COMPARISON BETWEEN ANALYTIC AND ALGEBRAIC CONSTRUCTIONS OF TOROIDAL COMPACTIFICATIONS OF PEL-TYPE SHIMURA VARIETIES

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Abstract. Using explicit identifications between algebraic and analytic theta functions, we compare the algebraic constructions of toroidal compactifications by Faltings–Chai and the author, with the analytic constructions of toroidal compactifications following Ash–Mumford–Rapport–Tai. As one of the applications, we obtain the corresponding comparison for Fourier–Jacobi expansions of holomorphic automorphic forms.

Motivating Question

In [23], the author constructed toroidal and minimal compactifications for moduli problems parameterizing abelian varieties with PEL structures, each being defined over some localization (with no ramified primes in the residue characteristics) of the ring of integers of the so-called reflex field. The construction uses the algebraic theory of degeneration of abelian varieties developed in [25] and [17]. There is no (complex) analytic argument involved.

On the other hand, abelian varieties over $\mathbb{C}$ with PEL structures can be parameterized explicitly by double coset spaces forming finite disjoint unions of quotients of Hermitian symmetric spaces by arithmetic groups. Any such analytic moduli space appears as a union of connected components in the analytification of the complex fibers of some algebraic moduli space.

According to the theory of [4], these double coset spaces admit canonical compactifications by finite disjoint unions of irreducible normal projective varieties. This is the analytic construction of minimal compactifications. Moreover, according to the theories developed in [20] and [2], the singularities of these projective varieties are resolved by complex algebraic spaces (namely Moishezon spaces) parameterized by certain combinatorial data. This is the analytic construction of toroidal compactifications.

Since the algebraic and analytic compactifications enjoy properties perfectly parallel to each other, it is tempting to conclude (as in [17, Ch. IV, Prop. 5.15]) that each of the analytic compactifications appears as a union of connected components in the analytification of the complex fiber of some algebraic compactification. (In general we cannot hope that they are identical due to technical restrictions in the

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definition of moduli problems incurred by the so-called failure of Hasse's principle. See [21, §8].)

In [16, Ch. VII, 4.4] and [28, 6.13], their explanations involve analytic constructions of partial compactifications of degenerating families of abelian varieties analogous to the algebraic ones. However, due to difficulties in patching over higher-dimensional cusps (and their intersections), it is not obvious that such an analytic construction is possible in reasonable generality. Moreover, even admitting this possibility, it is not obvious that the identification should be a simple one. For example, why should the analytic q-expansions be identified with the algebraic q-expansions without introducing some "periods"? A systematic answer to this question is desirable for practical reasons.

In this article, we propose an alternative explanation by comparing analytic and algebraic theta functions. Since the theory of degeneration of polarized abelian varieties (in [25] and [17]) is based on algebraic theta functions, and since the algebraic construction of toroidal compactifications (in [17] and [23]) is based on this theory of degeneration, we believe it is more natural (and logically simpler) to focus on the (canonical) spaces of theta functions than on (non-canonical) partial compactifications of the degenerating families.

The outline of our approach is as follows. First, we analyze the boundary structure of the analytic toroidal compactifications in terms of Siegel domains of the third kind, and write down tautological degeneration data over the formal completions along the boundary strata. (This means we are using the analytic q's as the algebraic q's right in the beginning.) The detailed steps are lengthy but straightforward. Next, by Mumford's construction, we obtain degenerating families along such completions. As a byproduct, we obtain comparison isomorphisms between analytic and algebraic objects over completions along the boundary strata. Finally, we show that, over the completions along boundary points (on the boundary strata), the pullbacks of generic fibers of the degenerating families coincide with the pullbacks of the universal families, using explicit bases given by theta functions. This allows us to patch the comparison isomorphisms (over completions) together over the whole compactifications.

Since the question makes sense only if the readers are reasonably familiar with both [2] and [23], we will make frequent references to them without repeating the definitions and arguments. (Although the notations might be slightly different, readers who are familiar with [17] and related works should have no problem in following [23].) On the other hand, since it is common that the readers might have their own choices of analytic coordinates in special cases, we shall be as explicit as possible when it comes to complex coordinates and theta functions. We hope this is a practical approach for potential users of this article.

Contents

We shall follow [23, Notations and Conventions] unless otherwise specified. (Although our references to [23] uses the numbering in the original version, the reader is advised to consult the errata and revision for corrections of typos and minor mistakes, and for improved exposition.)

Throughout the article, $\sqrt{-1}$ denotes some fixed choice of square root of -1 in $\mathbb{C}$, $z \mapsto z^c$ denotes the complex conjugation of $\mathbb{C}$, and the notation Im' denotes the *modified* imaginary part defined by $\mathrm{Im}'(z) = \frac{1}{2}(z - z^c)$ for any $z \in \mathbb{C}$. (In particular, the value of Im' is independent of the choice of $\sqrt{-1}$.) By symplectic isomorphisms between modules with symplectic pairings, we *always* mean isomorphisms between the modules matching the pairings up to an invertible scalar multiple. (These are often called symplectic similitudes, but our understanding is that the codomains of pairings are modules rather than rings, which ought to be matched as well.)

1. PEL-Type moduli problems

1.1. **Linear algebraic data.** Let $\mathcal{O}$ be an order in a finite-dimensional semisimple algebra over $\mathbb{Q}$ with positive involution * and center F. Here *positivity* of * means $\mathrm{Tr}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}/\mathbb{R}}(xx^\star) > 0$ for any $x \neq 0$ in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. We assume that $\mathcal{O}$ is mapped to itself under *.

Let us denote by $\mathbf{e} : \mathbb{C} \to \mathbb{C}^\times$ the homomorphism sending z to $\exp(z)$ for any $z \in \mathbb{C}$. Let $\mathbb{Z}(1) := \ker(\mathbf{e})$, which is a free $\mathbb{Z}$-module of rank one. Any square-root $\sqrt{-1}$ of -1 in $\mathbb{C}$ determines an isomorphism

$$(2\pi\sqrt{-1})^{-1} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}, \tag{1.1.1}$$

but there is no canonical isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$. For any commutative $\mathbb{Z}$-algebra R, we denote by $R(1)$ the module $R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}(1)$. When R is a subring of $\mathbb{R}$ containing (2π) and $(2\pi)^{-1}$, it is also convenient to use the isomorphism $-\sqrt{-1}$: $R(1) \xrightarrow{\sim} R$, the multiple of (1.1.1) by 2π. This modified isomorphism defines the same notion of positivity as the original isomorphism.

By a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h_0)$ (as in [23, Def. 1.2.1.3]), we mean the following data:

(1) An $\mathcal{O}$-lattice, namely a $\mathbb{Z}$-lattice L with the structure of an $\mathcal{O}$-module.
(2) An alternating pairing $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y \rangle = \langle x, b^\star y \rangle$ for any $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ satisfying:
   (a) For any $z \in \mathbb{C}$ and $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we have $\langle h_0(z)x, y \rangle = \langle x, h_0(z^c)y \rangle$, where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.
   (b) For any choice of $\sqrt{-1}$ in $\mathbb{C}$, the pairing $-\sqrt{-1}\,\langle \cdot, h_0(\sqrt{-1}) \cdot \rangle$: $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is symmetric and positive definite. (This last condition forces $\langle \cdot, \cdot \rangle$ to be nondegenerate.)

(In [23, Def. 1.2.1.3], h_0 was denoted by h.) The tuple $(\mathcal{O}, ^\star, L, \langle \cdot, \cdot \rangle, h_0)$ then gives us an integral version of the $(B, ^\star, V, \langle \cdot, \cdot \rangle, h_0)$ in [21] and related works.

The reductive group that will be associated with the geometric objects we study is as follows:

Definition 1.1.2 (cf. [23, Def. 1.2.1.5]). *Let a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h_0)$ be given as above. For any $\mathbb{Z}$-algebra R, set*

$$\mathrm{G}(R) := \left\{ (g, r) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \begin{array}{c} \langle gx, gy \rangle = r\langle x, y \rangle, \\ \forall x, y \in L \end{array} \right\}.$$

Namely, $\mathrm{G}(R)$ *is the group of symplectic automorphisms of* $L \underset{\mathbb{Z}}{\otimes} R$ *(cf.* [23, Def. 1.1.4.11]). *The assignment is functorial in* R *and defines a group functor* G *over* $\mathrm{Spec}(\mathbb{Z})$.

The projection to the second factor $(g, r) \mapsto r$ *defines a homomorphism* $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$*, which we call the* ***similitude character****. We shall often denote elements* (g, r) *in* G *by simply* g*, and denote by* $\nu(g)$ *the value of* r *when we need it, although one should keep in mind that* r *is not determined by* g*, for example, when* $L = \{0\}$.

Let $\Box$ be any set of rational primes. (It can be either an empty set, a finite set, or an infinite set.) Then we have definitions for $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^{\infty,\Box})$, $\mathrm{G}(\mathbb{A}^\infty)$, $\mathrm{G}(\mathbb{R})$, $\mathrm{G}(\mathbb{A}^\Box)$, $\mathrm{G}(\mathbb{A})$, $\mathrm{G}(\mathbb{Z})$, $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$, $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, $\mathrm{G}(\hat{\mathbb{Z}})$, $\Gamma(n) := \ker(\mathrm{G}(\mathbb{Z}) \to \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$, $\mathcal{U}^\Box(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}^\Box) \to \mathrm{G}(\hat{\mathbb{Z}}^\Box/n\hat{\mathbb{Z}}^\Box) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$ for any integer $n \geq 1$ prime-to-$\Box$, and $\mathcal{U}(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$.

Following Pink [27, 0.6], we define the neatness of open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as follows: Let us view $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as a *subgroup* of $\mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times \mathbf{G}_{\mathrm{m}}(\hat{\mathbb{Z}}^\Box)$. (Alternatively, we may take any faithful linear algebraic representation of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$.)

Then, for each rational prime $p > 0$ not in $\Box$, it makes sense to talk about *eigenvalues* of elements g_p in $\mathrm{G}(\mathbb{Z}_p)$, which are elements in $\bar{\mathbb{Q}}_p^\times$. Let $g = (g_p) \in \mathrm{G}(\hat{\mathbb{Z}}^\Box)$, with p running through rational primes such that $\Box \nmid p$. For each such p, let Γ_p be the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by eigenvalues of g_p. For any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, consider the subgroup $(\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\mathrm{tors}}$ of torsion elements of $\bar{\mathbb{Q}}^\times \cap \Gamma_p$, which is independent of the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Definition 1.1.3 ([23, Def. 1.4.1.8])**.** *We say that $g = (g_p)$ is* **neat** *if $\underset{p \notin \Box}{\cap} (\bar{\mathbb{Q}}^\times \cap \Gamma_p)_{\mathrm{tors}} = \{1\}$. We say that an open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ is* **neat** *if all its elements are neat.*

Remark 1.1.4. The usual Serre's lemma that no nontrivial root of unity can be congruent to 1 mod n if $n \geq 3$ shows that $\mathcal{H}$ is neat if $\mathcal{H} \subset \mathcal{U}^\Box(n)$ for some $n \geq 3$ such that $\Box \nmid n$.

Remark 1.1.5. Definition 1.1.3 makes no reference to the group $\mathrm{G}(\mathbb{Q})$ of rational elements. Nevertheless, if $\mathcal{H}$ is neat, then $\mathcal{H} \cap \mathrm{G}(\mathbb{Q})$ is neat as an arithmetic group, in the sense of [8, 17.1].

1.2. **Definition of moduli problems.** Let us fix the choice of a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h_0)$ as in the previous subsection. Let F_0 be the so-called *reflex field* defined as in [23, Def. 1.2.5.4]. We shall denote the ring of integers in F_0 by $\mathcal{O}_{F_0}$, and use similar notations for other number fields. This is in conflict with the notation of the order $\mathcal{O}$, but the precise interpretation will be clear from the context.

Let Disc be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (as in [23, Def. 1.1.1.6]; see also [23, Prop. 1.1.1.12]). Closely related to Disc is the invariant $\mathrm{I}_{\mathrm{bad}}$ for $\mathcal{O}$ defined in [23, Def. 1.2.1.17], which is either 2 or 1 depending on whether type D cases are involved.

Definition 1.2.1. *We say that a prime number p is* **bad** *if $p \,|\, \mathrm{I}_{\mathrm{bad}} \,\mathrm{Disc}\, [L^\# : L]$. We say a prime number p is* **good** *if it is not bad.*

Fix any choice of a set $\Box$ of *good primes*. We denote by $\mathbb{Z}_{(\Box)}$ the unique localization of $\mathbb{Z}$ (at the multiplicative subset of $\mathbb{Z}$ generated by nonzero integers prime-to-$\Box$) having $\Box$ as its set of height one primes, and denote by $\hat{\mathbb{Z}}^\Box$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^\Box$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. Let $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(\mathrm{Sch}/\mathsf{S}_0)$ be the category of schemes over S_0. For any open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, there is an associated moduli problem $\mathsf{M}_\mathcal{H}$ defined as follows:

Definition 1.2.2 (cf. [23, Def. 1.4.1.4])**.** *The moduli problem $\mathsf{M}_\mathcal{H}$ is defined by the category fibred in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ whose fiber over each S is the groupoid $\mathsf{M}_\mathcal{H}(S)$ described as follows: The objects of $\mathsf{M}_\mathcal{H}(S)$ are tuples $(G, \lambda, i, \alpha_\mathcal{H})$, where:*

1. *G is an abelian scheme over S.*
2. *$\lambda : G \to G^\vee$ is a polarization of degree prime to $\Box$.*
3. *$i : \mathcal{O} \to \mathrm{End}_S(G)$ defines an $\mathcal{O}$-structure of (G, λ).*
4. *$\underline{\mathrm{Lie}}_{G/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in* [23, Def. 1.3.4.2] *given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h_0)$.*
5. *$\alpha_\mathcal{H}$ is an (integral) level-$\mathcal{H}$ structure of (G, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle \cdot, \cdot \rangle)$ as in* [23, Def. 1.3.7.8].

The isomorphisms $(G, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (G', \lambda', i', \alpha'_{\mathcal{H}})$ *of* $\mathsf{M}_{\mathcal{H}}(S)$ *are given by* (*naive*) *isomorphisms* $f : G \xrightarrow{\sim} G'$ *such that* $\lambda = f^\vee \circ \lambda' \circ f$, $f \circ i(b) = i'(b) \circ f$ *for every* $b \in \mathcal{O}$, *and* $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ (*symbolically*).

Remark 1.2.3. The definition here using isomorphism classes is not as canonical as the ones proposed by Grothendieck and Deligne using quasi-isogeny classes (as in [21]). For the relation between their definitions and ours, see [23, §1.4]. We prefer our definition because it is better for the study of compactifications, and more concretely because it does not help (for our purpose) to realize theta functions as sections of line bundles on quasi-isogeny classes of abelian varieties.

Theorem 1.2.4 ([23, Thm. 1.4.1.12 and Cor. 7.2.3.10])**.** *The moduli problem* $\mathsf{M}_{\mathcal{H}}$ *is a smooth separated algebraic stack of finite type over* S_0. *It is representable by a quasi-projective scheme if the objects it parameterizes have no nontrivial automorphism, which is in particular the case when* $\mathcal{H}$ *is* **neat** (*as in Definition* 1.1.3).

We shall insist *from now on in this article* the following technical condition on PEL-type $\mathcal{O}$-lattices:

Condition 1.2.5 (cf. [23, Cond. 1.4.3.9])**.** *The PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot , \cdot \rangle, h_0)$ *is chosen such that the action of* $\mathcal{O}$ *on* L *extends to an action of some maximal order* $\mathcal{O}'$ *in* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *containing* $\mathcal{O}$.

By forgetting level structures, there is a canonical morphism from the moduli problem $\mathsf{M}_{\mathcal{H}}$ defined with $\Box = \emptyset$ to the characteristic zero fiber of any analogous moduli problem defined by a larger set $\Box$, and this canonical morphism is finite étale with open image because level structures are parameterized by isomorphisms between finite étale group schemes. For our purpose of comparison with the complex analytic construction, it is most natural to focus on the case $\Box = \emptyset$.

From now on, we shall assume that $\Box = \emptyset$, in which case $\mathcal{O}_{F_0,(\Box)} = F_0$. Then $\mathcal{H}$ is an open compact subgroup of $\mathrm{G}(\mathbb{A}^\infty)$, and the moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)}) = \mathrm{Spec}(F_0)$.

2. Complex abelian varieties

2.1. **Complex structures.** Let us fix a choice of a symplectic $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle)$ as in §1.1. The aim of this section is to understand the role played by the choices of the polarization h of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ (that makes $(L, \langle \cdot , \cdot \rangle, h)$ a PEL-type $\mathcal{O}$-lattice). (Keep in mind that $(L, \langle \cdot , \cdot \rangle, h)$ defines a polarized Hodge structure of weight -1.)

Definition 2.1.1. *Let* $\epsilon \in \{\pm 1\}$. *A* ϵ**-polarization** *of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$ *is an* $\mathbb{R}$*-algebra homomorphism* $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ *such that the assignment* $x + y\sqrt{-1} \mapsto h(x + y\epsilon\sqrt{-1})$, *for* $x, y \in \mathbb{R}$, *defines a polarization of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$.

Let us restate [26, Ch. II, Lem. 4.1] in our context as follows:

Lemma 2.1.2. *Let us fix an element* $\epsilon \in \{\pm 1\}$. *Consider the three sets formed respectively by the following three types of data on* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$:

(1) *An* ϵ**-polarization** h *of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle)$.

(2) *An* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$*-module* V*, together with an* $\mathcal{O}$*-module morphism* $\mathsf{i} : L \hookrightarrow V$ *and a Hermitian pairing* $H : V \times V \to \mathbb{C}$ *(which is* $\mathbb{C}$*-linear in the second variable) such that:*
  (a) $\operatorname{Im}' H(x, y) = \langle x, y\rangle$ *for every* $x, y \in L$*.*
  (b) *The Hermitian pairing* $\epsilon H : V \times V \to \mathbb{C}$ *is positive-definite.*

(3) *An* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$*-submodule* P *of* $L\underset{\mathbb{Z}}{\otimes}\mathbb{C}$ *such that:*
  (a) $\mathsf{P} = \mathsf{P}^{\perp} := \{x \in L\underset{\mathbb{Z}}{\otimes}\mathbb{C} : \langle x, y\rangle = 0, \forall\, y \in \mathsf{P}\}$.
  (b) *For any nonzero* x *in* P*, we have* $\epsilon\langle x^c, x\rangle > 0$*. Here* $c : L\underset{\mathbb{Z}}{\otimes}\mathbb{C} \to L\underset{\mathbb{Z}}{\otimes}\mathbb{C} : x \mapsto x^c$ *is the complex conjugation induced by the one of* $\mathbb{C}$*.*

Then the three sets are in bijections with each other under the following assignments:

- *From* (1) *to* (2)*, we define* $V := L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *with complex structure* $h(\sqrt{-1})$*, with* $\mathsf{i} : L \to V$ *being the canonical morphism, and set*

$$H(x, y) := \langle x, y\rangle - \sqrt{-1}\,\langle x, h(\sqrt{-1})y\rangle. \tag{2.1.3}$$

- *From* (1) *to* (3)*, we define* $\mathsf{P} := \{\sqrt{-1}\,x - h(\sqrt{-1})x : x \in L\underset{\mathbb{Z}}{\otimes}\mathbb{R}\}$*.*
- *From* (2) *to* (1)*, we define the complex structure* $h(\sqrt{-1})$ *on* $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *to be the one induced by the natural complex structure* $\sqrt{-1} : V \overset{\sim}{\to} V$*, under the* $\mathbb{R}$*-linear isomorphism* $\mathsf{i}_{\mathbb{R}} : L\underset{\mathbb{Z}}{\otimes}\mathbb{R} \to V$ *induced by* $\mathsf{i} : L \to V$*.*
- *From* (2) *to* (3)*, we define* P *to be the kernel of the* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$*-module morphism* $\mathsf{i}_{\mathbb{C}} : L\underset{\mathbb{Z}}{\otimes}\mathbb{C} \twoheadrightarrow V$*.*
- *From* (3) *to* (1)*, we define the complex structure* $h(\sqrt{-1})$ *on* $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *to be the one induced by the complex structure* $1 \otimes \sqrt{-1}$ *on* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}$*, under the composition of the canonical* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$*-module morphisms* $L\underset{\mathbb{Z}}{\otimes}\mathbb{R} \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{C} \twoheadrightarrow (L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}$*, which is an isomorphism because* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \cap \mathsf{P} = 0$*.*
- *From* (3) *to* (2)*, we define* $V := (L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}$*, define* $\mathsf{i} : L \to V$ *to be the composition of the canonical* $\mathcal{O}$*-module morphisms* $L \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{C} \twoheadrightarrow V = (L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}$*, and define the Hermitian pairing* H *by the same formula* (2.1.3)*.*

As in [26, p. 173], the reader is advised to fully master this lemma before moving on. We omit the proof because it is elementary and straightforward.

Definition 2.1.4. *An* $\mathbb{R}$*-algebra homomorphism* $h : \mathbb{C} \to \operatorname{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ *is called a* $\pm$*-polarization if it is an* ϵ*-polarization for some* $\epsilon \in \{\pm 1\}$*, in which case we shall denote* $\operatorname{sgn}(h) = \epsilon$*.*

Lemma 2.1.5. *If* h *is an* $\pm$*-polarization of* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$ *and* $g \in \mathrm{G}(\mathbb{R})$*, then the* $\mathbb{R}$*-algebra homomorphism* $g(h) : \mathbb{C} \to \operatorname{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ *defined by* $z \mapsto g \circ h(z) \circ g^{-1}$ *is again a* $\pm$*-polarization, and* $\operatorname{sgn}(g(h)) = \operatorname{sgn}(\nu(g))\operatorname{sgn}(h)$*.*

2.2. **Polarized abelian varieties.** Let h be a $\pm$-polarization of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$, which defines a complex structure $h(\sqrt{-1})$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. We shall denote by V_h the $\mathbb{C}$-vector space with underlying $\mathbb{R}$-vector space $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ and complex structure $h(\sqrt{-1})$. By (2.1.3), h defines a Hermitian pairing $H_h : V_h \times V_h \to \mathbb{C}$ such that $\operatorname{sgn}(h) H_h$ is positive definite and such that $(\operatorname{Im}' H_h)(L \times L) \subset \mathbb{Z}(1)$.

According to the Theorem of Appell–Humbert [24, §2], the following sets of data are in canonical bijection with each other:

(1) Isomorphism classes of line bundles on G_h.
(2) Pairs (H, α), where:
 (a) $H : V_h \times V_h \to \mathbb{C}$ is a Hermitian pairing on V_h such that $(\operatorname{Im}' H)(L \times L) \subset \mathbb{Z}(1)$.
 (b) $\alpha : L \to \mathbb{C}^\times$ is a map such that $\alpha_1(l_1 + l_2)\, \alpha_1(l_1)^{-1}\, \alpha_1(l_2)^{-1} = \mathbf{e}(\frac{1}{2} \operatorname{Im}' H(l_1, l_2))$ for any $l_1, l_2 \in L$.

Explicitly, a pair (H, α) as above defines an action of L on $V_h \times \mathbb{C}$ by sending $l \in L$ to the holomorphic map

$$V_h \times \mathbb{C} \to V_h \times \mathbb{C} : (x, w) \mapsto (x + l,\; w\, \alpha(l)\, \mathbf{e}(\tfrac{1}{4} H(l, l) + \tfrac{1}{2} H(l, x))),$$

covering the translation action of L on V_h. Then forming quotients by L defines a holomorphic map

$$\mathcal{L}(H, \alpha) := (V_h \times \mathbb{C})/L \to G_h = V_h/L,$$

giving $\mathcal{L}(H, \alpha)$ a structure of a holomorphic line bundle over G_h, with sections of $\mathcal{L}(H, \alpha)$ represented by functions $f : V_h \to \mathbb{C}$ (defining pairs $(x, f(x))$ on $V_h \times \mathbb{C}$) satisfying

$$f(x + l) = f(x)\, \alpha(l)\, \mathbf{e}(\tfrac{1}{4} H(l, l) + \tfrac{1}{2} H(l, x)).$$

By Lefschetz's theorem [24, §3, Cor. on p. 35], the complex torus $G_h := V_h/L$ is projective. In particular, G_h is an abelian variety. The ample line bundles on G_h correspond to pairs (H, α) such that H is positive definite.

