.*)

Let us introduce the following condition for Σ (cf. Lemma 7.3.1.7):

Condition 7.3.3.3. *For any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of cusp label, any vertex* ℓ_0 *of* $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *corresponding to a top-dimensional cone* σ_0*, we have*

$$\langle \ell_0, x \rangle < \langle \gamma \cdot \ell_0, x \rangle$$

for any $x \in \overline{\sigma}_0 \cap \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ *and any* $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$*.*

Let us now statement the main result of this section. Note that the running assumption on $\mathcal{H}$ in this section is indispensable because we need $\mathsf{M}_{\mathcal{H}}$ to be a scheme before we investigate whether its compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ could be a scheme for some choice of Σ.

Theorem 7.3.3.4. *Suppose* Σ *is chosen to be projective with a compatible collection* pol *of polarization functions as in Definition 7.3.1.3, suppose* $\jmath_{\mathcal{H},\mathsf{pol}}$ *is defined on* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *as in Definition 7.3.3.1, and suppose* $\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ *is defined on* $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ *as in Definition 7.3.3.2 for any integer* $d \geq 1$*. Then there exists an integer* $d_0 \geq 1$ *such that the following are true:*

1. *The canonical morphism* $\jmath^{\otimes d_0}_{\mathcal{H},\mathsf{pol}} \to \oint^*_{\mathcal{H}} \mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}$ *is an isomorphism, which induces a canonical proper dominant morphism*

$$\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}}) : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})$$

   by the universal property of the normalization of blow-up as in Definition 7.3.2.2.

2. *The canonical morphism* $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ *above is an isomorphism.*

If Condition 7.3.3.3 is satisfied, then the above two statements are true for any $d_0 \geq 3$*.*

The proof can be divided into three rather independent parts. The first is the following reduction step:

Reduction to the case where Condition 7.3.3.3 is satisfied. By Lemma 7.3.1.7, there exists an open compact subgroup $\mathcal{H}' \subset \mathcal{H}$ such that Condition 7.3.3.3 is satisfied by the cone decomposition $\Sigma^{(\mathcal{H}')} = \{\Sigma_{\Phi_{\mathcal{H}'}}\}$ and the compatible collection $\mathsf{pol}^{(\mathcal{H}')} = \{\mathsf{pol}_{\Phi_{\mathcal{H}'}}\}$ of polarization functions for $\mathsf{M}_{\mathcal{H}'}$ defined as follows: For any representative $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ of cusp labels at level $\mathcal{H}'$ whose $\mathcal{H}$-orbit determines a representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$ in its natural sense (by Convention 5.3.1.13), we have a canonical isomorphism $\mathbf{P}_{\Phi_{\mathcal{H}'}} = \mathbf{P}_{\Phi_{\mathcal{H}}}$ because their definition only involves the objects X and Y that are independent of levels. Then we define $\Sigma^{(\mathcal{H}')}$ (resp. $\mathsf{pol}^{(\mathcal{H}')}$) by taking the cone decomposition $\Sigma_{\Phi_{\mathcal{H}'}}$ (resp. polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}'}}$) labeled by $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ to be the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}}}$) labeled by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$.

By construction, the surjections $\Xi_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}}(\sigma) \twoheadrightarrow \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ are finite flat (with possible ramification along the boundary strata) whenever $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is induced by $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ and σ is a cone in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}'}} = \Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}'}} = \mathbf{P}_{\Phi_{\mathcal{H}}}$. Therefore, the canonical surjection $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (given by Proposition 6.4.2.9) is finite flat. It is the unique finite flat extension of the canonical (finite étale) surjection $\mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$. If we now assume that $\mathcal{H}'$ is normal in $\mathcal{H}$, then by explicit construction we can identify it as a quotient by the finite group $\mathcal{H}/\mathcal{H}'$. Moreover, we know that $\jmath_{\mathcal{H}',\mathsf{pol}^{(\mathcal{H}')}} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{*}\jmath_{\mathcal{H},\mathsf{pol}}$ by construction. Hence the assumptions of Proposition 7.3.2.3 are all satisfied.

Since Condition 7.3.3.3 is satisfied by $\Sigma^{(\mathcal{H}')}$ and $\mathsf{pol}^{(\mathcal{H}')}$ for $\mathsf{M}_{\mathcal{H}'}$, we see that statements 1 and 2 for $\mathsf{M}_{\mathcal{H}}$ are true for some (unknown) integer $d_0 \geq 1$ if the corresponding statements are true for $\mathsf{M}_{\mathcal{H}'}$ for any (particular) $d_0' \geq 1$. For simplicity, let us change the notations by replacing $\mathsf{M}_{\mathcal{H}'}$, $\Sigma^{(\mathcal{H}')}$, $\mathsf{pol}^{(\mathcal{H}')}$, and d_0' by respectively $\mathsf{M}_{\mathcal{H}}$, Σ, pol, and d_0. Then, to prove Theorem 7.3.3.4, it suffices to prove statements 1 and 2 for any $d_0 \geq 3$ when Condition 7.3.3.3 is satisfied by Σ and pol for $\mathsf{M}_{\mathcal{H}}$. □

Now let us prove statements 1 and 2 separately under the assumption that Condition 7.3.3.3 is satisfied.

Proof of statement 1 of Theorem 7.3.3.4. Assume that Condition 7.3.3.3 holds. To verify that $\jmath_{\mathcal{H},\mathsf{pol}} \to \oint_{\mathcal{H}}^{*} \mathcal{J}_{\mathcal{H},\mathsf{pol}}$ is an isomorphism, it suffices to verify the same statement along the completions of strict localizations at geometric points of $\mathsf{M}^{\min}_{\mathcal{H}}$. Let $\bar{x}$ be a geometric point along the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. According to Corollary 7.2.3.16, we may identify $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ with $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. According to Proposition 7.2.3.14, we have a canonical isomorphism $\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}} \cong \left[\prod_{\ell \in \mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}} \right]^{\Gamma_{\Phi_{\mathcal{H}}}}$ given by (7.2.3.15), where $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is any representative of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$. As in the proof of Proposition 7.2.3.14, the map (7.2.3.15) is obtained by taking the *common intersection* of the rings of regular functions over the various completions of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ along the fibers of the structural morphisms $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ over $\bar{x}$. The structural sheaf of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ can be written symbolically as $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} = \underset{\ell\in\sigma^\vee}{\hat{\bigoplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ (as an $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-sheaf of algebras), and the global sections of its completion along (the fiber over) $\bar{x}$ is isomorphic to $\underset{\ell\in\sigma^\vee}{\hat{\bigoplus}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ (as an $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}}$-sheaf of algebras).

Let $d \geq 1$ be any integer. Let us first identify the pullback $(\mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ of $\mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ to $(\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}}$. For each one-dimensional cone $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$, let s_σ be a $\mathbb{Z}_{\geq 0}$-generator of $\sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$. By definition, the order of $\jmath_{\mathcal{H},\mathsf{pol}}$ along the σ-stratum of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is given by the value of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ at s_σ. By definition,

$$\begin{aligned}\sigma_0^\vee \cap \mathbf{S}_{\Phi_{\mathcal{H}}} &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle > 0, \forall y \in \sigma\}\\ &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle \geq 1, \forall y \in \sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}\}\\ &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, s_\sigma\rangle \geq 1\}.\end{aligned}$$

Therefore, in $\underset{\ell\in\sigma^\vee}{\hat{\bigoplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, the sheaf of ideals defining the σ-stratum consists of elements whose nonzero terms are supported on those ℓ such that $\langle \ell, s_\sigma\rangle \geq 1$, and hence the pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ consists of elements whose nonzero terms are supported on those ℓ such that $\langle \ell, s_\sigma\rangle \geq d \cdot \mathsf{pol}_{\Phi_{\mathcal{H}}}(s_\sigma)$, or equivalently such that $\langle \ell, t_\sigma\rangle \geq d$ for $t_\sigma := (\mathsf{pol}_{\Phi_{\mathcal{H}}}(s_\sigma))^{-1} s_\sigma$. Note that t_σ is the unique boundary of the half-line $\sigma \cap K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ by definition of $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, and we have

$$\begin{aligned}K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} &= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1, \forall y \in K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}\\ &= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, t_\sigma\rangle \geq 1, \forall t_\sigma\},\end{aligned}$$

the first equality being the definition, and the second equality being true because the faces of the boundary of $\overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ are spanned by the t_σ's. Therefore, for each particular $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, the condition $\langle \ell, t_\sigma\rangle \geq d$ for all t_σ is equivalent to the condition that $\ell \in d \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. By *taking common intersections* of global sections over the completion along fibers over $\bar{x}$, we see that the sheaf of ideals $(\mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}} \subset \mathscr{O}_{(\mathsf{M}_{\mathcal{H}}^{\min})^\wedge_{\bar{x}}} \cong \left[\prod_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\right]^{\Gamma_{\Phi_{\mathcal{H}}}}$ consists of elements whose nonzero terms are supported on those $\ell \in d \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Now let us investigate the pullback of the canonical morphism $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}} \to \oint^*_{\mathcal{H}} \mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ under $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathsf{M}^{\min}_{\mathcal{H}}$. The goal is to show that the pullback is an isomorphism for any $d \geq 3$.

Since the strata corresponding to top-dimensional cones meet all the irreducible components, it suffices to show that the morphism is an isomorphism after pullback to the completion of any stratum corresponding to top-dimensional cones. Let σ_0 be any top-dimension cone in $\Sigma_{\Phi_{\mathcal{H}}}$, which corresponds to a vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (by statement 4 of Proposition 7.3.1.2). Then $(\sigma_0)^\perp = \{0\}$, $(\sigma_0)^\vee_0 = (\sigma_0)^\vee - \{0\}$, and hence the ideal of definition of $\hat{\bigoplus}_{\ell\in(\sigma_0)^\vee} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, name the sheaf of ideals defining the σ_0-stratum, consists of elements whose nonzero terms are supported on those nonzero ℓ in $(\sigma_0)^\vee$. By construction, the pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ consists of elements in $\hat{\bigoplus}_{\ell\in(\sigma_0)^\vee} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ whose nonzero terms are supported on those ℓ lying in $d\cdot\ell_0 + (\sigma_0)^\vee$, the translation of $(\sigma_0)^\vee$ by $d\cdot\ell_0$. In other words, it is the sheaf of invertible ideals generated by $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0)$.

Since ℓ_0 is dual to the top-dimensional cone σ_0 in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, the invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$ is relatively ample over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Since $d \geq 3$, by Lefschetz's theorem (see for example [99, §17, Thm., p. 163]), the sections of the invertible sheaf $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0))^\wedge_{\bar{x}} \cong (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)^{\otimes d})^\wedge_{\bar{x}}$ is generated by its global sections over $(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$, namely the sections of $(\underline{\mathrm{FJ}}^{(d\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}} = (\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0))^\wedge_{\bar{x}}$. In particular, the canonical morphism $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0))^\wedge_{\bar{x}} \to (\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*(\underline{\mathrm{FJ}}^{(d\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ is an isomorphism.

Let us write any section f of $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ as an infinite sum

$$f = \sum_{\ell\in d\cdot\ell_0+(\sigma_0)^\vee} f^{(\ell)},$$

where each $f^{(\ell)}$ is a section of $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$. Since f is $\Gamma_{\Phi_{\mathcal{H}}}$-invariant, we may decompose it into an infinite sum

$$f = \sum_{[\ell]\in(\ell_0+(\sigma_0)^\vee)/\Gamma_{\Phi_{\mathcal{H}}}} f^{[\ell]}$$

of subseries $f^{[\ell]} = \sum_{\ell\in[\ell]} f^{(\ell)}$, where each $[\ell]$ is by definition the orbit of some $\ell \in d\cdot\ell_0 + (\sigma_0)^\vee$.

Since the ideal of definition of the structure sheaf of $\hat{\bigoplus}_{\ell\in(\sigma_0)^\vee}(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$ consists of elements whose nonzero terms are supported on those nonzero ℓ in $(\sigma_0)^\vee$, we see that $f^{[d\cdot\ell_0]} = \sum_{\gamma\in\Gamma_{\Phi_{\mathcal{H}}}} f^{(\gamma\cdot(d\cdot\ell_0))}$ is a *leading subseries* of f in the sense that the sheaf of ideals generated by f and by $f^{[d\cdot\ell_0]}$ over $(\mathfrak{X}_{\Phi_{\mathcal{H}},\mathcal{H}}(\sigma_0))^\wedge_{\bar{z}}$ are the same. If Condition 7.3.3.3 is satisfied, then we have the stronger statement that $f^{(d\cdot\ell_0)}$ is a *leading term* of $f^{[d\cdot\ell_0]}$ in the sense that the sheaf of ideals generated by $f^{[d\cdot\ell_0]}$ and by $f^{(d\cdot\ell_0)}$ over $(\mathfrak{X}_{\Phi_{\mathcal{H}},\mathcal{H}}(\sigma_0))^\wedge_{\bar{x}}$ are the same. As a result, if $f_1,\ldots,f_k$ are any finite number of sections of $(\mathcal{J}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ generating $(\mathcal{J}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$, then the pullback of $(\mathcal{J}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ to $(\mathfrak{X}_{\Phi_{\mathcal{H}},\mathcal{H}}(\sigma_0))^\wedge_{\bar{x}}$ is generated by the leading terms $f_1^{(d\cdot\ell_0)},\ldots f_k^{(d\cdot\ell_0)}$ of respective $f_1,\ldots,f_k$, which are sections of the sheaf of invertible ideals $(\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*(\underline{\mathrm{FJ}}^{(d\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}} \cong (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0))^\wedge_{\bar{x}}$.

Comparing with the above description of the pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, we see that the pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}} \to \oint^*_{\mathcal{H}}\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ under $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathsf{M}^{\min}_{\mathcal{H}}$ is an isomorphism when $d\geq 3$. Since $\bar{x}$ is arbitrary, this proves statement 1 by taking any integer $d_0 \geq 3$. ☐

Proof of statement 2 of Theorem 7.3.3.4. Let $d_0 \geq 3$ be any integer such that statement 1 is satisfied. In this case, there is a canonical proper dominant morphism $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}}) : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}^{\min}_{\mathcal{H}})$, and our goal is to show that this morphism $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ is an isomorphism. As in the proof of statement 1, we may verify this by pulling back to the completion of the strict localizations of $\mathsf{M}^{\min}_{\mathcal{H}}$ at its various geometric points. Let us assume the same setting as in the proof of statement 1, with $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ a representative of a cusp label and $\bar{x}$ a geometric point on the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum of $\mathsf{M}^{\min}_{\mathcal{H}}$.

Then we have a composition of canonical proper dominant morphisms

$$(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \overset{\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})}{\to} (\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}^{\min}_{\mathcal{H}}))^\wedge_{\bar{x}} \overset{\text{can.}}{\to} \mathrm{NBl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}), \quad (7.3.3.5)$$

where each of the $(\,\cdot\,)^\wedge_{\bar{x}}$ stands for the pullback under $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathsf{M}^{\min}_{\mathcal{H}}$. If we can show that (7.3.3.5) is an isomorphism, then $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ will be forced to be an isomorphism. By Zariski's main theorem (Proposition 7.2.3.3), it suffices to show that there is a canonical morphism $\mathrm{NBl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}) \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$ whose pre-composition with (7.3.3.5) is the identity map on $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$.

As we have seen in the proof of statement 1, the sheaf of ideals $(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}} \subset \mathscr{O}_{(\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}}} \cong \left[\prod\limits_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\right]^{\Gamma_{\Phi_{\mathcal{H}}}}$ consists of elements whose nonzero terms are supported on those $\ell \in d_0\cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. For any vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional cone $\sigma_0 \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$ (by statement 4 of Proposition 7.3.1.2), the relative ampleness of the invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0\cdot\ell_0)$ on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ implies that there is a canonical isomorphism

$$\begin{aligned}(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}} &\cong \underline{\mathrm{Proj}}_{\mathscr{O}_{(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}}}\big(\mathop{\oplus}_{k\geq 0}((\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0\cdot\ell_0)))^\wedge_{\bar{x}}\big)\\ &= \underline{\mathrm{Proj}}_{\mathscr{O}_{(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}}}\big(\mathop{\oplus}_{k\geq 0}(\underline{\mathrm{FJ}}^{(k\cdot d_0\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\big).\end{aligned}$$

Since the graded algebra $\mathrm{Sym}((\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}) = \mathop{\oplus}\limits_{k\geq 0}\mathrm{Sym}^k((\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}})$ (which defines the blow-up $\mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$) contains $\mathop{\oplus}\limits_{k\geq 0}(\underline{\mathrm{FJ}}^{(k\cdot d_0\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ as a graded subalgebra, we see that there is a dominant morphism $\mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}}) \to (C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ such that its pre-composition with the canonical morphisms $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_0})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$ is the canonical morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_0})^\wedge_{\bar{x}} \to (C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$. By relating neighboring top-dimension cones by their faces, and hence relating any two cones in the cone decomposition by a sequence of mutual neighboring top-dimensional cones, we see that the morphism $\mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}}) \to (C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ is independent of the choice of σ_0, and such that its pre-composition with $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$ is the canonical morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to (C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ for *any* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$.

This shows that there are tautological data $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, (A,λ_A,i_A), and $(c_{\mathcal{H}},c^\vee_{\mathcal{H}})$ on $\mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$ (and hence on $\mathrm{NBl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$), whose pullback to $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathrm{Bl}_{\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ are the tautological data on $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}$ for any $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$. To construct a map from $\mathrm{Bl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\mathrm{min}}_{\mathcal{H}})^\wedge_{\bar{x}})$ to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$, it suffices to show that we can construct a tautological $\tau_{\mathcal{H}}$ on $\mathrm{NBl}_{\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ that verifies the condition in statement 6 of Theorem 6.4.1.1. Since there exists the tautological invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ for any $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$, the question is about the following condition:

Let V be any complete discrete valuation ring with valuation $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ and algebraically closed residue field, and let $\xi : \mathrm{Spec}(V) \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ be any morphism centered at a geometric point over $\bar{x}$. Then the condition is whether there exits some cone σ in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ such that $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \geq 0$ for any $\ell \in \sigma^\vee$. By definition of the normalization, it suffices to verify this condition for maps ξ from $\mathrm{Spec}(V)$ to $\mathrm{Bl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$.

By definition of $\mathrm{Bl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$, it has an open covering by affine open subschemes $\{U_f\}$ labeled by nonzero sections f of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$. Over each U_f, the section f of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ becomes a local generator of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$.

Let f be any nonzero section of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$, and let $\xi : \mathrm{Spec}(V) \to \mathrm{Bl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$ be any morphism with center a geometric point above $\bar{x}$ lying in the open subscheme U_f defined by f. By writing $f = \sum\limits_{\ell \in d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}} f^{(\ell)}$, where each $f^{(\ell)}$ is a section of $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$, and by comparing the values of $\upsilon(f^{(\ell)})$, we see that there is a (nonzero) *leading term* $f^{(\ell_0)}$ for some $\ell_0 \in d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, such that $\upsilon(f^{(\ell_0)}) \leq \upsilon(f^{(\ell)})$ for any $\ell \in d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (if we adopt the convention that $\upsilon(0) = +\infty$). In this case, $f^{(\ell_0)}$ is a generator of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$, and necessarily also a generator of the pullback of $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$. This forces $\upsilon(f^{(\ell_0)}) > 0$ because ξ is centered on a geometric point in the support of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$.

Note that any vertex of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ is of the form $d_0 \cdot \ell_{\mathrm{gen}}$ for some vertex ℓ_{gen} of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional cone $\sigma_{\mathrm{gen}} \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ of $\Sigma_{\Phi_{\mathcal{H}}}$. Take any vertex $d_0 \cdot \ell_{\mathrm{gen}}$ of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. Let $g^{(d_0 \cdot \ell_{\mathrm{gen}})}$ be a section of $(\underline{\mathrm{FJ}}^{(d_0 \cdot \ell_{\mathrm{gen}})}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$, namely a global section of $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_{\mathrm{gen}}))^\wedge_{\bar{x}}$. Then we obtain a section $g^{[d_0 \cdot \ell_{\mathrm{gen}}]} = \sum\limits_{\gamma \in \Gamma_{\Phi_{\mathcal{H}}}} g^{(\gamma \cdot (d_0 \cdot \ell_{\mathrm{gen}}))}$ with $g^{(\gamma \cdot (d_0 \cdot \ell_{\mathrm{gen}}))} := \gamma(g^{(d_0 \cdot \ell_{\mathrm{gen}})})$ for any $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$, and the fact that $f^{(\ell_0)}$ is a generator of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ shows in particular that $\upsilon(f^{(\ell_0)}) \leq \upsilon(g^{(d_0 \cdot \ell_{\mathrm{gen}})})$. Since $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\mathrm{gen}})$ is relatively ample and since $d_0 \geq 3$ by assumption, the sheaf $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_{\mathrm{gen}}))^\wedge_{\bar{x}} \cong (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\mathrm{gen}})^{\otimes d_0})^\wedge_{\bar{x}}$ is generated by its global sections by Lefschetz's theorem (see for example [99, §17, Thm., p. 163]). Hence we have arrived at the conclusion that

$$0 < \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)) \leq \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_{\mathrm{gen}})) \tag{7.3.3.6}$$

for any vertex $d_0 \cdot \ell_{\mathrm{gen}}$ of $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Suppose ℓ_0 is a point in the *interior* of $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. By convexity of the polarization function, ℓ_0 is a linear combination

$$\ell_0 = \sum_{1\leq i\leq k} r_i(d_0 \cdot \ell_i),$$

of vertices $d_0 \cdot \ell_i$ of $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ such that $r_i \in \mathbb{Q}_{\geq 0}$ for any $1 \leq i \leq k$ and $\sum\limits_{1\leq i\leq k} r_i > 1$. Let $N \geq 1$ be an integer such that $Nr_i \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i \leq k$. By (7.3.3.6), we have the relation

$$\begin{aligned} 0 < \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell_0)) &< \sum_{1\leq i\leq k} \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i \cdot \ell_0)) \\ &\leq \sum_{1\leq i\leq k} \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i(d_0 \cdot \ell_i))) = \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell_0)), \end{aligned}$$

which is impossible. Hence we see that ℓ_0 must be on the boundary of $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Suppose that the smallest face of the boundary $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ containing ℓ_0 is the span of the vertices $d_0 \cdot \ell_1, \ldots, d_0 \cdot \ell_k$ of $d_0 \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, where each ℓ_i is a vertex of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to some top-dimensional cone $\sigma_i \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$. Then ℓ_0 is a linear combination

$$\ell_0 = \sum_{1\leq i\leq k} r_i(d_0 \cdot \ell_i),$$

where $r_i \in \mathbb{Q}_{\geq 0}$ for any $1 \leq i \leq k$ and $\sum\limits_{1\leq i\leq k} r_i = 1$. Let $N \geq 1$ be an integer such that $Nr_i \in \mathbb{Z}_{\geq 0}$ for any $1 \leq i \leq k$. By (7.3.3.6), we have the relation

$$\begin{aligned} 0 < \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell_0)) &= \sum_{1\leq i\leq k} \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i \cdot \ell_0)) \\ &\leq \sum_{1\leq i\leq k} \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i \cdot (d_0 \cdot \ell_i))) = \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell_0)), \end{aligned}$$

which is possible only when

$$\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_i)) = \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)) = \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f^{(\ell_0)}))$$

for any $1 \leq i \leq k$. As a result, $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_j - d_0 \cdot \ell_i)) = 0$ for any $1 \leq i, j \leq k$, or equivalently $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_j - \ell_i)) = 0$ for any $1 \leq i, j \leq k$.