By [24, §9], we have the following facts about G_h:

(1) The dual abelian variety of G_h is isomorphic to $G_h^\vee := V_h/L^\#$.
(2) The alternating pairing $\langle\,\cdot\,,\,\cdot\,\rangle : L \times L \to \mathbb{Z}(1)$ defines a self-dual alternating pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{P}} : (L \times L^\#) \times (L \times L^\#) \to \mathbb{Z}(1)$ by

$$((l_1, l_2), (l_1', l_2')) \mapsto \langle l_1, l_2' \rangle - \langle l_1', l_2 \rangle.$$

Hence, the Hermitian pairing $H_{\mathcal{P}_h} : (V_h \times V_h) \times (V_h \times V_h) \to \mathbb{C}$ defined by

$$((x_1, x_2), (x_1', x_2')) \mapsto H_h(x_1, x_2') + H_h(x_2, x_1') = H_h(x_1, x_2') - H_h(x_1', x_2)^c$$

satisfies $\operatorname{Im} H_{\mathcal{P}_h} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{P}}$. Let $\alpha_{\mathcal{P}} : L \times L^\# \to \mathbb{C}^\times$ be the map defined by

$$\alpha_{\mathcal{P}}(l_1, l_2) := \mathbf{e}(\tfrac{1}{2} \langle l_1, l_2 \rangle).$$

Then

$$\alpha_{\mathcal{P}}(l_1 + l_1', l_2 + l_2')\, \alpha_{\mathcal{P}}(l_1, l_2)^{-1}\, \alpha_{\mathcal{P}}(l_1', l_2')^{-1} = \mathbf{e}(\tfrac{1}{2} \langle l_1, l_2' \rangle + \tfrac{1}{2} \langle l_1', l_2 \rangle)$$
$$= \mathbf{e}(\tfrac{1}{2} \langle l_1, l_2' \rangle - \tfrac{1}{2} \langle l_1', l_2 \rangle) = \mathbf{e}(\tfrac{1}{2} \operatorname{Im}' H_{\mathcal{P}_h}((l_1, l_2), (l_1', l_2'))),$$

(because $\langle l_1', l_2\rangle \in \mathbb{Z}(1)$), and

$$\alpha_{\mathcal{P}}(l_1,l_2)\,\mathbf{e}(\tfrac{1}{4}H_{\mathcal{P}_h}((l_1,l_2),(l_1,l_2))+\tfrac{1}{2}H_{\mathcal{P}_h}((l_1,l_2),(x,y)))$$
$$=\mathbf{e}(\tfrac{1}{2}\langle l_1,l_2\rangle+\tfrac{1}{4}[H_h(l_1,l_2)+H_h(l_2,l_1)]+\tfrac{1}{2}[H_h(l_1,y)+H_h(l_2,x)])$$
$$=\mathbf{e}(\tfrac{1}{2}H_h(l_1,l_2)+\tfrac{1}{2}H_h(l_1,y)+\tfrac{1}{2}H_h(l_2,x)).$$

(3) The Poincaré line bundle $\mathcal{P}_h$ on $G_h\times G_h^\vee$ is isomorphic to the line bundle $\mathcal{L}(H_{\mathcal{P}_h},\alpha_{\mathcal{P}})$ corresponding by the Theorem of Appell–Humbert to the pair $(H_{\mathcal{P}_h},\alpha_{\mathcal{P}})$ defined above. Explicitly, $\mathcal{L}(H_{\mathcal{P}_h},\alpha_{\mathcal{P}})$ is the quotient of $V_h\times V_h\times\mathbb{C}$ by the action of $L\times L^\#$ defined by sending $(l_1,l_2)\in L\times L^\#$ to the holomorphic map

$$V_h\times V_h\times\mathbb{C}\to V_h\times V_h\times\mathbb{C}:$$
$$(x,y,w)\mapsto(x+l_1,\ y+l_2,\ w\,\mathbf{e}(\tfrac{1}{2}H_h(l_1,l_2)+\tfrac{1}{2}H_h(l_1,y)+\tfrac{1}{2}H_h(l_2,x))).$$

The fiber of such a line bundle at any point y of V_h is isomorphic to the quotient of $V_h\times\mathbb{C}$ by the action of L defined by sending $l\in L$ to the holomorphic map

$$V_h\times\mathbb{C}\to V_h\times\mathbb{C}:(x,w)\mapsto(x+l,\ w\,\mathbf{e}(\tfrac{1}{2}H_h(l,y))).$$

Note that this is not exactly an action of the form given by the Theorem of Appell–Humbert. After the holomorphic change of coordinates

$$V_h\times\mathbb{C}\to V_h\times\mathbb{C}:(x,w)\mapsto(x,\ w\,\mathbf{e}(-\tfrac{1}{2}H_h(y,x))),$$

the action above becomes

$$(x,w)\mapsto(x+l,\ w\,\mathbf{e}(\tfrac{1}{2}H_h(l,y)-\tfrac{1}{2}H_h(y,x+l)+\tfrac{1}{2}H_h(y,x)))$$
$$=(x+l,\ w\,\mathbf{e}(\mathrm{Im}'\,H_h(l,y)))=(x+l,\ w\,\mathbf{e}(\langle l,y\rangle)),$$

which is the line bundle $\mathcal{L}(0,\alpha_y)$ with $\alpha_y(l)=\mathbf{e}(\langle l,y\rangle)$. This line bundle $\mathcal{L}(0,\alpha_y)$ depends only on the point $\bar{y}\in G_h^\vee=V_h/L^\#$ defined by y.

(4) Consider the homomorphism $\lambda_h:G_h=V_h/L\to G_h^\vee=V_h/L^\#$ induced by $\mathrm{sgn}(h)$ times the identity morphism $V_h\to V_h$. The pullback of $\mathcal{P}_h$ along the homomorphism $(\mathrm{Id}_{G_h},\lambda_h):G_h\to G_h\times G_h^\vee$ induced by the morphism $V_h\to V_h\times V_h:x\mapsto(x,\mathrm{sgn}(h)x)$ is by definition the quotient of $V_h\times\mathbb{C}$ by the action of L defined by sending $l\in L$ to the holomorphic map

$$V_h\times\mathbb{C}\to V_h\times\mathbb{C}:(x,w)\mapsto(x+l,\ w\,\mathbf{e}(\tfrac{1}{2}\,\mathrm{sgn}(h)H_h(l,l)+\mathrm{sgn}(h)H_h(l,x))).$$

This implies the following facts:

(a) $\mathcal{L}_h:=(\mathrm{Id}_{G_h},\lambda_h)^*\,\mathcal{P}_h$ is isomorphic to $\mathcal{L}(2\,\mathrm{sgn}(h)H_h,0)$.

(b) λ_h is a polarization, because $\mathcal{L}_h$ is ample by positive definiteness of $\mathrm{sgn}(h)H_h$. (See [23, Prop. 1.3.2.18].)

(c) The homomorphism $\lambda_{\mathcal{L}_h}:G_h\to G_h^\vee$, characterized by sending $\bar{x}\in G_h$ to the point of $G_h^\vee$ defining the isomorphism class of $T_{\bar{x}}^*\,\mathcal{L}_h\underset{\mathscr{O}_{G_h}}{\otimes}\mathcal{L}_h^{\otimes\,-1}$, is twice of λ_h.

(d) The kernel of λ_h is canonically isomorphic to $L^\#/L$, and the subgroup $K(\mathcal{L}_h)=\ker(\lambda_{\mathcal{L}_h})$ is canonically isomorphic to $\frac{1}{2}L^\#/L$.

One important feature of the polarization $\lambda_h:G_h\to G_h^\vee$ is the Weil pairing e^{λ_h} it defines. Let us first make explicit the canonical pairing

$$\mathrm{e}_{G_h[n]}:G_h[n]\times G_h^\vee[n]\to\boldsymbol{\mu}_{n,\mathbb{C}}$$

for any integer $n \geq 1$. Let $y \in \frac{1}{n}L^{\#}$ be a representative of an element $\bar{y}$ of $G_h^\vee[n] \cong \frac{1}{n}L^{\#}/L^{\#}$. The point $\bar{y}$ corresponds to the line bundle $\mathcal{L}(0,\alpha_y)$ with $\alpha_y(l) = \mathbf{e}(\langle l,y\rangle)$ for any $l \in L$, namely the quotient of $V_h \times \mathbb{C}$ by L defined by $(x,w) \mapsto (x+l,\ w\,\mathbf{e}(\langle l,y\rangle))$. If we pullback $\mathcal{L}(0,\alpha_y)$ under the multiplication $[n] : G_h \to G_h$ by n, then we obtain the quotient of $V_h \times \mathbb{C}$ by L defined by $(x,w) \mapsto (x+l,\ w\,\mathbf{e}(\langle nl,y\rangle)) = (x+l,w)$, which is the trivial line bundle on G_h. Hence we can interpret the line bundle $\mathcal{L}(0,\alpha_y)$ as the quotient of the trivial line bundle on G_h by the action of $G_h[n]$ on the trivial line bundle (covering its translation action on G_h) defined by $(x,w) \mapsto (x+\frac{1}{n}l,\ w\,\mathbf{e}(\langle l,y\rangle))$. According to the theory explained in [23, §5.2.4] (based on [24, §15, proof of Thm. 1]), this shows that the canonical pairing $\mathrm{e}_{G_h[n]}$ (with the sign convention there) can be identified with the pairing

$$\mathrm{e}_n : (\tfrac{1}{n}L/L) \times (\tfrac{1}{n}L^{\#}/L^{\#}) \to \boldsymbol{\mu}_{n,\mathbb{C}} : (x,y) \mapsto \mathbf{e}(n\langle x,y\rangle).$$

If $n|m$ for some integer $m \geq 1$, then we have $\mathrm{e}_n(x_n,y_n) = \mathrm{e}_m(x_m,y_m)^{\frac{m}{n}}$ if $x_n = \frac{m}{n}x_m$ and $y_n = \frac{m}{n}y_m$. This justifies the compatibility among levels, and defines the pairing

$$\mathrm{e} : (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \times (L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \to \mathrm{T}\,\mathbf{G}_{\mathrm{m},\mathbb{C}} : (x,y) \mapsto \mathbf{e}(\langle x,y\rangle)$$

realizing $\mathrm{e}_{G_h} : \mathrm{T}\,G_h \times \mathrm{T}\,G_h^\vee \to \mathrm{T}\,\mathbf{G}_{\mathrm{m},\mathbb{C}}$. (Note that the base extension of $\langle\,\cdot\,,\,\cdot\,\rangle$ from $\mathbb{Z}$ to $\hat{\mathbb{Z}}$ already incorporates the factors $\frac{m}{n}$ needed among levels.)

As a result, we see that the Weil pairing $\mathrm{e}^{\lambda_h} : \mathrm{T}\,G_h \times \mathrm{T}\,G_h \to \mathrm{T}\,\mathbf{G}_{\mathrm{m},\mathbb{C}}$ can be realized as

$$\mathrm{e} : (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \times (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \to \mathrm{T}\,\mathbf{G}_{\mathrm{m},\mathbb{C}} : (x,y) \mapsto \mathbf{e}(\langle x,y\rangle),$$

with finite level pairings $\mathrm{e}^{\lambda_h} : G_h[n] \times G_h[n] \to \boldsymbol{\mu}_{n,\mathbb{C}}$ realized as

$$\mathrm{e}_n : (\tfrac{1}{n}L/L) \times (\tfrac{1}{n}L/L) \to \boldsymbol{\mu}_{n,\mathbb{C}} : (x,y) \mapsto \mathbf{e}(n\langle x,y\rangle).$$

Remark 2.2.1. The sign convention of the Weil pairings is a choice. It has to be chosen to be compatible with all other choices we have made.

2.3. **PEL structures.** Let h be a $\pm$-polarization of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in §2.2. Then we obtain a complex abelian variety G_h and a polarization $\lambda_h : G_h \to G_h^\vee$.

Since the action of $\mathcal{O}$ on L commutes with the complex structure $h(\sqrt{-1})$, it defines an explicit $\mathcal{O}$-endomorphism structure $i_h : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{C}}(G_h)$ of (G_h,λ_h).

Since $\mathrm{Lie}_{G_h/\mathbb{C}} = V_h$ is isomorphic to the quotient of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ by the submodule $\mathsf{P} = \{\sqrt{-1}x - h(\sqrt{-1})x : x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}\}$, it is isomorphic to the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-submodule V_0 of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ on which $h(z)$ acts by $1 \otimes z$. In particular, $\mathrm{Lie}_{G_h/\mathbb{C}}$ satisfies the determinantal condition given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$, defined in the same way as in [23, Def. 1.3.4.2] using the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module structure of V_0. The determinantal condition given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ and by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ are the same if h is conjugate to h_0 by some element in $\mathrm{G}(\mathbb{R})^+ = \{g \in \mathrm{G}(\mathbb{R}) : \nu(g) > 0\}$.

We shall denote the set of $\mathrm{G}(\mathbb{R})$-conjugates of h_0 by X. If we denote by $\mathcal{U}_\infty = \mathrm{Cent}_{\mathrm{G}(\mathbb{R})}(h_0)$ the stabilizer of h_0 under the conjugation action of $\mathrm{G}(\mathbb{R})$, then X can be identified with the quotient $\mathrm{G}(\mathbb{R})/\mathcal{U}_\infty$. In particular, X has the structure of a real manifold. Moreover, by (3) of Lemma 2.1.2, the connected components of X can be embedded as open complex submanifolds of the projective variety $\mathrm{G}(\mathbb{C})/\mathrm{P}_{h_0}(\mathbb{C})$, where $\mathrm{P}_{h_0}(\mathbb{C})$ is the stabilizer of the totally isotropic complex subspace P_0 of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ defined by h_0. Therefore, X has the structure of a finite union of open complex submanifolds of $\mathrm{G}(\mathbb{C})/\mathrm{P}_{h_0}(\mathbb{C})$, compatible with its real manifold structure.

Since the n-torsion points of G_h are canonically isomorphic to $\frac{1}{n}L/L$ for any integer $n \geq 1$, we have canonical isomorphisms $\hat{\alpha}_h : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{T}\, G_h$ and $\mathbf{e} : \hat{\mathbb{Z}}(1) \xrightarrow{\sim} \mathrm{T}\, \mathbf{G}_{\mathrm{m},\mathbb{C}}$ matching $\langle \cdot\,, \cdot \rangle$ with the Weil pairing e^{λ_h}. (See Remark 2.2.1 and the explicit description of e^{λ_h} preceding it.) The $\mathcal{H}$-orbit of $\hat{\alpha}_h$ defines an integral level structure $\alpha_{h,\mathcal{H}}$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \langle \cdot\,, \cdot \rangle)$ of (G_h, λ_h, i_h) for any open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$. As a result, if $h \in \mathsf{X}$, the tuple $(G_h, \lambda_h, i_h, \alpha_{h,\mathcal{H}})$ defines an object of $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$, or a geometric point $\mathrm{Spec}(\mathbb{C}) \to \mathsf{M}_{\mathcal{H}}$.

Let us denote the canonical morphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \xrightarrow{\sim} \mathrm{V}\, G_h$ induced by $\hat{\alpha}_h$ by the same notation. Then, for any $h \in \mathsf{X}$ and $g \in \mathrm{G}(\mathbb{A}^\infty)$, the $\mathcal{H}$-orbit $[\hat{\alpha}_h \circ g]_{\mathcal{H}}$ of $\hat{\alpha}_h \circ g : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \xrightarrow{\sim} \mathrm{V}\, G_h$ defines a rational level structure of type $(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty, \langle \cdot\,, \cdot \rangle)$ of (G_h, λ_h, i_h). In particular, we obtain a map

$$\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(\mathbb{C}) \tag{2.3.1}$$

of underlying sets defined by sending (h, g) to $(G_h, \lambda_h, i_h, [\hat{\alpha}_h \circ g]_{\mathcal{H}})$. (Here the underlying set of the groupoid $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(\mathbb{C})$ is the set of isomorphism classes in it.)

To obtain objects of $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$, let us modify the tuples $(G_h, \lambda_h, i_h, [\hat{\alpha}_h \circ g]_{\mathcal{H}})$ by $\mathbb{Q}^\times$-isogenies as in the proof of [23, Prop. 1.4.3.3]. By [23, Lem. 1.3.5.2], the image of $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \xrightarrow{\hat{\alpha}_h} \mathrm{V}\, G_h$ corresponds to (the target of) a $\mathbb{Q}^\times$-isogeny $f : G_h \to G_{h,g}$. Under the pairing $\langle \cdot\,, \cdot \rangle$, we have $(g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}))^\# = \nu(g)^{-1} g(L^\# \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$. Then the source and target of the dual $\mathbb{Q}^\times$-isogeny $f^\vee : G_{h,g}^\vee \to G_h^\vee$ correspond to the open compact subgroups $\nu(g)^{-1} g(L^\# \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ and $L^\# \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$. The composition $(f^\vee)^{-1} \circ \lambda_h \circ f^{-1} : G_{h,g} \to G_{h,g}^\vee$ of $\mathbb{Q}^\times$-isogenies is positive because λ_h is, but it is not necessarily an isogeny. Take the unique $r \in \mathbb{Q}^\times_{>0}$ such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}$. Then we set $\lambda_{h,g} := r(f^\vee)^{-1} \circ \lambda_h \circ f^{-1}$, which is a positive isogeny, namely a polarization. Let $i_{h,g} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{C}}(G_{h,g})$ be the canonical structure defined by the $\mathcal{O}$-module structure of $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$, and let $\hat{\alpha}_{h,g} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{T}\, G_{h,g}$ be induced by the composition of $\hat{\alpha}_h \circ g : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \xrightarrow{\sim} \mathrm{V}\, G_h$ and $\mathrm{V}(f) : \mathrm{V}\, G_h \xrightarrow{\sim} \mathrm{V}\, G_{h,g}$. Let $\alpha_{h,g,\mathcal{H}}$ be the integral level-$\mathcal{H}$ structure defined by the $\mathcal{H}$-orbit of $\hat{\alpha}_{h,g}$. Then $(G_{h,g}, \lambda_{h,g}, i_{h,g}, \alpha_{h,g,\mathcal{H}})$ defines an object of $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$.

The isogeny $f : G_h \to G_{h,g}$ induces an isomorphism $f_* : H_1(G_h, \mathbb{Q}) \xrightarrow{\sim} H_1(G_{h,g}, \mathbb{Q})$. The pullback $(\mathrm{Id}_{G_h}, \lambda_h)^* \mathcal{P}_{G_h}$ is isomorphic to an ample line bundle which determines (by the Theorem of Appell–Humbert) a Hermitian pairing whose imaginary part is twice of the alternating pairing $\langle \cdot\,, \cdot \rangle$. If we set $L^{(g)} := H_1(G_{h,g}, \mathbb{Z})$, then the pullback $(\mathrm{Id}_{G_{h,g}}, \lambda_{h,g})^* \mathcal{P}_{G_{h,g}}$ determines similarly twice of an alternating pairing $\langle \cdot\,, \cdot \rangle^{(g)}$ on $L^{(g)}$, and we have a symplectic isomorphism $f_* : (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle \cdot\,, \cdot \rangle) \xrightarrow{\sim} (L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle \cdot\,, \cdot \rangle^{(g)})$ matching the pairings up to a multiple in $\mathbb{Q}^\times_{>0}$.

Lemma 2.3.2. *Under the map* (2.3.1)*, two points* (h_1, g_1) *and* (h_2, g_2) *define isomorphic objects in* $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$ *if and only if there exist elements* $\gamma \in \mathrm{G}(\mathbb{Q})$ *and* $u \in \mathcal{H}$ *such that* $(h_2, g_2) = (\gamma(h_1), \gamma g_1 u)$*. Hence we have a canonical injection*

$$\mathsf{Sh}_{\mathcal{H}} := \mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \to \mathsf{M}_{\mathcal{H}}(\mathbb{C}) \tag{2.3.3}$$

of underlying sets.

Proof. If (h_1, g_1) and (h_2, g_2) determine isomorphic objects in $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$, then the symplectic $\mathcal{O}$-lattices they determine are both isomorphic to some $(L', \langle\,\cdot\,,\,\cdot\,\rangle')$. The isogenies $f_1 : G_h \to G_{h_1,g_1}$ and $f_2 : G_h \to G_{h_2,g_2}$ induces symplectic isomorphisms $(f_1)_* : (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle) \overset{\sim}{\to} (L' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle')$ and $(f_2)_* : (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle) \overset{\sim}{\to} (L' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle')$, so that $(f_2)_* \circ (f_1)_*^{-1} : (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle) \overset{\sim}{\to} (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle\,\cdot\,,\,\cdot\,\rangle)$ defines an element γ of $\mathrm{G}(\mathbb{Q})$. This shows that the $\mathbb{Q}^\times$-isogeny $f_2 \circ f_1^{-1} : G_{h_1,g_1} \overset{\sim}{\to} G_{h_2,g_2}$ induces an isomorphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathrm{Lie}_{G_{h_1,g_1}/\mathbb{C}} \overset{\sim}{\to} \mathrm{Lie}_{G_{h_2,g_2}/\mathbb{C}} \cong L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ matching the complex structures by the relation $h_2(z)(x) = (\gamma \circ h_1(z) \circ \gamma^{-1})(x)$ for any $z \in \mathbb{C}$ and $x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, namely $h_2 = \gamma(h_1)$. Moreover, since the $\mathcal{H}$-orbit of $\hat{\alpha}_{h,g_2} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \overset{\sim}{\to} \mathrm{T}\, G_{h_2,g_2}$ is same as the $\mathcal{H}$-orbit of the composition of $\hat{\alpha}_{h,g_1} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \overset{\sim}{\to} \mathrm{T}\, G_{h_1,g_1}$ with $\mathrm{T}(f_2 \circ f_1^{-1}) : \mathrm{T}\, G_{h_1,g_1} \overset{\sim}{\to} \mathrm{T}\, G_{h_2,g_2}$, we see that $g_2\mathcal{H} = \gamma g_1 \mathcal{H}$. Since the converse is clear, we see that (h_1, g_1) and (h_2, g_2) define isomorphic objects in $\mathsf{M}_{\mathcal{H}}(\mathbb{C})$ if and only if $(h_2, g_2) = (\gamma h_1, \gamma g_1 u)$ for some $\gamma \in \mathrm{G}(\mathbb{Q})$ and $u \in \mathcal{H}$, as desired. □

2.4. **Variation of complex structures.** To give meaning to the injection in Lemma 2.3.2, we need to study how $(G_{h,g}, \lambda_{h,g}, i_{h,g}, \alpha_{h,g,\mathcal{H}})$ varies with h. For this purpose, it is convenient to have some local complex coordinates for X by realizing it as a complex analytic subspace of some complex manifold. A conventional choice of such an ambient complex manifold is the Siegel upper half space.

Definition 2.4.1. *A* $\mathbb{Z}$*-submodule* L_{MI} *of* L *is called* **maximally isotropic** *with respect to the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *if it satisfies the following conditions:*

1. *For any* $x, y \in L_{\mathrm{MI}}$*, we have* $\langle x, y\rangle = 0$*. In other words,* L_{MI} *is totally isotropic under the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$*.*
2. *If* $z \in L$ *satisfies* $\langle x, z\rangle = 0$ *for all* $x \in L_{\mathrm{MI}}$*, then* $z \in L_{\mathrm{MI}}$*.*

(*We shall suppress the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *from the statements when the context is clear.*)

Note that we do not require L_{MI} to be an $\mathcal{O}$-submodule.

Lemma 2.4.2. *Given any totally isotropic* $\mathbb{Z}$*-submodule* L' *of* L*, we can find a maximally isotropic* $\mathbb{Z}$*-submodule* L_{MI} *of* L *containing* L'*. In particular, maximal isotropic submodules* L_{MI} *of* L *exist* (*although they might not be* $\mathcal{O}$*-submodules*)*.*

Lemma 2.4.3. *Let* L_{MI} *be a maximal isotropic* $\mathbb{Z}$*-submodule of* L*. Let* $d = \mathrm{rk}_{\mathbb{Z}} L_{\mathrm{MI}}$ *and let* $\{e_i\}_{1\le i\le d}$ *be a free* $\mathbb{Z}$*-basis of* L_{MI}*. Then there exists elements* $\{f_i\}_{1\le i\le d}$ *of* $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *such that* $\langle e_i, f_j\rangle = 2\pi\sqrt{-1}\,\delta_{ij}$ *and* $\langle f_i, f_j\rangle = 0$ *for any* $1 \le i, j \le d$*.*

Lemma 2.4.4. *Let us fix choices of* $\{e_i\}_{1\le i\le d}$ *and* $\{f_i\}_{1\le i\le d}$ *as in Lemma* 2.4.3*. Let us also fix an element* $\epsilon \in \{\pm 1\}$*. Then the* ϵ*-polarizations of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$ *correspond bijectively to a complex analytic subset of the Siegel half space* $\mathsf{H}_d^\epsilon := \{x \in \mathrm{M}_d(\mathbb{C}) : {}^t x = x, \epsilon \,\mathrm{Im}\, x > 0\}$*.*

Proof. By Lemma 2.1.2, any ϵ-polarization h of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,,\cdot \rangle)$ defines a complex structure $h(\sqrt{-1})$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, or equivalently a group homomorphism i_h from L to a $\mathbb{C}$-vector space V_h inducing an isomorphism $\mathsf{i}_{h,\mathbb{R}} : L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \xrightarrow{\sim} V_h$.

Since $-\epsilon\sqrt{-1}\langle\,\cdot\,, h(\sqrt{-1})\,\cdot\,\rangle$ is positive definite, we see that $\langle e_i, h(\sqrt{-1})e_i\rangle \neq 0$ for any $1 \leq i \leq d$. Since $\langle e_i, e_j\rangle = 0$, this shows that $(L_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \cap h(\sqrt{-1})(L_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) = \{0\}$, and that the $\mathbb{C}$-span of $\mathsf{i}_h(L_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ in V_h is the whole space (by $\mathbb{R}$-dimension counting). We shall interpret this as an isomorphism $\mathbb{C}^{\oplus d} \xrightarrow{\sim} V$ sending the i-th standard basis vector to $\mathsf{i}_h(e_i)$. Similarly, we have an isomorphism $\mathbb{C}^{\oplus d} \xrightarrow{\sim} V$ defined by sending the i-th standard basis vector to $\mathsf{i}_{h,\mathbb{R}}(f_i)$. This allows us to define a matrix Ω_h in $\mathrm{M}_d(\mathbb{C})$ by setting $\mathsf{i}_h(f_i) = \sum\limits_{1\leq i\leq d} (\Omega_h)_{ij}\mathsf{i}_h(e_j)$ in V_h.

We claim that Ω_h is an element of H_d^ϵ. Let us define matrices A, B, and C in $\mathrm{M}_d(\mathbb{R})$ by writing $A := \mathrm{Re}\,\Omega_h$, $B := \mathrm{Im}\,\Omega_h$, and $2\pi\sqrt{-1}\,C_{ij} := \langle e_i, h(\sqrt{-1})e_j\rangle$, for any $1 \leq i, j \leq d$. Note that ϵC is (symmetric and) positive definite. Then, for any $1 \leq i \leq d$, we have $f_i = \sum\limits_k [A_{ik} + h(\sqrt{-1})B_{ik}]e_k$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. From

$$2\pi\sqrt{-1}\,\delta_{ij} = \langle e_i, f_j\rangle = \langle e_i, \sum_k [A_{jk} + h(\sqrt{-1})B_{jk}]e_k\rangle = 2\pi\sqrt{-1}\sum_k C_{ik}B_{jk},$$

we obtain $I = C\,{}^tB$, or $B = {}^tC^{-1}$, showing that B is symmetric, and that $\epsilon\,\mathrm{Im}\,\Omega_h = \epsilon B > 0$. From

$$\begin{aligned} 0 = \langle f_i, f_j\rangle &= \langle \sum_k [A_{ik} + h(\sqrt{-1})B_{ik}]e_k, \sum_l [A_{jl} + h(\sqrt{-1})B_{jl}]e_l\rangle \\ &= 2\pi\sqrt{-1}\sum_{k,l} [A_{ik}C_{kl}\,{}^tB_{lj} - B_{ik}C_{kl}\,{}^tA_{lj}] = 2\pi\sqrt{-1}\,(A_{ij} - {}^tA_{ij}), \end{aligned}$$

we obtain $A - {}^tA = 0$, showing that A and hence Ω_h are symmetric. This justifies the claim.

Using the basis $\{e_i\}_{1\leq i\leq d}$ and $\{f_i\}_{1\leq i\leq d}$, we can describe $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as $\mathbb{Q}^{\oplus d} \oplus \mathbb{Q}^{\oplus d}$ and hence $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ as $\mathbb{C}^{\oplus d} \oplus \mathbb{C}^{\oplus d}$. Then the matrix Ω_h determined by h allows us to identify $\mathsf{P}_h = \ker(L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \twoheadrightarrow V_h)$ (by $\mathbb{C}$-dimension counting) with the $\mathbb{C}$-span of the vectors $\{(v_i, v_i')\}_{1\leq i\leq d}$ in $\mathbb{C}^{\oplus d} \oplus \mathbb{C}^{\oplus d}$, where v_i is the column vector with entries $\{-(\Omega_h)_{ij}\}_{1\leq j\leq d}$, and v_i' is the i-th standard basis vector of $\mathbb{C}^{\oplus d}$, for each $1 \leq i \leq d$. This gives a holomorphic embedding of H_d^ϵ into the projective variety parameterizing totally isotropic subspaces of $\mathbb{C}$-dimension d in $\mathbb{C}^{\oplus d} \oplus \mathbb{C}^{\oplus d}$. By choosing a finite $\mathbb{Z}$-basis of $\mathcal{O}$ over $\mathbb{Z}$, the condition for such subspaces to be invariant under the action of $\mathcal{O}$ can be described by finitely many algebraic equations. As a result, we see that the collection of ϵ-polarizations of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot\,,\cdot\rangle)$ corresponds to points of a complex analytic subset of H_d^ϵ, identified locally as a complex submanifold of $\mathrm{G}(\mathbb{C})/\mathrm{P}_h(\mathbb{C})$, where $\mathrm{P}_h(\mathbb{C})$ is the stabilizer of the totally isotropic complex subspace P_h of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ defined by h. □

Corollary 2.4.5 (of the proof of Lemma 2.4.4)**.** *Using $\{\mathsf{i}_h(e_i)\}_{1\leq i\leq d}$ as a basis of V_h, the Hermitian pairing $H_h : V_h \times V_h \to \mathbb{C}$ defined by h is identified with the Hermitian pairing $\mathbb{C}^{\oplus d} \times \mathbb{C}^{\oplus d} \to \mathbb{C} : (x, y) \mapsto 2\pi\ {}^tx^c(\mathrm{Im}\,\Omega_h)^{-1}y$.*

Proof. This is because $H_h(\mathsf{i}_h(e_i),\mathsf{i}_h(e_j)) = -\sqrt{-1}\,\langle e_i, h(\sqrt{-1})e_j\rangle = 2\pi\, C_{ij}$. □

Now let us consider the tuple $(G_{h,g},\lambda_{h,g},i_{h,g},\alpha_{h,g,\mathcal{H}})$. By construction of the $\mathbb{Q}^\times$-isogeny $f : G_h \to G_{h,g}$, there is associated an $\mathcal{O}$-lattice $L^{(g)} := H_1(G_{h,g})$, together with a pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$, such that the symplectic $\mathcal{O}$-lattice $(L^{(g)}\underset{\mathbb{Z}}{\otimes}\mathbb{Q},\langle\,\cdot\,,\,\cdot\,\rangle^{(g)})$ is canonically isomorphic to $(L\underset{\mathbb{Z}}{\otimes}\mathbb{Q},\langle\,\cdot\,,\,\cdot\,\rangle)$ under f_*. We shall identify $L^{(g)}$ with an $\mathcal{O}$-lattice in $L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, and identify $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$ with $r\langle\,\cdot\,,\,\cdot\,\rangle$, where $r \in \mathbb{Q}^\times_{>0}$ is such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}$, so that the $\mathbb{Q}^\times$-isogeny $f : G_h \to G_{h,g}$ can be identified with $V_h/L \overset{\sim}{\to} V_h/L^{(g)}$.

Let $L^{(g)}_{\mathrm{MI}} := L^{(g)}\cap(L_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$. By definition, $L^{(g)}/L^{(g)}_{\mathrm{MI}}$ is torsion-free and can be identified with a submodule of $(L/L_{\mathrm{MI}})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. Using $\{\mathsf{i}_h(e_i)\}_{1\leq i\leq d}$ as a $\mathbb{C}$-basis of V_h, we have isomorphisms $V_h \cong \mathbb{C}^{\oplus d}$ and hence $V_h/\mathsf{i}_{h,\mathbb{R}}(L^{(g)}_{\mathrm{MI}}) \cong L^{(g)}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times$ is an algebraic torus over $\mathbb{C}$. The elements $\{f_i\}_{1\leq i\leq d}$ define a splitting of $L\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \twoheadrightarrow (L/L_{\mathrm{MI}})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, which might fail to induce a splitting of $L^{(g)} \twoheadrightarrow L^{(g)}/L^{(g)}_{\mathrm{MI}}$ in general. However, they do define a morphism $L^{(g)}/L^{(g)}_{\mathrm{MI}} \to L^{(g)}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times : x \mapsto \mathbf{e}(2\pi\sqrt{-1}\,\Omega_h(x))$, which varies holomorphically with h. Therefore, $G_{h,g} \cong \mathbb{C}^{\oplus d}/L \cong (L^{(g)}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times)/(L^{(g)}/L^{(g)}_{\mathrm{MI}})$ defines a family of abelian varieties varying holomorphically with h.

Let $(L^{(g)})^{\#}$ be the dual lattice of $L^{(g)}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$, and let $(L^{(g)})^{\#}_{\mathrm{MI}} := (L^{(g)})^{\#}\cap(L_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})$. Since $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} = r\langle\,\cdot\,,\,\cdot\,\rangle$ is $\mathbb{Z}(1)$-valued, $(L^{(g)})^{\#}_{\mathrm{MI}}$ contains $L^{(g)}_{\mathrm{MI}}$, and $(L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}$ is torsion-free and can be identified with a submodule of $(L/L_{\mathrm{MI}})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ containing $L^{(g)}/L^{(g)}_{\mathrm{MI}}$. The polarization λ_h is by definition induced by $\mathrm{sgn}(h)$ times the identity morphism $V_h \to V_h$, and hence the polarization $\lambda_{h,g} = r(f^\vee)^{-1}\circ\lambda_h\circ f^{-1}$ is also induced by $\mathrm{sgn}(h)$ times the identity morphism $V_h \to V_h$. By taking quotients by $L^{(g)}_{\mathrm{MI}}$ and $(L^{(g)})^{\#}_{\mathrm{MI}}$ respectively, we obtain an isogeny $L^{(g)}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times \to (L^{(g)})^{\#}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times$ of algebraic tori over $\mathbb{C}$. By taking further quotients by $L^{(g)}/L^{(g)}_{\mathrm{MI}}$ and $(L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}$ respectively, we obtain $\lambda_{h,g} : G_{h,g} \cong (L^{(g)}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times)/(L^{(g)}/L^{(g)}_{\mathrm{MI}}) \to G^\vee_{h,g} \cong ((L^{(g)})^{\#}_{\mathrm{MI}}\underset{\mathbb{Z}}{\otimes}\mathbb{C}^\times)/((L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}})$, varying holomorphically with h.

The remaining structures $i_{h,g}$, the Lie algebra condition, and $\alpha_{h,g,\mathcal{H}}$ vary holomorphically, because they are locally constant in nature (on top of $(G_{h,g},\lambda_{h,g})$). This shows that the tuples $(G_{h,g},\lambda_{h,g},i_{h,g},\alpha_{h,g,\mathcal{H}})$ are fibers of a holomorphic family over $\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$.

2.5. **PEL-type Shimura varieties.** Let X_0 be the connected component of X containing h_0, and let $\mathrm{G}(\mathbb{R})_0$ (resp. $\mathrm{G}(\mathbb{Q})_0$) denote its stabilizer in $\mathrm{G}(\mathbb{R})$ (resp. $\mathrm{G}(\mathbb{Q})$). Then $\mathrm{G}(\mathbb{R})_0$ (resp. $\mathrm{G}(\mathbb{Q})_0$) has finite index in $\mathrm{G}(\mathbb{R})$ (resp. $\mathrm{G}(\mathbb{Q})$).

Lemma 2.5.1. *The canonical map* $\mathrm{G}(\mathbb{Q})_0\backslash\mathsf{X}_0\times\mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \to \mathrm{G}(\mathbb{Q})\backslash\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ *is a bijection.*

Proof. The surjectivity of the map follows from density of $\mathrm{G}(\mathbb{Q})$ in $\mathrm{G}(\mathbb{R})$ (and by transitiveness of the action of $\mathrm{G}(\mathbb{Q})$ on X). If $(\gamma x_1, \gamma g_1) = (x_2, g_2 u)$ for some $x_1, x_2 \in \mathsf{X}_0$, $\gamma \in \mathrm{G}(\mathbb{Q})$, and $u \in \mathcal{H}$, then $\gamma \in \mathrm{G}(\mathbb{Q})_0$, and hence (x_1, g_1) and (x_2, g_2) define the same double coset in $\mathrm{G}(\mathbb{Q})_0\backslash\mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. This shows the injectivity of the map. □

By [7, Thm. 5.1], the cardinality of the double coset space $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ is finite. Since $\mathrm{G}(\mathbb{Q})_0$ is of finite index in $\mathrm{G}(\mathbb{Q})$, this shows that the double coset space $\mathrm{G}(\mathbb{Q})_0\backslash\mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ is also finite. Let $\{g_i\}_{i\in I}$ be (noncanonically) a finite set of elements in $\mathrm{G}(\mathbb{A}^\infty)$ such that $\mathrm{G}(\mathbb{A}^\infty) = \coprod_{i\in I} \mathrm{G}(\mathbb{Q})_0 g_i \mathcal{H}$. Then we have

$$\begin{aligned} \mathsf{Sh}_{\mathcal{H}} &= \mathrm{G}(\mathbb{Q})\backslash\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} = \mathrm{G}(\mathbb{Q})_0\backslash\mathsf{X}_0 \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \\ &= \coprod_{i\in I} \mathrm{G}(\mathbb{Q})_0\backslash\mathsf{X}_0 \times (\mathrm{G}(\mathbb{Q})_0 g_i \mathcal{H})/\mathcal{H} = \coprod_{i\in I} \Gamma^{(g_i)}\backslash\mathsf{X}_0, \end{aligned} \tag{2.5.2}$$

where $\Gamma^{(g_i)} := (g_i \mathcal{H} g_i^{-1}) \cap \mathrm{G}(\mathbb{Q})_0$, because $\gamma g_i \mathcal{H} = g_i \mathcal{H}$ if and only if $\gamma \in g_i \mathcal{H} g_i^{-1}$. Each of the groups $\Gamma^{(g_i)}$ is an arithmetic subgroup of $\mathrm{G}(\mathbb{Q})$, because $\mathrm{G}(\mathbb{Q})_0$ is of finite index in $\mathrm{G}(\mathbb{Q})$, because $g_i \mathcal{H} g_i^{-1}$ is commensurable with $\mathrm{G}(\hat{\mathbb{Z}})$ by open compactness of $\mathcal{H}$, and because $\mathrm{G}(\hat{\mathbb{Z}}) \cap \mathrm{G}(\mathbb{Q}) = \mathrm{G}(\mathbb{Z})$. Therefore, its image in $\mathrm{G}^{\mathrm{ad}}(\mathbb{Q})$ is also arithmetic. (The action of $\Gamma^{(g_i)}$ on X factors through its image in $\mathrm{G}^{\mathrm{ad}}(\mathbb{Q})$.) By [4, 10.11], we know that each of the quotient $\Gamma^{(g_i)}\backslash\mathsf{X}_0$ has a structure of the analytification of an irreducible normal quasi-projective variety over $\mathbb{C}$. This allows us to identify $\mathsf{Sh}_{\mathcal{H}}$ with the analytification of a quasi-projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$. (By abuse of language, varieties in this article are not necessarily connected.)

Remark 2.5.3. There is nowhere we need $(\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathsf{X})$ to be a Shimura datum.

Assumption 2.5.4. *We shall assume from now on that $\mathcal{H}$ is* **neat**. (*See Definition* 1.1.3.)

Remark 2.5.5. Assumption 2.5.4 is made only for simplicity of exposition, so that we can work with fine moduli spaces. Statements for non-neat level $\mathcal{H}$ can be obtained for coarse moduli spaces by taking quotients by finite groups.

Then $g_i \mathcal{H} g_i^{-1}$ is neat for every g_i, and hence (as already mentioned in Remark 1.1.5) $\Gamma^{(g_i)}$ is neat in the sense of [8, 17.1]. In particular, the action of $\Gamma^{(g_i)}$ on X_0 has no fixed point. This shows that the action of $\mathrm{G}(\mathbb{Q})$ on $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ has no fixed point, and that the holomorphic family over $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ with fibers $(G_h, \lambda_h, i_h, [\hat{\alpha}_h \circ g]_{\mathcal{H}})$ descends to a holomorphic family $(G_{\mathrm{hol}}, \lambda_{\mathrm{hol}}, i_{\mathrm{hol}}, [\hat{\alpha}_{\mathrm{hol}} \circ g]_{\mathcal{H}})$ over the (nonsingular) quasi-projective variety $\mathsf{Sh}_{\mathcal{H}} = \mathrm{G}(\mathbb{Q})\backslash\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$. We would like to show that this family is algebraic. That is, it is isomorphic (as a complex analytic space) to the analytification of an object of $\mathsf{M}_{\mathcal{H}}(\mathsf{Sh}_{\mathcal{H},\mathrm{alg}})$.

By Theorem 1.2.4, $\mathsf{M}_{\mathcal{H}}$ has the structure of a nonsingular quasi-projective variety over $\mathrm{Spec}(F_0)$, carrying a universal family $(G, \lambda, i, \alpha_{\mathcal{H}})$. The reflex field F_0 is by definition a subfield of $\mathbb{C}$. (See [23, Def. 1.2.5.4].) Let us denote the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}$ to $\mathbb{C}$ (under the canonical homomorphism $F_0 \hookrightarrow \mathbb{C}$) by $(G_{\mathbb{C}}, \lambda_{\mathbb{C}}, i_{\mathbb{C}}, \alpha_{\mathcal{H},\mathbb{C}}) \to \mathsf{M}_{\mathcal{H},\mathbb{C}}$. The fiber $(G_s, \lambda_s, i_s, \alpha_{\mathcal{H},s})$ over each point $s : \mathrm{Spec}(\mathbb{C}) \to \mathsf{M}_{\mathcal{H},\mathbb{C}}$ determines a PEL-type $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-module $(H_1(G_s, \mathbb{Q}), \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s}, h_0)$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s}$ is the pairing induced by λ_s. The isomorphism classes of such PEL-type $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-modules are locally constant. Let $\mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$ denote the open and closed subscheme of $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ consisting of the connected components over which $(H_1(G_s,\mathbb{Q}),\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s},h_0)$ is isomorphic to $(L\underset{\mathbb{Z}}{\otimes}\mathbb{Q},\langle\,\cdot\,,\,\cdot\,\rangle,h_0)$. Then $\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ is a nonsingular quasi-projective variety over $\mathbb{C}$. By abuse of notation, we shall denote the pullback of the universal family by $(G_\mathbb{C},\lambda_\mathbb{C},i_\mathbb{C},\alpha_{\mathcal{H},\mathbb{C}})\to\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, and denote its analytification by $(G_{\mathrm{an}},\lambda_{\mathrm{an}},i_{\mathrm{an}},\alpha_{\mathcal{H},\mathrm{an}})\to\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$.

Lemma 2.5.6. *There exists a holomorphic map* $F:\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}\to\mathsf{Sh}_\mathcal{H}$ *such that* $(G_{\mathrm{an}},\lambda_{\mathrm{an}},i_{\mathrm{an}},\alpha_{\mathcal{H},\mathrm{an}})\to\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ *is the pullback of* $(G_{\mathrm{hol}},\lambda_{\mathrm{hol}},i_{\mathrm{hol}},\alpha_{\mathcal{H},\mathrm{hol}})\to\mathsf{Sh}_\mathcal{H}$ *(as complex analytic spaces) under* F.

Proof. For any s of $\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, the fiber $(G_s,\lambda_s,i_s,\alpha_{\mathcal{H},s})$ of $(G_{\mathrm{an}},\lambda_{\mathrm{an}},i_{\mathrm{an}},\alpha_{\mathcal{H},\mathrm{an}})$ over s determines a PEL-type $\mathcal{O}$-module $(H_1(G_s,\mathbb{Q}),\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s},h_s)$. By definition of $\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, there exists a non-canonical isomorphism $(H_1(G_s,\mathbb{Q}),\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s},h_s)\cong(L\underset{\mathbb{Z}}{\otimes}\mathbb{Q},\langle\,\cdot\,,\,\cdot\,\rangle,h_0)$. Therefore, there exists some point (h,g) of $\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)$ such that $(H_1(G_s,\mathbb{Z}),\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s},h_s)\cong(L^{(g)},\langle\,\cdot\,,\,\cdot\,\rangle^{(g)},h)$, and hence $(G_s,\lambda_s,i_s,\alpha_{\mathcal{H},s})\cong(G_{h,g},\lambda_{h,g},i_{h,g},\alpha_{\mathcal{H},h,g})$. Since the isomorphism class of $(H_1(G_s,\mathbb{Q}),\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_s})$ is locally constant, the choice of g can be made locally constant on $\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$. For points s in a connected and simply-connected analytic open subset of $\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, if we fix the choice of g, then we can take h to vary holomorphically with s, because the family is the analytification of an algebraic family. The local assignment of $(h,g)\in\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)$ is not unique, but the induced local assignment with image in $\mathsf{Sh}_\mathcal{H}$ is unique by Lemma 2.3.2. Thus the assignments over analytic open sets patch together and determine the desired holomorphic map $F:\mathsf{M}_{\mathcal{H},\mathrm{an},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}\to\mathsf{Sh}_\mathcal{H}$. □

By [9, 3.10], any holomorphic map from a quasi-projective variety to $\mathsf{Sh}_\mathcal{H}$ is the analytification of a morphism of algebraic (quasi-projective) varieties. Therefore, F is the analytification of an algebraic morphism $F_{\mathrm{alg}}:\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}\to\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$. By Lemma 2.3.2, the algebraic morphism F_{alg} is a bijection on $\mathbb{C}$-valued points. Since both its source and target are nonsingular varieties, this forces F_{alg} to be an isomorphism. Since the map between the total manifolds $G_{\mathrm{an}}\to G_{\mathrm{hol}}$ is bijective and uniquely determined on the projective fibers over $\mathbb{C}$-valued points of the (isomorphic) base manifolds, this shows that $(G_{\mathrm{hol}},\lambda_{\mathrm{hol}},i_{\mathrm{hol}},\alpha_{\mathcal{H},\mathrm{hol}})\to\mathsf{Sh}_\mathcal{H}$ is uniquely isomorphic to the analytification of the pullback $(G_{\mathrm{alg}},\lambda_{\mathrm{alg}},i_{\mathrm{alg}},\alpha_{\mathcal{H},\mathrm{alg}})\to\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ of $(G_\mathbb{C},\lambda_\mathbb{C},i_\mathbb{C},\alpha_{\mathcal{H},\mathbb{C}})\to\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ under F_{alg}^{-1}.

Definition 2.5.7. *The PEL-type Shimura variety defined by* $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ *is the above quasi-projective variety* $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ *over* $\mathrm{Spec}(\mathbb{C})$*, embedded as an open and closed subscheme of* $\mathsf{M}_{\mathcal{H},\mathbb{C}}$.