On the other hand, suppose $d_0 \cdot \ell_{\mathrm{gen}}$ is any vertex of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, where ℓ_{gen} is a vertex ℓ_{gen} of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to some top-dimensional cone $\sigma_{\mathrm{gen}} \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$. In particular, σ_{gen} could be any of the top-dimensional cones $\sigma_{k+1}, \ldots, \sigma_l$ in $\Sigma_{\Phi_{\mathcal{H}}}$ other than $\sigma_1, \ldots, \sigma_k$ that have a common positive-dimensional face with any of $\sigma_1, \ldots, \sigma_k$. Then the condition $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)) \leq \upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_{\mathrm{gen}}))$ shows that $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0 \cdot \ell_{\mathrm{gen}} - d_0 \cdot \ell_i)) \geq 0$ for any $1 \leq i \leq k$, or equivalently $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\mathrm{gen}} - \ell_i)) \geq 0$ for any $1 \leq i \leq k$.

As a result, we see that $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \geq 0$ whenever $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$ is an element in

$$\sum_{1 \leq i,j \leq k} \mathbb{R}(\ell_j - \ell_i) + \sum_{1 \leq i \leq k < j \leq l} \mathbb{R}_{\geq 0}(\ell_j - \ell_i). \tag{7.3.3.7}$$

If we take σ_0 to be the unique *common face* σ_0 of $\sigma_1, \ldots, \sigma_k$, i.e. $\overline{\sigma}_0 = \overline{\sigma}_1 \cap \ldots \cap \overline{\sigma}_k$, then (7.3.3.7) is simply $\mathbb{R}_{\geq 0} \cdot \sigma_0^\vee$ by Lemma 7.3.1.8. This verifies the condition that there exists a cone σ in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ such that $\upsilon(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \geq 0$ for any $\ell \in \sigma^\vee$ if we simply take $\sigma = \sigma_0$.

By statement 6 of Theorem 6.4.1.1, there is a canonical morphism from $\mathrm{NBl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}) \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$, such that the tautological degeneration data over $\mathrm{NBl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$ are the pullback of those from $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$. On the other hand, the tautological degeneration data over $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$ are the pullback of those from $\mathrm{NBl}_{(\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$ under $\mathrm{NBl}_{\mathscr{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$, simply by construction. This forces the composition of two maps to the identity map on $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}}$, as desired. $\square$

Appendix A

Algebraic Spaces and Algebraic Stacks

The purpose of this appendix is to introduce the general concepts of algebraic spaces and algebraic stacks, which are useful generalizations for the study of moduli problems. (This appendix and the next one are reproduced from the first two chapters and the first two appendices of my master thesis presented to National Taiwan University [82] in the Spring of 2001.)

A.1 Some Category Theory

A.1.1 A Set-Theoretical Remark

In the standard axiomatic set theory (say, Zermelo-Frankel), some naive operations of sets are forbidden so that certain logical problems will not arise from these. For example, a collection formed by all sets should be called a *class*, but not a *set*.

However, mathematicians seldom need the full generality of axiomatic set theory. To avoid clumsy languages, a common solution is to introduce a *universe*, namely a large *set of sets* (using a naive terminology here) that is closed under all necessary operations, and to consider only those sets in this universe. Then, when we say that we are forming a set of certain sets, we are only forming a set of those corresponding sets in the universe. Hence no logical badness arises in such operations.

To be more precise:

Definition A.1.1.1. *A* **universe** *is a nonempty set* U *with the following axioms:*

1. *If* $x \in \mathsf{U}$ *and if* $y \in x$*, then* $y \in \mathsf{U}$*.*
2. *If* $x, y \in \mathsf{U}$*, then* $\{x, y\} \in \mathsf{U}$*.*
3. *If* $x \in \mathsf{U}$*, then the power set* 2^x*, namely the set formed by all subsets of* x*, is in* U*.*
4. *If* $(x_i)_{i \in I \in \mathsf{U}}$ *is a family of elements of* U*, then* $\underset{i \in I}{\cup} x_i \in \mathsf{U}$*.*

Remark A.1.1.2. It is immediate that U satisfies the following properties:

1. If $x \in \mathsf{U}$, then $\{x\}$ is in U
2. If $x \subset y$ and $y \in \mathsf{U}$, then $x \in \mathsf{U}$.
3. If $x, y \in \mathsf{U}$, then $x \cup y$ and $x \times y$ are in U.
4. If $(x_i)_{i \in I \in \mathsf{U}}$ is a family of elements of U, then $\prod\limits_{i \in I} x_i \in \mathsf{U}$.
5. If $x \in \mathsf{U}$, then $\mathrm{card}(x) < \mathrm{card}(\mathsf{U})$ (strictly). In particular, the relation $\mathsf{U} \in \mathsf{U}$ is impossible.

One may check that many other usual operations of sets are also closed in U under these axioms.

For a more detailed exposition of the theory of universe, one may consult [3, I, 0] or [3, I, Appendice: Univers (par. N. Bourbaki)].

Remark A.1.1.3. In what follows, we fix a pertinent choice of universe that is sufficient for our need, and will not bother to mention about our choice of the universe.

A.1.2 2-Categories and 2-Functors

Here we summarize some properties of 2-categories and 2-functors from [61] and [45].

Definition A.1.2.1. *A* **2-category** C *consists of the following data:*

1. *A set of objects* $\mathrm{Ob}\,\mathsf{C}$*.*

2. *For any two objects* $X, Y \in \operatorname{Ob} \mathsf{C}$ *a category* $\operatorname{Hom}_{\mathsf{C}}(X, Y)$*, written also as* $\operatorname{Hom}(X, Y)$*.*

3. *For any three objects* $X, Y, Z \in \operatorname{Ob} \mathsf{C}$ *a functor*

$$\mu_{X,Y,Z} : \operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$$

   such that:

   (a) *The categories* $\operatorname{Hom}(X, Y)$ *are pairwise disjoint.*

   (b) *For any* $X \in \operatorname{Ob} \mathsf{C}$ *there exists the identity morphism* $\mathrm{id}_X : X \to X$ *of* X*, which is determined uniquely by the condition that, for any three objects* $X, Y, Z \in \operatorname{Ob} \mathsf{C}$*,*

$$\mu_{X,X,Y}(\mathrm{id}_X, \,\cdot\,) = \mu_{X,Y,Y}(\,\cdot\,, \mathrm{id}_Y) = \mathrm{id}_{\operatorname{Hom}(X,Y)} \,.$$

   (c) *For any four objects* $X, Y, Z, T \in \operatorname{Ob} \mathsf{C}$*, we have the following* **associative law***:*

$$\mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\operatorname{Hom}(Z,T)}) = \mu_{X,Y,T} \circ (\mathrm{id}_{\operatorname{Hom}(X,Y)} \times \mu_{Y,Z,T})$$

For any two objects X, Y in C, we call an object f of $\operatorname{Hom}(X, Y)$ a *1-morphism* and write it simply as $f : X \to Y$. Let f, g be two objects of $\operatorname{Hom}(X, Y)$, a morphism $\alpha : f \to g$ is then called a *2-morphism*, represented in the form

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & Y. \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

Let X, Y, Z be three objects in C. For $f \in \operatorname{Ob}\operatorname{Hom}(X, Y)$ and $g \in \operatorname{Ob}\operatorname{Hom}(Y, Z)$ (resp. $\alpha \in \operatorname{Mor}\operatorname{Hom}(X, Y)$ and $\beta \in \operatorname{Mor}\operatorname{Hom}(Y, Z)$), we write simply $g \circ f$ (resp. $\beta \circ \alpha$) in place of $\mu_{X,Y,Z}(f, g) \in \operatorname{Ob}\operatorname{Hom}(X, Z)$ (resp. $\mu_{X,Y,Z}(\alpha, \beta) \in \operatorname{Mor}\operatorname{Hom}(X, Z)$).

Definition A.1.2.2. *Two objects* X *and* Y *of* C *are* **equivalent** *if there exist two 1-morphisms* $u : X \to Y$ *and* $v : Y \to X$ *and two invertible 2-morphisms* (*or 2-isomorphisms*) *such that*

$$\begin{aligned} \alpha &: v \circ u \to \mathrm{id}_X, \\ \beta &: u \circ v \to \mathrm{id}_Y \,. \end{aligned}$$

Definition A.1.2.3. *Consider the following diagram of 1-morphisms:*

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\
& Y \arrow[ur, "g"'] \arrow[u, Rightarrow, "\alpha"] &
\end{tikzcd}$$

If α is a 2-morphism of $g \circ f$ to h, which is an 2-isomorphism, then we say that the diagram **commutes***. Diagrams in other forms are defined to be commutative in the same way.*

On the other hand, a diagram of 2-morphisms will be called commutative only if the compositions are **equal***.*

Now we define the concept of a covariant 2-functor. (A contravariant 2-functor is defined in a similar way.)

Definition A.1.2.4. *Let C and C' be two 2-categories. A* **2-functor** *$F : \mathsf{C} \to \mathsf{C}'$ consists of a map*

$$F : \operatorname{Ob} \mathsf{C} \to \operatorname{Ob} \mathsf{C}'$$

and a functor

$$F : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$$

between any two objects $X, Y \in \operatorname{Ob} \mathsf{C}$ such that:

1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
2. $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$.
3. *For any diagram*

   $$X \xrightarrow{f} Y \xrightarrow{g} Z$$

   in C there exists a 2-isomorphism $\varepsilon_{g,f} : F(g) \circ F(f) \to F(g \circ f)$

   $$\begin{tikzcd}
   & F(Y) \arrow[dr, "F(g)"] \arrow[d, Rightarrow, "\varepsilon_{g,f}"] & \\
   F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"'] & & F(Z)
   \end{tikzcd}$$

   such that:

(a) $\varepsilon_{f,\mathrm{id}_X} = \varepsilon_{\mathrm{id}_Y,f} = \mathrm{id}_{F(f)}$.

(b) ε *is associative: The following diagram commutes:*

$$\begin{tikzcd}
F(h) \circ F(g) \circ F(f) \arrow[r, "\varepsilon_{h,g} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \varepsilon_{g,f}"'] & F(h \circ g) \circ F(f) \arrow[d, "\varepsilon_{h \circ g,f}"] \\
F(h) \circ F(g \circ f) \arrow[r, "\varepsilon_{h,g \circ f}"] & F(h \circ g \circ f)
\end{tikzcd}$$

4. *For any pair of 2-morphisms* $\alpha : f \to f'$ *and* $\beta : g \to g'$ *in* C*, we have* $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.

5. *For any pair of 2-morphisms* $\alpha : f \to f'$ *and* $\beta : g \to g'$ *in* C*, the following diagram commutes:*

$$\begin{tikzcd}
F(g) \circ F(f) \arrow[r, "F(\beta) \circ F(\alpha)"] \arrow[d, "\varepsilon_{g,f}"'] & F(g') \circ F(f') \arrow[d, "\varepsilon_{g',f'}"] \\
F(g \circ f) \arrow[r, "F(\beta \circ \alpha)"] & F(g' \circ f')
\end{tikzcd}$$

By a slight abuse of language, the last condition is usually written as $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. This equality does not make sense, but if we fix a choice for all the 2-isomorphisms ε, then there's a unique way to interpret this equality.

In the application of 2-categories to our stack theory, the 2-isomorphism ε is canonically defined. Hence we may say $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$ in a well-defined way.

Definition A.1.2.5. *Let* F *and* G *be two 2-functors from* C *to* C'*. A* **1-morphism** (*or a* **natural transformation**) φ *from* F *to* G *assigns to each object* A *in* C *a 1-morphism* $\varphi(A) : F(A) \to G(A)$ *in* C'*, and to each 1-morphism* $f : A \to B$ *in* C *a 2-morphism* $\varphi(f) : F(f) \to G(f)$*, such that for each 1-morphism* $g : A \to B$*, the diagram of 1-morphisms*

$$\begin{tikzcd}
F(A) \arrow[r, "\varphi(A)"] \arrow[d, "F(g)"'] & G(A) \arrow[d, "G(g)"] \\
F(B) \arrow[r, "\varphi(B)"] & G(B)
\end{tikzcd}$$

is commutative (up to 2-isomorphisms), and such that for each 2-morphism $\alpha : f \to g$ *of 1-morphisms, the diagram of 2-morphisms*

$$\begin{tikzcd}
F(f) \arrow[r, "\varphi(f)"] \arrow[d, "F(\alpha)"'] & G(f) \arrow[d, "G(\alpha)"] \\
F(g) \arrow[r, "\varphi(g)"] & G(g)
\end{tikzcd}$$

is also commutative.

Definition A.1.2.6. *Let* φ *and* ψ *be two 1-morphisms of functors from the 2-functor* F *to the 2-functor* G*. A 2-morphism* θ *from* φ *to* ψ *assigns to each object* A *in* C *a 2-morphism* $\theta(A) : \varphi(A) \to \psi(A)$ *between 1-morphisms, and to each 1-morphism* f *in* C *an identity* $\theta(f) : \varphi(f) \overset{=}{\to} \psi(f)$ *between 2-morphisms.*

The last statement asserts that a 2-morphism between two 1-morphisms φ and ψ exists only when $\varphi(f) = \psi(f)$ for each 1-morphism f in C.

For any two 2-functors F and G, the 1-morphisms and 2-morphisms from F to G defined above form a category $\mathsf{Hom}(F, G)$ whose objects are 1-morphisms of 2-functors and whose morphisms are 2-morphisms between 1-morphisms.

Remark A.1.2.7. Beware that it is practically not reasonable to define a (*1-*)*isomorphism* between 2-functors naively to be a 1-morphism with an inverse. Usually, it may be more realistic and more convenient to define it to be a 1-morphism with a *quasi-inverse* (defined in an obvious way analogous to the quasi-inverse of an equivalence of functor) such that their compositions are 2-isomorphic to the identity.

Given any (1-)category C, we may define a 2-category by making the set $\mathrm{Hom}(X, Y)$ into a category whose objects are elements of the $\mathrm{Hom}(X, Y)$ and whose morphisms are only identities.

Given a 2-category C, there are two ways of defining a 1-category. We have to make $\mathrm{Hom}(X, Y)$ into a set. The naive way is just to take the set of objects of $\mathrm{Hom}(X, Y)$, and we obtain the so-called underlying category of C. This has the problem that a 2-functor $F : \mathsf{C} \to \mathsf{C}'$ is not in general a functor of the underlying categories (because we only require the composition of 1-morphisms to be respected up to 2-isomorphisms).

The best way of constructing a 1-category from a 2-category is to define the set of morphisms between the objects X and Y as the set of isomorphism classes of objects of $\mathrm{Hom}(X,Y)$: two objects f and g of $\mathrm{Hom}(X,Y)$ are isomorphic if there exists a 2-isomorphism $\alpha : f \to g$ between them. We call the category obtained in this way the 1-category associated to C. Note that a 2-functor between 2-categories then becomes a functor between the associated 1-categories.

A.2 Grothendieck Topology

The main references of this section are [6] and [120]. Related topics may be found in [32, IV], [3] and [88, III].

We begin with an example.

Example A.2.1. (The topology in the usual sense.) Let X be a topological space, and let T denote the collection of all open subsets of X. T becomes a category if we define

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset & \text{if } U \not\subset V \\ \text{inclusion } U \to V & \text{if } U \subset V \end{cases}$$

for $U, V \in T$. X is a final object in the category T. The intersection $\cap U_i$ of finitely many $U_1,\ldots,U_n$ in T is equal to the product of the $U_1,\ldots,U_n$ in the category T. The union $\cup U_i$ of arbitrarily many U_i in T is equal to the direct sum of the U_i in the category T.

Grothendieck's generalization of the notion of topology consists of replacing the category T of open sets of a topological space X by an arbitrary category, in which for example $\mathrm{Hom}(U,V)$ may have more than one element, and of prescribing in addition for this category a system $\{U_i \xrightarrow{\phi_i} U\}$ of *coverings* of its objects. (For a brief introduction to the theory of categories and functors we need, see Section A.1.2.)

Definition A.2.2. *A* **topology** (*or* **site**) T *consists of a category* $\mathrm{cat}(T)$ *and of a set* $\mathrm{cov}(T)$ *of* **coverings**, *namely families* $\{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ *of morphisms in* $\mathrm{cat}(T)$, *such that the following properties hold:*

1. *For* $\{U_i \to U\}$ *in* $\mathrm{cov}(T)$ *and a morphism* $V \to U$ *in* $\mathrm{cat}(T)$, *all fiber products* $U_i \underset{U}{\times} V$ *exist, and* $\{U_i \underset{U}{\times} V \to V\}$ *is again in* $\mathrm{cov}(T)$.

2. *Given* $\{U_i \to U\} \in \mathrm{cov}(T)$ *and a family* $\{V_{ij} \to U_i\} \in \mathrm{cov}(T)$ *for all* $i \in I$*, the family* $\{V_{ij} \to U\}$*, obtained by composition of morphisms, also belongs to* $\mathrm{cov}(T)$*.*

3. *If* $\phi : U' \to U$ *is an isomorphism in* $\mathrm{cat}(T)$*, then* $\{U' \xrightarrow{\phi} U\} \in \mathrm{cov}(T)$*.*

Definition A.2.3. *A* **morphism** $f : T \to T'$ **of topologies** *is a functor* $f : \mathrm{cat}(T) \to \mathrm{cat}(T')$ *of the underlying categories with the following two properties:*

1. $\{U_i \xrightarrow{\phi_i} U\} \in \mathrm{cov}(T)$ *implies* $\{f(U_i) \xrightarrow{f(\phi_i)} f(U)\} \in \mathrm{cov}(T')$*.*

2. *For* $\{U_i \to U\} \in \mathrm{cov}(T)$ *and a morphism* $V \to U$ *in* $\mathrm{cat}(T)$ *the canonical morphism*
$$f(U_i \underset{U}{\times} V) \to f(U_i) \underset{f(U)}{\times} f(V)$$
is an isomorphism for all i*.*

Definition A.2.4. *Let* T *be a topology and let* C *denote the category of abelian groups or the category of sets (more generally: a category with products).*

A **presheaf** *on* T *with values in* C *is a contravariant functor* $F : T \to \mathsf{C}$*. (More precisely, a contravariant functor* $F : \mathrm{cat}(T) \to \mathsf{C}$*.)*

A **morphism** $f : F \to F$ **of presheaves** *with values in* C *is defined as a morphism of contravariant functors.*

A presheaf F *is a* **sheaf** *on* T *if for every covering* $\{U_i \to U\}$ *in* T *the diagram*
$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \underset{U}{\times} U_j)$$
is exact in C*, where the double arrow in the diagram means two morphisms of* $\prod\limits_i F(U_i)$ *to* $\prod\limits_{i,j} F(U_i \underset{U}{\times} U_j)$ *induced respectively by the projections* $U_i \underset{U}{\times} U_j \to U_i$ *and* $U_i \underset{U}{\times} U_j \to U_j$*. The exactness here means that any* $(x_i) \in \prod\limits_i F(U_i)$ *mapped by the two morphisms to a same image in* $\prod\limits_{i,j} F(U_i \underset{U}{\times} U_j)$ *is in the image of* $F(U) \to \prod\limits_i F(U_i)$*.* **Morphisms of sheaves** *are defined as morphisms of presheaves.*

Recall that a morphism $U \to V$ is surjective if $\mathrm{Hom}(V,Z) \to \mathrm{Hom}(U,Z)$ is injective for all $Z \in \mathsf{C}$. In a similar way, we define a family of morphisms $\{U_i \xrightarrow{f_i} U\}$ to be *surjective* if $\mathrm{Hom}(U,Z) \to \prod_i \mathrm{Hom}(U_i,Z)$ is injective for all $Z \in \mathsf{C}$.

Let (Sch/S) be the category whose objects are *schemes* over S.

Remark A.2.5. We must remark here about the term *scheme* used here. In certain old literatures such as [46] and [101], the term *scheme* is used to mean a *separated prescheme*, where a *prescheme* is a locally ringed space in which every point has an open neighborhood that is an affine scheme. (This is exactly the term *scheme* we use today.) It may be somewhat confusing, because this definition is not the same with those given in many standard introductory texts (such as [64], [116], etc), and is even already abandoned in [53]. (See in particular [53, Avant-propos].) However, the objects we are going to study later, such as *algebraic spaces* and *algebraic stacks* (see [76, II, 1.9], [28, Def. 4.5, 4.6], [85, Rem. 1.4(3)]), are generalizations of *quasi-separated preschemes*, namely those preschemes X whose associated diagonal morphisms $\Delta_X : X \to X \times X$ are quasi-compact (see [53, I, 6.13]). Hence we will adopt the convention that the term *scheme* is reserved for *separated preschemes*, whereas the term *prescheme* is used exactly as in the old literature. (We could have adopted the convention that schemes are just quasi-separated preschemes.) We hope that this resumption of old-fashioned convention will not bother the readers too much.

Consider three topologies on the category (Sch/S). If the coverings are collections of surjective families of finitely many morphisms that are *étale* (resp. *flat and of finite presentation*, resp. *flat*), then we say that the corresponding topology is the *étale* (resp. *fppf*, resp. *fpqc*) topology. Here *fppf* (resp. *fpqc*) is the abbreviate (in French) of *fidélement plate de présentation finie* (resp. *fidélement plate quasi-compacte*).

Note that although the phrase *faithfully flat* (which is *fidélement plate* in French) did not appear in the English description above, it is already guaranteed by the assumption that the family is surjective. (This is exactly what the word *faithful* means here.)

A.3 Algebraic Spaces

Fix any scheme S. Consider the category (Sch/S) of schemes over S with the *étale topology*. An *S-space* is then defined to be a sheaf of sets over the étale site (Sch/S). We denote by (Spc/S) the category of S-spaces, with morphisms the morphisms of sheaves.

The category (Sch/S) of S-schemes can be identified as a full subcategory of (Spc/S): To each S-scheme X, we associate the *functor of points*

$$U \mapsto X(U) = \mathrm{Hom}_{(\mathrm{Sch}/S)}(U, X)$$

over (Sch/S). By Grothendieck's descent theory [58, VIII, 5.1 and 5.3], this functor is a sheaf. Hence it defines a space. By Yoneda's lemma, morphisms between two such S-spaces are equivalent to morphisms between two schemes. Therefore the category of S-schemes is a full subcategory of S-spaces.

By abuse of language, we say that an S-space is an (S-)scheme if it comes from a scheme in this way. In particular, the S-scheme S corresponds to the final object of (Spc/S).

A morphism $f : X \to Y$ in (Spc/S) is called *schematic* if for all $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and all $y \in Y(U)$ (viewed as a morphism $U \to Y$ in (Spc/S)) the fiber product $U \underset{y,Y,f}{\times} X$ is a scheme.

$$\begin{tikzcd}
U \underset{y,Y,f}{\times} X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
U \arrow[r, "y"] & Y
\end{tikzcd}$$

Remark A.3.1. The terminology used here should be distinguished with the *representability* of a morphism. A morphism $f : X \to Y$ in (Spc/S) is called *representable* if for all $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and all $y \in Y(U)$ the fiber product $U \underset{y,Y,f}{\times} X$ is an *algebraic space*, which is defined in Definition A.3.2.