Remark 2.5.8. Some $\mathbb{Q}$-simple factor of the group $\mathrm{G}^{\mathrm{ad}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ might have compact $\mathbb{R}$-points. This might seem unpleasant, because then our PEL-type Shimura varieties might not qualify as Shimura varieties according to some definitions in the literature. (See for example [13, (2.1.1.3)].) However, such definitions (in the literature) are unnatural for the study of compactifications of Shimura varieties, because zero-dimensional boundary components always appear.

2.6. **Classical theta functions.** For each pair $(h,g) \in \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$, let us define $\mathcal{L}_{h,g} := (\mathrm{Id}_{G_{h,g}}, \lambda_{h,g})^* \mathcal{P}_{G_{h,g}}$, an ample line bundle on $G_{h,g}$ such that $\lambda_{\mathcal{L}_{h,g}} = 2\lambda_{h,g}$. As in §2.4, we shall identify $L^{(g)}$ with an $\mathcal{O}$-lattice in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, and identify $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}$ with $r\langle\,\cdot\,,\,\cdot\,\rangle$, where $r \in \mathbb{Q}^\times_{>0}$ such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}$, so that the $\mathbb{Q}^\times$-isogeny $f : G_h \to G_{h,g}$ can be identified with $V_h/L \to V_h/L^{(g)}$. Set

$$H_{h,g}(x,y) := \langle x,y\rangle^{(g)} - \sqrt{-1}\,\langle x, h(\sqrt{-1})y\rangle^{(g)} = rH_h(x,y).$$

Then the line bundle $\mathcal{L}_{h,g}$ can be realized as the quotient of $V_h \times \mathbb{C}$ by the action of $L^{(g)}$ defined by sending $l \in L^{(g)}$ to the holomorphic map

$$V_h \times \mathbb{C} \to V_h \times \mathbb{C} : (x,w) \mapsto (x+l, w\,\mathbf{e}(\tfrac{1}{2}\,\mathrm{sgn}(h) H_{h,g}(l,l) + \mathrm{sgn}(h) H_{h,g}(l,x))).$$

Using $\{\mathsf{i}_h(e_i)\}_{1\le i\le d}$ as a $\mathbb{C}$-basis of V_h, we can identify V_h with $\mathbb{C}^{\oplus\,d}$, and identify

$$H_{h,g}(x,y) = 2\pi r\,{}^t x^c (\operatorname{Im}\Omega_h)^{-1} y$$

for $x,y \in \mathbb{C}^{\oplus\,d}$ as in Corollary 2.4.5. Note that this function may not vary holomorphically with h for general x and y. A classical way to deal with this situation is to introduce the following (noncanonical) realization of the line bundle $\mathcal{L}_{h,g}$. Let $B_{h,g} : V_h \times V_h \to \mathbb{C}$ denote the symmetric $\mathbb{C}$-bilinear pairing such that $B_{h,g}(x,y) = H_{h,g}(x,y)$ for any $x,y \in L_{\mathrm{MI}}$. Then

$$B_{h,g}(x,y) = 2\pi r\,{}^t x (\operatorname{Im}\Omega_h)^{-1} y,$$

and hence

$$(B_{h,g} - H_{h,g})(x,y) = 2\pi r\,{}^t(x - x^c)(\operatorname{Im}\Omega_h)^{-1} y,$$

for $x,y \in \mathbb{C}^{\oplus\,d}$. If we write $x = x_1 + \Omega_h(x_2)$, where $x_1, x_2 \in L^{(g)}_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, then we obtain

$$(B_{h,g} - H_{h,g})(x,y) = 4\pi\sqrt{-1}\,r\,{}^t(\operatorname{Im}\Omega_h(x_2))(\operatorname{Im}\Omega_h)^{-1} y = 4\pi\sqrt{-1}\,r\,{}^t x_2 y.$$

This shows that:

Lemma 2.6.1. *The perfect pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times (L^{(g)})^\# \to \mathbb{Z}(1)$ induces a prefect pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{\mathrm{MI}} : L^{(g)}_{\mathrm{MI}} \times ((L^{(g)})^\# / (L^{(g)})^\#_{\mathrm{MI}}) \to \mathbb{Z}(1)$. By abuse of notation, we shall denote base extensions of this pairing by the same notation. For $l \in (L^{(g)})^\#$, the group homomorphism*

$$\xi_l : L^{(g)}_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{C} = \mathbb{C}^{\oplus\,d} \to \mathbb{C}^\times : x \mapsto \mathbf{e}(\tfrac{1}{2}(B_{h,g} - H_{h,g})(l,x))$$

satisfies

$$\xi_l(x) = \mathbf{e}(\langle x, l\rangle^{(g)}_{\mathrm{MI}}).$$

Hence, we have $\xi_l = \xi_{l+l_{\mathrm{MI}}}$ for any $l_{\mathrm{MI}} \in (L^{(g)})^\#_{\mathrm{MI}}$, and $\xi_l(x) = \xi_l(x + l_{\mathrm{MI}})$ for any $l_{\mathrm{MI}} \in L^{(g)}_{\mathrm{MI}}$. The assignment $l \mapsto \xi_l$ induces a group isomorphism

$$(L^{(g)})^\# / (L^{(g)})^\#_{\mathrm{MI}} \xrightarrow{\sim} \operatorname{Hom}(L^{(g)}_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times, \mathbb{C}^\times)$$

where the target is the character group of the algebraic torus $L^{(g)}_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times$ over $\mathbb{C}$. We shall denote ξ_l by $\xi_{\bar{l}}$ if $\bar{l}$ is the image of l under the canonical surjection $(L^{(g)})^\# \twoheadrightarrow (L^{(g)})^\# / (L^{(g)})^\#_{\mathrm{MI}}$.

After the holomorphic change of coordinates

$$V_h \times \mathbb{C} \to V_h \times \mathbb{C} : (x, w) \mapsto (x,\ w\, \mathbf{e}(-\tfrac{1}{2}\operatorname{sgn}(h) B_{h,g}(x,x))),$$

the line bundle $\mathcal{L}_{h,g}$ becomes the quotient of $V_h \times \mathbb{C}$ by the action of $L^{(g)}$ defined by sending $l \in L^{(g)}$ to the holomorphic map $V_h \times \mathbb{C} \to V_h \times \mathbb{C}$ defined by

$$\begin{aligned}(2.6.2)\quad (x,w) \mapsto (x+l,\ w\,&\mathbf{e}(\tfrac{1}{2}\operatorname{sgn}(h)H_{h,g}(l,l) + \operatorname{sgn}(h)H_{h,g}(l,x))\\ &\mathbf{e}(-\tfrac{1}{2}\operatorname{sgn}(h)(B_{h,g}(x+l,x+l) - B_{h,g}(x,x))))\\ = (x+l,\ w\,&\mathbf{e}(-\tfrac{1}{2}\operatorname{sgn}(h)(B_{h,g}-H_{h,g})(l,l) - \operatorname{sgn}(h)(B_{h,g}-H_{h,g})(l,x))).\end{aligned}$$

If $l \in L^{(g)}_{\mathrm{MI}}$, then $(B_{h,g} - H_{h,g})(l,l) = (B_{h,g} - H_{h,g})(l,x) = 0$ by Lemma 2.6.1. Therefore, any holomorphic section $f(x)$ of the modified line bundle is periodic under translation by $L^{(g)}_{\mathrm{MI}}$, so that

$$f(x) = \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}}\, \xi_{\bar{l}}(x)$$

for some uniquely determined coefficients $c_{\bar{l}} \in \mathbb{C}$.

For any $\bar{l}' \in L^{(g)}/L^{(g)}_{\mathrm{MI}}$, take any representative $l' \in L^{(g)}$ mapping to $\bar{l}'$ under the canonical surjection, we obtain

$$\begin{aligned} f(x+l') &= f(x)\,\mathbf{e}(-\tfrac{1}{2}\operatorname{sgn}(h)(B_{h,g}-H_{h,g})(l',l'))\,\mathbf{e}(-\operatorname{sgn}(h)(B_{h,g}-H_{h,g})(l',x))\\ &= f(x)\,\xi_{-\operatorname{sgn}(h)\bar{l}'}(l')\,\xi_{-2\operatorname{sgn}(h)\bar{l}'}(x).\end{aligned}$$

Comparing the coefficients of $\xi_{\bar{l}}(x)$ in

$$f(x+l) = \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}}\, \xi_{\bar{l}}(x+l') = \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}}\, \xi_{\bar{l}}(l')\, \xi_{\bar{l}}(x)$$

and

$$\begin{aligned} &f(x)\,\xi_{-\operatorname{sgn}(h)\bar{l}'}(l')\,\xi_{-2\operatorname{sgn}(h)\bar{l}'}(x)\\ &\quad= \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}}\,\xi_{-\operatorname{sgn}(h)\bar{l}'}(l')\,\xi_{\bar{l}-2\operatorname{sgn}(h)\bar{l}'}(x)\\ &\quad= \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}+2\operatorname{sgn}(h)\bar{l}'}\,\xi_{-\operatorname{sgn}(h)\bar{l}'}(l')\,\xi_{\bar{l}}(x),\end{aligned}$$

we obtain the relation

$$(2.6.3)\qquad c_{\bar{l}+2\operatorname{sgn}(h)\bar{l}'} = c_{\bar{l}}\,\xi_{\bar{l}+\operatorname{sgn}(h)\bar{l}'}(l').$$

Lemma 2.6.4. *Any infinite sum* $f(x) = \sum_{\bar{l} \in (L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}} c_{\bar{l}}\, \xi_{\bar{l}}(x)$ *satisfying the relation* (2.6.3) *converges absolutely and uniformly over compact subsets of* V_h.

Proof. By Lemma 2.6.1, if we treat l and l' as elements on $L^{(g)}_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, then

$$\xi_{\bar{l}+\operatorname{sgn}(h)\bar{l}'}(l') = \mathbf{e}(\tfrac{1}{2}(B_{h,g} - H_{h,g})(l + \operatorname{sgn}(h)l', l')).$$

If we treat $\bar{l}$ and $\bar{l}'$ as elements of $(L/L_{\mathrm{MI}}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, with basis given by $\{f_i\}_{i\in I}$, then we can rewrite the above formula as

$$\xi_{\bar{l}+\operatorname{sgn}(h)\bar{l}'}(l') = \mathbf{e}(2\pi\sqrt{-1}\, r\, {}^t(l + \operatorname{sgn}(h)l')\Omega_h l').$$

Since $\operatorname{sgn}(h) \operatorname{Im} \Omega_h$ is positive-definite, and since

$$\begin{aligned}\operatorname{Im}({}^t(l+\operatorname{sgn}(h)l')\Omega_h l') &= {}^t(l+\operatorname{sgn}(h)l')(\operatorname{Im}\Omega_h)l' \\ &= \operatorname{sgn}(h)\,{}^t(\tfrac{1}{2}l+\operatorname{sgn}(h)l')(\operatorname{Im}\Omega_h)(\tfrac{1}{2}l+\operatorname{sgn}(h)l') - \operatorname{sgn}(h)\,{}^t(\tfrac{1}{2}l)(\operatorname{Im}\Omega_h)(\tfrac{1}{2}l),\end{aligned}$$

we see that $\|\operatorname{Im}[{}^t(l+\operatorname{sgn}(h)l')\Omega_h l']\|$ and $\|l'\|^2$ are comparable up to a constant ratio when $\|l'\| \to \infty$. Now the rest of the proof is elementary. □

Corollary 2.6.5. *Let $\{\bar{l}^{(j)}\}_{j\in J}$ be a complete set of representatives of*

$$[(L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}] \,/\, [2(L^{(g)}/L^{(g)}_{\mathrm{MI}})].$$

Then, for each $j \in J$, the infinite sum

$$\theta^{(j)}_{h,g}(x) := \sum_{\bar{l}\in L^{(g)}/L^{(g)}_{\mathrm{MI}}} \xi_{\bar{l}^{(j)}+\operatorname{sgn}(h)\bar{l}}(\Omega_h(\bar{l}))\, \xi_{\bar{l}^{(j)}+2\operatorname{sgn}(h)\bar{l}}(x)$$

converges absolutely and uniformly over compact subsets of V_h, and defines a holomorphic function over V_h, varying holomorphically with respect to h. The collection $\{\theta^{(j)}_{h,g}(x)\}_{j\in J}$ forms a $\mathbb{C}$-basis of $\Gamma(G_{h,g}, \mathcal{L}_{h,g})$. In particular, we have

$$\dim_{\mathbb{C}} \Gamma(G_{h,g}, \mathcal{L}_{h,g}) = [((L^{(g)})^{\#}/(L^{(g)})^{\#}_{\mathrm{MI}}) : 2(L^{(g)}/L^{(g)}_{\mathrm{MI}})] = [\tfrac{1}{2}(L^{(g)})^{\#} : L^{(g)}]^{1/2}.$$

3. Analytic toroidal compactifications

3.1. **Rational boundary components.** Here we assume that the reader is familiar with the notion of rational boundary components of Hermitian symmetric spaces. (See for example the summaries in [4] or [11].)

Lemma 3.1.1. *Let us fix a choice of an element $g \in \mathrm{G}(\mathbb{A}^\infty)$. Let $L^{(g)}$ denote the $\mathcal{O}$-lattice in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $L^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ corresponds naturally to the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$-submodule $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$. Consider the five sets formed respectively by the following five types of data on $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$:*

1. *A rational boundary component of X_0 (as in [4, 3.5]). (For compatibility with formation of products, it is necessary to include X_0 itself as a rational boundary component.)*
2. *An $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule $\mathtt{V}_{-2}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ that is totally isotropic under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$.*
3. *An increasing filtration $\mathtt{V} = \{\mathtt{V}_{-i}\}_{i\in\mathbb{Z}}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ satisfying the following conditions:*
   - (a) *$\mathtt{V}_{-3} = 0$ and $\mathtt{V}_0 = L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.*
   - (b) *Each graded piece $\mathrm{Gr}^{\mathtt{V}}_{-i} := \mathtt{V}_{-i}/\mathtt{V}_{-i-1}$ is an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-module. (In this case, the filtration $\mathtt{V}$ is* **admissible**.*)*
   - (c) *$\mathtt{V}_{-1}$ and $\mathtt{V}_{-2}$ are annihilators of each other under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$. (In this case, the filtration $\mathtt{V}$ is* **symplectic**.*)*
4. *An $\mathcal{O}$-sublattice $\mathtt{F}^{(g)}_{-2}$ of $L^{(g)}$, with $L^{(g)}/\mathtt{F}^{(g)}_{-2}$ torsion-free, that is totally isotropic under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$.*
5. *An increasing filtration $\mathtt{F}^{(g)} = \{\mathtt{F}^{(g)}_{-i}\}_{i\in\mathbb{Z}}$ of $L^{(g)}$ satisfying the following conditions:*
   - (a) *$\mathtt{F}^{(g)}_{-3} = 0$ and $\mathtt{F}^{(g)}_0 = L^{(g)}$.*

(b) *Each graded piece* $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-i} := \mathtt{F}^{(g)}_{-i}/\mathtt{F}^{(g)}_{-i-1}$ *is an* $\mathcal{O}$*-lattice, admitting an splitting* $\varepsilon^{(g)} : \mathrm{Gr}^{\mathtt{F}^{(g)}} := \bigoplus_{-i\in\mathbb{Z}} \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-i} \overset{\sim}{\to} L^{(g)}$. (*In this case, the filtration* $\mathtt{F}^{(g)}$ *is* **admissible**.)

(c) $\mathtt{F}^{(g)}_{-1}$ *and* $\mathtt{F}^{(g)}_{-2}$ *are annihilators of each other under the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$. (*In this case, the filtration* $\mathtt{F}^{(g)}$ *is* **symplectic**.)

(*We allow parabolic subgroups to be the whole group, and we allow totally isotropic submodules to be zero.*) *Then the five sets are in canonical bijections with each other.*

Proof. As explained in [4, 3.5], the rational boundary components of X_0 correspond bijectively to the rational parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ each of whose images in the $\mathbb{Q}$-simple quotients of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is either a maximal proper parabolic subgroup or the whole group. For simplicity, let us call temporarily such rational parabolic subgroups *maximal*. Given any such rational parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, the action of the Lie algebra of its unipotent radical defines an isotropic filtration $\mathtt{V}$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By maximality of the parabolic subgroup, we see that $\mathtt{V}$ is determined by its largest totally isotropic filtered piece. Now the equivalences among the maximal rational parabolic subgroups and the remaining objects in the lemma is elementary. □

For each $g \in \mathrm{G}(\mathbb{A}^\infty)$, let $L^{(g)}$ denote the $\mathcal{O}$-lattice in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $L^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ corresponds naturally to the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$-submodule $g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty$. Then the assignment

$$\begin{aligned} \mathtt{V}_{-2} &\mapsto \mathtt{V} = \{\mathtt{V}_{-i}\}_{i\in\mathbb{Z}} \\ &\mapsto \mathtt{F}^{(g)} := \{\mathtt{F}^{(g)}_{-i} := \mathtt{V}_{-i} \cap L^{(g)}\}_{i\in\mathbb{Z}} \\ &\mapsto \mathtt{Z}^{(g)} := \{\mathtt{Z}^{(g)}_{-i} := g^{-1}(\mathtt{F}^{(g)}_{-i} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}} = \{(g^{-1}(\mathtt{V}_{-i} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty)) \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})\}_{i\in\mathbb{Z}} \end{aligned}$$

defines an injection from the set of rational boundary components of X_0 to the set of fully symplectic admissible filtrations of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. (See [23, Def. 5.2.7.1].)

The action of $\mathrm{G}(\mathbb{Q})$ on $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ induces an action of $\mathrm{G}(\mathbb{Q})$ on $\{\mathtt{V}\} \times \mathrm{G}(\mathbb{A}^\infty)$.

Definition 3.1.2. *A rational boundary component of* $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ *is a* $\mathrm{G}(\mathbb{Q})$*-orbit of some pair* $(\mathtt{V}, g)$.

By the explicit definition above, pairs in the $\mathrm{G}(\mathbb{Q})$-orbit of $(\mathtt{V}, g)$ define the same fully symplectic admissible filtration of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. This induces a map from the set of rational boundary components of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ to the set of fully symplectic admissible filtrations of $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. However, this map is generally far from injective.

For example, if $u \in \mathrm{G}(\hat{\mathbb{Z}})$ is an element preserving $\mathtt{V}_{-2,\mathbb{A}^\infty} := \mathtt{V}_{-2} \underset{\mathbb{Q}}{\otimes} \mathbb{A}^\infty$, then $(\mathtt{V}, g)$ and $(\mathtt{V}, gu)$ define the same filtration $\mathtt{Z}^{(g)} = \mathtt{Z}^{(gu)}$. For the purpose of studying toroidal compactifications, it is important to distinguish $(\mathtt{V}, g)$ and $(\mathtt{V}, gu)$ by supplying a rigidification on the rational structure of $\mathtt{V}_{-2}$. For each given $(\mathtt{V}, g)$, let us define a torus argument $\Phi^{(g)} = (X^{(g)}, Y^{(g)}, \phi^{(g)}, \varphi^{(g)}_{-2}, \varphi^{(g)}_0)$ for $\mathtt{Z}^{(g)}$ as follows:

(1) $X^{(g)} := \mathrm{Hom}_{\mathbb{Z}}(\mathtt{F}_{-2}^{(g)}, \mathbb{Z}(1)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}}, \mathbb{Z}(1))$.
(2) $Y^{(g)} := \mathrm{Gr}_0^{\mathtt{F}^{(g)}} = \mathtt{F}_0^{(g)}/\mathtt{F}_{-1}^{(g)}$.
(3) $\phi^{(g)} : Y^{(g)} \hookrightarrow X^{(g)}$ is equivalent to the nondegenerate pairing

$$\langle \cdot, \cdot \rangle_{20}^{(g)} : \mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}} \times \mathrm{Gr}_0^{\mathtt{F}^{(g)}} \to \mathbb{Z}(1)$$

induced by $\langle \cdot, \cdot \rangle^{(g)} : L^{(g)} \times L^{(g)} \to \mathbb{Z}(1)$, with the sign convention $\langle x, y\rangle_{20}^{(g)} = \phi^{(g)}(y)(x)$.
(4) $\varphi_{-2}^{(g)} : \mathrm{Gr}_{-2}^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1))$ is the composition

$$\mathrm{Gr}_{-2}^{\mathtt{Z}^{(g)}} \overset{\mathrm{Gr}_{-2}(g)}{\xrightarrow{\sim}} \mathrm{Gr}_{-2}^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)).$$

(5) $\varphi_0^{(g)} : \mathrm{Gr}_0^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ is the composition

$$\mathrm{Gr}_0^{\mathtt{Z}^{(g)}} \overset{\mathrm{Gr}_0(g)}{\xrightarrow{\sim}} \mathrm{Gr}_0^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} Y^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}.$$

Finally, by Condition 1.2.5 and the fact that maximal orders over Dedekind domains are *hereditary* ([29, Thm. 21.4 and Cor. 21.5]), for any $(\mathtt{V}, g)$, the associated filtration $\mathtt{F}^{(g)}$ of $L^{(g)}$ is split by some splitting $\varepsilon^{(g)} : \mathrm{Gr}^{\mathtt{F}^{(g)}} \xrightarrow{\sim} L^{(g)}$. Each splitting $\varepsilon^{(g)}$ defines by base extension a splitting $\varepsilon^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} : \mathrm{Gr}^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} \xrightarrow{\sim} L^{(g)} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}} = g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}})$, and hence by composition with $\mathrm{Gr}(g)$ and g^{-1} a splitting $\delta^{(g)} : \mathrm{Gr}^{\mathtt{Z}^{(g)}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$. This defines an assignment

$$(\mathtt{V}, g, \varepsilon^{(g)}) \mapsto (\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)}).$$

Let us define two triples $(\mathtt{V}, g, \varepsilon^{(g)})$ and $(\mathtt{V}', g', (\varepsilon^{(g)})')$ to be equivalent if $\mathtt{V} = \mathtt{V}'$ and $g = g'$, and define two triples $(\mathtt{Z}, \Phi, \delta)$ and $(\mathtt{Z}', \Phi', \delta')$ to be equivalent if $\mathtt{Z} = \mathtt{Z}'$ and if the torus arguments $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ and $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ are equivalent in the sense that there exists some pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ matching the remaining data. By definition, the equivalence classes $[(\mathtt{V}, g, \varepsilon^{(g)})]$ of triples $(\mathtt{V}, g, \varepsilon^{(g)})$ correspond exactly to the pairs $(\mathtt{V}, g)$ they define by forgetting the splitting $\varepsilon^{(g)}$. On the other hand, let us denote by $[(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})]$ the equivalence class defined by $(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$, and let us call them the *cusp labels* for $(L, \langle \cdot, \cdot \rangle, h_0)$.

Now we have the assignment $(\mathtt{V}, g) \mapsto [(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})]$ induced by the assignment $(\mathtt{V}, g, \varepsilon^{(g)}) \mapsto (\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$. This assignment is still not injective in general, but will suffice for our purpose.

For any $\mathbb{Q}$-algebra R, let us write $\mathtt{V}_{-i,R} := \mathtt{V}_{-i} \underset{\mathbb{Q}}{\otimes} R$ and $\mathrm{Gr}_{-i,R}^{\mathtt{V}} := \mathtt{V}_{-i,R}/\mathtt{V}_{-i-1,R}$. Similarly, for any $\mathbb{Z}$-algebra R, let us write $\mathtt{F}_{-i,R}^{(g)} := \mathtt{F}_{-i}^{(g)} \underset{\mathbb{Z}}{\otimes} R$ and $\mathrm{Gr}_{-i,R}^{\mathtt{F}^{(g)}} := \mathtt{F}_{-i,R}^{(g)}/\mathtt{F}_{-i-1,R}^{(g)}$.

To each boundary component represented by $(\mathtt{V}, g)$, the symplectic filtration $\mathtt{V}$ induces a symplectic lattice $(\mathrm{Gr}_{-1}^{\mathtt{V}}, \langle \cdot, \cdot \rangle_{11})$, and the associated symplectic filtration $\mathtt{F}^{(g)}$ on $L^{(g)}$ induces a symplectic lattice $(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}, \langle \cdot, \cdot \rangle_{11}^{(g)})$. It is clear that $(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \langle \cdot, \cdot \rangle_{11}^{(g)}) \cong (\mathrm{Gr}_{-1}^{\mathtt{V}}, \langle \cdot, \cdot \rangle_{11})$.

Any $h \in \mathsf{X}$ defines a complex structure $h(\sqrt{-1})$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, inducing an isomorphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \xrightarrow{\sim} V_h = (L \underset{\mathbb{Z}}{\otimes} \mathbb{C})/\mathsf{P}_h$. Since $\mathtt{F}^{(g)}_{-2,\mathbb{R}}$ is totally isotropic, and since $-\operatorname{sgn}(h)\sqrt{-1}\,\langle\,\cdot\,,h(\sqrt{-1})\,\cdot\,\rangle$ is positive definite, we have $\mathtt{F}^{(g)}_{-2,\mathbb{R}} \cap h(\sqrt{-1})(\mathtt{F}^{(g)}_{-2,\mathbb{R}}) = \{0\}$. Then h defines a $\mathbb{C}$-linear embedding $\mathtt{F}^{(g)}_{-2,\mathbb{C}} \hookrightarrow V_h$, such that the composition $\mathtt{F}^{(g)}_{-2,\mathbb{R}} \overset{h(\sqrt{-1})}{\to} L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \twoheadrightarrow \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{R}}$ is an isomorphism of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules. By abuse of notation, we shall denote the image of the above embedding $\mathtt{F}^{(g)}_{-2,\mathbb{C}} \hookrightarrow V_h$ as $\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}$. Let $(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp := \{x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x,y\rangle = 0, \forall y \in \mathtt{F}^{(g)}_{-2,h(\mathbb{C})}\}$. Then we obtain an orthogonal direct sum

$$(3.1.3)\quad (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}}) \overset{\perp}{\oplus} ((\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp}),$$

which induces an isomorphism

$$(3.1.4)\qquad ((\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11,\mathbb{R}})$$

of symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules. Since $h(\sqrt{-1})$ preserves $\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}$, the relation

$$\langle h(\sqrt{-1})x, h(\sqrt{-1})y\rangle = \langle x,y\rangle$$

for every $x,y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ shows that $h(\sqrt{-1})$ also preserves $(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp$. As a result, the restriction of $h(\sqrt{-1})$ defines a complex structure on $(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp$, which corresponds by the isomorphism (3.1.4) (and Lemma 2.1.2) to a $\operatorname{sgn}(h)$-polarization h_{-1} on $(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{-1,\mathbb{R}})$, such that

$$(3.1.5)\quad ((\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp}, h|_{(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11,\mathbb{R}}, h_{-1})$$

is an isomorphism of polarized symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules. If $\operatorname{sgn}(h) = 1$, then $\operatorname{sgn}(h_{-1}) = 1$. Hence the triple $(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}, h_{-1})$ is a PEL-type $\mathcal{O}$-lattice. (This is in particular the case for $h = h_0$.)

Lemma 3.1.6. *With notations as in* [23, Rem. 5.2.7.2] (*with h there replaced with h_0 here*), *the PEL-type $\mathcal{O}$-lattice $(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}, (h_0)_{-1})$ qualifies as a* (*noncanonical*) *choice of $(L^{\mathsf{Z}^{(g)}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}^{(g)}}, h_0^{\mathsf{Z}^{(g)}})$, so that $(\mathrm{Gr}^{\mathsf{Z}^{(g)}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11}) \cong (\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\hat{\mathbb{Z}}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11})$ and $(\mathrm{Gr}^{\mathsf{Z}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11,\mathbb{R}}, (h_0)_{-1}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11}, (h_0)_{-1})$. (See Remark 3.1.7 below for the justification of notations.) In particular, at any neat level $\mathcal{H}$, the scheme $\mathsf{M}^{\mathsf{Z}^{(g)}_{\mathcal{H}}}_{\mathcal{H}}$ can be identified with the moduli problem defined by $(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}, (h_0)_{-1})$ at a suitable level* ($\mathcal{H}''_{-1}$, *to be introduced in* §3.4 *below*).

Remark 3.1.7. The notation $(h_0)_{-1}$ appeared twice in the second isomorphism in Lemma 3.1.6. Nevertheless, their constructions are identical because we have to use $\mathtt{F}^{(g)}_{-2,\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(X^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}(1)) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ to define $(h_0)_{-1}$ for $(\mathrm{Gr}^{\mathsf{Z}^{(g)}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11,\mathbb{R}})$ in [23, Prop. 5.1.2.2]. This is why we allow such an identification.

3.2. **Siegel domains of third kind.** In this section, let us fix a choice of a triple $(\mathtt{V}, g, \varepsilon^{(g)})$ inducing a rational boundary component of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$. Let $\mathtt{F}^{(g)}$ be associated with $(\mathtt{V}, g)$ as in §3.1. Let us define the group functor $\mathrm{G}^{(g)}$ by $(L^{(g)}, \langle \cdot\,, \cdot \rangle^{(g)})$ as in Definition 1.1.2, with the canonical identification $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ matching $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathrm{G}^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ functorially.

Definition 3.2.1. *With settings as above, let us define the following subquotients (i.e., quotients of subgroups) of $\mathrm{G}^{(g)}(R)$, for any $\mathbb{Z}$-algebra R:*

$$\mathrm{P}_{\mathtt{F}^{(g)}}(R) := \{(p, r) \in \mathrm{G}^{(g)}(R) : p(\mathtt{F}^{(g)}) = \mathtt{F}^{(g)}\},$$
$$\mathrm{P}'_{\mathtt{F}^{(g)}}(R) := \{(p, r) \in \mathrm{P}^{(g)}(R) : \mathrm{Gr}_{-2}(p) = r \,\mathrm{Id}_{\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2}} \text{ and } \mathrm{Gr}_0(p) = \mathrm{Id}_{\mathrm{Gr}^{\mathtt{F}^{(g)}}_0}\},$$
$$\mathrm{Z}_{\mathtt{F}^{(g)}}(R) := \{(p, r) \in \mathrm{P}_{\mathtt{F}^{(g)}} : \mathrm{Gr}_{-1}(p) = \mathrm{Id}_{\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}} \text{ and } r = 1\},$$
$$\mathrm{U}_{\mathtt{F}^{(g)}}(R) := \{(p, r) \in \mathrm{P}_{\mathtt{F}^{(g)}} : \mathrm{Gr}(p) = \mathrm{Id}_{\mathrm{Gr}^{\mathtt{F}^{(g)}}} \text{ and } r = 1\},$$
$$\mathrm{G}_{h,\mathtt{F}^{(g)}}(R) := \left\{ \begin{matrix} (p_{-1}, r_{-1}) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,R}) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \exists (p, r) \in \mathrm{P}_{\mathtt{F}^{(g)}} \text{ s.t. } \mathrm{Gr}_{-1}(p) = p_{-1} \text{ and } r = r_{-1} \end{matrix} \right\},$$
$$\mathrm{G}_{l,\mathtt{F}^{(g)}}(R) := \left\{ \begin{matrix} (p_{-2}, p_0) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,R}) \times \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,R}) : \\ \exists (p, r) \in \mathrm{Z}_{\mathtt{F}^{(g)}}(R) \text{ s.t. } \mathrm{Gr}_{-2}(p) = p_{-2} \text{ and } \mathrm{Gr}_0(p) = p_0 \end{matrix} \right\},$$
$$\mathrm{U}_{2,\mathtt{F}^{(g)}}(R) := \left\{ \begin{matrix} p_{20} \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,R}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,R}) : \\ \exists (p, 1) \in \mathrm{U}_{\mathtt{F}^{(g)}}(R) \text{ s.t. } (\varepsilon^{(g)})^{-1} \circ p \circ \varepsilon^{(g)} = \begin{pmatrix} 1 & & p_{20} \\ & 1 & \\ & & 1 \end{pmatrix} \end{matrix} \right\},$$
$$\mathrm{U}_{1,\mathtt{F}^{(g)}}(R) := \left\{ \begin{matrix} (p_{21}, p_{10}) \in \begin{matrix} \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,R}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,R}) \\ \times \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,R}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,R}) \end{matrix} : \\ \exists (p, 1) \in \mathrm{U}_{\mathtt{F}^{(g)}}(R) \text{ s.t. } (\varepsilon^{(g)})^{-1} \circ p \circ \varepsilon^{(g)} = \begin{pmatrix} 1 & p_{21} & p_{20} \\ & 1 & p_{10} \\ & & 1 \end{pmatrix}, \text{ some } p_{20} \end{matrix} \right\}.$$

Lemma 3.2.2. *By definition, there are natural inclusions $\mathrm{U}_{2,\mathtt{F}^{(g)}} \subset \mathrm{U}_{\mathtt{F}^{(g)}} \subset \mathrm{Z}_{\mathtt{F}^{(g)}} \subset \mathrm{P}_{\mathtt{F}^{(g)}} \subset \mathrm{G}^{(g)}$ and $\mathrm{U}_{\mathtt{F}^{(g)}} \subset \mathrm{P}'_{\mathtt{F}^{(g)}} \subset \mathrm{P}_{\mathtt{F}^{(g)}}$, and natural exact sequences:*

$$1 \to \mathrm{Z}_{\mathtt{F}^{(g)}} \to \mathrm{P}_{\mathtt{F}^{(g)}} \to \mathrm{G}_{h,\mathtt{F}^{(g)}} \to 1,$$
$$1 \to \mathrm{U}_{\mathtt{F}^{(g)}} \to \mathrm{Z}_{\mathtt{F}^{(g)}} \to \mathrm{G}_{l,\mathtt{F}^{(g)}} \to 1,$$
$$1 \to \mathrm{U}_{2,\mathtt{F}^{(g)}} \to \mathrm{U}_{\mathtt{F}^{(g)}} \to \mathrm{U}_{1,\mathtt{F}^{(g)}} \to 1,$$
$$1 \to \mathrm{P}'_{\mathtt{F}^{(g)}} \to \mathrm{P}_{\mathtt{F}^{(g)}} \to \mathrm{G}_{l,\mathtt{F}^{(g)}} \to 1,$$
$$1 \to \mathrm{U}_{\mathtt{F}^{(g)}} \to \mathrm{P}'_{\mathtt{F}^{(g)}} \to \mathrm{G}_{h,\mathtt{F}^{(g)}} \to 1.$$

The symplectic isomorphisms (3.1.3) and (3.1.5) define an assignment $h \mapsto h_{-1}$ from $\pm$-polarizations of $(L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,, \cdot \rangle^{(g)})$ to $\pm$-polarizations of $(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \langle \cdot\,, \cdot \rangle^{(g)})$, satisfying $\mathrm{sgn}(h) = \mathrm{sgn}(h_{-1})$, which is equivariant with respect to the action of the subgroup $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})$ of $\mathrm{G}(\mathbb{R}) = \mathrm{G}^{(g)}(\mathbb{R})$. Recall that $\mathsf{X} = \mathrm{G}(\mathbb{R})\, h_0 = \mathrm{G}(\mathbb{R})/\mathcal{U}_\infty$, where $\mathcal{U}_\infty = \mathrm{Cent}_{\mathrm{G}(\mathbb{R})}(h_0)$ is known to be maximal compact modulo center. Therefore, by Iwasawa decomposition, $\mathsf{X} = \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})\, h_0$. If we set $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0 := \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R}) \cap \mathrm{G}(\mathbb{R})_0$ and $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q})_0 := \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \cap \mathrm{G}(\mathbb{Q})_0$, then we have $\mathsf{X}_0 = \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0\, h_0$. Let us set $\mathsf{X}^{\mathtt{F}^{(g)}} := \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})\, ((h_0)_{-1}, \mathrm{sgn}(h_0))$, with $\mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})$ acting on $\mathrm{sgn}(h_0)$ by the signs of the similitudes. Let $\mathsf{X}^{\mathtt{F}^{(g)}}_0$ be the connected component of $\mathsf{X}^{\mathtt{F}^{(g)}}$ containing

$((h_0)_{-1}, \mathrm{sgn}(h_0))$, and let $\mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})_0$ be the subgroup of $\mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})$ stabilizing $\mathsf{X}_0^{\mathtt{F}^{(g)}}$. Since $\mathrm{sgn}(h_0)$ is a constant on $\mathsf{X}_0^{\mathtt{F}^{(g)}}$, we shall also write $\mathsf{X}_0^{\mathtt{F}^{(g)}} = \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})_0\,(h_0)_{-1}$. Then we have:

Lemma 3.2.3. *The assignment $h \mapsto h_{-1}$ defines an $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0$-equivariant morphism*

$$\mathsf{X}_0 = \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0\, h_0 \twoheadrightarrow \mathsf{X}_0^{\mathtt{F}^{(g)}} = \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{R})_0\,(h_0)_{-1}$$

of complex manifolds (with $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0$-actions).

The composition of $\mathcal{O}$-equivariant morphisms

$$\mathrm{Gr}^{\mathtt{F}^{(g)}} \overset{\varepsilon^{(g)}}{\underset{\sim}{\to}} L^{(g)} \overset{\text{can.}}{\to} L\underset{\mathbb{Z}}{\otimes}\mathbb{R} \overset{(3.1.3)}{\underset{\sim}{\to}} \mathtt{F}^{(g)}_{-2,h(\mathbb{C})} \oplus (\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^{\perp} \overset{\text{can.}\oplus(3.1.5)}{\underset{\sim}{\to}} \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,\mathbb{C}} \oplus \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}.$$

defines in particular a collection of morphisms

$$\begin{aligned} &\varepsilon^{(g)}_{22}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,h(\mathbb{C})}, && \varepsilon^{(g)}_{12}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}},\\ &\varepsilon^{(g)}_{21}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,h(\mathbb{C})}, && \varepsilon^{(g)}_{11}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}},\\ &\varepsilon^{(g)}_{20}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,h(\mathbb{C})}, && \varepsilon^{(g)}_{10}(h) : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}. \end{aligned}$$

By construction, we know:

Lemma 3.2.4. *The morphisms $\varepsilon^{(g)}_{22}(h)$ and $\varepsilon^{(g)}_{11}(h)$ are the canonical embeddings, and $\varepsilon^{(g)}_{12}(h) = 0$, all being independent of h. The polarization h is completely determined by the morphisms h_{-1}, $\varepsilon^{(g)}_{21}(h)$, $\varepsilon^{(g)}_{20}(h)$, and $\varepsilon^{(g)}_{10}(h)$, together with the sign $\mathrm{sgn}(h)$ that it defines.*

It remains to understand how the morphisms $\varepsilon^{(g)}_{21}(h)$, $\varepsilon^{(g)}_{20}(h)$, and $\varepsilon^{(g)}_{10}(h)$ vary with h, or equivalently with the action of $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})$. By Lemma 3.2.2, we shall analyze step by step the actions of $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{R})$, $\mathrm{U}_{\mathtt{F}^{(g)}}(\mathbb{R})$, $\mathrm{Z}_{\mathtt{F}^{(g)}}(\mathbb{R})$, and $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})$.

Lemma 3.2.5. *Let us identify both $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{R})$ and $\{\varepsilon^{(g)}_{20}(h)\}_{h\in\mathsf{X}}$ canonically as subsets of the group*

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{C}}) \overset{\text{can.}}{\underset{\sim}{\to}} \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,\mathbb{R}}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{C}}),$$

and identify both $\mathrm{U}_{1,\mathtt{F}^{(g)}}(\mathbb{R})$ and $\{(\varepsilon^{(g)}_{21}(h), \varepsilon^{(g)}_{10}(h))\}_{h\in\mathsf{X}}$ canonically as subsets of the group

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,\mathbb{R}}) \times \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}})\\ &\qquad \overset{\text{can.}}{\underset{\sim}{\to}} \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,\mathbb{R}}) \times \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{R}}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}). \end{aligned}$$

Then the following are true:

1. *The group $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{R})$ acts by translations on $\{\varepsilon^{(g)}_{20}(h)\}_{h\in\mathsf{X}}$.*
2. *The group $\mathrm{U}_{1,\mathtt{F}^{(g)}}(\mathbb{R})$ acts by translations on $\{(\varepsilon^{(g)}_{21}(h), \varepsilon^{(g)}_{10}(h))\}_{h\in\mathsf{X}}$.*
3. *For each $h \in \mathsf{X}$, $\mathrm{Im}\,\varepsilon^{(g)}_{20}(h) \in \mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{R})$.*

Proof. Only (3) is not obvious. We need to show that, if we define $(p,1) \in \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \times \mathbf{G}_{\mathrm{m}}(\mathbb{R})$ by $(\varepsilon^{(g)})^{-1} \circ p \circ \varepsilon^{(g)} = \begin{pmatrix} 1 & & \mathrm{Im}\,\varepsilon^{(g)}_{20}(h) \\ & 1 & \\ & & 1 \end{pmatrix}$, then $(p,1)$ defines an element of $\mathrm{G}^{(g)}(\mathbb{R})$. That is, we need to show that $\langle px, py\rangle = \langle x, y\rangle$ for any $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. Let us write $x = \varepsilon^{(g)}(x_{-2}, x_{-1}, x_0)$ and $y = \varepsilon^{(g)}(y_{-2}, y_{-1}, y_0)$. Then $px = \varepsilon^{(g)}(x_{-2} + \operatorname{Im}\varepsilon_{20}^{(g)}(h)(x_0), x_{-1}, x_0)$ and $py = \varepsilon^{(g)}(y_{-2} + \operatorname{Im}\varepsilon_{20}^{(g)}(h)(y_0), y_{-1}, y_0)$. Using the decomposition (3.1.3), we write $\varepsilon^{(g)}(0,0,x_0) = x_1 + h(\sqrt{-1})x_2 + x_3$ for some $x_1, x_2 \in \mathtt{F}_{-2,\mathbb{R}}^{(g)}$ and $x_3 \in (\mathtt{F}_{-2,h(\mathbb{C})}^{(g)})^\perp$, and we write $\varepsilon^{(g)}(0,0,y_0) = y_1 + h(\sqrt{-1})y_2 + y_3$ for some $y_1, y_2 \in \mathtt{F}_{-2,\mathbb{R}}^{(g)}$ and $y_3 \in (\mathtt{F}_{-2,h(\mathbb{C})}^{(g)})^\perp$. Then $\operatorname{Im}\varepsilon_{20}^{(g)}(h)(x_0) = x_2$ and $\operatorname{Im}\varepsilon_{20}^{(g)}(h)(y_0) = y_2$. Since $(\mathtt{F}_{-2,\mathbb{R}}^{(g)})^\perp = \mathtt{F}_{-2,\mathbb{R}}^{(g)} + (\mathtt{F}_{-2,h(\mathbb{C})}^{(g)})^\perp$, we have

$$\begin{aligned}\langle px, py\rangle - \langle x, y\rangle &= \langle \varepsilon^{(g)}(\operatorname{Im}\varepsilon_{20}^{(g)}(h)(x_0), 0, 0),\ \varepsilon^{(g)}(0,0,y_0)\rangle \\ &\quad + \langle \varepsilon^{(g)}(0,0,x_0),\ \varepsilon^{(g)}(\operatorname{Im}\varepsilon_{20}^{(g)}(h)(x_0), 0, 0)\rangle \\ &= \langle x_2,\ y_1 + h(\sqrt{-1})y_2 + y_3\rangle + \langle x_1 + h(\sqrt{-1})x_2 + x_3,\ y_2\rangle \\ &= \langle x_2,\ h(\sqrt{-1})y_2\rangle + \langle h(\sqrt{-1})x_2,\ y_2\rangle = 0,\end{aligned}$$

as desired. □

Combining Lemmas 3.2.4 and 3.2.5, we obtain:

Proposition 3.2.6. *If we consider* X_0 *as a subset of the set of* $\mathcal{O}$*-equivariant complex structures on* $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$*, so that* $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})\mathsf{X}_0$ *makes sense, then the morphisms*

$$\begin{aligned}\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})\,\mathsf{X}_0 \to \mathsf{X}_2^{\mathtt{F}^{(g)}} &:= \{(\varepsilon_{20}^{(g)}(h),\ (\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)),\ h_{-1})\}_{h\in\mathsf{X}_0} \\ \to \mathsf{X}_1^{\mathtt{F}^{(g)}} &:= \{((\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)),\ h_{-1})\}_{h\in\mathsf{X}_0} \\ \to \mathsf{X}_0^{\mathtt{F}^{(g)}} &= \{h_{-1}\}_{h\in\mathsf{X}_0}\end{aligned} \tag{3.2.7}$$

of real manifolds defined by the assignments

$$\begin{aligned}h &\mapsto (\varepsilon_{20}^{(g)}(h),\ (\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)),\ h_{-1}) \\ &\mapsto ((\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)),\ h_{-1}) \\ &\mapsto h_{-1}\end{aligned}$$

make the first morphism in (3.2.7) *a bijection, the second morphism in* (3.2.7) *a torsor under* $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})$*, and the third morphism in* (3.2.7) *a torsor under* $\mathrm{U}_{1,\mathtt{F}^{(g)}}(\mathbb{R})$*. We shall denote the second and third morphisms in* (3.2.7) *by* π_2 *and* π_1*, respectively. The composition* $\pi_1 \circ \pi_2$ *will be denoted by* π*.*

Corollary 3.2.8. *There is a canonical* $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})_0$*-equivariant isomorphism*

$$\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})\,\mathsf{X}_0 \cong \mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C}) \times \mathrm{U}_{1,\mathtt{F}^{(g)}}(\mathbb{R}) \times \mathsf{X}_0^{\mathtt{F}^{(g)}}$$

of real manifolds (*with* $\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{R})$*-actions*)*, which is up to a sign convention* (*and with reversed order*) *the isomorphism* $D(F) \cong U(F)_\mathbb{C} \times V(F) \times F$ *in* [2, p. 235]*. Under this isomorphism, the subset* $\{\operatorname{Im}\varepsilon_{20}^{(g)}(h)\}_{h\in\mathsf{X}_0}$ *of* $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{R})$ *corresponds to the set* $C(F)$ *in* [2, p. 227]*.* (*See also* [2, p. 233].) *The morphisms among the connected components induced by our* π_2*,* π_1*, and* π *in Proposition* 3.2.6 *correspond respectively to the morphisms* π'_F*,* p_F*, and* π_F *in* [2, p. 237]*.*

Lemma 3.2.9. *The second projection*

$$\operatorname{Hom}_\mathcal{O}(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}, \mathrm{Gr}_{-2,\mathbb{R}}^{\mathtt{F}^{(g)}}) \times \operatorname{Hom}_\mathcal{O}(\mathrm{Gr}_{0}^{\mathtt{F}^{(g)}}, \mathrm{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}}) \twoheadrightarrow \operatorname{Hom}_\mathcal{O}(\mathrm{Gr}_{0}^{\mathtt{F}^{(g)}}, \mathrm{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}})$$

induces an isomorphism

$$\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}_0^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}).$$

Proof. By definition, an element

$$(p_{21}, p_{10}) \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2,\mathbb{R}}^{\mathsf{F}^{(g)}}) \times \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}})$$

defines an element of $\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R})$ if and only if there exists $(p,1) \in \mathrm{U}_{\mathsf{F}^{(g)}}$ such that $(\varepsilon^{(g)})^{-1} \circ p \circ \varepsilon^{(g)} = \begin{pmatrix} 1 & p_{21} & p_{20} \\ & 1 & p_{10} \\ & & 1 \end{pmatrix}$ for some p_{20}. Suppose $x = \varepsilon^{(g)}(x_{-2}, x_{-1}, x_0)$ and $y = \varepsilon^{(g)}(y_{-2}, y_{-1}, y_0)$. Then

$$\begin{aligned} \langle px, py \rangle^{(g)} - \langle x, y \rangle^{(g)} &= \langle p_{21}(x_{-1}), y_0 \rangle_{20}^{(g)} + \langle x_{-1}, p_{10}(y_0) \rangle_{11}^{(g)} - \langle p_{21}(y_{-1}), x_0 \rangle_{20}^{(g)} \\ &- \langle y_{-1}, p_{10}(x_0) \rangle_{11}^{(g)} + \langle p_{20}(x_0), y_0 \rangle_{20}^{(g)} - \langle p_{20}(y_0), x_0 \rangle_{20}^{(g)} + \langle p_{10}(x_0), p_{10}(y_0) \rangle_{11}^{(g)}. \end{aligned}$$

Since $\langle \cdot, \cdot \rangle_{20}^{(g)} : \mathrm{Gr}_{-2,\mathbb{R}}^{\mathsf{F}^{(g)}} \times \mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}} \to \mathbb{R}(1)$ and $\langle \cdot, \cdot \rangle_{11}^{(g)} : \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}} \times \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}} \to \mathbb{R}(1)$ are perfect pairings, setting $x_0 = 0$ shows that p_{21} is determined by p_{10} if $(p,1) \in \mathrm{U}_{\mathsf{F}^{(g)}}$. Conversely, for any given p_{10} in $\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}_0^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}})$, there exists p_{21} such that $\langle p_{21}(x_{-1}), y_0 \rangle_{20}^{(g)} + \langle x_{-1}, p_{10}(y_0) \rangle_{11}^{(g)} = 0$ for any $x_{-1} \in \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}^{(g)}}$ and $y_0 \in \mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{Z}^{(g)}}$. To verify that (p_{21}, p_{10}) defines an element of $\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R})$, we claim that there exists $p_{20} \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2,\mathbb{R}}^{\mathsf{F}^{(g)}})$ such that

$$\langle p_{20}(x_0), y_0 \rangle_{20}^{(g)} - \langle p_{20}(y_0), x_0 \rangle_{20}^{(g)} + \langle p_{10}(x_0), p_{10}(y_0) \rangle_{11}^{(g)} = 0$$

for any $x_0, y_0 \in \mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{Z}^{(g)}}$. Since $\mathrm{char}(\mathbb{R}) \neq 2$, the alternating pairing $\langle p_{10}(\,\cdot\,), p_{10}(\,\cdot\,) \rangle_{11}^{(g)}$ can be written as the difference between a pairing and its transpose. Hence the claim follows from the perfectness of $\langle \cdot, \cdot \rangle_{20}^{(g)} : \mathrm{Gr}_{-2,\mathbb{R}}^{\mathsf{F}^{(g)}} \times \mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}} \to \mathbb{R}(1)$. □

Corollary 3.2.10. *If we equip* $\pi_1^{-1}(h_{-1})$ *with the complex structure defined by* h_{-1} *for every* $h_{-1} \in \mathsf{X}^{\mathsf{F}^{(g)}}$*, then the torsor* $\pi_1 : \mathsf{X}_1^{\mathsf{F}^{(g)}} \to \mathsf{X}_0^{\mathsf{F}^{(g)}}$ *under* $\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R})$ *becomes a complex vector bundle over* $\mathsf{X}_0^{\mathsf{F}^{(g)}}$ *such that the translation actions of* $\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R})$ *on the fibers vary holomorphically with* h_{-1}*.*

Remark 3.2.11. Corollary 3.2.10 corresponds to the statements in [2, p. 238].