The properties of S-schemes that are stable under base change extend naturally to schematic morphisms of S-spaces. Hence we may say a morphism in (Spc/S) is schematic and étale, schematic and quasi-compact, etc.

Definition A.3.2. *An S-algebraic space is an S-space X such that:*

1. (**Quasi-separateness**, *cf. Remark A.2.5:*) *The diagonal morphism $\Delta_X : X \to X \underset{S}{\times} X$ is schematic and quasi-compact.*

2. *There exists a scheme X' and a morphism of S-spaces $X' \xrightarrow{\pi} X$ (automatically schematic, see the proof of Proposition A.5.23) that is étale and surjective.*

We denote the full subcategory of S-algebraic spaces by $(\text{Alg-Spc}\,/S)$. The subcategory $(\text{Sch}\,/S)$ of S-schemes is naturally contained in $(\text{Alg-Spc}\,/S)$.

An *equivalence relation* $X_\bullet$ in $(\text{Spc}\,/S)$ is given by the data consisting of two spaces X_0, X_1, and a monomorphism of S-spaces (namely a monomorphism of sheaves)

$$X_1 \xrightarrow{\delta} X_0 \underset{S}{\times} X_0$$

such that for any object U in $(\text{Sch}\,/S)$

$$X_1(U) \xrightarrow{\delta(U)} X_0(U) \underset{S}{\times} X_0(U)$$

is the graph of an equivalence relation in the category of sets. A *quotient for the equivalence relation* $X_\bullet$ is defined by the cokernel of the diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\mathrm{pr}_1 \circ \delta"] \arrow[r, shift right, "\mathrm{pr}_2 \circ \delta"'] & X_0. \end{tikzcd}$$

Any such S-space quotient Q exists, and X_1 can be canonically identified with the fiber product $X_0 \underset{Q}{\times} X_0$. Conversely, for all epimorphism $X_0 \twoheadrightarrow Q$ in $(\text{Spc}\,/S)$, Q is identified with the S-space quotient of the equivalence relation given by the naturally morphism $X_0 \underset{Q}{\times} X_0 \to X_0 \underset{S}{\times} X_0$.

Proposition A.3.3 ([76, II, 1.3])**.** *An S-space is an S-algebraic space if and only if it is a quotient for an equivalence relation $X_\bullet$ in* $(\text{Spc}\,/S)$ *such that X_0 and X_1 are S-schemes, such that $\mathrm{pr}_1 \circ\delta$ and $\mathrm{pr}_2 \circ\delta$ are étale morphisms, and such that δ is a quasi-compact monomorphism (of S-schemes). To be more precise:*

1. *If X is an S-algebraic space and if $\pi : X' \to X$ is a morphism as in Definition A.3.2, then the projections $\mathrm{pr}_1, \mathrm{pr}_2 : X' \underset{X}{\times} X' \to X'$ are étale, the morphism $(\mathrm{pr}_1, \mathrm{pr}_2) : X' \underset{X}{\times} X' \to X' \underset{S}{\times} X'$ is quasi-compact, and X is the quotient of the equivalence relation defined by this morphism.*

2. If $X_\bullet$ is an equivalence relation in (Sch/S) *with* $\mathrm{pr}_1 \circ \delta$, $\mathrm{pr}_2 \circ \delta$ *étale and δ quasi-compact, then the S-space quotient of $X_\bullet$ is algebraic, and the canonical morphism of X_0 into the quotient is schematic, étale, and surjective.*

Hence we may define alternatively:

Definition A.3.4. *An S-space X is called an* $(S\text{-})$*algebraic space if it is the quotient S-space for an equivalence relation described above such that X_0 and X_1 are S-schemes, such that* $\mathrm{pr}_1 \circ \delta$ *and* $\mathrm{pr}_2 \circ \delta$ *are étale* (*morphisms of S-schemes*), *and such that δ is a quasi-compact morphism* (*of S-schemes*).

Hence an algebraic space can be viewed as the quotient of a scheme by an étale equivalence relation (whose precise definition is described above).

In lower dimensions (whose definition will be made precise in Definition A.3.1.6), there are few examples of algebraic spaces that are not schemes. We have the following theorem by Artin:

Theorem A.3.5 ([11, Thm. 2.7])**.** *Every 1-dimensional algebraic space* (**algebraic curve**) *is a scheme. Every nonsingular 2-dimensional algebraic space* (**algebraic surface**) *is a scheme. Every algebraic space with a group structure is a scheme. However, there exist singular 2-dimensional algebraic spaces and nonsingular 3-dimensional algebraic spaces that are not schemes.*

See [64, Appendix B] for an example of an algebraic space (or rather, a *Moishezon space*) that is not a scheme.

A.3.1 Properties of an Algebraic Space

A large part of the notions and results of the theory of schemes is generalized to the theory of algebraic spaces by Artin [11] and Knutson [76]. Since the study of algebraic stacks will be based on algebraic spaces, we summarize here many of the important definitions and results from [76].

Proposition A.3.1.1 ([76, II, 1.4])**.** *Let X_1 and X_2 be algebraic spaces, and let $\pi_1 : X_1' \to X_1$ and $\pi_2 : X_2' \to X_2$ be étale surjections, where X_1' and X_2' are schemes. Then π_1 and π_2 are automatically schematic, and $X_1' \underset{X_1}{\times} X_1'$ and $X_2' \underset{X_2}{\times} X_2'$ are representable by schemes. Let g and h be morphisms in the* *following diagram of solid arrows*

$$\begin{tikzcd}
X_1' \underset{X_1}{\times} X_1' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[d, "g"] & X_1' \arrow[r, "\pi_1"] \arrow[d, "h"] & X_1 \arrow[d, dotted, "f"] \\
X_2' \underset{X_2}{\times} X_2' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X_2' \arrow[r, "\pi_2"] & X_2
\end{tikzcd}$$

where $h \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ g$ *and* $h \circ \mathrm{pr}_2 = \mathrm{pr}_2 \circ g$. *Then there is a unique morphism* $f : X_1 \to X_2$ *with* $\pi_2 \circ h = f \circ \pi_1$. *Conversely, every morphism* $f : X_1 \to X_2$ *is induced in this way for some choice of* X_1', X_2', g, h.

Hence the study of morphisms between algebraic spaces is equivalent to the study of morphisms between certain étale surjections of them from schemes.

Proposition A.3.1.2 ([76, II, 1.5]). *Disjoint unions and fiber products exist in the category of algebraic spaces.*

For an algebraic space X, the diagonal morphism $X \to X \times X$ is schematic. Hence we may naturally extend those properties of schemes that are defined by imposing conditions on the diagonal morphisms of schemes to algebraic spaces. More precisely, consider any commutative diagram of the form

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & X \arrow[d] \\
U \arrow[r] & X \times X,
\end{tikzcd}$$

where U is a scheme and where V is the scheme that is the fiber product $U \underset{X \times X}{\times} X$. Let us recall how we say that a schematic morphism has a certain property. In this case, it amounts to consider the properties of the morphism $V \to U$ between schemes.

Definition A.3.1.3. *With notations as above, we define* X *to be* **locally separated** (*resp.* **separated**) *if* $V \to U$ (*i.e.* $X \to X \times X$) *is a quasi-compact* (*resp. closed*) *immersion for any morphism* $U \to X \times X$.

For the above definition, it is sufficient to check for a *single* $U = X' \times X'$ where X' is a scheme and where $X' \to X$ is an étale surjection [76, II, 1.8].

Definition A.3.1.4. *A morphism $f : X_1 \to X_2$ between algebraic spaces is* **étale** *if there are étale surjections $X_1' \to X_1$ and $X_2' \to X_2$ from schemes and an étale morphism h (cf. Proposition A.3.1.1) such that the following diagram commutes:*

$$\begin{tikzcd}
X_1' \arrow[r] \arrow[d, "h"'] & X_1 \arrow[d, "f"] \\
X_2' \arrow[r] & X_2
\end{tikzcd}$$

Definition A.3.1.5. *An algebraic space X is* **quasi-compact** *if X admits a surjection $X' \to X$ with X' a quasi-compact scheme. A map $f : X \to Y$ of algebraic spaces is* **quasi-compact** *if for every étale map $U \to Y$, with U a quasi-compact scheme, $U \underset{Y}{\times} X$ is quasi-compact.*

An étale (resp. quasi-compact) surjection of algebraic spaces is then obviously defined to be a surjection (in the language of category) that is étale (resp. quasi-compact).

These definitions enable us to extend the descent theory of schemes [58, VIII] to algebraic spaces [76, II, 3].

Definition A.3.1.6. *Let "P" be a property of schemes that is local in nature in the étale topology, i.e, if $X' \to X$ is an étale surjection of schemes, then X has property "P" if and only if X' has property "P". An algebraic space X is therefore defined to have property "P" if and only if there is an étale surjection $X' \to X$ from a scheme X such that X' has property "P".*

Thus we are able to speak of **locally noetherian** *algebraic spaces,* **reduced** *algebraic spaces,* **nonsingular** *algebraic spaces,* **normal** *algebraic spaces and* **n-dimensional** *algebraic spaces.*

Definition A.3.1.7. *Let "P" be a property of morphisms of schemes that is local in nature on the target in the étale topology. That is, for any morphism $f : X \to Y$, the morphism f has property "P" if and only if for any étale surjection $U \to Y$ and any étale surjection $V \to U \underset{Y}{\times} X$, the induced map $f' : V \to U$ has property "P".*

$$\begin{tikzcd}
V \arrow[r] \arrow[dr, "f'"'] & U \underset{Y}{\times} X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& U \arrow[r] & Y
\end{tikzcd}$$

This definition clearly extends to the case where X and Y are algebraic spaces, in which case we require that $U \to Y$ and $V \to U \underset{Y}{\times} X$ are respectively étale surjections from schemes U and V.

Hence we can speak of morphisms of algebraic spaces that are **surjective**, **flat**, **faithfully flat**, **universally open**, **locally of finite presentation**, **locally of finite type** *and* **locally quasi-finite**.

Definition A.3.1.8. *A morphism $f : X \to Y$ of algebraic spaces is* **of finite type** *if it is locally of finite type and quasi-compact. The morphism f is* **of finite presentation** *if it is locally of finite presentation, quasi-compact, and the induced map $X \to X \underset{Y}{\times} X$ is quasi-compact. The morphism f is* **quasi-finite** *if it is locally quasi-finite and quasi-compact.*

It is immediate from the local nature of the definition that these properties of morphisms are stable in the étale topology.

Definition A.3.1.9. *Let "P" be a property of morphisms of schemes satisfying effective descent.* (*See [58, VIII] and [50, IV, 2.6 and 2.7].*) *Namely* (*in a simplified form; see Remark A.3.1.10 below for more detail*), *a morphism $f : X \to Y$ has property "P" if for any scheme Y' and morphism $Y' \to Y$, the induced morphism $f' : X \underset{Y}{\times} Y' \to Y'$ has property "P".*

Then "P" extends to a property of morphisms of algebraic spaces satisfying effective descent in the following way: A morphism $f : X \to Y$ of algebraic spaces has property "P" if and only if for any scheme Y' and morphism $Y' \to Y$, the fiber product $X \underset{Y}{\times} Y'$ is a scheme and the induced morphism $f' : X \underset{Y}{\times} Y' \to Y'$ has property "P".

Then we can speak about **open immersions**, **closed immersions**, **immersions**, **affine morphisms**, **quasi-affine morphisms** *and* **reduced closed immersions of schemes**. *We will also use the terms* **open subspace**, **closed subspace** *and* **subspace**. *Note in particular that if $f : X \to Y$ is a morphism of algebraic spaces that is an open immersion, closed immersion, immersion, affine morphism or quasi-affine morphism, and Y is a scheme, then so is X. In particular, a subspace* (*under this definition*) *of a scheme is a scheme.*

Remark A.3.1.10. Originally in [58, VIII] and [50, IV, 2.6 and 2.7], the general condition should be in the following more instructive form:

Let $S' \to S$ be a fpqc surjection of preschemes, $f : X \to Y$ a morphism between two S-preschemes X, Y, and $f' : X' \to Y'$ between the pullbacks $X' = X \underset{S}{\times} S'$, $Y' = Y \underset{S}{\times} S'$.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"'] & Y & S
\end{tikzcd}$$

A property satisfying effective descent is a property "P" such that f has property "P" if and only if f' has property "P".

For such a property, if $Y' \to Y$ is also fpqc, then we may replace S by Y (and hence $X' = X \underset{Y}{\times} Y'$) and verify the condition for a simpler diagram:

$$\begin{tikzcd}
X' = X \underset{Y}{\times} Y' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

For the situation now, since the maps considered are étale surjections of schemes, the assumptions on $S' \to S$ or $Y' \to Y$ are automatically verified. Furthermore, the properties that we are going to consider such as being an immersion are also stable under base change (see [46, I, 4.4.1]). Hence it suffices to define simply in the form above.

Definition A.3.1.11. *A morphism $f : X \to Y$ of algebraic spaces is* **quasi-separated** (*resp.* **locally separated**, *resp.* **separated**) *if the induced map $X \to X \underset{Y}{\times} X$ is quasi-compact* (*resp. a quasi-compact immersion, resp. a closed immersion*).

Proposition A.3.1.12 ([76, II, 3.10])**.** *The classes of quasi-separated, locally separated and separated maps of algebraic spaces are stable in the étale topology. Also, for any morphism $f : X \to Y$ such that X and Y are algebraic spaces and such that Y is separated, X is quasi-separated* (*resp. locally separated, resp. separated*) *if and only if the morphism f is quasi-separated* (*resp. locally separated, resp. separated*).

Definition A.3.1.13. *A morphism $f : X \to Y$ of algebraic spaces is* **proper** *if f is separated, of finite type, and* **universally closed** (*to be defined later in Definition A.3.3.8*).

A.3.2 Quasi-Coherent Sheaves on an Algebraic Space

As described above, the extension of the definition of étaleness to morphisms between algebraic spaces and the corresponding properties enable us to speak of the étale site of (Alg-Spc) in an obvious manner. For any sheaf F on the étale site (Sch), we may extend it in a unique way to the étale site (Alg-Spc). Namely, for any algebraic space X with an étale surjection $X' \to X$ by a scheme X', we define

$$F(X) = \ker(F(X') \rightrightarrows F(X' \underset{X}{\times} X')).$$

In particular, we may extend the *structural sheaf of rings* $\mathscr{O}$ on $(\mathrm{Sch}\,/S)$ (which assigns to each $\mathrm{Spec}(R)$ the ring R) to the étale site of algebraic spaces. The *sheaf of units* $\mathscr{O}^{\times}$ and *sheaf* $\mathscr{N}$ *of nilpotent elements* of $\mathscr{O}$ are extended in a similar way.

For a particular algebraic space X, the restriction of $\mathscr{O}$ to the local étale site $(\text{Alg-Spc}\,/X)$ is the *structural sheaf* of X, denoted by $\mathscr{O}_X$. For example, for any scheme U with $U \to X$, the structural sheaf $\mathscr{O}_X$ assigns to U the structural sheaf $\mathscr{O}_X(U) = \mathscr{O}_U$ of U. Hence the idea of the structural sheaf of an algebraic space generalizes the idea of studying the ring of functions of a scheme by localizations.

Definition A.3.2.1. *Let X be an algebraic space and $\mathscr{O}_X$ its structural sheaf of rings. A sheaf of $\mathscr{O}_X$-modules F is* **quasi-coherent** (*resp.* **coherent**, *resp.* **locally free of rank** r) *if for any morphism $i : U \to X$, with U a scheme, the induced sheaf i^*F* (*which is defined in a natural way to be the sheaf over U assigning $i^*F(V) = F(V)$ to each $V \to U$*) *is quasi-coherent* (*resp. coherent, resp. locally free of rank r*) *in the usual sense* (*of sheaf of modules on schemes*).

In the above definition, it is sufficient to consider a single *epimorphism* from a scheme to X.

A.3.3 Points and the Zariski Topology of an Algebraic Space

Let X be an algebraic space. Consider morphisms of algebraic spaces of the form $i : p = \mathrm{Spec}(k) \to X$ (with k a field) that are injective in the sense of category theory. Two such morphisms $i_1 : p_1 \to X$ and $i_2 : p_2 \to X$ are considered equivalent if there is an isomorphism $e : p_1 \to p_2$ with $i_2 \circ e = i_1$.

Definition A.3.3.1. *A* **point** *of X is defined to be an equivalence class of such morphisms of algebraic spaces. By abuse of language, we say that p is in X and write $p \in X$.*

The **residue field** *$k(p)$ of X at p is defined to be the field k above.*

A **geometric point** *$j : q \to X$ is any morphism of algebraic spaces with $q = \operatorname{Spec}(\bar{k})$ for some separably closed field $\bar{k}$. (Note that j need not be injective. Hence a geometric point is usually not a point.)*

Proposition A.3.3.2 ([76, II, 6.2])**.** *Let $f : q \to X$ be a morphism of algebraic spaces with $q = \operatorname{Spec}(k)$ for some field k. Then there is a point p of X such that f factors through $q \to p \to X$.*

Theorem A.3.3.3 ([76, II, 6.4])**.** *Let X be an algebraic space and $p \to X$ a point of X. Then there is an affine scheme U and an étale morphism $U \to X$ such that $p \to X$ factors through $p \to U \to X$.*

Definition A.3.3.4. *A point $p \to X$ is* **scheme-like** *if there is an affine scheme U and an open immersion $U \to X$ such that $x \to X$ factors through $p \to U \to X$.*

The difference between Definition A.3.3.4 and the conclusion of Theorem A.3.3.3 is, the étale map $U \to X$ of Theorem A.3.3.3 from an affine scheme U to X is required to be an *open immersion.*

The following proposition is immediate from the definition.

Proposition A.3.3.5. *Let X be an algebraic space. Then there is an open subspace U of X such that U is a scheme, and a point $p \to X$ is in U if and only if p is scheme-like. The algebraic space X is a scheme if and only if all its points are scheme-like.*

We may sometimes say that *U is the open subspace where X is a scheme.*

Definition A.3.3.6. *Let X be an algebraic space. The* **associated underlying topological space** *$|X|$ of X is defined to be the collection of points of X (modulo equivalence relations of points). The set $|X|$ is given a topological structure by taking a subset $U \subset |X|$ to be open if $U = |Y|$ for some open subspace Y of X. This topology on $|X|$ is called the* **Zariski topology***.*

Proposition A.3.3.7 ([76, II, 6.10])**.** *$|X|$ is a topological space and there is a one-one correspondence between open subspaces of X and open subsets of $|X|$, and a one-one correspondence between reduced closed subspaces of X and closed subsets of $|X|$. Also, $X \mapsto |X|$ is a functor.*

Definition A.3.3.8. *X is irreducible if and only if $|X|$ is, and the definitions of* **topologically dense subspace***,* **surjective map***,* **universally open map***,* **open map***,* **immersion***,* **closed map***, and* **universally closed** *can be made via the underlying topological space.* (*If it is already defined, then the previous definition is compatible with the one given here. See [76, II, 6.11].*) *For example, a map is defined to be* **universally closed** *if for any algebraic space X' with $X' \to X$, the induced map $|Y \underset{X}{\times} X'| \to |X'|$ is closed.*

Proposition A.3.3.9 ([76, II, 6.7])**.** *Let X be an algebraic space. Then X is a scheme almost everywhere. That is, the open subspace of scheme-like points is topologically dense.*

In fact, the Krull dimension of the *bad locus* of X (namely the set of points where X is not scheme-like) is smaller than the one of X. Hence we have a picture in mind that an algebraic space is formed by adding something of lower dimension to a scheme *at infinity* (see [11, p. 18]).

This will not be used later in our text. However, we believe that having such a geometric picture in mind should be helpful.

A.4 Category Fibred in Groupoids

Recall that a *groupoid* is a category whose morphisms are all isomorphisms.

Definition A.4.1. *Let S be any scheme. A* **category fibred in groupoids** *over* (Sch/S) *is a category X with a morphism $p : \mathsf{X} \to (\mathrm{Sch}/S)$* (*called the* **structural morphism**) *such that:*

1. *For any morphism $V \xrightarrow{\phi} U$ in (Sch/S) and any object x of X such that $p(x) = U$, there exists a morphism $y \xrightarrow{f} x$ in X such that $p(f) = \phi$.*

$$\begin{tikzcd}
y \arrow[r, dotted] \arrow[d, dotted, "f"', ""{name=F}] & V \arrow[d, "\phi", ""'{name=P}] \\
x \arrow[r] & U
\arrow[from=F, to=P, dotted, shorten=2mm]
\end{tikzcd}$$

2. For any diagram

$$\begin{tikzcd}
y \arrow[dr, "f"] & \\
& x \\
z \arrow[ur, "g"'] &
\end{tikzcd}$$

in X *whose image in* (Sch/S) *is*

$$\begin{tikzcd}
V \arrow[dr, "\phi"'] & \\
& U \\
W \arrow[ur, "\psi"] &
\end{tikzcd}$$

and for any morphism $V \xrightarrow{\chi} W$ *in* (Sch/S) *such that* $\phi = \psi \circ \chi$*, there exists a* **unique** *morphism* $y \xrightarrow{h} z$ *in* X *such that* $f = g \circ h$ *and* $p(h) = \chi$*.*

$$\begin{tikzcd}
y \arrow[rr] \arrow[dr, "f"] \arrow[dd, dotted, "h"'] & & V \arrow[dr, "\phi"] \arrow[dd, "\chi"' near start] & \\
& x \arrow[rr] & & U \\
z \arrow[rr] \arrow[ur, "g"'] & & W \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Remark A.4.2. In [85], this is called an *S-groupoid*. However, it is quite confusing to deal with both *groupoids* and *category fibred in groupoids*. Hence we avoid to call a category fibred in groupoids in such an abbreviated way.

For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$, we denote by X_U or $\mathsf{X}(U)$ the fiber of X over U whose objects are those $x \in \mathrm{Ob}\,\mathsf{X}$ such that $p(x) = U$ and whose morphisms are those $f \in \mathrm{Mor}\,\mathsf{X}$ such that $p(f) = \mathrm{id}_U$. Then we see from the definition that the category X_U is a groupoid.

From the definition, we see that for any morphism $V \xrightarrow{\phi} U$ in (Sch/S) and any object $x \in \mathsf{X}_U$, there exists an object $y \xrightarrow{f} x$ (unique up to an isomorphism) such that $p(y) = V$ and $p(f) = \phi$. Let us we specify once for each $V \xrightarrow{\phi} U$ and each $x \in \mathsf{X}_U$ a choice of $y \xrightarrow{f} x$, and write it as $\phi^* x \to x$ or $x_V \to x$. Moreover, for each $f \in \operatorname{Mor} \mathsf{X}_U$, we denote by $\phi^* f$ or $f|_V$ the unique morphism g in the diagram

$$\begin{tikzcd}
\phi^* x' \arrow[r] \arrow[d, "g"'] & x' \arrow[d, "f"] \\
\phi^* x \arrow[r] & x.
\end{tikzcd}$$

Then we have defined a functor $\phi^* : \mathsf{X}_U \to \mathsf{X}_V$ for any morphism $V \xrightarrow{\phi} U$ in (Sch/S). We call this functor the *functor of base change* by ϕ and denote it also by $\cdot|_V$. For any two morphisms $W \xrightarrow{\psi} V$ and $V \xrightarrow{\phi} U$ in (Sch/S), we have a canonical isomorphism (up to 2-isomorphisms) between the two functors $\psi^* \circ \phi^*$ and $(\phi \circ \psi)^*$.