3.3. **Arithmetic quotients.** Let us define the following subquotients of $\Gamma_{\mathcal{H}}^{(g)}$:

$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} := \Gamma_{\mathcal{H}}^{(g)} \cap \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q}) = (g\mathcal{H}g^{-1}) \cap \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q})_0,$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\prime} := \Gamma_{\mathcal{H}}^{(g)} \cap \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q}) = (g\mathcal{H}g^{-1}) \cap \mathrm{P}'_{\mathtt{F}^{(g)}}(\mathbb{Q})_0,$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{Z}} := \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \cap \mathrm{Z}_{\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h,+} := \text{image of } \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \text{ under the homomorphism } \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \twoheadrightarrow \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h} := \text{image of } \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\prime} \text{ under the homomorphism } \mathrm{P}'_{\mathtt{F}^{(g)}}(\mathbb{Q}) \twoheadrightarrow \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l} := \text{image of } \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \text{ under the homomorphism } \mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \twoheadrightarrow \mathrm{G}_{l,\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l,-} := \text{image of } \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{Z}} \text{ under the homomorphism } \mathrm{Z}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \twoheadrightarrow \mathrm{G}_{l,\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} := \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \cap \mathrm{U}_{\mathtt{F}^{(g)}}(\mathbb{Q}),$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2} := \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \cap \mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{Q})$$
$$\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_1} := \text{image of } \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \text{ under the homomorphism } \mathrm{U}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \twoheadrightarrow \mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{Q})$$

Remark 3.3.1. The groups $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}}$ and $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2}$ correspond to the groups Γ_F and $U(F)_{\mathbb{Z}}$ in [2, p. 248]. However, the group $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l}$ might differ from the group $\bar{\Gamma}_F$ there by the quotient of a finite subgroup that acts trivially by conjugation on $\mathrm{U}_2(\mathbb{C})$. Such a difference is harmless in the theory of toroidal compactifications, because the admissibility of cone decompositions is only determined by the conjugation actions of the groups on $\mathrm{U}_2(\mathbb{C})$.

Lemma 3.3.2. *By definition, there are natural inclusions* $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2} \subset \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \subset \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{Z}} \subset \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \subset \Gamma_{\mathcal{H}}^{(g)}$ *and* $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \subset \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\prime} \subset \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}}$*, and natural exact sequences:*

$$1 \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{Z}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h,+} \to 1,$$
$$1 \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\prime} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h} \to 1,$$
$$1 \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\prime} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l} \to 1,$$
$$1 \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{Z}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l,-} \to 1,$$
$$1 \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}} \to \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_1} \to 1.$$

Lemma 3.3.3. *The splitting* $\varepsilon^{(g)} : \mathrm{Gr}^{\mathtt{F}^{(g)}} \xrightarrow{\sim} L^{(g)}$ *defines an isomorphism*

$$\mathrm{P}_{\mathtt{F}^{(g)}}(\mathbb{Q})/\mathrm{U}_{\mathtt{F}^{(g)}}(\mathbb{Q}) \cong \mathrm{G}_{l,\mathtt{F}^{(g)}}(\mathbb{Q}) \times \mathrm{G}_{h,\mathtt{F}^{(g)}}(\mathbb{Q})$$

mapping $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}}/\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}}$ *isomorphically to a subgroup of* $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l} \times \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h,+}$ *containing* $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l,-} \times \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h}$*. The two projections then induce isomorphisms* $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l}/\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l,-} \cong (\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)}}/\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}})/(\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},l,-} \times \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h}) \cong \Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h,+}/\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h}$*. When* $\mathcal{H} = \mathcal{U}(n)$*, we have* $\Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},l} = \Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},l,-}$ *and* $\Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},h,+} = \Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},h}$*, and hence the above mapping defines an isomorphism* $\Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)}}/\Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},\mathrm{U}} \cong \Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},l} \times \Gamma_{\mathcal{U}(n)}^{\mathtt{F}^{(g)},h}$.

Now we have the following diagram of canonical morphisms:

$$\begin{array}{ccccc}
\mathsf{X}_2^{\mathsf{F}^{(g)}} & \xrightarrow{\pi_2} & \mathsf{X}_1^{\mathsf{F}^{(g)}} & \xrightarrow{\pi_1} & \mathsf{X}_0^{\mathsf{F}^{(g)}} \\
{\scriptstyle \text{quot. by } \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2}} \downarrow & & \downarrow & & \downarrow \\
\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}} & \longrightarrow & \mathsf{X}_1^{\mathsf{F}^{(g)}} & \longrightarrow & \mathsf{X}_0^{\mathsf{F}^{(g)}} \\
{\scriptstyle \text{quot. by } \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_1}} \downarrow & & \downarrow & & \downarrow \\
\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}} & \longrightarrow & \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}} & \longrightarrow & \mathsf{X}_0^{\mathsf{F}^{(g)}} \\
{\scriptstyle \text{quot. by } \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h}} \downarrow & & \downarrow & & \downarrow \\
\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}}) & \longrightarrow & \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}}) & \longrightarrow & \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}} \\
{\scriptstyle \text{quot. by } \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},l}} \downarrow & & \downarrow & & \downarrow \\
\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}} & \longrightarrow & (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}} / \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}}) \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}}) & \longrightarrow & \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}
\end{array}$$

(For the bottom-right vertical arrow, we use the isomorphism $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},l} / \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},l,-} \cong (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}} / \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}}) / (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},l,-} \times \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h}) \cong \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} / \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h}$ in Lemma 3.3.3.)

3.4. **The morphism $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}} \to \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}$.** Let $\mathrm{Gr}_{-1} : \mathrm{P}_{\mathsf{F}^{(g)}} \to \mathrm{G}_{h,\mathsf{F}^{(g)}}$ be canonical homomorphism defined by taking Gr_{-1} (and keeping the similitude ν). Let $\mathcal{H}_{-1} := \mathrm{Gr}_{-1}((g\mathcal{H}g^{-1}) \cap \mathrm{P}_{\mathsf{F}^{(g)}}(\mathbb{A}^\infty))$, $\mathcal{H}'_{-1} := \mathrm{Gr}_{-1}((g\mathcal{H}g^{-1}) \cap \mathrm{P}'_{\mathsf{F}^{(g)}}(\mathbb{A}^\infty))$, and $\mathcal{H}''_{-1} := \mathrm{Gr}_{-1}((g\mathcal{H}g^{-1}) \cap (\mathrm{G}_{l,\mathsf{F}^{(g)}}(\mathbb{Q}) \ltimes \mathrm{P}'_{\mathsf{F}^{(g)}}(\mathbb{A}^\infty)))$ for any open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, which satisfy $\mathcal{H}'_{-1} \subset \mathcal{H}''_{-1} \subset \mathcal{H}_{-1}$. By the same arguments in §2.4, there is a holomorphic family $(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}})$ over

$$\mathsf{Sh}_{\mathcal{H}}^{\mathsf{F}^{(g)}} := \mathrm{G}_{h,\mathsf{F}^{(g)}}(\mathbb{Q}) \backslash \mathsf{X}^{\mathsf{F}^{(g)}} \times \mathrm{G}_{h,\mathsf{F}^{(g)}}(\mathbb{A}^\infty) / \mathcal{H}''_{-1}$$

defined by varying the complex structure h_{-1} on the real torus $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}^{(g)}} / \mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$ with polarization given by $\langle \cdot, \cdot \rangle_{11}$. Let us denote by $\mathsf{M}_{\mathcal{H},\mathbb{C}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ the pullback of the moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ (see Lemma 3.1.6) to $\mathbb{C}$, which is representable by a quasi-projective variety. Let us define the subvariety $\mathsf{M}_{\mathcal{H},\mathbb{C},L \otimes_{\mathbb{Z}} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ of $\mathsf{M}_{\mathcal{H},\mathbb{C}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ using the symplectic $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$-modules $(\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}, \langle \cdot, \cdot \rangle^{(g)})$, and denote by $(A_{\mathbb{C}}, \lambda_{A_{\mathbb{C}}}, i_{A_{\mathbb{C}}}, \varphi_{-1,\mathcal{H},\mathbb{C}}) \to \mathsf{M}_{\mathcal{H},\mathbb{C},L \otimes_{\mathbb{Z}} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ the pullback of the universal family. Similarly, let $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}^{(g)}}$ be the finite étale covering of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ classifying the additional structure $(\varphi_{-2,\mathcal{H}}^{(g),\sim}, \varphi_{0,\mathcal{H}}^{(g),\sim})$ inducing $(\varphi_{-2,\mathcal{H}}^{(g)}, \varphi_{0,\mathcal{H}}^{(g)})$ and $\varphi_{-1,\mathcal{H}}$ (see [23, erratum for Def. 5.4.2.6]), and denote its pullback to $\mathbb{C}$ by $\mathsf{M}_{\mathcal{H},\mathbb{C}}^{\Phi_{\mathcal{H}}^{(g)}}$. Let $\mathsf{M}_{\mathcal{H},\mathbb{C},L \otimes_{\mathbb{Z}} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}} \to \mathsf{M}_{\mathcal{H},\mathbb{C},L \otimes_{\mathbb{Z}} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ be the pullback of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}^{(g)}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$. Then there is also a tautological pair $(\varphi_{-2,\mathcal{H},\mathbb{C}}^{(g),\sim}, \varphi_{0,\mathcal{H},\mathbb{C}}^{(g),\sim})$ inducing $(\varphi_{-2,\mathcal{H}}^{(g)}, \varphi_{0,\mathcal{H}}^{(g)})$ and $\varphi_{-1,\mathcal{H},\mathbb{C}}$ over $\mathsf{M}_{\mathcal{H},\mathbb{C},L \otimes_{\mathbb{Z}} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}}$. Then, by the same argument in §2.5,

$(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}}) \to \mathsf{Sh}_{\mathcal{H}}^{\mathsf{F}^{(g)}}$ is uniquely isomorphic to the analytification of $(A_{\mathbb{C}}, \lambda_{A_{\mathbb{C}}}, i_{A_{\mathbb{C}}}, \varphi_{-1,\mathcal{H},\mathbb{C}}) \to \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$.

Since $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} = \mathcal{H}'_{-1} \cap \mathrm{G}_{h,\mathsf{F}^{(g)}}(\mathbb{Q})_0$ and $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} = \mathcal{H}''_{-1} \cap \mathrm{G}_{h,\mathsf{F}^{(g)}}(\mathbb{Q})_0$, and since $\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)},h} = \Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)},h,+} = \Gamma_{\mathcal{U}(n)}^{\mathsf{Z}^{(g)},h}$ for all $n \geq 1$ such that $\mathcal{U}(n) \subset \mathcal{H}$, the construction of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}^{(g)}}$ as the quotient of $\coprod \mathsf{M}_n^{\Phi_n^{(g)}}$ (with the disjoint union running over representatives $(\mathsf{Z}_n^{(g)}, \Phi_n^{(g)}, \delta_n^{(g)})$, with the same $(X^{(g)}, Y^{(g)}, \phi^{(g)})$, in $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$) by $\mathcal{H}/\mathcal{U}(n)$, and the construction of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ as a quotient of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}^{(g)}}$ by $\Gamma_{\Phi_{\mathcal{H}}^{(g)}} \cong \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},l}$, show that

$$\mathsf{Sh}_{\mathcal{H},0}^{\Phi^{(g)}} := \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}} \to \mathsf{Sh}_{\mathcal{H},0}^{\mathsf{F}^{(g)}} := \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}$$

is the pullback of the analytification $\mathsf{M}_{\mathcal{H},\mathrm{an},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}} \to \mathsf{M}_{\mathcal{H},\mathrm{an},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ of $\mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}} \to \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$ under $\mathsf{Sh}_{\mathcal{H},0}^{\mathsf{F}^{(g)}} \hookrightarrow \mathsf{Sh}_{\mathcal{H}}^{\mathsf{F}^{(g)}} \cong \mathsf{M}_{\mathcal{H},\mathrm{an},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$. Let us (abusively) denote the pullback of the holomorphic family over $\mathsf{Sh}_{\mathcal{H}}^{\mathsf{F}^{(g)}}$ by (the same notation)

$$(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}}) \to \mathsf{Sh}_{\mathcal{H},0}^{\mathsf{F}^{(g)}} = \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}},$$

and also by

$$(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}}) \to \mathsf{X}_0^{\mathsf{F}^{(g)}}$$

the further pullback to $\mathsf{X}_0^{\mathsf{F}^{(g)}}$. Let us denote by

$$(\varphi_{-2,\mathcal{H},\mathrm{hol}}^{(g),\sim}, \varphi_{0,\mathcal{H},\mathrm{hol}}^{(g),\sim}) \to \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}$$

the pullback of $(\varphi_{-2,\mathcal{H},\mathbb{C}}^{(g),\sim}, \varphi_{0,\mathcal{H},\mathbb{C}}^{(g),\sim}) \to \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}}$. By construction, over each $h_{-1} \in \mathsf{X}_0^{\mathsf{F}^{(g)}}$, the pullback $(\varphi_{-2,\mathcal{H},h_{-1},g}^{(g),\sim}, \varphi_{0,\mathcal{H},h_{-1},g}^{(g),\sim})$ of $(\varphi_{-2,\mathcal{H},\mathrm{hol}}^{(g),\sim}, \varphi_{0,\mathcal{H},\mathrm{hol}}^{(g),\sim})$ is (up to isomorphism) the $\mathcal{H}$-orbit of the canonical tuple $((\varphi_{-2}^{(g)}, \varphi_0^{(g)}), \varphi_{-1,h_{-1},g})$ above the $\mathcal{H}$-orbit $\varphi_{-1,\mathcal{H},h_{-1},g}$ of $\varphi_{-1,h_{-1},g}$. For later references, let us define $\mathsf{Sh}_{\mathcal{H},0,\mathrm{alg}}^{\Phi^{(g)}}$ (resp. $\mathsf{Sh}_{\mathcal{H},0,\mathrm{alg}}^{\mathsf{F}^{(g)}}$) to be the connected component of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\Phi^{(g)}} \cong \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\Phi_{\mathcal{H}}^{(g)}}$ (resp. $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}^{\mathsf{F}^{(g)}} \cong \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_{\mathcal{H}}^{(g)}}$) whose analytification is $\mathsf{Sh}_{\mathcal{H},0}^{\Phi^{(g)}} = \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}$ (resp. $\mathsf{Sh}_{\mathcal{H},0}^{\mathsf{F}^{(g)}} = \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},h,+} \backslash \mathsf{X}_0^{\mathsf{F}^{(g)}}$).

Lemma 3.4.1. *The fiber* $(A_{h_{-1},g}, \lambda_{A_{h_{-1},g}}, i_{A_{h_{-1},g}}, \varphi_{-1,\mathcal{H},h_{-1},g})$ *over* $h_{-1} \in \mathsf{X}_0^{\mathsf{F}^{(g)}}$ *of* $(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}}) \to \mathsf{X}_0^{\mathsf{F}^{(g)}}$ *can be described (up to isomorphism) as follows:*

(1) $A_{h_{-1},g}$ *is the complex torus* $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}} / \mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$ *with complex structure given by* h_{-1}.

(2) $\lambda_{A_{h_{-1},g}} : A_{h_{-1},g} \to A_{h_{-1},g}^{\vee}$ *is the homomorphism* $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}} / \mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}} \to \mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}} / (\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}})^{\#}$*, where* $(\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}})^{\#}$ *is the dual lattice of* $\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$ *with respect to the pairing* $\langle \cdot, \cdot \rangle_{11}^{(g)}$. *(We are using* $\mathrm{sgn}(h_{-1}) = \mathrm{sgn}(h_0) = 1$ *here.)*

(3) $i_{A_{h_{-1},g}} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{C}}(A_{h_{-1},g})$ *is the* $\mathcal{O}$*-endomorphism structure of* $(A_{h_{-1},g}, \lambda_{A_{h_{-1},g}})$ *induced by the* $\mathcal{O}$*-lattice structure of* $\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$.

(4) $\varphi_{-1,\mathcal{H},h_{-1},g}$ *is the* $\mathcal{H}$*-orbit of the canonical isomorphism* $\varphi_{-1,h_{-1},g}$: $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\hat{\mathbb{Z}}} \xrightarrow{\sim} \mathrm{T}\, A_{h_{-1},g}$ *matching* $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11}$ *with the* $\lambda_{A_{h_{-1},g}}$*-Weil pairing of* $A_{h_{-1},g}$. (*See Remark* 2.2.1.)

Remark 3.4.2. The fiber-wise description in Lemma 3.4.1 determines the holomorphic family $(A_{\mathrm{hol}}, \lambda_{A_{\mathrm{hol}}}, i_{A_{\mathrm{hol}}}, \varphi_{-1,\mathcal{H},\mathrm{hol}}) \to \Gamma^{\mathtt{F}^{(g)},h,+}_{\mathcal{H}} \backslash \mathsf{X}^{\mathtt{F}^{(g)}}_0$ uniquely (up to isomorphism). Similarly, the fiber-wise description in the paragraph preceding Lemma 3.4.1 determines $(\varphi^{(g),\sim}_{-2,\mathcal{H},\mathrm{hol}}, \varphi^{(g),\sim}_{0,\mathcal{H},\mathrm{hol}}) \to \Gamma^{\mathtt{F}^{(g)},h}_{\mathcal{H}} \backslash \mathsf{X}^{\mathtt{F}^{(g)}}_0$.

3.5. **The morphism** $\Gamma^{\mathtt{F}^{(g)},h}_{\mathcal{H}} \backslash (\Gamma^{\mathtt{F}^{(g)},\mathrm{U}_1}_{\mathcal{H}} \backslash \mathsf{X}^{\mathtt{F}^{(g)}}_1) \to \Gamma^{\mathtt{F}^{(g)},h}_{\mathcal{H}} \backslash \mathsf{X}^{\mathtt{F}^{(g)}}_0$**.** For each $h \in \mathsf{X}_0$, the isomorphisms (3.1.3) and (3.1.4) define a holomorphic exact sequence

$$0 \to \mathtt{F}^{(g)}_{-2,\mathbb{C}} \to L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \to 0, \tag{3.5.1}$$

where $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ and $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}$ are equipped with the complex structures h and h_{-1}, respectively. If we form the quotient of $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ by $\mathtt{F}^{(g)}_{-1}$, we obtain a holomorphic exact sequence

$$0 \to T_g(\mathbb{C}) := \mathrm{Hom}_{\mathbb{Z}}(X^{(g)}, \mathbb{C}^\times) \to G^\natural_{h,g} := (L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/\mathtt{F}^{(g)}_{-1} \to A_{h_{-1},g} \to 0. \tag{3.5.2}$$

Similarly, if we form the quotient of $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ by $(\mathtt{F}^{(g)})^{\#}_{-1} := \mathtt{F}^{(g)}_{-1,\mathbb{Q}} \cap (L^{(g)})^{\#}$, we obtain

$$0 \to T^\vee_g(\mathbb{C}) := \mathrm{Hom}_{\mathbb{Z}}(Y^{(g)}, \mathbb{C}^\times) \to G^{\vee,\natural}_{h,g} := (L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/(\mathtt{F}^{(g)})^{\#}_{-1} \to A^\vee_{h_{-1},g} \to 0. \tag{3.5.3}$$

Lemma 3.5.4. *The holomorphic extensions* (3.5.2) *and* (3.5.3) *are canonically algebraizable.*

Proof. We shall prove the statement for (3.5.2) because the same argument works for (3.5.3).

Given any algebraic character χ of T_g, let us denote by $G^\natural_{h,g,\chi}$ the push-out of $G^\natural_{h,g}$ by $-\chi : T_g(\mathbb{C}) \to \mathbb{C}^\times$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_g(\mathbb{C}) & \longrightarrow & G^\natural_{h,g} & \longrightarrow & A_{h_{-1},g} & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle -\chi} & & \Big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & G^\natural_{h,g,\chi} & \longrightarrow & A_{h_{-1},g} & \longrightarrow & 0 \end{array}$$

Then $G^\natural_{h,g,\chi}$ is algebraic over $A_{h_{-1},g}$, because the cocycles defining it as a $\mathbf{G}_{\mathrm{m}}$-torsor also define a line bundle, and line bundles over abelian varieties are uniquely algebraizable by [30, §12, Thm. 3]. In other words, it is a semi-abelian variety.

Let $\{\chi_i\}_{1\leq i\leq r}$ be any $\mathbb{Z}$-basis of $X^{(g)} \cong X(T_g)$. Then there is a canonical isomorphism $G^\natural_{h,g} \cong \prod\limits_{1\leq i\leq r} G^\natural_{h,g,\chi_i}$ (fiber product over $A_{h_{-1},g}$) showing that $G^\natural_{h,g}$ is also algebraizable. This algebraic structure is independent of the choice of the basis $\{\chi_i\}_{1\leq i\leq r}$, because for any different $\mathbb{Z}$-basis $\{\chi'_i\}_{1\leq i\leq r}$, the change of bases induces an algebraic isomorphism $\prod\limits_{1\leq i\leq r} G^\natural_{h,g,\chi_i} \xrightarrow{\sim} \prod\limits_{1\leq i\leq r} G^\natural_{h,g,\chi'_i}$. □

Lemma 3.5.5. *For any $h_{-1} \in \mathsf{X}_0^{\mathsf{F}^{(g)}}$, and any integer $n \geq 1$, there exist canonical isomorphisms*
(3.5.6)
$$\mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2,\mathbb{R}})/n\,\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}X^{(g)}, A^{\vee}_{h_{-1},g})^{\circ}$$
and
(3.5.7)
$$\mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{R}}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}})/n\,\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}Y^{(g)}, A_{h_{-1},g})^{\circ}.$$

(*Here the right-hand sides of the homomorphisms are the fiber-wise geometric identity components of* $\mathrm{Hom}_{\mathcal{O}}(\frac{1}{n}X^{(g)}, A^{\vee}_{h_{-1},g})$ *and* $\mathrm{Hom}_{\mathcal{O}}(\frac{1}{n}Y^{(g)}, A_{h_{-1},g})$, *respectively, as in* [23, Prop. 5.2.3.8].)