Note that the groupoids form a 2-category (see Section A.1.2) (Gr) whose objects are groupoids, whose 1-morphisms are functors between groupoids, and whose 2-morphisms the natural transformations between them. Then the above statements allow us to associate to the category fibred in groupoids X a 2-functor F from (Sch/S) (viewed as a 2-category in the canonical way) to (Gr), by assigning to each U in (Sch/S) the groupoid $\mathsf{F}(U) = \mathsf{X}_U$ in (Gr), and by assigning to each morphism $\phi : U \to V$ in (Sch/S) the functor (1-morphism) $\phi^* : \mathsf{X}_V \to \mathsf{X}_U$ between groupoids. Two different choices made above in the construction of the base change functors may result in two different 2-functors, but the 2-functors thus defined must be canonically isomorphic. More precisely, as an example, if for $x \in \operatorname{Ob} \mathsf{X}_U$ and $\phi : V \to U$ we make different choices

$$y \xrightarrow{f} x$$
$$y' \xrightarrow{f'} x$$

and construct different functors F and F' such that

$$\mathsf{F}(\phi^*)(x) = y$$
$$\mathsf{F}'(\phi^*)(x) = y'.$$

Then by the definition of the category fibred in groupoids, there is a unique isomorphism $y' \to y$. Hence the 2-functors F and F' are canonically isomorphic.

Definition A.4.3. *A* **morphism** $f : \mathsf{X} \to \mathsf{Y}$ *of categories fibred in groupoids over* S *is a functor between the categories, such that* $p_{\mathsf{Y}} \circ f = p_{\mathsf{X}}$.

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, "f"] \arrow[dr, "p_{\mathsf{X}}"'] & & \mathsf{Y} \arrow[dl, "p_{\mathsf{Y}}"] \\
& S, &
\end{tikzcd}$$

where p_{X} *and* p_{Y} *are structural morphisms of respectively* X *and* Y.

We denote by $\mathsf{Hom}_S(\mathsf{X}, \mathsf{Y})$ the category whose objects are morphisms of categories fibred in groupoids and whose morphisms are natural transformations.

Definition A.4.4. *A morphism* $f : \mathsf{X} \to \mathsf{Y}$ *of categories fibred in groupoids is a* **monomorphism** (*resp. an* **isomorphism**) *if, for all* $U \in \mathrm{Ob}(\mathrm{Sch}/S)$*, the functor* $f_U : \mathsf{X}_U \to \mathsf{Y}_U$ *is fully faithful* (*resp. an equivalence of categories*) (*see Remark A.1.2.7*)*. If* f *is an isomorphism of categories fibred in groupoids, then we say that* X *and* Y *are isomorphic.*

The categories fibred in groupoids over (Sch/S) also form a 2-category $(\text{Ct-F-Gr}/S)$: the objects are categories fibred in groupoids, the 1-morphisms are the morphisms defined above between the underlying categories of the categories fibred in groupoids that are compatible with the structural morphisms, and the 2-morphisms are natural transformations between the functors defined to be 1-morphisms.

Any sheaf F on (Sch/S) is naturally identified with a category fibred in groupoids over (Sch/S), by considering for any scheme U the set $F(U)$ to be a category whose morphisms are all identities. Then $F(U)$ is a groupoid for any U, and the conditions of being a category fibred in groupoids are naturally verified. It is immediate that the category of spaces, algebraic spaces and schemes are all natural full sub-2-categories of $(\text{Ct-F-Gr}/S)$.

Example A.4.5. (Fiber product of categories fibred in groupoids) Consider the following diagram:

$$\begin{tikzcd}
& \mathsf{X}'' \arrow[d, "F''"] \\
\mathsf{X}' \arrow[r, "F'"] & \mathsf{Y}
\end{tikzcd}$$

where F and F' are two 1-morphisms in $(\text{Ct-F-Gr}/S)$. We are going to construct the fiber product $\mathsf{X}' \underset{F', \mathsf{Y}, F''}{\times} \mathsf{X}''$.

For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$, the fiber $(\mathsf{X}' \underset{F',\mathsf{Y},F''}{\times} \mathsf{X}'')_U$ consist of objects in the form

$$(x', x'', g)$$

where $x' \in \mathrm{Ob}\,\mathsf{X}'_U$, where $x'' \in \mathrm{Ob}\,\mathsf{X}''_U$, and where g is a morphism $F'(x') \to F''(x'')$ in Y_U. A morphism from (x'_1, x''_1, g_1) to (x'_2, x''_2, g_2) in this fiber is a pair

$$(x'_1 \xrightarrow{f'} x'_2, x''_1 \xrightarrow{f''} x''_2),$$

where $f' \in \mathrm{Mor}\,\mathsf{X}'_U$ and $f'' \in \mathrm{Mor}\,\mathsf{X}''_U$, such that

$$g_2 \circ F'(f') = F''(f'') \circ g_1.$$

Finally, for any morphism $V \xrightarrow{\phi} U$ in (Sch/S), we have

$$\phi^*(x', x'', g) = (\phi^* x', \phi^* x'', \phi^* g)$$

and

$$\phi^*(f', f'') = (\phi^* f', \phi^* f'').$$

A.5 Stacks

Definition A.5.1. *A* **stack** X *over S is a category fibred in groupoids over* (Sch/S) *satisfying:*

1. (**Prestack**:) *For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and any $x, y \in \mathrm{Ob}\,\mathsf{X}_U$, the presheaf*

$$\begin{aligned} \underline{\mathrm{Isom}}(x, y) : (\mathrm{Sch}/U) &\to (\mathrm{Sets}) \\ (V \to U) &\mapsto \mathrm{Hom}_{\mathsf{X}_V}(x|_V, y|_V) \end{aligned}$$

   is a sheaf over (Sch/S).

2. *For any covering family $(V_i \xrightarrow{\phi_i} U)_{i \in I}$ in (Sch/S), the* **descent datum** *relative to the covering family is* **effective**.

Remark A.5.2. Let us explain condition 2 explicitly. Write $V_{ij} = V_i \underset{U}{\times} V_j$ and $V_{ijk} = V_i \underset{U}{\times} V_j \underset{U}{\times} V_k$ for any $i, j, k \in I$, and write for any object or morphism the pullback in V_{ij} (resp. V_{ijk}) as $\cdot|_{V_{ij}}$ (resp. $\cdot|_{V_{ijk}}$). A *descent datum* for the covering family $(V_i \xrightarrow{\phi_i} U)_{i\in I}$ in $(\mathrm{Sch}\,/S)$ consists of the following information: for any $x_i \in \mathrm{Ob}\,\mathsf{X}_{V_i}$, a morphism $f_{ij} : x_i|_{V_{ij}} \overset{\sim}{\to} x_j|_{V_{ij}}$ in $\mathsf{X}_{V_{ij}}$, and we have the so-called *cocycle condition*

$$(f_{ik}|_{V_{ijk}}) = (f_{jk}|_{V_{ijk}}) \circ (f_{ij}|_{V_{ijk}})$$

in $\mathsf{X}_{V_{ijk}}$. A descent datum $(x_i, f_{ij})_{i\in I}$ is called *effective* if there exist an object $x \in \mathrm{Ob}\,\mathsf{X}_U$ and isomorphisms

$$f_i : (x|_{V_i}) \overset{\sim}{\to} x_i$$

in X_{V_i} for all $i \in I$, such that

$$(f_j|_{V_{ij}}) = f_{ij} \circ (f_i|_{V_{ij}})$$

for all $i, j \in I$. Condition 1 of Definition A.5.1 assures that the object x and the morphisms f_i are canonically determined up to (2-)isomorphisms.

Alternatively, we may define stacks in the following language of 2-category. Let us recall (in page 629) how a 2-functor (up to canonical isomorphisms) is associated to a category fibred in groupoids.

Definition A.5.3. *A* **stack** F *over* S *is a* **sheaf of groupoids** *on* $(\mathrm{Sch}\,/S)$*, i,e. a 2-functor (presheaf)* $\mathsf{F} : (\mathrm{Sch}\,/S) \to (\mathrm{Gr})$ *satisfying the following axioms: Let* $(V_i \to U)_{i\in I}$ *be a covering of* U *in the site* $(\mathrm{Sch}\,/S)$*.*

1. *If* x *and* y *are two objects of* $\mathsf{F}(U)$*, and* $f_i : x|_{V_i} \to y|_{V_i}$ *are morphisms such that* $f_i|_{V_{ij}} = f_j|_{V_{ij}}$*, then there is a morphism* $f : x \to y$ *such that* $f|_{V_i} = f_i$*.*
2. *If* x *and* y *are two objects of* $\mathsf{F}(U)$ *(where* $\mathsf{F}(U)$ *is a groupoid now), and* $f : x \to y$ *and* $g : x \to y$ *are two morphisms such that* $f|_{V_i} = g|_{V_i}$*, then* $f = g$*.*
3. *If* x_i *are objects of* $\mathsf{F}(V_i)$ *and* $f_{ij} : x_i|_{V_{ij}} \to x_j|_{V_{ij}}$ *are morphisms satisfying the* **cocycle condition** $(f_{ik}|_{V_{ijk}}) = (f_{jk}|_{V_{ijk}}) \circ (f_{ij}|_{V_{ijk}})$*, then there are* $x \in \mathsf{F}(U)$ *and* $f_i : x|_{V_i} \overset{\sim}{\to} x_i$ *such that* $(f_j|_{V_{ij}}) = f_{ij} \circ (f_i|_{V_{ij}})$*.*

It is clear that the two definitions are equivalent to each other.

Remark A.5.4. The terminology *sheaf of groupoids* is the analogue of *sheaf of sets* used in the definition of a space (in Section A.3). Namely, for a sheaf of sets F, we require each $F(U)$ to be a set; whereas for a sheaf of groupoids F, we require each $\mathsf{F}(U)$ to be a groupoid.

We denote by (St/S) the full sub-2-category of $(\text{Ct-F-Gr}/S)$ whose objects are S-stacks. Stacks are called *champs* in French.

Definition A.5.5. *A* **commutative diagram** *of stacks is a diagram*

$$\begin{tikzcd}
\mathsf{X} \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathsf{Z} \\
& \mathsf{Y} \arrow[ur, "g"'] \arrow[u, Rightarrow, "\alpha"] &
\end{tikzcd}$$

such that $\alpha : g \circ f \to h$ is an isomorphism of functors.

Definition A.5.6. *A* **morphism of stacks** *$f : \mathsf{X} \to \mathsf{Y}$ is a morphism of categories fibred in groupoids. The morphism f is a* **monomorphism** *(resp. an* **isomorphism***) if it is a monomorphism (resp. an isomorphism) of categories fibred in groupoids (defined as in Definition A.4.4). If f is an isomorphism of categories fibred in groupoids, then we say that the stacks F and G are isomorphic.*

The category $\mathsf{Hom}_S(\mathsf{X}, \mathsf{Y})$ is defined in a same way either as the category of morphisms of stacks or as the category of morphisms of categories fibred in groupoids.

Lemma A.5.7. *Let X be an S-stack and U an S-scheme. The functor*

$$u : \mathsf{Hom}_S(U, \mathsf{X}) \to \mathsf{X}(U)$$

sending a morphism of stacks $f : (\mathrm{Sch}/U) \to \mathsf{X}$ to $f(\mathrm{id}_U)$ is an equivalence of categories.

Proof. Follows from Yoneda's lemma. □

This useful observation that we will use very often means that an object of X lying over U is equivalent to a morphism (of stacks) from U to X.

The following construction is the analogue of a sheaf associated to a presheaf for the prestacks:

Lemma A.5.8. *For any S-prestack X there is associated canonically an S-stack $\widetilde{\mathsf{X}}$ with an 1-morphism of S-groupoids*

$$\iota : \mathsf{X} \to \widetilde{\mathsf{X}}$$

that induces for any S-stack Y an equivalence of categories

$$\mathsf{Hom}_S(\widetilde{\mathsf{X}}, \mathsf{Y}) \xrightarrow{\iota^*} \mathsf{Hom}_S(\mathsf{X}, \mathsf{Y}).$$

Proof. Let us we describe the construction of $\widetilde{\mathsf{X}}$:

For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$, the objects of the category $\widetilde{\mathsf{X}}$ are the triples

$$\widetilde{x} = (U' \to U, x', f''),$$

where $\{U' \to U\}$ is a covering family (of only one element) in (Sch/S), where x' an object of $\mathsf{X}_{U'}$, and where f'' a descent datum for x' relative to $U' \to U$. For $\widetilde{x}_1, \widetilde{x}_2 \in \mathrm{Ob}\,\widetilde{\mathsf{X}}_U$, a morphism from $\widetilde{x}_1$ to $\widetilde{x}_2$ in $\widetilde{\mathsf{X}}_U$ is a morphism (with obvious notations)

$$g' : x'_1|_{U'_1 \underset{U}{\times} U'_2} \xrightarrow{\sim} x'_2|_{U'_1 \underset{U}{\times} U'_2}$$

in $\mathsf{X}_{U'_1 \underset{U}{\times} U'_2}$ compatible with the descent data f''_1 and f''_2. For any morphism $\phi : V \to U$ in (Sch/S), the functor of base change $\phi^* : \widetilde{\mathsf{X}}_U \to \widetilde{\mathsf{X}}_V$ is defined in an obvious way. Finally, for any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and all $x \in \mathrm{Ob}\,\mathsf{X}_U$, we have

$$\iota(x) = (U \overset{\mathrm{id}_U}{\to} U, x, \mathrm{id}_U).$$

The lemma is then proved by routine verifications. □

Remark A.5.9. For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$, the functor $\iota_U : \mathsf{X}_U \to \widetilde{\mathsf{X}}_U$ induced by ι is fully faithful. That is, ι is a monomorphism of categories fibred in groupoids. If X is an S-stack, then $\iota : \mathsf{X} \to \widetilde{\mathsf{X}}$ is an equivalence of categories (an isomorphism of S-stacks).

The sub-2-category (St/S) of $(\text{Ct-F-Gr}/S)$ is stable under taking arbitrary projective limits, and in particular stable under taking fiber products. However, it is not stable under taking inductive limits, but we may consider the associated S-stacks of the prestacks defined in a natural way by taking the inductive limits. (The process of taking inductive limits is stable in $(\text{Ct-F-Gr}/S)$.)

Example A.5.10. Let us demonstrate here the sum $\coprod_{i \in I} \mathsf{X}_i$ of a family $(\mathsf{X}_i)_{i \in I}$ of S-stacks. For any $U \in \mathrm{Ob}(\mathrm{Sch}/S)$, the objects of the fiber $(\coprod_{i \in I} \mathsf{X}_i)_U$ are

$$(U = \coprod_{i \in I} U_i, (x_i)_{i \in I})$$

where $U = \coprod_{i \in I} U_i$ is a decomposition of U into a disjoint union indexed by $i \in I$ of the objects U_i of (Sch/S), and where $x_i \in \mathrm{Ob}\,\mathsf{X}_{i,U_i}$ for each $i \in I$. A morphism in the fiber is a couple

$$(U = \coprod_{i \in I} U_i, (x'_i \xrightarrow{f_i} x''_i)),$$

where $U = \coprod\limits_{i \in I} U_i$ is a decomposition of U as above, and where $x'_i \xrightarrow{f_i} x''_i$ is a morphism in X_{i,U_i} for each $i \in I$. Finally, for each morphism $V \xrightarrow{\phi} U$ in $(\mathrm{Sch}\,/S)$, we have

$$\phi^*(U = \coprod_{i \in I} U_i, (x_i)_{i \in I}) = (V = \coprod_{i \in I} \phi^{-1}(U_i), (\phi^* x_i)_{i \in I})$$

and $\phi^* : \mathrm{Mor}\,\mathsf{X}_U \to \mathrm{Mor}\,\mathsf{X}_V$ is defined in an obvious way.

The category $(\mathrm{Spc}\,/S)$ are naturally a full subcategory of $(\mathrm{St}\,/S)$ by viewing 1-functors as 2-functors in the canonical way. The categories $(\text{Alg-Spc}\,/S)$ and $(\mathrm{Sch}\,/S)$ are then both subcategories of $(\mathrm{St}\,/S)$.

Definition A.5.11. *A stack* Y *is a* **substack** *of* X *if it is a full subcategory of* X *and*

1. *If an object x of X is in Y, then all objects isomorphic to x are also in Y.*
2. *For all morphisms of schemes $f : V \to U$, if x is in $\mathsf{Y}(U)$, then f^*x is in $\mathsf{Y}(V)$.*
3. *Let $\{V_i \to U\}$ be a covering of U in the site $(\mathrm{Sch}\,/S)$. Then x is in $\mathsf{Y}(U)$ if and only if $x|_{V_i}$ is in $\mathsf{Y}(V_i)$ for all i.*

Definition A.5.12. *Let $f : \mathsf{Z} \to \mathsf{X}$ be a (1-)morphism of S-stacks. We say that f is an* **epimorphism** *if, for any $U \in \mathrm{Ob}(\mathrm{Sch}\,/S)$ and any $x \in \mathrm{Ob}\,\mathsf{X}_U$, there is a covering $U' \to U$ in the site $(\mathrm{Sch}\,/S)$ and $z' \in \mathrm{Ob}\,\mathsf{Z}_{U'}$ such that $f_{U'}(z')$ is isomorphic to $x|_{U'}$ in $\mathsf{X}_{U'}$.*

Then we have the following proposition:

Proposition A.5.13 ([85, 3.7])**.** *Let $f : \mathsf{Z} \to \mathsf{X}$ be an 1-morphism of S-stacks, and Y a full subcategory of X such that, for any $U \in \mathrm{Ob}(\mathrm{Sch}\,/S)$, the objects of Y_U are those $x \in \mathrm{Ob}\,\mathsf{X}_U$ that are* **"locally in the essential image of** f**"** *in the following sense: There exists a covering $U' \to U$ in the* *site* (Sch/S), *an object* $z' \in \mathsf{Z}_{U'}$, *and an isomorphism from* $f_{U'}(z')$ *to* $x|_{U'}$ *in* $\mathsf{X}_{U'}$. *Then* Y *is a substack of* X, *and* f *factors through*

$$\mathsf{Z} \xrightarrow{\bar{f}} \mathsf{Y} \xrightarrow{i} \mathsf{X},$$

where $\bar{f}$ *is an epimorphism and the inclusion* i *is a monomorphism* (*defined as in Definitions A.5.12 and A.5.6*). *Moreover,* f *is a monomorphism if and only if* $\bar{f}$ *is an isomorphism, and* f *is an epimorphism if and only if* $\mathsf{Y} = \mathsf{X}$. *Finally, if we have another factorization of* f *to the composition of an epimorphism* $\bar{f}' : \mathsf{Z} \to \mathsf{Y}'$ *and a monomorphism* $i' : \mathsf{Y}' \to \mathsf{X}$, *then we have a commutative diagram* (*up to 2-isomorphisms*)

$$\begin{tikzcd}
\mathsf{Z}' \arrow[r, "\bar{f}'"] \arrow[dr] & \mathsf{Y}' \arrow[r, "i'"] \arrow[d, "\bar{i}'"] & \mathsf{X} \\
& \mathsf{Y} \arrow[ur] &
\end{tikzcd}$$

in (St/S) (*where* Y *is the same substack* Y', *and where the 1-morphism* $\bar{i}'$ *induced by* $i' : \mathsf{Y}' \to \mathsf{X}$ *is an 1-isomorphism*).

Corollary A.5.14. *A 1-morphism in* (St/S) *is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Definition A.5.15. *A stack* X *is said to be* **representable by an algebraic space** (*resp.* **scheme**) *if there is an algebraic space* (*resp. scheme*) X *such that the stack associated to* X *is isomorphic to* X.

Suppose "P" is a property of algebraic spaces (*resp. schemes*) *and* X *is a representable stack. Then we say that* X *has property "P" if and only if* X *has property "P".* (*See Definition A.3.1.6.*)

Remark A.5.16. Later on in our text, if not particularly stated, an algebraic stack will be called representable if it is representable by an algebraic space.

Remark A.5.17. The existence of an algebraic space (or a scheme) X such that the stack associated to X is isomorphic to X means that, for each $U \in \mathrm{Ob}\,\mathsf{X}_U$, the isomorphism classes of objects in X_U are one-one corresponded to elements of $X(U)$, and the morphisms between them are also one-one corresponded. To be more precise, if x and y are isomorphic objects (not necessarily different) in X_U, then since they are corresponded to the same element in $X(U)$, any morphism between them must be sent into the identity.

Since an equivalence of categories is a fully faithful functor, there must be only one isomorphism between x and y in X_U. In particular, if $x = y$, then we see that x cannot have any automorphism other than the identity id_x of x.

Definition A.5.18. *A morphism of stacks $f : \mathsf{X} \to \mathsf{Y}$ is called* **representable** *if for all objects U in (Sch/S) and morphisms $U \to \mathsf{Y}$, the fiber product stack $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ is representable by an algebraic space.*

Suppose "P" is a property of morphisms of algebraic spaces that is local in nature on the target for the étale topology on (Sch/S) and stable under arbitrary base change. Then we say that f has property "P" if for every $U \to \mathsf{Y}$, the pullback morphism $U \underset{\mathsf{Y}}{\times} \mathsf{X} \to U$ (of algebraic spaces) has property "P".

Remark A.5.19. Here is a list of such properties summarized from [85, 3.10] (cf. Definition A.3.1.7 and Definition A.3.3.8):

1. surjective, radicial, and universally bijective [46, I, 3.6.1, 3.6.4 and 3.7.6];
2. universally open, universally closed, separated, quasi-compact, locally of finite type, locally of finite presentation, of finite type, of finite presentation, being an immersion, being an open immersion, being a closed immersion, being an open immersion with dense image, affine, quasi-affine, entire, finite, quasi-finite, and proper [50, IV, 2.5.1], [52, IV, 17.7.5];
3. fibers are geometrically connected, geometrically reduced, and geometrically irreducible [50, IV, 4.5.6, and IV, 4.6.10];
4. locally of finite type and of relative dimension $\leq d$, and locally of finite type and of pure relative dimension d [50, IV, 5.5.1, and IV, 4.1.4];
5. flat, unramified, smooth, and étale [50, IV, 2.2.13], [52, IV, 17.7.4].

Lemma A.5.20. *Let $f : \mathsf{X} \to \mathsf{Y}$ be a representable morphism of S-stacks and $\mathsf{Y}' \to \mathsf{Y}$ an arbitrary morphism of S-stacks. Then the morphism $f' : \mathsf{X}' = \mathsf{X} \underset{\mathsf{Y}}{\times} \mathsf{Y}' \to \mathsf{Y}'$ induced by base change is representable.*

Moreover, let "P" be any property that is local in nature on the target for the topology chosen on the étale site (Sch/S) and stable under arbitrary base change. If f has property "P", then f' has property "P" too.

Proof. This is immediate from the definitions. ☐

Lemma A.5.21 ([85, 3.12(a)])**.** *Let $f : \mathsf{X} \to \mathsf{Y}$ be a representable morphism of S-stacks. If Y is representable by an algebraic space, then so is X.*

Remark A.5.22. Suppose $\mathsf{X} \to \mathsf{Y}$ is a morphism of stacks, and X, Y are representable by algebraic spaces X, Y respectively. Then there is a canonically induced morphism $X \to Y$ formed by sending each isomorphism class of $\mathsf{X}(U)$ to the corresponding isomorphism class in $\mathsf{Y}(U)$ for each $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ (see Remark A.5.17). Recall that diagrams of stacks are commutative only up to 2-isomorphisms. However, if the stacks involved in a commutative diagram of stacks are all representable, then the induced diagram of algebraic spaces must be also commutative, since the isomorphism classes are indeed one-one corresponded under 2-isomorphisms.

Let $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ be the obvious diagonal morphism. A morphism from a scheme U to $\mathsf{X} \underset{S}{\times} \mathsf{X}$ is equivalent to two objects X_1, X_2 of X_U. By taking the fiber product, we obtain:

$$\begin{tikzcd}
\underline{\mathrm{Isom}}_U(X_1, X_2) \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\
U \arrow[r, "(X_1, X_2)"] & \mathsf{X} \underset{S}{\times} \mathsf{X}
\end{tikzcd}$$

Proposition A.5.23. *Let X be an S-stack. The following are equivalent:*

1. *The morphism $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ is representable.*
2. *The S-stack $\underline{\mathrm{Isom}}_U(X_1, X_2)$ is representable for any S-scheme U, and any $X_1, X_2 \in \mathrm{Ob}\,\mathsf{X}_U$.*
3. *For any S-algebraic space X, any morphism $X \to \mathsf{X}$ is representable.*
4. *For any S-scheme U, any morphism $U \to \mathsf{X}$ is representable.*
5. *For any S-schemes U, V and any morphisms $U \to \mathsf{X}$ and $V \to \mathsf{X}$, the fiber product $U \underset{\mathsf{X}}{\times} V$ is representable.*

Proof. The implications 1 ⇔ 2 and 3 ⇒ 4 ⇔ 5 follow immediately from the definitions.

1 ⇒ 5: Assume that Δ_{X} is representable. One checks easily that the following diagram is cartesian; namely, $U \times_{\mathsf{X}} V$ is a fiber product in the following commutative diagram:

$$\begin{tikzcd}
U \times_{\mathsf{X}} V \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\
U \times_{S} V \arrow[r] & \mathsf{X} \times_{S} \mathsf{X}
\end{tikzcd}$$

Then $U \times_{\mathsf{X}} V$ is representable.

5 ⇒ 1: Consider the following diagram:

$$\begin{tikzcd}
U \times_{\mathsf{X} \times_{S} \mathsf{X}} \mathsf{X} \arrow[r] \arrow[d] & U \times_{\mathsf{X}} U \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\
U \arrow[r, "\Delta_U"] & U \times_{S} U \arrow[r] & \mathsf{X} \times_{S} \mathsf{X}
\end{tikzcd}$$

Both squares are cartesian and $U \times_{\mathsf{X}} U$ is representable by hypothesis. Hence $U \times_{\mathsf{X} \times_{S} \mathsf{X}} \mathsf{X}$ is representable.

5 ⇒ 3: For any S-scheme U and any morphism $U \to \mathsf{X}$, the induced morphism $X \times_{\mathsf{X}} U \to X$ is representable by the following diagram:

$$\begin{tikzcd}
V \times_{\mathsf{X}} U \arrow[r] \arrow[d] & X \times_{\mathsf{X}} U \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r] & X \arrow[r] & \mathsf{X}
\end{tikzcd}$$

Hence $X \times_{\mathsf{X}} U$ is representable by Lemma A.5.21. □

A.6 Algebraic Stacks

There are different definitions of an algebraic stack in the existing literature that are not equivalent to each other. (See [28], [14], and [85].) The following is the definition taken from [85].

Definition A.6.1. *An* **algebraic stack** *is a stack* X *such that:*

1. (**Quasi-separateness**, *cf. Remark A.2.5:*) *The diagonal morphism* Δ_{X} *is representable, quasi-compact and separated.* (*To be more precise, for any* $U \in \mathrm{Ob}(\mathrm{Sch}\,/S)$ *and any* $x, y \in \mathrm{Ob}\,\mathsf{X}_U$*, we require that the presheaf* $\underline{\mathrm{Isom}}(x, y)$ *of isomorphisms from* x *to* y *is an algebraic space quasi-compact and separated over* U*. The separateness here is automatic for schemes and algebraic spaces, because the diagonal morphisms are monomorphisms for schemes and algebraic spaces by definition* (*see Definition A.3.2*)*, and monomorphisms are automatically separated* (*see [46, I, 55.1]*)*.*)
2. *There exist an* S*-algebraic space* X *and a morphism* (*called an* **atlas** *or a* **presentation** *of* X) *of* S*-stacks* $p : X \to \mathsf{X}$ (*automatically representable by Proposition A.5.23*) *that is surjective* (*defined as in Definition A.5.12*) *and smooth.*

However, for the purpose of studying moduli spaces of abelian schemes with additional structures, it is usually desirable to consider the following narrower definition:

Definition A.6.2. *A* **Deligne-Mumford** S**-stack** *is an algebraic* S*-stack that admits an étale presentation.* (*That is, in Definition A.6.1, p is moreover an étale morphism.*)

Remark A.6.3. From now on, if not particularly mentioned, we will refer algebraic stacks to Deligne-Mumford stacks, and use the term Artin stacks to call the more general algebraic stacks that we originally defined. If it is absolutely necessary to make the distinction clear, then we will also specify Deligne-Mumford stacks explicitly. A presentation of a Deligne-Mumford stack X will be understood to be an étale and surjective morphism $p : X \to \mathsf{X}$ from an algebraic space X to the stack X. (This is slightly different with the definition given in [28]. The presentation is assumed to be from a *scheme* there.)

We denote by (Alg-St $/S$) the full subcategory of algebraic stacks in (St $/S$). An S-stack associated to an S-algebraic space is clearly an algebraic (Deligne-Mumford) stack. Hence (Alg-Spc $/S$) is a sub-2-category of (Alg-St $/S$).

Proposition A.6.4 ([85, 4.4]). *An algebraic stack* X *over* S *is representable if and only if the diagonal 1-morphism*

$$\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X} \tag{A.6.5}$$

is a monomorphism.

Remark A.6.6. Suppose a stack X has the property that in each category X_U the only automorphisms are the identity morphisms (cf. Remark A.5.17). Then the set $X(U)$ of isomorphism classes of objects in X_U is a contravariant functor from $(\mathrm{Sch}\,/S)$ to $(\mathrm{Sets}\,/S)$, which is a sheaf by the definitions of a stack. Therefore X is a *space* (defined as in Definition A.3) which is canonically isomorphic to X. In fact, under this assumption, one can verify that the diagonal morphism (A.6.5) of X is a monomorphism (see [85, 2.4.1.1]). Hence we see that X is representable.

For the remaining parts of this section, the reader is encouraged to compare the definitions and properties with the corresponding ones of algebraic spaces exposed in Section A.3.

A.6.1 Properties of Algebraic Stacks

Let us recall how a representable stack or a representable morphism between stacks is said to have a property "P" (in Definition A.5.15 and Definition A.5.18). The properties of an algebraic stack are characterized by its presentation:

Definition A.6.1.1. *Let "P" be a property of algebraic spaces that is local in nature for the étale topology.* (*See Definition A.3.1.6.*) *Then we say that an* S*-algebraic stack* X *has property "P" if and only if for one* (*and then for all*) *presentation* $p : X \to \mathsf{X}$*, the algebraic space* X *has property "P".* (*See Definition A.3.1.6.*)

Definition A.6.1.2. *Let "P" be a property of morphisms of algebraic spaces that is local in nature for the étale topology.* (*See Definition A.3.1.7.*) *Suppose* $f : \mathsf{X} \to \mathsf{Y}$ *is a morphism of algebraic stacks. Then we say that* f *has property "P" if and only if for one* (*and then for all*) *commutative diagram of stacks*

$$\begin{tikzcd}
X' \arrow[r, "p'"] \arrow[dr, "f'"'] & Y \underset{\mathsf{Y}}{\times} \mathsf{X} \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "f"] \\
& Y \arrow[r, "p"'] & \mathsf{Y}
\end{tikzcd} ,$$

where p and p' are presentations for respectively Y *and* $Y \underset{\mathsf{Y}}{\times} \mathsf{X}$, f' *has property* P.

Definition A.6.1.3. *An algebraic stack is called* **quasi-compact** *if there exists a presentation* $p : X \to \mathsf{X}$ *with* X *quasi-compact. A morphism* $f : \mathsf{X} \to \mathsf{Y}$ *of algebraic stacks is called* **quasi-compact** *if for any morphism from a scheme* Y *into* Y, *the fiber product* $Y \underset{\mathsf{Y}}{\times} \mathsf{X}$ *is a quasi-compact algebraic stack over* Y.

Lemma A.6.1.4. *In Definition A.6.1.3 it suffices to require that there exists a smooth surjection* $f : Y \to \mathsf{X}$ *from a quasi-compact algebraic space* Y.

Proof. Suppose $p : X \to \mathsf{X}$ is any presentation of X, with an open covering of X by affine open subschemes X_α. Then $Y \underset{\mathsf{X}}{\times} X_\alpha$ are open in $Y \underset{\mathsf{X}}{\times} X$, and their images W_α in Y are again open because étale maps (or smooth maps if X is an Artin stack) are open. By assumption, Y is quasi-compact, so there is a finite subset α_i of all possible α's such that W_{α_i} cover Y, and hence cover X by surjectivity of f. As a result, $Y \underset{\mathsf{X}}{\times} X_{\alpha_i}$ cover X, and hence X_{α_i} cover X as well. Therefore we may replace X by the finite union of X_{α_i} and obtain a quasi-compact presentation of X. □

Definition A.6.1.5. *We define a morphism* $f : \mathsf{X} \to \mathsf{Y}$ *to be of* **finite type**, *if it is quasi-compact and locally of finite type; to be* **of finite presentation**, *if it is quasi-compact and locally of finite presentation. An algebraic stack is* **noetherian**, *if it is quasi-compact, quasi-separated and locally noetherian.*

Definition A.6.1.6. *An algebraic stack* X *is* **separated** *if the* (*representable*) *diagonal morphism* Δ_{X} *is universally closed* (*hence proper, since it is separated and of finite type*).

A morphism $f : \mathsf{X} \to \mathsf{Y}$ *of algebraic stacks is* **separated** (*resp.* **quasi-separated**) *if for any morphism* $U \to \mathsf{X}$ *from a separated scheme* U, *the fiber product* $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ *is separated* (*resp. quasi-separated*).

Definition A.6.1.7. *A morphism* $f : \mathsf{X} \to \mathsf{Y}$ *is said to be* **proper** *if it is separated, of finite type and universally closed.*

Let $f : \mathsf{X} \to S$ be a morphism of finite type from an algebraic stack X to a noetherian scheme S. Assume the diagonal map $\mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ is separated and quasi-compact.

Theorem A.6.1.8 (valuative criterion for separateness; see [85, 7.8])**.** *The morphisms f is separated if and only if, for any complete discrete valuation ring R with algebraically closed residue field and any commutative diagram*

$$\begin{tikzcd}
 & \mathsf{X} \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & S
\end{tikzcd}$$

any isomorphism between the restrictions of g_1 and g_2 to the generic point of $\operatorname{Spec}(R)$ *can be extended to an isomorphism between g_1 and g_2.*

Theorem A.6.1.9 (valuative criterion for properness; see [85, 7.12])**.** *Suppose f is separated* (*see Theorem A.6.1.8 above*)*. Then f is proper if and only if, for any discrete valuation ring R with field of fractions K and any commutative diagram*

$$\begin{tikzcd}
 & & \mathsf{X}\ , \arrow[d, "f"] \\
\operatorname{Spec}(K) \arrow[urr, "g"] \arrow[r] & \operatorname{Spec}(R) \arrow[r] & S
\end{tikzcd}$$

there exists a finite extension K' of K such that g extends to $\operatorname{Spec}(R')$*, where R' is the integral closure of R in K':*

$$\begin{tikzcd}
 & & \mathsf{X}\ . \arrow[dd, "f"] \\
\operatorname{Spec}(K') \arrow[r] \arrow[d] & \operatorname{Spec}(R') \arrow[ur, dotted] \arrow[d] & \\
\operatorname{Spec}(K) \arrow[uurr, bend left, "g"] \arrow[r] & \operatorname{Spec}(R) \arrow[r] & S
\end{tikzcd}$$

Remark A.6.1.10. To prove a given f is proper, it suffices to verify the above criterion under the additional hypothesis that R is complete and has an algebraically closed residue field. Furthermore, if there is an *open dense subset* U of X (defined as in Definition A.6.3.4), then it is enough to test only those g's that factor through U. (See [85, 7.12.4].)

Definition A.6.1.11. *The* **disjoint union** $\mathsf{X} = \coprod_{i \in I} \mathsf{X}_i$ *of a family* $(\mathsf{X}_i)_{i \in I}$ *of stacks is the stack a section of which over a scheme U consists of a decomposition $U = \coprod_{i \in I} U_i$ of U and a section of x_i over U_i for each i.*

The empty stack $\emptyset$ is the algebraic stack represented by the empty scheme. A stack is *connected* if it is nonempty and is not the disjoint union of two nonempty stacks.

Proposition A.6.1.12 ([28, Prop. 4.14]; see [85, 4.9])**.** *A locally noetherian algebraic stack is in one and only one way the disjoint union of a family of connected algebraic stacks* (*called its* **connected components**)*.*

We denote by $\pi_0(\mathsf{X})$ the set of connected components of the locally noetherian algebraic stack X. If $p : X \to \mathsf{X}$ is étale and surjective, then $\pi_0(\mathsf{X})$ is the cokernel of the two maps, namely a quotient of the equivalence relation of sets defined by the two maps.

$$\pi_0(X \underset{\mathsf{X}}{\times} X) \rightrightarrows \pi_0(X) \to \pi_0(\mathsf{X}).$$

Definition A.6.1.13. *A substack* Y *of* X *is called* **open** (*resp.* **closed***, resp.* **locally closed**) *if the inclusion morphisms* $\mathsf{Y} \to \mathsf{X}$ *is representable and is an open immersion* (*resp. closed immersion, resp. locally closed immersion*)*.*

Definition A.6.1.14. *An algebraic stack* X *is* **irreducible** *if it is nonempty and for any two open substacks* Y_1 *and* Y_2 *in* X*, their intersection* $\mathsf{Y}_1 \cap \mathsf{Y}_2$ *is nonempty.*

A.6.2 Quasi-Coherent Sheaves on Algebraic Stacks

Definition A.6.2.1. *Let* X *be an algebraic stack. The* **étale site** X_{et} *of* X *is the category with objects the étale morphisms from schemes*

$$u : U \to \mathsf{X}$$

where a morphism from (U, u) *to* (V, v) *is a morphism of schemes* $\phi : U \to V$ *plus a 2-morphism between the 1-morphisms* $u : U \to \mathsf{X}$ *and* $v \circ \phi : U \to \mathsf{X}$*. A collection of morphisms* $\phi_i : (U_i, u_i) \to (U, u)$ *is a covering family if the underlying family of morphisms of schemes is surjective.*

Remark A.6.2.2. This definition is due to [28]. Alternatively, one may define the objects to be the étale morphisms from *algebraic spaces*. Then this is identical with the étale site over an algebraic space introduced by Knutson [76] when X is representable by an algebraic space. However, the categories of sheaves of sets on the étale site of an algebraic stack defined in both ways are equivalent. Hence the difference between the two definitions can be ignored, since the étale site is mainly used for the study of sheaves on it.

The site X_{et} is in a natural way ringed (whose meaning is made clear by the following definitions). When we speak of *sheaves* on X we mean sheaves on X_{et}.

Definition A.6.2.3. *A* **quasi-coherent sheaf** $\mathscr{F}$ *on the algebraic stack* X *is a sheaf consisting of the following set of data:*

1. *For each morphism* $U \to \mathsf{X}$ *where* U *is a scheme, a quasi-coherent sheaf* $\mathscr{F}_U$ *on* U*.*
2. *For each commutative diagram*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & V \arrow[d] \\ & \mathsf{X} \end{tikzcd}$$

an isomorphism $\phi_f : \mathscr{F}_U \xrightarrow{\sim} f^*\mathscr{F}_V$ *satisfying the* **cocycle condition***. Namely, for any commutative diagram*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & V \arrow[r, "g"] \arrow[d] & W \arrow[dl] \\ & \mathsf{X} & \end{tikzcd}$$

we have $\phi_{g\circ f} = \phi_f \circ f^*\phi_g$*.*

We say that $\mathscr{F}$ *is* **coherent** (*resp.* **of finite type***, resp.* **of finite presentation**, **locally free**) *if* $\mathscr{F}_U$ *is coherent* (*resp. of finite type, resp. of finite presentation, resp. locally free*) *for all* U*.*

A morphism of quasi-coherent sheaves $h : \mathscr{F} \to \mathscr{F}'$ *is a collection of morphisms of sheaves* $h_U : \mathscr{F}_U \to \mathscr{F}'_U$ *compatible with all the isomorphisms* ϕ*.*

Definition A.6.2.4. *Let* X *be an algebraic stack. The* **structural sheaf** $\mathscr{O}_{\mathsf{X}}$ *is defined by taking* $(\mathscr{O}_{\mathsf{X}})_U = \mathscr{O}_U$*.*

Remark A.6.2.5. Since a sheaf on a scheme can be obtained by gluing the restriction of an affine covering, it is enough to consider affine schemes.

Proposition A.6.2.6 ([28]; see [85, 14.2.4])**.** *Let* X *be an algebraic stack. Then the functor that to any algebraic stack* $f : \mathsf{T} \to \mathsf{X}$ *associates the* $\mathscr{O}_{\mathsf{X}}$*-sheaf of algebras* $f_*\mathscr{O}_{\mathsf{T}}$ *induces an equivalence between the following two categories:*

1. *The category of algebraic stacks* **schematic** *and affine over* X*.* (*This is defined in an obvious way analogous to schematic morphisms of algebraic spaces. More precisely, a morphism* $f : \mathsf{X} \to \mathsf{Y}$ *is* **schematic** *if for any morphism* $U \to \mathsf{Y}$ *from a scheme* U*, the fiber product* $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ *is representable by a* **scheme**.)

2. *The dual of the category of quasi-coherent* $\mathscr{O}_{\mathsf{X}}$*-algebras.*

Definition A.6.2.7. *Let* $\mathscr{F}$ *be a quasi-coherent sheaf of* $\mathscr{O}_{\mathsf{X}}$*-algebras on an algebraic stack* X*. For each étale morphisms* $u : U \to \mathsf{X}$*, with* U *an affine scheme, let* $\mathscr{A}_U$ *be the integral closure of* $\Gamma(U, (\mathscr{O}_{\mathsf{X}})_U) = \Gamma(U, \mathscr{O}_U)$ *in* $\mathscr{F}_U$*. By [48, II, 6.3.4], the* $\mathscr{A}_U$ *for variable* U *are the sections over* U *of a quasi-coherent sheaf* $\mathscr{A}$ *on* X*, which will be called the* **integral closure** *of* $\mathscr{O}_{\mathsf{X}}$ *in* $\mathscr{F}$*.*

Definition A.6.2.8. *Let* $f : \mathsf{T} \to \mathsf{X}$ *be schematic and affine. The algebraic stack associated by Proposition A.6.2.6 to the integral closure of* $\mathscr{O}_{\mathsf{X}}$ *in* $f^*\mathscr{O}_{\mathsf{T}}$ *will be called the* **normalization** *of* X *with respect to* T*. Its formation is compatible with any étale base change.*

A.6.3 Points and the Zariski Topology of an Algebraic Stack

Let X be an algebraic stack. Consider the set of pairs (x, k), where k is a field over S and where $x : \operatorname{Spec}(k) \to \mathsf{X}$ is an object in $\mathsf{X}_{\operatorname{Spec}(k)}$. (This is slightly different from the definition of a point of algebraic spaces in Definition A.3.3.1. Here we do not assume that the morphism is injective (in the sense of category theory).) We define two elements (x, k) and (x', k') of this set to be *equivalent* if there is a third field k'' with coverings $\operatorname{Spec}(k'') \to \operatorname{Spec}(k)$ and $\operatorname{Spec}(k'') \to \operatorname{Spec}(k')$ in the étale site (Sch/S) such that the induced objects $x|_{\mathrm{Spec}(k)}$ and $x'|_{\mathrm{Spec}(k')}$ are isomorphic in $\mathsf{X}(\mathrm{Spec}(k''))$.

$$\begin{tikzcd}
\mathrm{Spec}(k'') \arrow[r] \arrow[d] & \mathrm{Spec}(k') \arrow[d, "x'"] \\
\mathrm{Spec}(k) \arrow[r, "x"'] & \mathsf{X}
\end{tikzcd} \tag{A.6.3.1}$$

It is clear that this is an equivalence relation.

Definition A.6.3.2. *A* **point** *of the algebraic stack* X *is an equivalence class in the set defined above. The set of points of* X*, called the* **associated underlying topological space**, *is denoted by* $|\mathsf{X}|$.

Suppose $x : \mathrm{Spec}(k) \to \mathsf{X}$ is a representative of a point of X, and $f : \mathsf{X} \to \mathsf{Y}$ is a morphism. Then we have obviously a point of Y, the equivalence class of $f \circ x : \mathrm{Spec}(k) \to \mathsf{X} \to \mathsf{Y}$. Hence we have a map $|f| : |\mathsf{X}| \to |\mathsf{Y}|$, which we often denote by f if there's no confusion.

Definition A.6.3.3. *Let* X *be an algebraic stack. The* **Zariski topology** *on* $|\mathsf{X}|$ *is defined by taking open sets to be subsets of the form* $U = |\mathsf{Y}|$ *for some open substack* Y.

Definition A.6.3.4. *As in the case of algebraic spaces, we may define a morphism* $f : \mathsf{X} \to \mathsf{Y}$ *between algebraic stacks to be open, closed, with dense image, universally closed, etc, via the map* $|f| : |\mathsf{X}| \to |\mathsf{Y}|$ *between their underlying topological space.*

A.6.4 Coarse Moduli Spaces

The results in this section are quoted from [30, I, §8]. The justifications for some of the claims can be found in [71], with explanations supplied by [25].

Let X be an algebraic stack over S.

Definition A.6.4.1. *A* **coarse moduli space** *of* X *is an algebraic space* $[\mathsf{X}]$ *over* S*, with a* S*-morphism* $\pi : \mathsf{X} \to [\mathsf{X}]$ *such that:*

1. *Any* S*-morphism from* X *to an algebraic space* Z *over* S *factors through* π *to be a morphism from* $[\mathsf{X}]$ *to* Z.
2. *If* $\bar{s} : \mathrm{Spec}(k) \to S$ *is a geometric point of* S (*where* k *is algebraically closed*)*, then* π *induces a bijection between the set of isomorphism classes of objects in* X *over* $\bar{s}$ (*namely* S*-morphisms from* s *to* X) *and* $[\mathsf{X}](\bar{s})$.

Remark A.6.4.2. The term *coarse* here is used to be distinguished with the term *fine* in a *fine moduli space* for X, namely an algebraic space that *represents* the algebraic stack.