Proof. Suppose $f \in \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2,\mathbb{R}})$. Since $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{20}$ and $\langle\,\cdot\,,\,\cdot\,\rangle^{(g)}_{11}$ are nondegenerate, there exists a unique element ${}^t f \in \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{R}}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}})$ such that
$$\langle f(x_{-1}), y_0\rangle^{(g)}_{20} = \langle x_{-1},\, {}^t f(y_0)\rangle^{(g)}_{11},$$
for any $x_{-1} \in \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}}$ and $y_0 \in \mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{R}}$. Let us identify $X^{(g)}$ as a subgroup of $Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{Q}}$ by $\phi^{(g)} : Y^{(g)} \hookrightarrow X^{(g)}$. Then, for each $\frac{1}{n}\chi \in \frac{1}{n}X^{(g)}$, the quotient of $\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C}$ by $\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1}$ defined by sending $l_{-1} \in \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1}$ to the holomorphic map
$$\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C} \to \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C} :$$
$$(x, w) \mapsto (x + l_{-1},\ w\,\mathbf{e}(\langle f(l_{-1}), \tfrac{1}{n}\chi\rangle^{(g)}_{20})) = (x + l_{-1},\ w\,\mathbf{e}(\langle l_{-1},\ {}^t f(\tfrac{1}{n}\chi)\rangle^{(g)}_{11}))$$
corresponds (by the theory and conventions set up in §2.2) to the point of $A^{\vee}_{h_{-1},g}$ represented by ${}^t f(\frac{1}{n}\chi)$. If f lies in $n\,\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2})$, then $\langle f(l_{-1}), \frac{1}{n}\chi\rangle^{(g)}_{20} \in \mathbb{Z}(1)$ and hence $\mathbf{e}(\langle f(l_{-1}), \frac{1}{n}\chi\rangle^{(g)}_{20}) = 1$ for any $\frac{1}{n}\chi \in \frac{1}{n}X^{(g)}$. Thus the assignment $f \mapsto (\frac{1}{n}\chi \mapsto {}^t f(\chi))$ induces a well-defined homomorphism (3.5.6).

Suppose $f \in \mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{R}}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}})$. For each $\frac{1}{n}y \in \frac{1}{n}Y^{(g)}$, the quotient of $\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C}$ by $(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})^{\#}$ defined by sending $l_{-1} \in (\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})^{\#}$ to the holomorphic map
$$\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C} \to \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C} : (x, w) \mapsto (x + l_{-1},\ w\,\mathbf{e}(\langle l_{-1}, f(\tfrac{1}{n}y)\rangle^{(g)}_{11}))$$
corresponds to the point of $A_{h_{-1},g} \cong (A^{\vee}_{h_{-1},g})^{\vee}$ represented by $f(\frac{1}{n}y)$. If f lies in $n\,\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0}, \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})$, then $\mathbf{e}(\langle l_{-1}, f(\frac{1}{n}y)\rangle^{(g)}_{11}) = 1$ for any $y \in Y^{(g)}$. Thus the assignment $f \mapsto (\frac{1}{n}y \mapsto f(\frac{1}{n}y))$ induces a well-defined homomorphism (3.5.7).

It is straightforward to verify that these two homomorphisms (3.5.6) and (3.5.7) are isomorphisms. □

Lemma 3.5.8. *For any $x_{-1} \in \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}}$ and $y_0 \in \mathrm{Gr}^{\mathsf{F}^{(g)}}_{0,\mathbb{R}}$, we have* $\langle x_{-1}, y_0\rangle^{(g)}_{10} = \langle \varepsilon^{(g)}(0, x_{-1}, 0),\ \varepsilon^{(g)}(0, 0, y_0)\rangle^{(g)} = \langle \varepsilon^{(g)}_{21}(x_{-1}), y_0\rangle^{(g)}_{20} + \langle x_{-1}, \varepsilon^{(g)}_{10}(y_0)\rangle^{(g)}_{11}$.

Lemma 3.5.9. *Suppose $h \in \mathsf{X}_0$ and $\pi(h) = h_{-1}$.*

(1) *If* $n = 1$, *then the two isomorphisms* (3.5.6) *and* (3.5.7) *send* $-\varepsilon_{21}^{(g)}(h)$ *and* $\varepsilon_{10}^{(g)}(h)$ *to the extension classes* $c_{h,g}$ *and* $c_{h,g}^{\vee}$ *of* $G_{h,g}^{\natural}$ *in* (3.5.2) *and of* $G_{h,g}^{\vee,\natural}$ *in* (3.5.2), *respectively.*

(2) *If* $n \geq 1$, *let* $c_{n,h,g} : \frac{1}{n}X^{(g)} \to A_{h_{-1},g}^{\vee}$ *and* $c_{n,h,g}^{\vee} : \frac{1}{n}Y^{(g)} \to A_{h_{-1},g}$ *denote the image of* $-\varepsilon_{21}^{(g)}(h)$ *and* $\varepsilon_{10}^{(g)}(h)$ *under* (3.5.6) *and* (3.5.7), *respectively. Let* $\varphi_{-1,n,h_{-1},g} : \frac{1}{n}\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}} / \operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}} \to A_{h_{-1},g}[n]$ *denote the canonical isomorphism, and let* $\tilde{\varphi}_{-1,n,h_{-1},g} : \frac{1}{n}\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}} \to A_{h_{-1},g}[n]$ *denote the composition of the canonical morphism* $\frac{1}{n}\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}} \to \frac{1}{n}\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}} / \operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ *with* $\varphi_{-1,n,h_{-1},g}$. *Let* $\phi_n^{(g)} : \frac{1}{n}Y^{(g)} \hookrightarrow \frac{1}{n}X^{(g)}$ *be the homomorphism canonically determined by* $\phi^{(g)}$. *Then these data satisfy the relation*

$$\text{(3.5.10)} \quad \mathrm{e}_{A_{h_{-1},g}[n]}(\tilde{\varphi}_{-1,n,h_{-1},g}(\tfrac{1}{n}l_{-1}), (\lambda_{A_{h_{-1},g}} c_{n,h,g}^{\vee} - c_{n,h,g}\phi_n^{(g)})(\tfrac{1}{n}y)) = \mathbf{e}(n\langle \tfrac{1}{n}l_{-1}, \tfrac{1}{n}y\rangle_{10})$$

for every $\frac{1}{n}l_{-1} \in \frac{1}{n}\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ *and every* $\frac{1}{n}y \in \frac{1}{n}Y^{(g)}$.

Proof. Since the isomorphisms (3.1.3) and (3.1.4) are compatible with the quotients by $\mathtt{F}_{-2}^{(g)}$, we have a holomorphic isomorphism $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/\mathtt{F}_{-2}^{(g)} \cong \operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g(\mathbb{C})$. The quotient $G_{h,g}^{\natural} = ((L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/\mathtt{F}_{-2}^{(g)})/\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ is then isomorphic to the quotient of $\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g(\mathbb{C})$ by $\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ defined by sending $l_{-1} \in \operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ to the holomorphic map

$$\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g(\mathbb{C}) \to \operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g(\mathbb{C}) : (x, w) \mapsto (x + l_{-1},\ w + \varepsilon_{21}^{(g)}(h)(l_{-1})).$$

If we push-out $T_g(\mathbb{C})$ by $-\chi : T_g(\mathbb{C}) \to \mathbb{C}^{\times}$, we obtain the quotient of $\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times}$ by $\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ defined by sending $l_{-1} \in \operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}}$ to the holomorphic map

$$\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times} \to \operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times} : (x, w) \mapsto (x + l_{-1},\ w\,\mathbf{e}(-\langle \varepsilon_{21}^{(g)}(h)(l_{-1}), \chi\rangle_{20}^{(g)})).$$

This shows that (3.5.6) maps $-\varepsilon_{21}^{(g)}(h)$ to the extension class of $G_{h,g}^{\natural}$ in (3.5.2).

Similarly, the quotient $G_{h,g}^{\vee,\natural} = ((L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/(\mathtt{F}_{-2,\mathbb{Q}}^{(g)} \cap (L^{(g)})^{\#}))/(\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ is isomorphic to the quotient of $\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g^{\vee}(\mathbb{C})$ by $(\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ defined by sending $l_{-1} \in (\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ to the holomorphic map

$$\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g^{\vee}(\mathbb{C}) \to \operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times T_g^{\vee}(\mathbb{C}) : (x, w) \mapsto (x + l_{-1},\ w + \varepsilon_{21}^{(g)}(h)(l_{-1})).$$

If we push-out $T_g^{\vee}(\mathbb{C})$ by $-y : T_g^{\vee}(\mathbb{C}) \to \mathbb{C}^{\times}$, we obtain the quotient of $\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times}$ by $(\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ defined by sending $l_{-1} \in (\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ to the holomorphic map

$$\operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times} \to \operatorname{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathbb{C}^{\times} : (x, w) \mapsto (x + l_{-1},\ w\,\mathbf{e}(-\langle \varepsilon_{21}^{(g)}(h)(l_{-1}), y\rangle_{20}^{(g)})).$$

By Lemma 3.5.8, $\langle \varepsilon_{21}^{(g)}(h)(l_{-1}),\, y\rangle_{20}^{(g)} + \langle l_{-1},\, \varepsilon_{10}^{(g)}(h)(y)\rangle_{11}^{(g)} = \langle l_{-1}, y\rangle_{10}^{(g)} \in \mathbb{Z}(1)$ for $l_{-1} \in (\operatorname{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ and $y \in Y^{(g)} = \operatorname{Gr}_0^{\mathtt{F}^{(g)}}$. Hence $\mathbf{e}(-\langle \varepsilon_{21}^{(g)}(h)(l_{-1}),\, y\rangle_{20}^{(g)}) = \mathbf{e}(\langle l_{-1},\, \varepsilon_{10}^{(g)}(h)y\rangle_{11}^{(g)})$. This shows that (3.5.7) maps $\varepsilon_{10}^{(g)}(h)$ to the extension class of $G_h^{\vee,\natural}$ in (3.5.3).

Since $(\lambda_{A_{h_{-1},g}} c^\vee_{n,h,g} - c_{n,h,g}\phi^{(g)}_n)(\frac{1}{n}y) \in A_{h_{-1},g}[n]$ can be represented by the point $\varepsilon^{(g)}_{10}(h)(\frac{1}{n}y) - (-{}^t(\varepsilon^{(g)}_{21}(h))(\frac{1}{n}y))$ in $\frac{1}{n}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})^\#$, we have the relation

$$\begin{aligned}&\mathrm{e}_{A_{h_{-1},g}[n]}(\tilde{\varphi}_{-1,n,h_{-1},g}(\tfrac{1}{n}l_{-1}),\,(\lambda_{A_{h_{-1},g}} c^\vee_{n,h,g} - c_{n,h,g}\phi^{(g)}_n)(\tfrac{1}{n}y))\\&\quad= \mathbf{e}(n(\langle \tfrac{1}{n}l_{-1},\,\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y)\rangle^{(g)}_{11} + \langle \varepsilon^{(g)}_{21}(h)(\tfrac{1}{n}l_{-1}),\,\tfrac{1}{n}y\rangle^{(g)}_{20})) = \mathbf{e}(n\langle \tfrac{1}{n}l_{-1},\tfrac{1}{n}y\rangle^{(g)}_{10})\end{aligned}$$

by using Lemma 3.5.8 again, which is the relation (3.5.10). □

By definition, $\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{U}(1)} = \mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{Z})$ is equal to

$$\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R}) \cap (\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1},\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2}) \times \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0},\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})).$$

For any integer $n \geq 1$, $\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{U}(n)}$ is equal to

$$\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R}) \cap n(\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1},\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-2}) \times \mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathsf{F}^{(g)}}_{0},\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})).$$

Therefore, over each $h_{-1} \in \mathsf{X}^{\mathsf{F}^{(g)}}_0$, the fiber of the morphism $\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{U}(n)} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_1 \to \mathsf{X}^{\mathsf{F}^{(g)}}_0$ is canonically identified with an analytic subset of

$$\mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}Y^{(g)}, A_{h_{-1},g})^\circ \times \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}X^{(g)}, A^\vee_{h_{-1},g})^\circ.$$

This subset is closed by compactness of $\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{R})/\mathrm{U}_{1,\mathsf{F}^{(g)}}(\mathbb{Z})$, and hence is a subvariety by Chow's theorem [24, p. 33]. Moreover, by Lemma 3.5.9, this subvariety lies in some connected component of the fiber product

$$\mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}Y^{(g)}, A_{h_{-1},g})^\circ \underset{\mathrm{Hom}_{\mathcal{O}}(\frac{1}{n}Y^{(g)}, A^\vee_{h_{-1},g})^\circ}{\times} \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}X^{(g)}, A^\vee_{h_{-1},g})^\circ$$

defined by the structural morphisms induced by $\phi^{(g)}_n : \frac{1}{n}Y^{(g)} \hookrightarrow \frac{1}{n}X^{(g)}$ and $\lambda_{A_{h_{-1},g}} : A_{h_{-1},g} \to A^\vee_{h_{-1},g}$, as in [23, §6.2.3]. By comparing dimensions, this subvariety is exactly the connected component. In particular, it varies holomorphically with h_{-1}. If $n \geq 3$, then it descends under quotient by $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{U}(n)}$ to an algebraic subfamily of the algebraic family of fiber products of abelian varieties. Let us summarize this as follows:

Lemma 3.5.11. *For any integer $n \geq 3$, consider the fiber product*

$$\ddot{C}_{\Phi^{(g)}_n} := \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}Y^{(g)}, A)^\circ \underset{\mathrm{Hom}_{\mathcal{O}}(\frac{1}{n}Y^{(g)}, A^\vee)^\circ}{\times} \mathrm{Hom}_{\mathcal{O}}(\tfrac{1}{n}X^{(g)}, A^\vee)^\circ$$

of abelian schemes over $\mathsf{M}^{\mathsf{Z}^{(g)}_n}_n$, and let $C_{\Phi^{(g)}_n,\delta^{(g)}_n}$ be the subscheme of $\ddot{C}_{\Phi^{(g)}_n}$ defined in [23, §6.2.3]. *By abuse of notation, let us denote by $C_{\Phi^{(g)}_n,\delta^{(g)}_n,\mathbb{C}}$ the pullback of $C_{\Phi^{(g)}_n,\delta^{(g)}_n} \to \mathsf{M}^{\mathsf{Z}^{(g)}_n}_n$ under $\mathsf{Sh}^{\mathsf{F}^{(g)}}_{\mathcal{U}(n),0,\mathrm{alg}} \hookrightarrow \mathsf{M}^{\mathsf{Z}^{(g)}_n}_{n,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \to \mathsf{M}^{\mathsf{Z}^{(g)}_n}_n$. Then the analytic morphism $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{U}(n)} \backslash (\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{U}(n)} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_1) \to \Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{U}(n)} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_0$ can be canonically identified as the analytification of $C_{\Phi^{(g)}_n,\delta^{(g)}_n,\mathbb{C}} \to \mathsf{Sh}^{\mathsf{F}^{(g)}}_{\mathcal{U}(n),0,\mathrm{alg}}$ (matching the relation* (3.5.10) *with the tautological relation over $C_{\Phi^{(g)}_n,\delta^{(g)}_n,\mathbb{C}}$; cf.* [23, Thm. 5.2.3.13 and §6.2.3]). *If we denote by $(c_{n,\mathrm{hol}}, c^\vee_{n,\mathrm{hol}})$ the tautological pair of homomorphisms on $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{U}(n)} \backslash (\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{U}(n)} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_1)$ whose pullback to a point $((\varepsilon^{(g)}_{21}(h), \varepsilon^{(g)}_{10}(h)),\, h_{-1})$ of $\mathsf{X}^{\mathsf{F}^{(g)}}_1$ is* *the pair* $(c_{n,h,g}, c^\vee_{n,h,g})$ *we have constructed, then* $(c_{n,\mathrm{hol}}, c^\vee_{n,\mathrm{hol}})$ *is identified with the analytification of the pullback of the tautological pair* $(c_n, c^\vee_n)$ *over* $C_{\Phi^{(g)}_n, \delta^{(g)}_n}$.

For a general neat open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, we have a tautological pair $(c_{\mathcal{H},\mathrm{hol}}, c^\vee_{\mathcal{H},\mathrm{hol}})$ over $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}} \backslash (\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{H}} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_1)$ defined by étale descent from data defined at principal levels $\mathcal{U}(n)$ as above. The precise definitions can be stated in the same way as in [23, §5.3.1]. Since the action of $\Gamma^{(g)}_{\mathcal{H}} / \Gamma^{(g)}_{\mathcal{U}(n)} \xrightarrow{\sim} \mathcal{H}/\mathcal{U}(n)$ is compatible on the analytic and algebraic sides because they are determined by their actions on the level structures, we obtain:

Corollary 3.5.12. *For any neat open compact subgroup* $\mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}})$*, let us define* $C_{\Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}}, \mathbb{C}} \to \mathsf{Sh}^{\Phi^{(g)}}_{\mathcal{H},\mathrm{alg}}$ *as in* [23, §6.2.4; see also the errata] *and in Lemma* 3.5.11 *above. Then the analytic morphism* $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}} \backslash (\Gamma^{\mathsf{F}^{(g)},\mathrm{U}_1}_{\mathcal{H}} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_1) \to \Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_0$ *can be canonically identified as the analytification of* $C_{\Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}}, \mathbb{C}} \to \mathsf{Sh}^{\Phi^{(g)}}_{\mathcal{H},0,\mathrm{alg}}$*. Under this identification, the tautological pair* $(c_{\mathcal{H},\mathrm{hol}}, c^\vee_{\mathcal{H},\mathrm{hol}})$ *is identified with the analytification of the pullback of the tautological pair* $(c_{\mathcal{H}}, c^\vee_{\mathcal{H}})$ *over* $C_{\Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}}}$.

3.6. **The morphism** $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}} \backslash (\Gamma^{\mathsf{F}^{(g)},\mathrm{U}}_{\mathcal{H}} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_2) \to \Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}} \backslash \mathsf{X}^{\mathsf{F}^{(g)}}_0$**.** For each $h \in \mathsf{X}_0$, consider the semi-abelian variety $G^\natural_{h,g} = (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) / \mathsf{F}^{(g)}_{-1}$. The abelian variety $G_{h,g} = (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) / L^{(g)}$ is by definition the quotient of $G^\natural_{h,g}$ by $Y^{(g)} = \mathrm{Gr}^{\mathsf{F}^{(g)}}_0 = L^{(g)} / \mathsf{F}^{(g)}_{-1}$. Therefore the essential datum is the ($\mathcal{O}$-equivariant) *period homomorphism* $\iota_{h,g} : Y^{(g)} \to G^\natural_{h,g}$.

Lemma 3.6.1. *The composition of* $\iota_{h,g}$ *with the canonical homomorphism* $G^\natural_{h,g} \to A^\vee_{h_{-1},g}$ *coincides with the homomorphism* $c^\vee_{h,g} : Y^{(g)} \to A^\vee_{h_{-1},g}$.

Proof. Since the homomorphism $\iota_{h,g} : Y^{(g)} \to G^\natural_{h,g} = (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) / \mathsf{F}^{(g)}_{-1}$ can be induced by $y \mapsto \varepsilon^{(g)}(0,0,y)$, its composition with $G^\natural_{h,g} \to A^\vee_{h_{-1},g} = \mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1,\mathbb{R}} / (\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1})^\#$ can be induced by $y \mapsto \varepsilon^{(g)}_{10}(y)$. Then the lemma follows from Lemma 3.5.9. □

For any $y \in Y^{(g)}$ and $\chi \in X^{(g)}$, the image of $\iota_{h,g}(y)$ under the push-out $G^\natural_{h,g} \to G^\natural_{h,g,\chi}$ by $-\chi : T_g(\mathbb{C}) \to \mathbb{C}^\times$ defines a point of the fiber of $G^\natural_{h,g,\chi}$ over $c^\vee_{h,g}(y) \in A_{h_{-1},g}$. Under the identification of the $\mathbf{G}_{\mathrm{m}}$-torsor $G^\natural_{h,g,\chi}$ with the point $c_{h,g}(\chi)$ of $A^\vee_{h_{-1},g}$, this point determines (and is determined by) a section $\tau_{h,g}(y,\chi)$ of $(c^\vee_{h,g}(y), c_{h,g}(\chi))^* \, \mathcal{P}^{\otimes -1}_{A_{h_{-1},g}}$.

Lemma 3.6.2. *The collection of sections* $\{\tau_{h,g}(y,\chi)\}_{y \in Y^{(g)}, \chi \in X^{(g)}}$ *is bimultiplicative and determines a trivialization of* $\mathbf{G}_{\mathrm{m}}$*-biextensions*

$$\tau_{h,g} : \mathbf{1}_{Y^{(g)} \times X^{(g)}} \xrightarrow{\sim} (c^\vee_{h,g} \times c_{h,g})^* \, \mathcal{P}^{\otimes -1}_{A_{h_{-1},g}}.$$

Proof. The linearity in $y \in Y^{(g)}$ follows from the linearity of $\iota_{h,g}$. The linearity in $\chi \in X^{(g)}$ (and compatibility with linearity in $Y^{(g)}$) follows from the very definition of push-outs, and from the way we define tensor products of $\mathbf{G}_{\mathrm{m}}$-torsors. □

Lemma 3.6.3. *For any* $((\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)),\ h_{-1}) \in \mathsf{X}_1^{\mathsf{F}^{(g)}}$*, and any integer* $n \geq 1$*, there exist a canonical isomorphism*

$$\begin{aligned}(3.6.4)\quad & \mathrm{Hom}_{\mathbb{R}}(\mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2,\mathbb{C}}^{\mathsf{F}^{(g)}})/n\,\mathrm{Hom}(\mathrm{Gr}_{0}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2}^{\mathsf{F}^{(g)}}) \\ & \qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{G}_{\mathrm{m}}\text{-BIEXT}}(\mathbf{1}_{\frac{1}{n}Y^{(g)}\times X^{(g)}}, (c_{n,h,g}^{\vee}\times c_{h,g})^{*}\,\mathcal{P}_{A_{h_{-1},g}}^{\otimes -1})\end{aligned}$$

sending the subgroup $\mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2,\mathbb{C}}^{\mathsf{F}^{(g)}})/n\,\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}_{0}^{\mathsf{F}^{(g)}}, \mathrm{Gr}_{-2}^{\mathsf{F}^{(g)}})$ *to the trivializations of* $\mathbf{G}_{\mathrm{m}}$*-biextensions annihilated by* $(b\times \mathrm{Id}_{X^{(g)}})^{*}-(\mathrm{Id}_{\frac{1}{n}Y^{(g)}}\times b^{\star})^{*}$ *for every* $b\in\mathcal{O}$*.*

Proof. Let us realize $\mathbf{1}_{\frac{1}{n}Y^{(g)}\times X^{(g)}}$ as $Y^{(g)}\times X^{(g)}\times\mathbb{C}^{\times}$, and realize the $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{P}_{A_{h_{-1},g}}$ (rather than as a line bundle) as the quotient of $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}$ by the action of $\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}\times(\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}})^{\#}$ defined by sending $(l_1,l_2)\in\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}\times(\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}})^{\#}$ to the holomorphic map

$$\begin{aligned}&\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}\to\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}:\\ &(x,y,w)\mapsto(x+l_1,\ y+l_2,\ w\,\mathbf{e}(\tfrac{1}{2}H_{h_{-1},g}(l_1,l_2)+\tfrac{1}{2}H_{h_{-1},g}(l_1,y)+\tfrac{1}{2}H_{h_{-1},g}(l_2,x))),\end{aligned}$$

where $H_{h_{-1},g}:\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\to\mathbb{C}$ is defined by $\langle\,\cdot\,,\,\cdot\,\rangle_{11}^{(g)}$ and h_{-1} as in Lemma 2.1.2.