If S is noetherian and X is separated of finite type over S, then we can show that X admits a coarse moduli space $[\mathsf{X}]$. Here are some of its properties:

1. Let $x : \mathrm{Spec}(k) \to \mathsf{X}$ be a geometric point of X, $\mathscr{O}^h_{\mathsf{X},x}$ is the *strict local ring* of X at x, $\mathscr{O}^h_{[\mathsf{X}],\pi(x)}$ the strict local ring of $[\mathsf{X}]$ at $\pi(x)$, and $\mathrm{Aut}(x)$ the group of automorphisms of the object x in X over $\mathrm{Spec}(k)$. The map π induces an isomorphism

   $$\mathrm{Spec}(\mathscr{O}^h_{\mathsf{X},x})/\,\mathrm{Aut}(x) \xrightarrow{\sim} \mathrm{Spec}(\mathscr{O}^h_{[\mathsf{X}],\pi(x)}).$$

   Suppose $H \subset \mathrm{Aut}(x)$ is a subgroup of the automorphisms of the object x in X over k, which extends to one morphism of the object $\mathrm{Spec}(\mathscr{O}^h_{\mathsf{X},x}) \to \mathsf{X}$ of X over $\mathrm{Spec}(\mathscr{O}^h_{\mathsf{X},x})$. Then the group $\mathrm{Aut}(x)/H$ acts effectively on $\mathscr{O}^h_{\mathsf{X},x}$.

2. If $u : U \to \mathsf{X}$ is étale surjective, then $[\mathsf{X}]$ is the quotient of U by the equivalence relation $U \underset{\mathsf{X}}{\times} U$.

3. The formation of a coarse moduli space does not in general commute with all base changes. However, it commutes with flat base changes, and with all base changes when π is étale. Moreover, if $[\mathsf{X}']$ is a coarse moduli scheme of $\mathsf{X}' := \mathsf{X} \underset{S}{\times} S'$, the morphism

   $$[\mathsf{X}'] \to [\mathsf{X}] \underset{S}{\times} S'$$

   is *radicial* (or *universally injective*. Namely, for each field K, the induced morphism is injective (see [46, I, 3.5.4])).

Appendix B

Deformations and Artin's Criterion

In this appendix we review the basic notions of infinitesimal deformations, which can be generalized in the language of 2-categories, and give a proof of Artin's criterion for algebraic stacks using his theory of algebraization.

B.1 Infinitesimal Deformations

Let Λ be a complete noetherian local ring with residue field k.

Notations B.1.1. *We denote by C_Λ the category of Artinian local Λ-algebras with residue field k and by $\hat{\mathsf{C}}_\Lambda$ the category of complete noetherian local Λ-algebras with residue field k.*

For example, $\Lambda[t]/(t^i)$ are objects in C_Λ and $\Lambda[[t]]$ is an object in $\hat{\mathsf{C}}_\Lambda$, and $\Lambda[[t]] = \varprojlim \Lambda[t]/(t^i)$. (This suggests the use of our notations in these definitions. See [113] and [55].)

A covariant functor F from C_Λ to (Sets) extends to $\hat{\mathsf{C}}_\Lambda$ by the formula

$$\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^{n+1})$$

for $R \in \hat{\mathsf{C}}_\Lambda$ with maximal ideal $\mathfrak{m}$. Conversely, a covariant functor F from $\hat{\mathsf{C}}_\Lambda$ into (Sets) induces by restriction a functor $F|_{\mathsf{C}_\Lambda} : \mathsf{C}_\Lambda \to \text{(Sets)}$. For any covariant functor F from $\hat{\mathsf{C}}_\Lambda$ into (Sets), there is a canonical map

$$F(R) \to \hat{F}(R) = \varprojlim F(R/\mathfrak{m}^{n+1}). \tag{B.1.2}$$

In general, we do not know whether it is a bijection or not.

For any R in $\hat{\mathsf{C}}_\Lambda$ with maximal ideal m, we set

$$h_R(A) = \operatorname{Hom}(R, A)$$

to define a functor h_R on C_Λ.

Lemma B.1.3. *If F is any functor on C_Λ, and R is in $\hat{\mathsf{C}}_\Lambda$, then we have a canonical isomorphism*

$$\hat{F}(R) \xrightarrow{\sim} \operatorname{Hom}(h_R, F). \tag{B.1.4}$$

Proof. Let $\hat{\xi} = \varprojlim \xi_n$ be in $\hat{F}(R)$, where $\{\xi_n \in F(R/\mathfrak{m}^{n+1})\}_{n\geq 0}$ is a compatible system of elements (i.e. ξ_{n+1} induces ξ_n in $F(R/\mathfrak{m}^{n+1})$ for any $n \geq 0$). Each

$$u : R \to A$$

factors through

$$u_n : R/\mathfrak{m}^{n+1} \to A$$

for some n, and we assign to each $u \in h_R(A)$ the element $F(u_n)(\xi_n)$ of $F(A)$. Conversely, for any natural transformation $h_R \to F$, let ξ_n be the image of the canonical map

$$(R \to R/\mathfrak{m}^{n+1}) \in h_R(R/\mathfrak{m}^{n+1})$$

in $F(R/\mathfrak{m}^{n+1})$. Then

$$\{\xi_n \in F(R/\mathfrak{m}^{n+1})\}_{n=0,1,\ldots}$$

form a compatible system of elements by the functorial property of the natural transformation $h_R \to F$.

$$\begin{tikzcd}
h_R(R/\mathfrak{m}^{n+2}) \arrow[r] \arrow[d] & F(R/\mathfrak{m}^{n+2}) \arrow[d] \\
h_R(R/\mathfrak{m}^{n+1}) \arrow[r] & F(R/\mathfrak{m}^{n+1})
\end{tikzcd}$$

$$\begin{tikzcd}
(R \to R/\mathfrak{m}^{n+2}) \arrow[r, mapsto] \arrow[d, mapsto] & \xi_{n+1} \arrow[d, mapsto] \\
(R \to R/\mathfrak{m}^{n+1}) \arrow[r, mapsto] & \xi_n
\end{tikzcd}$$

Thus we get an element $\hat{\xi} = \varprojlim \xi_n$ in $\hat{F}(R)$. It is clear that this association is an inverse to the previous assignment $\hat{F}(R) \to \mathrm{Hom}(h_R, F)$. Hence we have indeed an isomorphism (B.1.4). $\square$

After the above identification, we are ready to make the following definitions.

Definition B.1.5. *A covariant functor F from C_Λ to* (Sets) *is called* **prorepresentable** *if there exist $R \in \mathrm{Ob}\,\hat{\mathsf{C}}_\Lambda$ and $\hat{\xi} \in \hat{F}(R)$ that induce an isomorphism of functors over C_Λ:*

$$\hat{\xi} : h_R(A) = \mathrm{Hom}(R, A) \xrightarrow{\sim} F(A). \tag{B.1.6}$$

The algebra R is then uniquely determined.

Definition B.1.7. *Let F be a covariant functor from $\hat{\mathsf{C}}_\Lambda$ to* (Sets)*. Suppose $F|_{\mathsf{C}_\Lambda}$ is prorepresentable. Then we say that it is* **effectively prorepresentable** *if there exists a $\xi \in F(R)$ (with R as above) that induces an isomorphism as the $\hat{\xi}$ in (B.1.6) through the canonical map (B.1.2). (Note that $F(R)$ and $\hat{F}(R)$ are not the same in general.)*

Let F be a covariant functor from $\hat{\mathsf{C}}_\Lambda$ into (Sets), and consider an element

$$\xi_0 \in F(k). \tag{B.1.8}$$

Definition B.1.9. *By an* **infinitesimal deformation** *of ξ_0, we mean an element $\eta \in F(A)$ where $A \in \mathrm{Ob}\,\mathsf{C}_\Lambda$ is an Artinian local Λ-algebra with residue field k, and η induces $\xi_0 \in F(k)$ by functoriality.*

Definition B.1.10. *A* **formal deformation** *of ξ_0 is an element*

$$\hat{\xi} = \varprojlim \xi_n \in \hat{F}(R) = \varprojlim F(R/m^{i+1})$$

where $R \in \mathrm{Ob}\,\hat{\mathsf{C}}_\Lambda$ is a complete noetherian local Λ-algebra with maximal ideal $\mathfrak{m}$ and residue field k, and where $\{\xi_n \in F(R/\mathfrak{m}^{n+1})\}_{n \geq 0}$ is a compatible system of elements (i.e. ξ_{n+1} induces ξ_n in $F(R/\mathfrak{m}^{n+1})$ for any $n \geq 0$) with the element ξ_0 given above.

Definition B.1.11. *A formal deformation $\hat{\xi} = \varprojlim \xi_n \in \hat{F}(R)$ of ξ_0 is said to be* **effective** *if there is an element $\xi \in F(R)$ that induces ξ_n for each n.*

Definition B.1.12. *A formal deformation $\hat{\xi} \in \hat{F}(R)$ is said to be* **versal** (*resp.* **universal**) *if it has the following property:*

Let $A' \to A$ be any surjection of Artinian local Λ-algebras in C_Λ, $\eta' \in F(A')$ any infinitesimal deformation of ξ_0 and $\eta \in F(A)$ the infinitesimal deformation of ξ_0 induced by η'. Then any homomorphism $R \to A$ that induces $\eta \in F(A)$ by $\hat{\xi} \in \hat{F}(R)$ and (B.1.4) can be embedded (resp. uniquely embedded, namely embedded by a uniquely determined homomorphism $R \to A'$) in a commutative diagram

$$\begin{tikzcd} & A' \arrow[d] \\ R \arrow[ru] \arrow[r] & A \end{tikzcd}$$

such that η' is induced by the homomorphism $R \to A'$.

More precisely, if the $(n+1)$-th power of the maximal ideal of A' is zero, then any homomorphism $R \to A'$ factors through $R/\mathfrak{m}^{n+1} \to A'$. (And so does any homomorphism $R \to A$.) The above condition means that if $\eta \in F(A)$ is induced functorially by $\xi_n \in F(R/\mathfrak{m}^{n+1})$ through $R/\mathfrak{m}^{n+1} \to A$, then we have a (resp. a uniquely) completed diagram

$$\begin{tikzcd} & A' \arrow[d] \\ R/\mathfrak{m}^{n+1} \arrow[ru] \arrow[r] & A \end{tikzcd}$$

inducing functorially the diagram

$$\begin{tikzcd} & \eta' . \arrow[d, mapsto] \\ \xi_n \arrow[ru, mapsto] \arrow[r, mapsto] & \eta \end{tikzcd}$$

In case that $F(k)$ has only one element ξ_0. The existence of a formal universal deformation of ξ_0 is equivalent with the prorepresentability of F. (This is slightly weaker than the prorepresentability defined in [55], which considers also Artinian local rings whose residue fields are finite extensions of k. Nevertheless, we will deal with finite extensions directly in our text. See Section B.1.1 for more information.)

B.1.1 Structure of Complete Local Rings

In our study of deformation, we may need to consider the case where the field k is a finite field extension of a residue field $k(s)$ of S, and a functor F will be defined on the category of noetherian $\mathscr{O}_S$-algebras with residue field k. Under these assumptions, we would like to find a complete noetherian local $\mathscr{O}_S$-algebra Λ with residue field k such that a complete local noetherian ring is an $\mathscr{O}_S$-algebra with residue field k if and only if it is a Λ-algebra. If this is possible, then F can be naturally defined (by restriction) to be a functor from $\hat{\mathsf{C}}_\Lambda$ to (Sets) as before, and we are justified to study the restricted functor only.

This problem is solved by Cohen's structural theorems on complete local rings. We first consider the case where the characteristic of the complete local ring is equal to its residue field.

Definition B.1.1.1. *A local ring* $(R, \mathfrak{m}, k)$ *is called* **equicharacteristic** *if* $\operatorname{char}(R) = \operatorname{char}(k)$.

For such an equicharacteristic local ring, a subfield $k' \subset R$ *is called a* **coefficient field** *if* k' *is mapped isomorphically to* k *under the natural map* $R \to R/\mathfrak{m} = k$*, or equivalently, if* $R = k' + \mathfrak{m}$.

Theorem B.1.1.2. *Let* $(R, \mathfrak{m}, k)$ *be a complete equicharacteristic local ring. Then there is a coefficient field* $k' \subset R$*. If the maximal ideal* $\mathfrak{m}$ *has a minimal generator of* n *elements, then* R *is a homomorphic image of the formal power series ring* $k[[X_1, \ldots, X_n]]$.

Proof. The original proof can be found in [23, Thm. 9]. For references that are more accessible, see [91, Thm. 28.3], [36, Thm. 7.7] or [102, Thm. 31.1]. □

On the other hand, if $\operatorname{char}(R) \neq \operatorname{char}(k)$, then necessarily $\operatorname{char}(k) = p$ for some prime number p. In this case, it is not possible to have such a coefficient field, because the units of k must be sent to zero in R.

Remark B.1.1.3. Since we do not assume that R is an integral domain, it is possible that $\operatorname{char}(R) = p^n$ for some $n > 1$. In this case, there must be an element $x \in R$ such that $p^n x = 0$ but $p^{n-1}x \neq 0$. If k is isomorphic to a subfield k' of R, then the units of k are mapped to units of R. Let u be any of such units in R. Then $pu = 0$ and $p^{n-1}x = (p^{n-1}x)(uu^{-1}) = (pu)(p^{n-2}xu^{-1}) = 0$, which is absurd.

However, it is still possible to have a so-called *coefficient ring* $R_0 \subset R$, where R_0 is a complete local ring with maximal ideal pR_0 and $R = R_0 + \mathfrak{m}$. Namely, $k = R/\mathfrak{m} \cong R_0/pR_0$. More precisely, the coefficient ring is a homomorphic image of a *p-ring.* (See Definition B.1.1.4, Remark B.1.1.12 and Theorem B.1.1.13 below.)

Definition B.1.1.4. *A* ***p-ring*** *is a discrete valuation ring of characteristic zero whose maximal ideal is generated by the prime number* p*.*

The most simple example of a p-ring is the ring of p-adic integers $\mathbb{Z}_p$.

Theorem B.1.1.5 (see [91, Thm. 29.1])**.** *Let* (A, tA, k) *be a discrete valuation ring and* k' *an extension field of* k*, where* t *is the uniformizer of* A*, then there exists a discrete valuation ring* (A', tA', k') *containing* (A, tA, k) *with the same uniformizer* t*.*

Remark B.1.1.6. In [47, 0_{III}, 10.3.1], an analogous result is given in the case of noetherian local ring:

Proposition B.1.1.7. *Let* $(R, \mathfrak{m}, k)$ *be a local noetherian ring,* k' *a field extension of* k*. Then there is a local homomorphism from* R *to a noetherian local ring* $(R', \mathfrak{m}', k')$ *such that* R' *is a flat* R*-module.*

We remark also that [49, 0_{IV}, 19] contains a systematic treatment on structural theorems of noetherian local rings having deep applications to the study of certain local geometric properties such as formal smoothness, formal étaleness, etc.

Remark B.1.1.8. It is mentioned in [23] that this fundamental theorem is due to Hasse and Schmidt [65, Thm. 20], and that a particular simple proof is given by Mac Lane [89, Thm. 2].

Corollary B.1.1.9. *For any given field* K *of characteristic* p*, there is a* p*-ring* A_K *having* K *as its residue field.*

Proof. This is immediate by applying Theorem B.1.1.5 to the ring of p-adic integers $\mathbb{Z}_p$. ☐

Theorem B.1.1.10 (see [91, Thm. 29.2] or [23, Thm. 11])**.** *Let* $(R, \mathfrak{m}, K)$ *be a complete local ring,* (A, tA, k) *a* p*-ring, and* $\phi_0 : k \to K$ *a field homomorphism. Then there exists a local homomorphism* $\phi : A \to R$ *that induces* ϕ_0 *on the residue fields.*

Then the following corollary is immediate:

Corollary B.1.1.11. *A complete $p-$ring is uniquely determined up to an isomorphism by its residue field.*

Remark B.1.1.12. Let $(R, \mathfrak{m}, k)$ be a complete local ring with unequal characteristic, and let $p = \operatorname{char}(k)$. By Theorem B.1.1.5, there is a p-ring A_0 such that $A_0/pA_0 = k$. Let A be the completion of A_0, then A is a complete p-ring with residue field k. By Theorem B.1.1.10, there is a local homomorphism $\phi : A \to R$ that induces an isomorphism on the residue fields. If we set $R_0 = \phi(A)$, then R_0 is clearly a coefficient ring of R. If R has characteristic zero, then ϕ is injective and $R_0 \cong A$. If R has characteristic p^n, then $R_0 \cong A/p^nA$.

Theorem B.1.1.13 (see [23, Thm. 12] or [102, Thm. 31.1])**.** *Let $(R, \mathfrak{m}, k)$ be a complete local ring with* $\operatorname{char}(k) = p$*. Then there is a coefficient ring $R_0 \subset R$, which is a homomorphic image of a complete p-ring with residue field k. If the maximal ideal $\mathfrak{m}$ has a minimal generator of n elements, then R is a homomorphic image of the formal power series ring $R_0[[X_1, \dots, X_n]]$. If $p \notin \mathfrak{m}^2$, then it is a homomorphic image of $R_0[[X_1, \dots, X_{n-1}]]$ with only $n-1$ variables.*

The proof of Theorem B.1.1.5 requires Zorn's lemma. Hence we do not know the explicit construction of the extension. Alternatively, we shall consider the *Witt vectors*, which give an explicit construction of the unique complete p-ring with residue field k when k is *perfect* of characteristic p.

It is described in [114, II, 6] how to construct the Witt vectors $W(A)$ as a projective limit of $W_n(A)$ over for any commutative ring A. The upshot is the following:

Theorem B.1.1.14 (see [114, II, 6, Thm. 8])**.** *If k is a perfect field of characteristic p, then $W(k)$ is a complete p-ring with residue field k.*

Then it follows from Corollary B.1.1.11 that any complete p-ring with residue field k is isomorphic to $W(k)$.

Example B.1.1.15. $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$.

Now return to our problem:

Lemma B.1.1.16. *Let S be a scheme and s a point of S, k a finite field extension of the residue field $k(s)$ of S. Then there is a complete noetherian* *local $\mathscr{O}_S$-algebra Λ with residue field k such that a complete local noetherian ring R with residue field k fits into the diagram*

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[rd] & \mathrm{Spec}(R) \arrow[d, dotted] \\
 & S
\end{tikzcd}$$

with solids maps given by the natural residue maps if and only if it is a Λ-algebra.

In particular, if the characteristic of k (namely of $k(s)$) is zero, then we may take

$$\Lambda = k.$$

If the characteristic k is a prime number p, then we take Λ to be the completion of the ring

$$\hat{\mathscr{O}}_{S,s} \underset{A_k(s)}{\otimes} A_k,$$

where $A_k(s)$ and A_k are the (uniquely determined) complete p-rings with residue field $k(s)$ and k respectively (given by Corollary B.1.1.9).

Proof. If the characteristic of k is zero, then by Theorem B.1.1.2, any complete local ring with residue field k contains a subfield k' isomorphic to k, and hence is automatically a k-algebra. Therefore we may simply take Λ to be k.

Otherwise, if the characteristic of k is a prime p. By Theorem B.1.1.10, for any complete local ring R with residue field k, there is a homomorphism $A_k \to R$ making R into an A_k-algebra. Similarly, there is a homomorphism $A_{k(s)} \to \hat{\mathscr{O}}_{S,s}$ making the ring $\hat{\mathscr{O}}_{S,s}$ into an $A_{k(s)}$-algebra. (Here A_k and $A_{k(s)}$ are the unique p-rings with residue field k and $k(s)$ respectively as described in the statements of the theorem.) Now the homomorphism $\mathscr{O}_S \to R$ factors through $\hat{\mathscr{O}}_{S,s} \to R$ by the universal property of complete local rings, and we have a homomorphism $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k \to R$ by the universal property of tensor products. Hence it suffices to take Λ to be the completion of the ring $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k$. ☐

The following corollary of Lemma B.1.1.16 is a restatement of the remarks made in the beginning of this section.

Corollary B.1.1.17. *Let F be a functor*

$$F : (\mathrm{Sch}/S)^{\circ} \to (\mathrm{Sets}),$$

and k a finite field extension of a residue field $k(s)$ of S. Then there is a complete noetherian local $\mathscr{O}_S$-algebra Λ with residue field k such that a complete noetherian local ring with residue field k is an $\mathscr{O}_S$-algebra if and only if it is a Λ-algebra. Hence F defines a functor by restriction, again denoted by F (*with slight abuse of notation*)*:*

$$F : \hat{\mathsf{C}}_{\Lambda} \to (\mathrm{Sets}).$$

This will be used implicitly in Theorem B.2.5, Theorem B.2.6, Theorem B.2.1.12 (Artin's algebraization theorems) and explicitly in Theorem B.3.10. Here we say that it will be used *implicitly* because the notions of deformation were defined by Artin in a more general way in [9]. Hence Lemma B.1.1.16 and Corollary B.1.1.17 will be used to justify their compatibility with our more restrictive definitions. Lemma B.1.1.16 and Corollary B.1.1.17 will be used *explicitly* in Theorem B.3.10, where we need to associate deformation functors defined in the sense of Section B.1 to categories fibred in groupoids.

B.2 Existence of Algebraization

Let us recall the map (B.1.2):

$$F(R) \to \hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n).$$

Sometimes, for proving that a prorepresentable functor is effectively prorepresentable, we may prove directly that the map (B.1.2) is bijective. For example, consider the case if the versal formal deformation is prorepresented by a formal scheme that is *algebraizable* in the sense that the formal scheme can be obtained by completing a single noetherian scheme along a closed subscheme. (See [47, III, 5.4.2, 5.2.1] or [64, Ch. II, Exer. 9.3.2].) Grothendieck's existence theorem of formal sheaves (see [47, III, 5.1.4], or see Section 2.3.1) provides us with a general criterion which is useful for deducing the algebraizability of such formal schemes.

However, for the effectiveness of prorepresentability, we do not really need (B.1.2) to be bijective. Suppose that there is a versal formal deformation as described in the last section. To conclude the effectiveness, it will be enough to show that the map (B.1.2) has a *dense* image. For then, given any $\{\xi_n\} \in \varprojlim F(R/\mathfrak{m}^{n+1})$ induced from an element in the versal formal deformation, there is a $\xi' \in F(R)$ that induces $\xi_1 \in F(R/\mathfrak{m}^2)$ (because of the density assumption). Since the formal deformation is versal, we may lift the identify map of $R/\mathfrak{m}^2$ successively to diagrams

$$\begin{tikzcd}
R/\mathfrak{m}^{n+2} \arrow[r, "\varphi_{n+1}"] \arrow[d] & R/\mathfrak{m}^{n+2} \arrow[d] \\
R/\mathfrak{m}^{n+1} \arrow[r, "\varphi_n"'] & R/\mathfrak{m}^{n+1}
\end{tikzcd}$$

sending

$$\begin{tikzcd}
\xi_{n+1} \arrow[r, mapsto] \arrow[d, mapsto] & \xi'_{n+1} \arrow[d, mapsto] \\
\xi_n \arrow[r, mapsto] & \xi'_n
\end{tikzcd}$$

for each n, where ξ'_{n+1} and ξ'_n are elements induced by ξ'. Hence there is a homomorphism $\varphi : R \to R$ with the properties that it sends $\{\xi_n\} \mapsto \{\xi'_n\}$ and that φ is identity on $R/\mathfrak{m}^2$.

Lemma B.2.1. *Let R be a complete local ring, $\mathfrak{m}$ its maximal ideal, and $\varphi : R \to R$ an endomorphism that induces the identity on $R/\mathfrak{m}^2$. Then φ is an automorphism.*

Proof. We should prove that it is one-one and onto. By hypothesis, $\varphi(\mathfrak{m}) = \mathfrak{m}$, $\varphi(\mathfrak{m}^2) = \mathfrak{m}^2$, and φ is the identity on $\mathfrak{m}/\mathfrak{m}^2$. For any element in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, by writing it as a product of n elements in $\mathfrak{m}/\mathfrak{m}^2$, it is immediate that φ is again the identity on $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. Now since R is complete, for any $r \neq 0$ in R, there must be an n such that $r \in \mathfrak{m}^n - \mathfrak{m}^{n+1}$. In particular, $\varphi(r) \neq 0$. Hence φ is one-one.

It remains to prove that φ is onto. We are going to construct a converging sequence $\{s_i\}$ in R such that $\varphi(s_i) \equiv r \mod \mathfrak{m}^{i+1}$ for all i. For $i = 1$, by hypothesis, we may take $s_1 = r$. For larger i, suppose that there is already an $s_{i-1} \in R$ such that $\varphi(s_{i-1}) \equiv r \mod \mathfrak{m}^i$, then $r - \varphi(s_{i-1}) \in \mathfrak{m}^i$. Since φ is the identity on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, there is an $a_i \in \mathfrak{m}^i$ such that $\varphi(a_i) \equiv r - \varphi(s_{i-1})$ mod $\mathfrak{m}^{i+1}$. Now it suffices to take $s_i = s_{i-1} + a_i$. The sequence $\{s_i\}$ then converges to an element $s \in R$ such that $\varphi(s) = r$. ☐

Return to our case, since φ^{-1} exists, the inverse image $F(\varphi^{-1})(\xi') \in F(R)$ of ξ' is the required element.

Artin's theory [7] on solutions of analytic equations provided another possible approach to effectiveness. In [9], he suggests that if the functor is a quotient of a representable functor X of finite type over S, so that there is a surjective map $\varphi : X \to F$ of sheaves for the étale topology, then we can conclude the density of the image of (B.1.2) from an approximation theorem for formal solutions of equations similar to [7, (1.2)].

As a result, for a functor F from (Sch/S) to (Sets) where S is a scheme or an algebraic space over a field or an *excellent Dedekind domain*, Artin gave two fundamental theorems concerning the existence and uniqueness of the algebraization of an effective versal formal deformation of some element $\xi_0 \in F(k')$, with k' a field of finite type over $\mathscr{O}_S$. (Note that the notions of deformations are defined in this case after Section B.1.1.)

Following Grothendieck [51, IV, 8.14.2] and Artin [10, 4.4], we make the following definition:

Definition B.2.2. *A (contravariant) functor*

$$F : (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$$

is **locally of finite presentation** *over S if*

$$\varinjlim F(\mathrm{Spec}(A_i)) = F(\varprojlim \mathrm{Spec}(A_i)) \tag{B.2.3}$$

for any filtrating projective system of S-schemes $\mathrm{Spec}(A_i)$*.*

Proposition B.2.4 (see [51, IV, 8.14.2] and [10, 4.4])**.** *If F is represented by a scheme or an algebraic space over S, then F is locally of finite presentation (as a scheme or an algebraic space) if and only if F satisfies (B.2.3).*

Return to the problem of algebraization, the follow theorems are given by Artin in [9]:

Theorem B.2.5 (existence of algebraization; see [9, Thm. 1.6])**.** *Let S be a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain. Consider a (contravariant) functor*

$$F : (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$$

locally of finite presentation over S. (See Definition B.2.1.9.) Let $s \in S$ be a point whose residue field $k(s)$ is of finite type over $\mathscr{O}_S$, k' a finite field extension of $k(s)$, and $\xi_0 \in F(k')$. Suppose that an effective formal versal deformation (R, ξ) of ξ_0 exists, where R is a complete noetherian local $\mathscr{O}_S$-algebra with residue field k' and $\xi \in F(R)$. (Our language of deformation theory is applicable in this case, by choosing a suitable Λ in this case using Lemma B.1.1.16 and Corollary B.1.1.17.) Then there is an S-scheme X of finite type, a closed point $x \in X$ with residue field $k(x) = k'$, and an element $\widetilde{\xi} \in F(X)$, such that the triple $(X, x, \widetilde{\xi})$ is a versal deformation of ξ_0. More precisely, there is an isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ such that $\widetilde{\xi}$ induces ξ through the composition

$$\mathrm{Spec}(R) \xrightarrow{\sim} \mathrm{Spec}(\hat{\mathscr{O}}_{X,x}) \to X.$$

The isomorphism is unique if (R, ξ) is universal.

Theorem B.2.6 (uniqueness of algebraization; see [9, Thm. 1.7])**.** *With notations as in Theorem B.2.5, suppose that the element $\xi \in F(R)$ is uniquely determined by the set $\{\xi_n\}$ of its truncations (namely images of ξ induced by maps $R \to R/m^{n+1}$). Then the triple $(X, x, \widetilde{\xi})$ is unique up to local isomorphism for the étale topology, in the following sense: If $(X', x', \widetilde{\xi}')$ is another algebraization, then there is a third one $(X'', x'', \widetilde{\xi}'')$ and a diagram*

$$\begin{tikzcd} & X'' \arrow[dl, "f"'] \arrow[dr, "f'"] & , \\ X & & X', \end{tikzcd}$$

where f and f' are étale morphisms, which sends

$$\begin{tikzcd} & x'' \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ x & & x' \end{tikzcd}$$

and

$$\begin{tikzcd} & \widetilde{\xi}'' \arrow[dl] \arrow[dr] & . \\ \widetilde{\xi} & & \widetilde{\xi}' \end{tikzcd}$$

B.2.1 Generalization from Sets to Groupoids

The above theorems of algebraization consider the case where F is a contravariant functor from (Sch/S) to (Sets). However, in our later application, we will need to consider contravariant 2-functors from (Sch/S) to the 2-category (Gr) of groupoids. For such a 2-functor

$$\mathsf{F} : (\mathrm{Sch}/S)^\circ \to (\mathrm{Gr}),$$

we may associate canonically a functor

$$\overline{\mathsf{F}} : (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$$

such that, for each scheme U, $\overline{\mathsf{F}}(U)$ is the set of isomorphism classes of $\mathsf{F}(U)$. The functorial properties of $\overline{\mathsf{F}}$ are verified by the 2-functorial properties of F. Furthermore, we have a canonical morphism (of 2-functors)

$$\mathsf{F} \to \overline{\mathsf{F}}$$

defined by sending each object of $\mathsf{F}(U)$ to the isomorphism class containing it in $\overline{\mathsf{F}}(U)$.

The above theorems concerned the restriction of the functors to the categories of spectra of certain Artinian local rings or complete noetherian local rings over $\mathscr{O}_S$, which are special cases of C_Λ or $\hat{\mathsf{C}}_\Lambda$ with suitably chosen Λ (cf. Lemma B.1.1.16 and Corollary B.1.1.17).

To generalize the notions of deformations studied in Section B.1, we may consider a functor in the form

$$\mathsf{F} : \hat{\mathsf{C}}_\Lambda \to (\mathrm{Gr}).$$

Let

$$\xi_0 \in \mathrm{Ob}\,\mathsf{F}(k)$$

be an object in the groupoid $\mathsf{F}(k)$.

Let $R \in \hat{\mathsf{C}}_\Lambda$ be any complete noetherian ring with residue field k.

Definition B.2.1.1. *A* **formal deformation** *of ξ_0 is an object $\hat{\xi} = \varprojlim \xi_n$ with ξ_0 given above in the projective limit $\hat{\mathsf{F}}(R) = \varprojlim \mathsf{F}(R/\mathfrak{m}^{n+1})$, where $\{\xi_n \in \mathrm{Ob}\,\mathsf{F}(R/\mathfrak{m}^{n+1})\}_{n=0,1,\ldots}$ is a projective system compatible up to 2-isomorphisms.*

Definition B.2.1.2. *A formal deformation* $\hat{\xi} = \varprojlim \xi_n$ *is called* **effective***, if there is an object* $\xi \in F(R)$ *inducing objects isomorphic to* ξ_n *in* $\mathsf{F}(R/\mathfrak{m}^{n+1})$ *for all* n*.*

The reason to define the above notions in such a loose way is that, for a 2-functor, this is already the best possible. Even for a canonical diagram of the form

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ R/\mathfrak{m}^{i+1} \arrow[r] & R/\mathfrak{m}^{j+1}, \end{tikzcd}$$

we only have a diagram of 1-morphisms

$$\begin{tikzcd} \mathsf{F}(R) \arrow[d] \arrow[dr] & \\ \mathsf{F}(R/\mathfrak{m}^{i+1}) \arrow[r] & \mathsf{F}(R/\mathfrak{m}^{j+1}) \end{tikzcd}$$

commutative up to 2-isomorphism. That is, for an object $\eta \in \operatorname{Ob} \mathsf{F}(R)$, the induced images of η through

$$\mathsf{F}(R) \to \mathsf{F}(R/\mathfrak{m}^{j+1})$$

and through

$$\mathsf{F}(R) \to \mathsf{F}(R/\mathfrak{m}^{i+1}) \to \mathsf{F}(R/\mathfrak{m}^{j+1})$$

are only required to be *isomorphic*, but not necessarily *equal*.

Let h_R be the functor assigning to each Artinian local Λ-algebra A the set $h_R(A)$ of homomorphisms $R \to A$. As in (B.1.4), we would like to have a canonical *isomorphism* from $\hat{\mathsf{F}}(R)$ to $\mathsf{Hom}(h_R, \mathsf{F})$. However, this is not always possible. In general we can only show the existence of an *equivalence of categories* such that the isomorphism classes of objects and the morphisms between isomorphism classes of objects in the two categories are one-one corresponded.

Lemma B.2.1.3. *There is an equivalence of categories*

$$\hat{\mathsf{F}}(R) \to \mathsf{Hom}(h_R, \mathsf{F}) \qquad \text{(B.2.1.4)}$$

between the categories $\hat{\mathsf{F}}(R)$ *and* $\mathsf{Hom}(h_R, \mathsf{F})$*.*

Proof. Let $\hat{\xi} = \varprojlim \xi_n$ be an object in $\hat{\mathsf{F}}(R)$. Any morphism

$$u : R \to A,$$

where A is an Artinian local ring in $\operatorname{Ob} \mathsf{C}_\Lambda$, must factor through

$$u_n : R/\mathfrak{m}^{n+1} \to A$$

for some n, and we assign to u the object $\mathsf{F}(u_n)(\xi_n)$ induced by ξ_n in $\mathsf{F}(A)$.

The above assignment depends on the way we factor the homomorphism (cf. Remark B.2.1.5). In general, the object we assign in $\mathsf{F}(A)$ is not determined canonically. Therefore, to define a morphism from h_R to F, we must choose once for all possible homomorphisms of the form $R \to A$ the corresponding ways we factor the homomorphisms.

Let us adopt the following rule: For any Artinian local ring A in $\operatorname{Ob} \mathsf{C}_\Lambda$, consider the least n such that $\mathfrak{m}^{n+1} = 0$ holds for the maximal ideal $\mathfrak{m}$ of A, and such that $R \to A$ factors through $R/\mathfrak{m}^{n+1} \to A$. (Note that any morphism $h_R \to \mathsf{F}$ defined by another choice of factorizations must be canonically isomorphic to this morphism.) Hence the object-level assignment of the functor (B.2.1.4) is complete.

The morphism-level assignment is then canonical. Any morphism

$$f : \hat{\xi} \to \hat{\eta}$$

in $\hat{\mathsf{F}}(R)$ is given by a series of morphisms

$$f_n : \xi_n \to \eta_n$$

in $\mathsf{F}(R/\mathfrak{m}^{n+1})$. For any A in $\operatorname{Ob} \mathsf{C}_\Lambda$ with the n chosen as above, and for any morphism $u : R \to A$, the morphism from $\mathsf{F}(u_n)(\xi_n)$ to $\mathsf{F}(u_n)(\eta_n)$ is canonically defined by $\mathsf{F}(u_n)(f_n)$.

Conversely, for any natural transformation $h_R \to \mathsf{F}$, define for each n the object $\xi_n \in \operatorname{Ob} \mathsf{F}(R/\mathfrak{m}^{n+1})$ to be the image induced by the canonical map

$$(R \to R/\mathfrak{m}^{n+1}) \in h_R(R/\mathfrak{m}^{n+1}).$$

Then the commutative diagram (up to 2-isomorphisms)

$$\begin{tikzcd}
h_R(R/\mathfrak{m}^{n+2}) \arrow[r] \arrow[d] & \mathsf{F}(R/\mathfrak{m}^{n+2}) \arrow[d] \\
h_R(R/\mathfrak{m}^{n+1}) \arrow[r] & \mathsf{F}(R/\mathfrak{m}^{n+1})
\end{tikzcd}$$

inducing

$$\begin{tikzcd}
(R \to R/\mathfrak{m}^{n+2}) \arrow[r, mapsto] \arrow[d, mapsto] & \xi_{n+1} \arrow[d, mapsto] \\
(R \to R/\mathfrak{m}^{n+1}) \arrow[r, mapsto] & \xi_n
\end{tikzcd}$$

clearly gives a projective system

$$\{\xi_n \in \operatorname{Ob} \mathsf{F}(R/\mathfrak{m}^{n+1})\}_{n=0,1,\dots}$$

compatible up to 2-isomorphisms.

One may check that this association gives a quasi-inverse to the previous association. For any morphism from h_R to F induced by an element $\hat{\xi} \in \hat{\mathsf{F}}(R)$ through (B.2.1.4), it is always true that the above converse induces an element in $\hat{\mathsf{F}}(R)$ isomorphic to $\hat{\xi}$. Hence the two categories in (B.2.1.4) are equivalent. □

Remark B.2.1.5. Let $R \to A$ be a homomorphism from R to an Artinian local ring A in $\operatorname{Ob} \mathsf{C}_\Lambda$. The morphism $R \to A$ may factor through both $R/\mathfrak{m}^{n+1} \to A$ and $R/\mathfrak{m}^{n'+1} \to A$ for different n and n', and the objects $\mathsf{F}(u_n)(\xi_n)$ and $\mathsf{F}(u_{n'})(\xi_{n'})$ in $\mathsf{F}(A)$ induced respectively are isomorphic but generally not equal to each other. Anyhow, the following diagram

$$\begin{tikzcd}
F(R/\mathfrak{m}^{n'}) \arrow[d] \arrow[dr, "F(u_{n'})"] & \\
F(R/\mathfrak{m}^{n}) \arrow[r, "F(u_n)"'] & F(A)
\end{tikzcd}$$

is still commutative in this case, since a diagram of 1-morphisms in the theory of 2-categories and 2-functors are required to be commutative only up to 2-isomorphisms.

Remark B.2.1.6. We do not consider the notion of *prorepresentability* or *effective prorepresentability* here. The readers who are interested in the generalizations in this case should note that, since there is an ambiguity in defining the notion of an *isomorphism* between the 2-functor F and the functor h_R (cf. Remark A.1.2.7), the *right* generalizations must be defined in terms of equivalences of categories.

The notions of *versal* and *universal* formal deformations can be generalized naturally in the following way.

Definition B.2.1.7. *A formal deformation $\hat{\xi}$ is* **versal** *(resp.* **universal***) if it has the following property: Let $A' \to A$ be any surjection of Artinian local Λ-algebras in C_Λ, $\eta' \in \operatorname{Ob}\mathsf{F}(A')$ any infinitesimal deformation of ξ_0 and $\eta \in \operatorname{Ob}\mathsf{F}(A)$ the infinitesimal deformation of ξ_0 induced by η'. Then any homomorphism $R \to A$ that induces $\eta \in F(A)$ by $\hat{\xi} \in \hat{\mathsf{F}}(R)$ and (B.2.1.4) can be embedded (resp. uniquely embedded, namely via a uniquely determined homomorphism $R \to A'$) in a commutative diagram*

$$\begin{tikzcd} & A' \arrow[d] \\ R \arrow[ur] \arrow[r] & A \end{tikzcd}$$

such that η' is isomorphic to the object induced by the homomorphism $R \to A'$ (through $\hat{\xi}$ and (B.2.1.4)).

More precisely, if the $(n+1)$-th power of the maximal ideal of A' is zero, then any homomorphism $R \to A'$ factors through $R/\mathfrak{m}^{n+1} \to A'$. (And so does any homomorphism $R \to A$.) The above condition means that if $\eta \in \operatorname{Ob}\mathsf{F}(A)$ is induced functorially by $\xi_n \in \operatorname{Ob}\mathsf{F}(R/\mathfrak{m}^{n+1})$ through $R/\mathfrak{m}^{n+1} \to A$, then we have a (resp. uniquely) completed diagram

$$\begin{tikzcd} & A' \arrow[d] \\ R/\mathfrak{m}^{n+1} \arrow[ur] \arrow[r] & A \end{tikzcd}$$

inducing functorially a commutative diagram (up to 2-isomorphism)

$$\begin{tikzcd} & \mathsf{F}(A') \arrow[d] \\ \mathsf{F}(R/\mathfrak{m}^{n+1}) \arrow[ur] \arrow[r] & \mathsf{F}(A) \end{tikzcd}$$

such that η' is isomorphic to the object induced by ξ_n through $\mathsf{F}(R/\mathfrak{m}^{n+1}) \to \mathsf{F}(A')$.

Remark B.2.1.8. The above definitions may looked quite clumsy, since all the statements we made are forced to allow an ambiguity up to isomorphisms in the groupoids. However, by passing the 2-functor F to the functor $\overline{\mathsf{F}}$ through

$$\pi : \mathsf{F} \to \overline{\mathsf{F}}$$

defined in the beginning of this section, we see that any formal deformation $\hat{\xi} = \varprojlim \xi_n$ of some $\xi_0 \in \operatorname{Ob} \mathsf{F}(k)$ is mapped to a formal deformation $\pi(\hat{\xi}) = \varprojlim \pi(\xi_n)$ in $\varprojlim \overline{\mathsf{F}}(R/\mathfrak{m}^{n+1})$ of $\pi(\xi_0) \in \overline{\mathsf{F}}(k)$. And if $\hat{\xi}$ is versal (resp. universal), then $\pi(\hat{\xi})$ is versal (resp. universal) too. That is, the above definitions are compatible with the formation of $\pi : \mathsf{F} \to \overline{\mathsf{F}}$, and one may check that the existence of an effective formal deformation is equivalent for the 2-functor F and for the functor $\overline{\mathsf{F}}$ associated with F.

Now turn to the case of a 2-functor associated to a category fibred in groupoids F over $(\mathrm{Sch}\,/S)$. That is, a 2-functor

$$\mathsf{F} : (\mathrm{Sch}\,/S)^\circ \to (\mathrm{Gr})$$

assigning to each U in $(\mathrm{Sch}\,/S)$ the fiber groupoid F_U. For an object $\xi_0 \in \operatorname{Ob} \mathsf{F}(k)$ with k an $\mathscr{O}_S$-field of finite type, the existence of an *effective formal deformation* of ξ_0 is to say that there is an object $\xi \in \operatorname{Ob} \mathsf{F}(R)$ where R is a complete noetherian $\mathscr{O}_S$-algebra with maximal ideal $\mathfrak{m}$ and residue field k, such that ξ induces an object isomorphic to ξ_0 in $\operatorname{Ob} \mathsf{F}(k)$. By viewing ξ_0, ξ as morphisms $\mathrm{Spec}(k) \to \mathsf{F}$, $\mathrm{Spec}(R) \to \mathsf{F}$ respectively, the above statement amounts to say that, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[dr] & \mathrm{Spec}(R) \arrow[d] \\
 & \mathsf{F}.
\end{tikzcd}$$

(Note that such a diagram of groupoids is defined to be commutative only up to 2-isomorphisms.) The truncations $\xi_n \in \operatorname{Ob} \mathsf{F}(R/\mathfrak{m}^{n+1})$ of ξ are defined simply as the compositions:

$$\begin{tikzcd}
\mathrm{Spec}(R/\mathfrak{m}^{n+1}) \arrow[d] \arrow[ddr, bend left, "\xi_n"] & \\
\mathrm{Spec}(R/\mathfrak{m}) = \mathrm{Spec}(k) \arrow[d] \arrow[dr, "\xi_0"] & \\
\mathrm{Spec}(R) \arrow[r, "\xi"] & \mathsf{F}.
\end{tikzcd}$$

To say that the effective formal deformation is *versal* (resp. *universal*), it amounts to say that, for any diagram of solid arrows

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[r, leftrightarrow] \arrow[d] & \operatorname{Spec}(A) \arrow[d, hook] \\
\mathsf{F} & \operatorname{Spec}(A') \arrow[l] \arrow[ul, dotted]
\end{tikzcd}$$

where $A' \to A$ is a surjection of Artinian local $\mathscr{O}_S$-algebras with residue field k, there is a dotted arrow (resp. a unique dotted arrow) $\operatorname{Spec}(A') \to \operatorname{Spec}(R)$ making the diagram commute.

Following Grothendieck [51, IV, 8.14.2] and Artin [10, 4.4], we make the following definition for categories fibred in groupoids (cf. Definition B.2.2):

Definition B.2.1.9. *A category* X *fibred in groupoids over* $(\mathrm{Sch}\,/S)$ *is* **locally of finite presentation** *if for any filtrating projective system of* S*-schemes* $\operatorname{Spec}(A_i)$*, the canonical functor*

$$\varinjlim \mathsf{X}(\operatorname{Spec}(A_i)) \to \mathsf{X}(\varprojlim \operatorname{Spec}(A_i)) \tag{B.2.1.10}$$

defines an equivalence of categories.

Proposition B.2.1.11 (see [85, 4.15])**.** *An algebraic stack* X *is locally of finite presentation over* S (*as an algebraic stack*) *if and only if it satisfies* (*B.2.1.10*).

Theorem B.2.5 can be reformulated in the following form, to be used in the proof of Theorem B.3.8 in the next section.

Theorem B.2.1.12 (modified version of Theorem B.2.5)**.** *Let* S *be a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain. Consider a category* F *fibred in groupoids over* $(\mathrm{Sch}\,/S)$ *which is locally of finite presentation.* (*See Definition B.2.1.9.*) *Let* $s \in S$ *be a point whose residue field* $k(s)$ *is of finite type over* $\mathscr{O}_S$*,* k' *a finite field extension of* $k(s)$*, and* $\xi_0 \in \operatorname{Ob} \mathsf{F}(k')$*. Suppose that an effective formal versal deformation* (R, ξ) *of* ξ_0 *exists, where* R *is a complete noetherian local* $\mathscr{O}_S$*-algebra with residue field* k' *and* $\xi \in \operatorname{Ob} \mathsf{F}(R)$*.* (*Our language of deformation is again applicable in this case, for the same reason mentioned in Theorem B.2.5.*) *Then there is an* S*-scheme* X *of finite type, a closed point* $x \in X$ *with residue field* $k(x) = k'$*, and an element* $\widetilde{\xi} \in \operatorname{Ob} \mathsf{F}(X)$*, such that the triple* $(X, x, \widetilde{\xi})$ *is a versal deformation of* ξ_0*. More precisely, there is an isomorphism* $\hat{\mathscr{O}}_{X,x} \cong$

R such that $\widetilde{\xi}$ induces an object in $\operatorname{Ob}\mathsf{F}(R)$ *isomorphic to* ξ *through the composition*

$$\operatorname{Spec}(R) \xrightarrow{\sim} \operatorname{Spec}(\hat{\mathscr{O}}_{X,x}) \to X.$$

The isomorphism is unique if (R, ξ) *is universal.*

Proof. To proof the theorem, we consider the (contravariant) functor

$$\overline{\mathsf{F}} : (\mathrm{Sch}\,/S)^\circ \to (\mathrm{Sets})$$

associated to the category F fibred in groupoids, with the canonical surjection

$$\pi : \mathsf{F} \to \overline{\mathsf{F}}$$

as before. Consider the effective versal deformation $\eta \in \overline{\mathsf{F}}(R)$ of $\pi(\xi_0) \in \overline{\mathsf{F}}(k')$ given by the image $\pi(\xi)$ of ξ in $\overline{\mathsf{F}}(R)$. It is immediate that $\overline{\mathsf{F}}$ satisfies all the requirements of Theorem B.2.5, and hence there is a triple $(X, x, \widetilde{\eta})$ with $\widetilde{\eta} \in \overline{\mathsf{F}}(X)$, and an isomorphism (which is unique if the formal versal deformation is universal) $\hat{\mathscr{O}}_{X,x} \cong R$, such that $\widetilde{\eta}$ induces η through the composition

$$\operatorname{Spec}(R) \xrightarrow{\sim} \operatorname{Spec}(\hat{\mathscr{O}}_{X,x}) \to X.$$

Now the element $\widetilde{\eta} \in \overline{\mathsf{F}}(X)$ is an equivalence class of objects in $\mathsf{F}(X)$. Let us take any object $\widetilde{\xi}$ in $\operatorname{Ob}\mathsf{F}(X)$ in the class of $\widetilde{\eta}$. Since $\widetilde{\eta}$ induces η in $\overline{\mathsf{F}}(R)$, the object induced by $\widetilde{\xi}$ in $\operatorname{Ob}\mathsf{F}(R)$ must be isomorphic to ξ. Then it is clear that the triple $(X, x, \widetilde{\xi})$ satisfies our requirement. □

B.3 Artin's Criterion for Algebraic Stacks

Artin proved a criterion which is useful for proving that a functor is representable by an algebraic space. The same proof provides the following criterion for verifying a stack is algebraic, given in [30]. The theorem that we are going to generalize is [9, Thm. 3.4]. The interested reader may also consult the other parts of [9], the article [12], and the book [13].

Before the proof, we need some preparations. Throughout this section, we assume that S be a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain.

Definition B.3.1. *Let* X *be a category fibred in groupoids over* (Sch/S)*,* X *a scheme over* S*, and* $\xi : X \to \mathsf{X}$ *a morphism (i.e.* $\xi \in \mathrm{Ob}\,\mathsf{X}(X)$*). Let* x *be a point of* X*. We say that* ξ *is* **formally étale** *at* x *if for every commutative diagram of solid arrows*

$$\begin{tikzcd} X \arrow[d, "\xi"'] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] \\ \mathsf{X} & Z \arrow[l] \arrow[ul, dotted, "f"'] \end{tikzcd} \tag{B.3.2}$$

where Z *is the spectrum of an Artinian local* $\mathscr{O}_S$*-algebra, where* Z_0 *is a closed subscheme of* Z *defined by a nilpotent ideal, and where* f_0 *is a map sending* Z_0 *to* x *set theoretically, there exists a unique dotted arrow* f *making the diagram commutative.*

Remark B.3.3. Here *set theoretically* means that the unique closed point of Z_0 defined by the maximal ideal is sent to the point x. Note that any extension f of f_0 should also map Z set theoretically to x.

Remark B.3.4. By [52, IV, 17.14.1 and 17.14.2], the property of being étale and being formally étale are equivalent for morphisms locally of finite presentation. A special case of this fact can also been found in [13, I, Prop. 1.1], and a discussion of one of the two directions of the proof can be found in [94, Rem. 3.22]. This fact is generalized to the case of a morphism locally of finite presentation from a scheme to an algebraic spaces by [9, Lem. 3.3].

Here we give an equivalence between the property of being étale and being formally étale in the following case:

Proposition B.3.5. *Let* X *be a* S*-scheme,* X *a category fibred in groupoids, and* $\xi : X \to \mathsf{X}$ *a morphism. Assume that* ξ *is a representable morphism that is locally of finite representation. Namely, for every* S*-scheme* U *and morphism* $U \to \mathsf{X}$ *the fiber product* $X \underset{\mathsf{X}}{\times} U$ *is representable by an algebraic space locally of finite presentation over* U*. Let* $x \in X$*. Then* ξ *is formally étale at* x *if and only if the following condition holds:*

Let $U \to \mathsf{X}$ *be any morphism from a scheme* U *to* X*, and let* C *be the inverse image of* x *in* $X \underset{\mathsf{X}}{\times} U$*. Then the projection* $X \underset{\mathsf{X}}{\times} U \to U$ *is étale at every point of* C*.*

$$\begin{tikzcd} X \arrow[d, "\xi"'] & X \underset{\mathsf{X}}{\times} U \arrow[l] \arrow[d] \\ \mathsf{X} & U \arrow[l] \end{tikzcd}$$

Before the proof of Proposition B.3.5, we need some technical preparation. For convenience of the reader, we quote the following proposition:

Proposition B.3.6 ([53, I, 6.2.6(v)])**.** *If the composition of two morphism of schemes $f : X \to Y$ and $g : Y \to Z$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Then we have the following lemma:

Lemma B.3.7. *Let X be an algebraic space and U, V two schemes. Suppose that we have morphisms $V \to X \to U$ of algebraic spaces such that $V \to X$ is étale surjective and such that $X \to U$ is locally of finite presentation. Then $V \to X$ is locally of finite presentation.*

Proof. Consider the following diagram for any étale surjection $W \to X$ from a scheme W:

$$\begin{tikzcd}
V \arrow[d] & V \underset{X}{\times} W \arrow[l] \arrow[d] \arrow[ddl, bend right] \\
X \arrow[d] & W \arrow[l] \arrow[dl] \\
U &
\end{tikzcd}$$

Here $X \to U$ is locally of finite presentation. Hence $V \underset{X}{\times} W \to U$ and $W \to U$ are locally of finite presentation over W (for the property of being locally of finite presentation is stable in the étale topology). Applying Proposition B.3.6 to the composition $V \underset{X}{\times} W \to W \to U$, we see that $V \underset{X}{\times} W \to W$ is locally of finite presentation. As a result, the morphism $V \to X$ is locally of finite presentation by definition. $\square$

Proof of Proposition B.3.5. If the condition of the lemma is satisfied (for any scheme U), then we take $U = Z$. Any map $f_0 : Z_0 \to X$ with set theoretical image a closed point $x_0 \in X$ induces a morphism $Z_0 \to X \underset{\mathsf{X}}{\times} Z$ with set theoretical image y for some closed point y of $X \underset{\mathsf{X}}{\times} Z$ whose image in X is x_0. Consider an affine scheme V and an étale morphism $V \to X \underset{\mathsf{X}}{\times} Z$ as in Theorem A.3.3.3 such that $y \to X \underset{\mathsf{X}}{\times} Z$ factors through $y \to V \to X \underset{\mathsf{X}}{\times} Z$. Since $V \to X \underset{\mathsf{X}}{\times} Z$ is étale and locally of finite presentation (by Lemma

B.3.7), and since being étale is equivalent to being formally étale in this case (as explained in Remark B.3.4), the morphism from the closed point of Z_0 to y extends to a morphism from Z_0 to V. Therefore we have a diagram (of solid arrows):

$$\begin{tikzcd}
X \arrow[d, "\xi"'] & X \underset{\mathsf{X}}{\times} Z \arrow[l] \arrow[d] & V \arrow[l] \arrow[d] & Z_0 \arrow[l] \arrow[d, hook] \\
\mathsf{X} & Z \arrow[l] \arrow[r, equal] & Z & Z, \arrow[l, "\mathrm{id}_Z"'] \arrow[ul, dotted]
\end{tikzcd}$$

By hypothesis, $V \to Z$ is étale at the preimage of y and locally of finite presentation. Hence again by Remark B.3.4, we have an induced morphism $Z \to V$ (dotted in the diagram) making the diagram commute. The composition

$$Z \to V \to X \underset{\mathsf{X}}{\times} Z \to X$$

then gives the desired morphism f. If f' is another such morphism, then by universal property of the fiber product $X \underset{\mathsf{X}}{\times} Z$ and by the formally étaleness of $V \to X \underset{\mathsf{X}}{\times} Z$, the morphism f' extends uniquely to a morphism $Z \to V$ making the diagram commutes. Then f' and f must be the same, by the formally étaleness of $V \to Z$. This shows that ξ is formally étale.

Conversely, if U is any scheme and $U \to \mathsf{X}$ a morphism as described in the condition. Let $y \in C$ be any point whose image in X is x. By Theorem A.3.3.3, there is an affine scheme V and an étale morphism $V \to X \underset{\mathsf{X}}{\times} U$ such that $y \to X \underset{\mathsf{X}}{\times} U$ factors through $y \to V \to X \underset{\mathsf{X}}{\times} U$. Consider the diagram of solid arrows

$$\begin{tikzcd}
X \arrow[d, "\xi"'] & X \underset{\mathsf{X}}{\times} U \arrow[l] \arrow[d] & V \arrow[l] \arrow[d] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] , \\
\mathsf{X} & U \arrow[l] \arrow[r, equal] & U & Z \arrow[l] \arrow[ul, dotted, "f"'] \arrow[ull, dotted] \arrow[ulll, dotted]
\end{tikzcd}$$

where f_0 is a map sending Z_0 set theoretically to y. The composition

$$Z_0 \to V \to X \underset{\mathsf{X}}{\times} U \to X$$

defines a morphism from Z_0 to x set theoretically. Hence by formally étaleness of ξ, there is a morphism from $Z \to X$ extending the composition. By universal property of the fiber product $X \underset{\mathsf{X}}{\times} U$, this morphism induces a morphism $Z \to X \times_{\mathsf{X}} U$.

$$\begin{tikzcd}
V \arrow[d] & Z_0 \arrow[l] \arrow[d, hook] \\
X \times_{\mathsf{X}} U & Z \arrow[l] \arrow[ul, dotted]
\end{tikzcd}$$

The morphism $V \to X \times_{\mathsf{X}} U$ is étale, locally of finite presentation by Lemma B.3.7, and hence formally étale by Remark B.3.4. Therefore we have an induced morphism $f : Z \to V$ making the above diagram commute. By Remark B.3.4, and since the above Z and Z_0 may be chosen independent of U and V, this means that $V \to U$ is étale at y. Hence $X \times_{\mathsf{X}} U \to U$ is étale at y by definition. □

Theorem B.3.8 (Artin's criterion)**.** *Let S be a scheme of finite type over a field or an excellent Dedekind domain. Let X be a category fibred in groupoids over* (Sch/S)*. Then X is an algebraic stack locally of finite type over S if and only if the following conditions hold:*

1. *X is a stack for the étale topology.*
2. *X is locally of finite presentation.* (*See Definition B.2.1.9.*)
3. *Let $\xi, \eta \in \mathrm{Ob}\,\mathsf{X}(U)$ be two 1-morphisms of an S-scheme U of finite type over S into X. Then $\mathrm{Isom}_U(\xi, \eta)$ is an algebraic space locally of finite type over S.*
4. *For any field k_0 of finite type over S with a 1-morphism $i : \mathrm{Spec}(k_0) \to \mathsf{X}$, there exist a complete noetherian local ring R, a morphism j from the spectrum of a finite separable extension k_0' of k_0 into the close point s of $\mathrm{Spec}(R)$, and a commutative diagram*

$$\begin{tikzcd}
\mathrm{Spec}(k_0') \arrow[r] \arrow[d, "j"] & \mathrm{Spec}(k_0) \arrow[d, "i"] \\
\mathrm{Spec}(R) \arrow[r, "\xi"] & \mathsf{X}
\end{tikzcd} \tag{B.3.9}$$

   with ξ formally étale at s.

5. *If ξ is a 1-morphism from an S-scheme U of finite type to X and ξ is formally étale at a point u of finite type over S, then ξ is formally étale in a neighborhood of u.*

Proof. We first prove the necessity. Let X be an algebraic stack locally of finite type over S.

Condition 1 is trivial by definition, and condition 3 follows from Proposition A.5.23. Condition 2 is also automatic, because X is locally of finite type over S, where S is locally of finite type over a field or an excellent Dedekind domain. Therefore X is locally of finite presentation as an algebraic stack over S, and hence at the same time locally of finite presentation as a category fibred in groupoids over S by Proposition B.2.1.11.

To verify condition 4, let k_0 be any field of finite type over S with a 1-morphism $i : U = \operatorname{Spec}(k_0) \to S$. Let $X \to \mathsf{X}$ be the presentation of X. Consider the algebraic space U' representing the fiber product $X \underset{\mathsf{X}}{\times} U$, and take any point $\operatorname{Spec}(k_0') \to U'$. Then since U' is étale and of finite presentation over $U = \operatorname{Spec}(k_0)$, we see that k_0' is a finite separable extension of k_0. By Theorem A.3.3.3, the morphism $\operatorname{Spec}(k_0') \to X$ must factor through an affine scheme V with $V \to X$ an étale morphism, so that we have the following diagram:

$$\begin{tikzcd}
\operatorname{Spec}(k_0') \arrow[r] \arrow[d] & U' = X \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & U = \operatorname{Spec}(k_0) \arrow[d] \\
V \arrow[r] & X \arrow[r] & \mathsf{X}
\end{tikzcd}$$

Since X is locally of finite type over S, V is also locally of finite type over S (cf. Lemma B.3.7, with identical proof), and we may replace V by an affine neighborhood of $\operatorname{Spec}(k_0')$ that is of finite type over S. Then by the hypothesis of S, V is noetherian. By taking the formal completion of $\mathscr{O}_V$ with respect to the image s of $\operatorname{Spec}(k_0')$, we get the desired complete noetherian local ring $R = \hat{\mathscr{O}}_{V,s}$ inducing diagram (B.3.9).

Let ξ be a 1-morphism from an S-scheme U of finite type to X, which is formally étale at a point u of finite type over S. Again consider the fiber product and the following diagram:

$$\begin{tikzcd}
U' = X \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & U \arrow[d, "\xi"] \\
X \arrow[r] & \mathsf{X}
\end{tikzcd}$$

By Lemma B.3.5, the morphism $U' = X \underset{\mathsf{X}}{\times} U \to X$ is étale at every point whose image through $U' \to U$ is u. Since being étale is an open condition, we see that $U' \to X$ is étale at a neighborhood of a point in U' whose image through $U' \to U$ is u. Now since $X \to \mathsf{X}$ is étale, the induced projection $U' \to U$ is étale, and hence open. Then $U \to \mathsf{X}$ is étale at an open neighborhood of u and condition 5 is verified.

Conversely, suppose that all the conditions are satisfied. We are going to construct a representable morphism $X \to \mathsf{X}$ that is étale and surjective.

Let p be any point of X of finite type over S. This amounts to say that there is a field k_0^p of finite type over S with a 1-morphism $i^p : U = \mathrm{Spec}(k_0^p) \to S$. Condition 4 can be interpreted as the existence of an effective universal formal deformation (R^p, ξ^p) of $\xi_0^p \in \mathrm{Ob}\,\mathsf{X}(k^p)$, where R^p is complete noetherian local with residue field k^p (see Section B.2.1). Here we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k_0'^p) \arrow[r] \arrow[d] & \mathrm{Spec}(k_0^p)\ , \arrow[d] \\
s = \mathrm{Spec}(k^p) \arrow[r] & S,
\end{tikzcd}$$

where $\mathrm{Spec}(k_0'^p) \to \mathrm{Spec}(k_0^p)$ and $\mathrm{Spec}(k_0^p) \to S$ are of finite type. Hence $s = \mathrm{Spec}(k^p) \to S$ must be of finite type. Then since X is locally of finite presentation by condition 2, we may apply Theorem B.2.1.12 to see that the pair (R^p, ξ^p) is algebraizable, say by $(X^p, x^p, \widetilde{\xi^p})$, with properties such as a S-scheme of finite type, etc, described in Theorem B.2.5. Moreover, by the hypothesis of condition 4, $X^p \to \mathsf{X}$ is formally étale.

Thus by condition 5, we may replace X^p by an open neighborhood of x^p so that $\xi^p : X^p \to \mathsf{X}$ is formally étale at every point. By condition 3, $X^p \to \mathsf{X}$ is representable and étale. Namely, $X^p \underset{\mathsf{X}}{\times} U \to U$ is étale for every scheme $U \to \mathsf{X}$.

Now we claim that we may take X to be the amalgamation $\coprod\limits_p X^p$ for representatives p in each equivalence class of points of finite type of X, so that $X \to \mathsf{X}$ is our desired presentation of X.

Let U be any scheme with $U \to \mathsf{X}$, which we may assume to be of finite type. Then the projection $X \underset{\mathsf{X}}{\times} U \to U$ is étale. To show that the projection $X \to \mathsf{X}$ is surjective, it suffices to show that every point u of U of finite type over S is in the image, since these points are dense. Write $u = \mathrm{Spec}(k)$ for some k. Let $p = \mathrm{Spec}(k_0^p)$ be the representative of the equivalence class of points of finite type equivalent to u, which is used in forming the amalgamation $X = \coprod_p X^p$. The point p is equivalent further to the point $x^p = \operatorname{Spec}(k^p)$ of X^p. Take a separable field extension k' of both k^p and k, such that there is a commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec}(k') \arrow[r] \arrow[d] & u = \operatorname{Spec}(k) \arrow[d] \\
\operatorname{Spec}(k^p) \arrow[r] & \mathsf{X}
\end{tikzcd}$$

The two morphisms $\operatorname{Spec}(k') \to \operatorname{Spec}(k^p) \to X^p$ and $\operatorname{Spec}(k') \to u \to U$ induces a morphism from $\operatorname{Spec}(k')$ to the algebraic space $X^p \underset{\mathsf{X}}{\times} U$. By Proposition A.3.3.2, this factors through a point of $X^p \underset{\mathsf{X}}{\times} U$, whose image in the scheme U is some point u'. Then the morphism $\operatorname{Spec}(k') \to U$ factors though both the points u and u', which means that $u' = u$. Hence $X^p \underset{\mathsf{X}}{\times} U \to U$ covers u as desired. This completes the proof. $\square$

Theorem B.3.10. *If the residue fields of finite type of S are perfect, then, to establish the result of Theorem B.3.8, we may replace condition 4 of Theorem B.3.8 by the following one:*

Let $s \in S$ and k_0 be a finite extension of $k(s)$. Suppose that $u : \operatorname{Spec}(k_0) \to S$ is of finite type. Then by hypothesis, $k(s)$ is perfect and $k_0/k(s)$ is separable. By Lemma B.1.1.16, there exists a unique complete local ring Λ_{k_0} with residue field k_0, together with a morphism $\bar{u} : \operatorname{Spec}(\Lambda_{k_0}) \to S$ that is formally étale and extends to u through composition with $\operatorname{Spec}(k_0) \to \operatorname{Spec}(\Lambda_{k_0})$. (*This follows from the construction of Λ_{k_0} in Lemma B.1.1.16, the universal property of complete p-rings, and the separability of the residue field extension.*) *For $\xi_0 \in \operatorname{Ob}\mathsf{X}(k_0)$, we denote by $\mathsf{D}(\xi_0)$ the following category over the dual of $\hat{\mathscr{C}}_{\Lambda_{k_0}}$: For $A \in \operatorname{Ob}\hat{\mathscr{C}}_{\Lambda_{k_0}}$, an object of $\mathsf{D}(\xi_0)(A)$ is an object ξ of $\mathsf{X}(A)$, with an isomorphism*

$$(\text{image of } \xi \text{ in } \mathsf{X}(k_0)) \xrightarrow{\sim} \xi_0. \tag{B.3.11}$$

Then the condition to replace is the following one:

4′. For k_0 and ξ_0 as above, the covariant functor $D := \overline{\mathsf{D}}$ from $\hat{\mathscr{C}}_{\Lambda_{k_0}}$ to (Sets) *defined by*

$$A \mapsto \{\text{isomorphism classes in } \mathsf{D}(\xi_0)(A)\}$$

is effectively prorepresentable (*by a complete noetherian local ring R, and any $\xi \in \operatorname{Ob}\mathsf{X}(R)$ satisfying the previous restriction* (*B.3.11*)).

This condition may be verified after any scalar extension from k_0 to a finite extension.

Proof. Recall that, in our proof of Theorem B.3.8, condition 4 is equivalent to the existence of a certain effective universal deformation of the 2-functor associated to the category X fibred in groupoids, which is needed for applying Theorem B.2.1.12. However, the proof of Theorem B.2.1.12 merely requires the existence of an effective universal deformation of the functor associated canonically to the 2-functor associated to X, which is equivalent to the effective prorepresentability of the deformation functor defined above. Hence the statements of the theorem are proved. $\square$

Theorem B.3.12. *If S is of finite type over a field or over an excellent Dedekind domain having infinitely many points, and all those possible complete local rings R in Theorem B.3.10 are normal and of the same Krull dimension, then we may suppress condition 5 in Theorem B.3.8.*

Proof. We first remark that, under these hypotheses, the points of finite type of any scheme U of finite type over S are all closed points.

If ξ is a 1-morphism from an S-scheme U of finite type to X and ξ is formally étale at a point u of finite type over S, then U is normal at u [50, IV, 7.8.3]. Hence we may replace U by an open neighborhood of u that is entire, and we claim that ξ is then étale. (A morphism $X \to Y$ is *entire* if there is a covering $\{Y_\alpha = \mathsf{Spec}(B_\alpha)\}$ of Y by affine open sets such that, for all α, $f^{-1}(Y_\alpha)$ is an affine scheme $\mathrm{Spec}(A_\alpha)$, where every element of A_α is integral over B_α. See [48, II, 6.1.1] and [46, 0_{I}, 1.0.5].) It suffices to show that ξ is formally étale at every closed point $u' \in U$. Let u' be any closed point (which is of finite type by the above remark) with $u' \to \mathsf{X}$ induced by $U \to \mathsf{X}$, and let (X, x, η) be as in condition 4 of Theorem 4 and in Theorem B.2.1.12 (i.e. $x \to \mathsf{X}$ is equivalent to $u' \to \mathsf{X}$). Note that η is formally étale at x. We may replace X by a neighborhood of x and assume that it is also entire and unramified over X. The Krull dimension of U and X are both equal to some d, by assumption.

Now let w be the (unique) point of $X \underset{\mathsf{X}}{\times} U$ lying over x and u'.

$$\begin{tikzcd}
w \arrow[rr] \arrow[dr] \arrow[dd] & & u' \arrow[d] \\
& X \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & U \arrow[d, "\xi"] \\
x \arrow[r] & X \arrow[r, "\eta"'] & \mathsf{X}
\end{tikzcd}$$

Since η is formally étale at x, $X \underset{\mathsf{X}}{\times} U \to U$ is étale at w (by Proposition B.3.5). Hence the dimension of $(X \underset{\mathsf{X}}{\times} U)$ at w is also d. Since $X \underset{\mathsf{X}}{\times} U \to X$ is unramified and U is entire, it follows that this morphism is étale. Therefore ξ is formally étale at u'. $\square$

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Index