Proceeding as in the proof of Lemma 3.5.9, we can realize $G_{h,g,\chi}^{\natural}$ as the quotient of $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}$ by $\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$ defined by sending $l\in\mathrm{Gr}_{-1}^{\mathsf{F}^{(g)}}$ to the holomorphic map

$$\begin{aligned}&\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}\to\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}:\\ &\qquad(x,w)\mapsto(x+l,\ w\,\mathbf{e}(-\langle\varepsilon_{21}^{(g)}(h)(l),\chi\rangle_{20}^{(g)}))=(x+l,\ w\,\mathbf{e}(\langle l,-{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi)\rangle_{20}^{(g)})).\end{aligned}$$

This corresponds to the point of $A_{h_{-1},g}^{\vee}$ represented by $-{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi)$. However, as explained in §2.2, this realization differs from the fiber of the above realization of $\mathcal{P}_{A_{h_{-1},g}}$ at the points of $A_{h_{-1},g}\times A_{h_{-1},g}^{\vee}$ by the holomorphic change of coordinates

$$\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}\to\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}:(x,w)\mapsto(x,w\,\mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(-{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi),x))).$$

Suppose we have any $f\in\mathrm{Hom}_{\mathbb{R}}(\mathrm{Gr}_{0,\mathbb{R}}^{\mathsf{F}^{(g)}},\mathrm{Gr}_{-2,\mathbb{C}}^{\mathsf{F}^{(g)}})$. Then the bimultiplicative morphism

$$\begin{aligned}&\tfrac{1}{n}Y^{(g)}\times X^{(g)}\times\mathbb{C}^{\times}\to\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{F}^{(g)}}\times\mathbb{C}^{\times}:\\ &\qquad(\tfrac{1}{n}y,\chi,w)\mapsto(\varepsilon_{10}^{(g)}(h)(\tfrac{1}{n}y),\ -{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi),\\ &\qquad\qquad w\,\mathbf{e}(\langle f(\tfrac{1}{n}y),\chi\rangle_{20}^{(g)})\,\mathbf{e}(\tfrac{1}{2}H_{h_{-1},g}(-{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi),\varepsilon_{10}^{(g)}(h)(\tfrac{1}{n}y))))\end{aligned}$$

induces a morphism

$$\mathbf{1}_{\frac{1}{n}Y^{(g)}\times X^{(g)}}\to(c_{n,h,g}^{\vee}\times c_{h,g})^{*}\,\mathcal{P}_{A_{h_{-1},g}}^{\otimes -1}$$

of $\mathbf{G}_{\mathrm{m}}$-biextensions (which is then automatically an isomorphism).

If f lies in $n\,\mathrm{Hom}(\mathrm{Gr}_{0}^{\mathsf{F}^{(g)}},\mathrm{Gr}_{-2}^{\mathsf{F}^{(g)}})$, then $\langle f(\frac{1}{n}y),\chi\rangle_{20}^{(g)}$ lies in $\mathbb{Z}(1)$, and hence $\mathbf{e}(\langle f(y),\chi\rangle_{20}^{(g)}=1$ for any $y\in\frac{1}{n}Y^{(g)}$ and $\chi\in X^{(g)}$. Thus the assignment above is well-defined and induces the desired homomorphism (3.6.4). It is straightforward to verify that the homomorphism (3.6.4) is an isomorphisms.

If f lies in $\mathrm{Hom}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{R}},\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,\mathbb{R}})$, then the relation

$$\langle f(b\tfrac{1}{n}y),\chi\rangle^{(g)}_{20}=\langle bf(\tfrac{1}{n}y),\chi\rangle^{(g)}_{20}=\langle f(\tfrac{1}{n}y),b^{\star}\chi\rangle^{(g)}_{20}$$

shows that the trivialization of $\mathbf{G}_{\mathrm{m}}$-biextensions defined by f is annihilated by $(b\times\mathrm{Id}_{X^{(g)}})^{*}-(\mathrm{Id}_{\frac{1}{n}Y^{(g)}}\times b^{\star})^{*}$ for every $b\in\mathcal{O}$, as desired. □

Lemma 3.6.5. *For any $h\in\mathsf{X}_0$ and any $x_0,y_0\in\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,\mathbb{R}}$, we have*

$$\begin{aligned}\langle x_0,y_0\rangle^{(g)}_{00}&=\langle\varepsilon^{(g)}(0,0,x_0),\varepsilon^{(g)}(0,0,y_0)\rangle^{(g)}\\&=\langle\varepsilon^{(g)}_{20}(h)(x_0),y_0\rangle^{(g)}_{20}-\langle\varepsilon^{(g)}_{20}(h)(y_0),x_0\rangle^{(g)}_{20}+\langle\varepsilon^{(g)}_{10}(x_0),\varepsilon^{(g)}_{10}(y_0)\rangle^{(g)}_{11}.\end{aligned}$$

Proof. Simply substitute $\varepsilon^{(g)}(0,0,x_0)=x_1+h(\sqrt{-1})x_2+x_3$ and $\varepsilon^{(g)}(0,0,y_0)=y_1+h(\sqrt{-1})y_2+y_3$ for some $x_1,x_2,y_1,y_2\in\mathtt{F}^{(g)}_{-2,\mathbb{R}}$ and $x_3,y_3\in(\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^{\perp}$. □

Lemma 3.6.6. *Suppose $h\in\mathsf{X}_0$ is mapped to $(\varepsilon^{(g)}_{20}(h),\ (\varepsilon^{(g)}_{21}(h),\varepsilon^{(g)}_{10}(h)),\ h_{-1})$ in $\mathsf{X}^{\mathtt{F}^{(g)}}_{2}$.*

(1) *If $n=1$, then the isomorphism* (3.6.4) *sends $\varepsilon^{(g)}_{20}(h)$ to the trivialization $\tau_{h,g}$ defined in Lemma* 3.6.2.

(2) *If $n\geq1$, let $\tau_{n,h,g}:\mathbf{1}_{\frac{1}{n}Y^{(g)}\times X^{(g)}}\xrightarrow{\sim}(c^{\vee}_{n,h,g}\times c_{h,g})^{*}\mathcal{P}^{\otimes-1}_{A_{h_{-1},g}}$ denote the image of $\varepsilon^{(g)}_{20}(h)$ under* (3.6.4). *Then $\tau_{n,h,g}$ satisfies the relation*

$$\tau_{n,h,g}(\tfrac{1}{n}y,\phi^{(g)}(y'))\,\tau_{n,h,g}(\tfrac{1}{n}y',\phi^{(g)}(y))^{-1}=\mathbf{e}(n\langle\tfrac{1}{n}y,\tfrac{1}{n}y'\rangle^{(g)}_{00})\tag{3.6.7}$$

for every $\frac{1}{n}y,\frac{1}{n}y'\in\frac{1}{n}Y^{(g)}$, where the left-hand side denotes the image of $\tau_{n,h,g}(\frac{1}{n}y,\phi^{(g)}(y'))\otimes\tau_{n,h,g}(\frac{1}{n}y',\phi^{(g)}(y))^{-1}$ under the canonical morphism

$$(c^{\vee}_{n,h,g}(\tfrac{1}{n}y),c_{h,g}(\phi^{(g)}(y')))^{*}\mathcal{P}^{\otimes-1}_{A_{h_{-1},g}}\otimes(c^{\vee}_{n,h,g}(\tfrac{1}{n}y'),c_{h,g}(\phi^{(g)}(y)))^{*}\mathcal{P}_{A_{h_{-1},g}}\xrightarrow{\sim}\mathbb{C}^{\times},$$

induced by the canonical symmetry of

$$(c^{\vee}_{n,h,g}\times(c_{h,g}\phi^{(g)}))^{*}\mathcal{P}^{\otimes-1}_{A_{h_{-1},g}}\cong(c^{\vee}_{n,h,g},c^{\vee}_{n,h,g})^{*}(\mathrm{Id}_{A_{h_{-1},g}}\times\lambda_{A_{h_{-1},g}})^{*}\mathcal{P}^{\otimes-n}_{A_{h_{-1},g}}.$$

Proof. The construction of $\tau_{h,g}$ assigns to any $y\in Y^{(g)}$ the point of $G^{\natural}_{h,g,\chi}$ represented by $(\varepsilon^{(g)}_{10}(y),\ \mathbf{e}(\langle\varepsilon^{(g)}_{20}(h)(y),\chi\rangle^{(g)}_{20}))\in\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}\times T_g(\mathbb{C})$. As in the proof of Lemma 3.6.3, this corresponds to the section $\tau_{h,g}(y,\chi)$ of $(c^{\vee}_{h,g}(y)\times c_{h,g}(\chi))^{*}\mathcal{P}^{\otimes-1}_{A_{h_{-1},g}}$ represented by the point

$$\begin{aligned}&(\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y),\ -{}^{t}(\varepsilon^{(g)}_{21}(h))(\chi),\\&\qquad\mathbf{e}(\langle\varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y),\chi\rangle^{(g)}_{20})\,\mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(-{}^{t}(\varepsilon^{(g)}_{21}(h))(\chi),\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y))))\end{aligned}$$

of $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}\times\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}}\times\mathbb{C}^{\times}$. Therefore, the trivialization $\tau_{h,g}$ determined by the sections $\{\tau_{h,g}(y,\chi)\}_{y\in Y^{(g)},\chi\in X^{(g)}}$ agrees with image of $\varepsilon^{(g)}_{20}(h)$ under (3.6.4).

To verify the relation (3.6.7), let us first replace the representative

$$\begin{aligned}&(\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y),\ -{}^{t}(\varepsilon^{(g)}_{21}(h))(y'),\\&\qquad\mathbf{e}(\langle\varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y),y'\rangle^{(g)}_{20})\,\mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(-{}^{t}(\varepsilon^{(g)}_{21}(h))(y'),\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y))))\end{aligned}$$

of the section $\tau_{n,h,g}(\frac{1}{n}y, \phi^{(g)}(y'))$ of $(c^\vee_{n,h,g}(\frac{1}{n}y) \times c_{h,g}(\phi^{(g)}(y')))^* \mathcal{P}^{\otimes -1}_{A_{h_{-1}},g}$ with

$$(\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y),\ \varepsilon^{(g)}_{10}(h)(y'),$$
$$\mathbf{e}(\langle \varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y), y'\rangle^{(g)}_{20})\, \mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(\varepsilon^{(g)}_{10}(h)(y'), \varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y)))).$$

Then, by switching y and y' in the above expression, we obtain a representative of the section $\tau_{n,h,g}(\frac{1}{n}y', \phi^{(g)}(y))$ of $(c^\vee_{n,h,g}(\frac{1}{n}y') \times c_{h,g}(\phi^{(g)}(y)))^* \mathcal{P}^{\otimes -1}_{A_{h_{-1}},g}$. With these realizations, the canonical isomorphism

$$(c^\vee_{n,h,g}(\tfrac{1}{n}y), c_{h,g}(\phi^{(g)}(y')))^* \mathcal{P}^{\otimes -1}_{A_{h_{-1}},g} \xrightarrow{\sim} (c^\vee_{n,h,g}(\tfrac{1}{n}y'), c_{h,g}(\phi^{(g)}(y)))^* \mathcal{P}^{\otimes -1}_{A_{h_{-1}},g},$$

induced by the canonical symmetry of $(c^\vee_{n,h,g}, c^\vee_{n,h,g})^*(\mathrm{Id}_{A_{h_{-1}},g} \times \lambda_{A_{h_{-1}},g})^* \mathcal{P}^{\otimes -n}_{A_{h_{-1}},g}$ is simply

$$(\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y), \varepsilon^{(g)}_{10}(h)(y'), w) \mapsto (\varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y'), \varepsilon^{(g)}_{10}(h)(y), w)$$

for every $w \in \mathbb{C}^\times$. Therefore, the relation (3.6.7) is reduced to the verification that

$$\begin{aligned}
&[\, \mathbf{e}(\langle \varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y), y'\rangle^{(g)}_{20})\, \mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(\varepsilon^{(g)}_{10}(h)(y'), \varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y)))\,] \\
&\quad [\, \mathbf{e}(\langle \varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y'), y\rangle^{(g)}_{20})\, \mathbf{e}(-\tfrac{1}{2}H_{h_{-1},g}(\varepsilon^{(g)}_{10}(h)(y), \varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y')))\,]^{-1} \\
&= \mathbf{e}(\langle \varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y), y'\rangle^{(g)}_{20} - \langle \varepsilon^{(g)}_{20}(h)(\tfrac{1}{n}y'), y\rangle^{(g)}_{20} + n\langle \varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y), \varepsilon^{(g)}_{10}(h)(\tfrac{1}{n}y')\rangle^{(g)}_{11}) \\
&= \mathbf{e}(n\langle \tfrac{1}{n}y, \tfrac{1}{n}y'\rangle^{(g)}_{00}),
\end{aligned}$$

which is valid by Lemma 3.6.5. □

Lemma 3.6.8. *For any $\mathbb{Z}$-algebra R, the elements p_{20} of the group $\mathrm{U}_{2,\mathtt{F}^{(g)}}(R)$ (as a subset of $\mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,R}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,R})$) corresponds canonically to pairings*

$$p^*_{20} : (Y^{(g)} \underset{\mathbb{Z}}{\otimes} R) \times (X^{(g)} \underset{\mathbb{Z}}{\otimes} R) \to R(1)$$

satisfying the condition that $p^*_{20}(y, \phi^{(g)}(y')) = p^*_{20}(y', \phi^{(g)}(y))$ and $p^*_{20}(by, \chi) = p^*_{20}(y, b^\star\chi)$ for any $y, y' \in Y^{(g)} \underset{\mathbb{Z}}{\otimes} R$, $\chi \in X^{(g)} \underset{\mathbb{Z}}{\otimes} R$, and $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Proof. Suppose we have $p_{20} \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{0,R}, \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-2,R})$. Let $p \in \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} R)$ be the element such that $(\varepsilon^{(g)})^{-1} \circ p \circ \varepsilon^{(g)} = \begin{pmatrix} 1 & & p_{20} \\ & 1 & \\ & & 1 \end{pmatrix}$. Let us define $p^*_{20}(y, \chi) := \chi(p_{20}(y))$. In order for p to preserve the pairing $\langle \cdot\,, \cdot \rangle^{(g)}$, we need to show for any $z = \varepsilon^{(g)}(z_{-2}, z_{-1}, z_0)$ and $w = \varepsilon^{(g)}(w_{-2}, w_{-1}, w_0)$ in $L^{(g)} \underset{\mathbb{Z}}{\otimes} R$ that

$$\begin{aligned}
0 = \langle pz, pw\rangle^{(g)} - \langle z, w\rangle^{(g)} &= \langle p_{20}(z_0), w_0\rangle^{(g)}_{20} - \langle p_{20}(w_0), z_0\rangle^{(g)}_{20} \\
&= p^*_{20}(z_0, \phi^{(g)}(w_0)) - p^*_{20}(w_0, \phi^{(g)}(z_0)).
\end{aligned}$$

In order for p to preserve the $\mathcal{O}$-structure, we need p_{20} to be $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-equivariant, which is equivalent to the condition that $p^*_{20}(by, x) = p^*_{20}(y, b^\star x)$, because the action of $b \in \mathcal{O}$ on $\mathtt{F}^{(g)}_{-2}$ defines the action of $b^\star \in \mathcal{O}$ on $X^{(g)} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathtt{F}^{(g)}_{-2}, \mathbb{Z}(1))$. Thus, the condition for $(p, 1)$ to define an element of $\mathrm{G}^{(g)}(R)$ is equivalent to the two conditions in Lemma 3.6.8. □

Let us define for any integer $n \geq 1$ the abelian groups

$$\ddot{\mathbf{S}}_{\Phi_n^{(g)}} := ((\tfrac{1}{n}Y^{(g)}) \underset{\mathbb{Z}}{\otimes} X^{(g)}) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (b\tfrac{1}{n}y) \otimes \chi - (\tfrac{1}{n}y) \otimes (b^\star \chi) \end{pmatrix}_{\substack{y, y' \in Y^{(g)}, \\ \chi \in X^{(g)}, b \in \mathcal{O}}} \tag{3.6.9}$$

and set $\mathbf{S}_{\Phi_n^{(g)}} := \ddot{\mathbf{S}}_{\Phi_n^{(g)}, \mathrm{free}}$, the free quotient of $\ddot{\mathbf{S}}_{\Phi_n^{(g)}}$. (See [23, (6.2.3.5) and Conv. 6.2.3.26].) For a general torus argument $\Phi_{\mathcal{H}}^{(g)} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ at level $\mathcal{H}$, there is a recipe in [23, Lem. 6.2.4.4] that gives a corresponding free abelian group $\mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}$, such that $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_n^{(g)}} / \mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}, \mathbb{Z}(1)) \cong \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_2} / \Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}_2}$ when $\mathcal{U}(n) \subset \mathcal{H}$ for some integer $n \geq 1$.

Corollary 3.6.10. *There are* (*compatible*) *canonical isomorphisms:*

$$\begin{aligned}
&\mathrm{U}_{2, \mathrm{F}^{(g)}}(\mathbb{Z}) = \Gamma_{\mathcal{U}(1)}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_1^{(g)}}, \mathbb{Z}(1)), \\
&\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_n^{(g)}}, \mathbb{Z}(1)), \text{any integer } n \geq 1, \\
&\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}, \mathbb{Z}(1)), \text{any level } \mathcal{H}, \\
&\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \backslash \mathrm{U}_{2, \mathrm{F}^{(g)}}(\mathbb{C}) \overset{\text{can.}}{\overset{\sim}{\to}} \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}, \mathbb{C}^\times) \overset{\text{can.}}{\overset{\sim}{\to}} E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C}).
\end{aligned}$$

The canonical action of $\mathrm{U}_{2, \mathrm{F}^{(g)}}(\mathbb{C})$ on $\mathsf{X}_2^{\mathsf{F}^{(g)}}$ defines a holomorphic action of $E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})$ on $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_2} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}} \to \mathsf{X}_1^{\mathsf{F}^{(g)}}$, which is transitive and faithful on each fiber, and descends to $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}}) \to \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}})$ when $\mathcal{H}$ is neat.

Lemma 3.6.11. *For any integer $n \geq 3$, let $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}}$ be the $E_{\Phi_n^{(g)}}$-torsor over $C_{\Phi_n^{(g)}, \delta_n^{(g)}}$ defined in* [23, §6.2.3]. *By abuse of notation, let us denote by $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}}$ the pullback of $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}} \to C_{\Phi_n^{(g)}, \delta_n^{(g)}} \to \mathsf{M}_n^{\mathsf{Z}_n^{(g)}}$ under $\mathsf{Sh}_{\mathcal{U}(n), 0, \mathrm{alg}}^{\mathsf{F}^{(g)}} \hookrightarrow \mathsf{M}_{n, \mathbb{C}, L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}^{\mathsf{Z}_n^{(g)}} \to \mathsf{M}_n^{\mathsf{Z}_n^{(g)}}$. Then the analytic morphism $\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}}) \to \Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}})$ can be canonically identified as the analytification of $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}} \to C_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}}$* (*matching the relation* (3.6.7) *with the tautological relation over $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}}$; cf.* [23, Thm. 5.2.3.13 and §6.2.3]), *realizing the action of $E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})$ on $\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}})$ as the analytification of the $E_{\Phi_n^{(g)}}$-torsor structure of $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}} \to C_{\Phi_n^{(g)}, \delta_n^{(g)}, \mathbb{C}}$. If we denote by $\tau_{n, \mathrm{hol}}$ the tautological datum on $\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{U}(n)}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}})$ whose pullback to a point $(\varepsilon_{20}^{(g)}(h), (\varepsilon_{21}^{(g)}(h), \varepsilon_{10}^{(g)}(h)), h_{-1})$ of $\mathsf{X}_2^{\mathsf{F}^{(g)}}$ is the datum $\tau_{n,h,g}$ we have constructed, then $\tau_{n, \mathrm{hol}}$ is identified with the analytification of the pullback of the tautological datum τ_n over $\Xi_{\Phi_n^{(g)}, \delta_n^{(g)}}$.*

As in the case of $(c_{\mathcal{H}, \mathrm{hol}}, c_{\mathcal{H}, \mathrm{hol}}^\vee)$, for a general neat open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, we have a tautological datum over $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}})$ defined by étale descent from data defined at principal levels $\mathcal{U}(n)$ as above. Then we obtain:

Corollary 3.6.12. *For any neat open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, let us define $\Xi_{\Phi_{\mathcal{H}}^{(g)}, \delta_{\mathcal{H}}^{(g)}, \mathbb{C}} \to C_{\Phi_{\mathcal{H}}^{(g)}, \delta_{\mathcal{H}}^{(g)}, \mathbb{C}}$ as in* [23, §6.2.4] *and in Lemma* 3.6.11 *above. Then the analytic morphism $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}}) \to \Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, h} \backslash (\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)}, \mathrm{U}_1} \backslash \mathsf{X}_1^{\mathsf{F}^{(g)}})$ can be canonically* *identified as the analytification of* $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}} \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}$*, realizing the action of* $E_{\Phi^{(g)}_{\mathcal{H}}}(\mathbb{C})$ *on* $\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}}\backslash(\Gamma^{\mathsf{F}^{(g)},\mathrm{U}}_{\mathcal{H}}\backslash\mathsf{X}^{\mathsf{F}^{(g)}}_2)$ *as the analytification of the* $E_{\Phi^{(g)}_{\mathcal{H}}}$*-torsor structure of* $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}} \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}$*. Under this identification, the tautological datum* $\tau_{\mathcal{H},\mathrm{hol}}$ *is identified with the analytification of the pullback of the tautological datum* $\tau_{\mathcal{H}}$ *over* $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}$*.*

3.7. **Toroidal compactifications.**

Lemma 3.7.1. *If* $h \in \mathsf{X}_0$*, then* $\mathrm{G}_{l,\mathsf{F}^{(g)}}(\mathbb{R})(\mathrm{Im}\,\varepsilon^{(g)}_{20}(h))$ *is the subset of elements* p_{20} *in* $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{R})$ *that correspond under Lemma* 3.6.8 *to pairings*

$$p^*_{20} : (Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (X^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \cong (Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}(1)$$

such that $-\sqrt{-1}\,p^*_{20}$ *is positive definite.* (*We are using* $\mathrm{sgn}(h) = \mathrm{sgn}(h_0) = 1$.)

Proof. With notations as in the proof of Lemma 3.6.5, we obtain

$$\begin{aligned} -\sqrt{-1}\,\langle \mathrm{Im}\,\varepsilon^{(g)}_{20}(h)(x_0), y_0\rangle^{(g)}_{20} &= -\sqrt{-1}\,\langle x_2, y_1 + h(\sqrt{-1})y_2 + y_3\rangle^{(g)}_{20} \\ &= -\sqrt{-1}\,\langle x_2, h(\sqrt{-1})y_2\rangle^{(g)}, \end{aligned}$$

which is symmetric and positive definite. Since $\mathrm{G}_{l,\mathsf{F}^{(g)}}(\mathbb{R})$ acts on the pairings in $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{R})$ by automorphisms of $Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, the action is transitive on positive definite pairings by the classification of real positive involutions in [21, §2]. □

Corollary 3.7.2. *For any* $\mathcal{H}$ *and any* $\mathbb{R}$*-module isomorphism* $\mathbb{R}(1) \to \mathbb{R}$ *sending* $\sqrt{-1}$ *to a positive number, the isomorphism* $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbf{S}_{\Phi^{(g)}_1}, \mathbb{R}(1)) \xrightarrow{\sim} (\mathbf{S}_{\Phi^{(g)}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ *induced by the isomorphism* $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{Z}) = \Gamma^{\mathsf{F}^{(g)},\mathrm{U}_2}_{\mathcal{U}(1)} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi^{(g)}_1}, \mathbb{Z}(1))$ *in Corollary* 3.6.10 *maps the subset* $\{\mathrm{Im}\,\varepsilon^{(g)}_{20}(h)\}_{h\in\mathsf{X}_0}$ *of* $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{R})$ *to the cone* $\mathbf{P}^+_{\Phi^{(g)}_{\mathcal{H}}}$ *of* $(\mathbf{S}_{\Phi^{(g)}_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ *defined in* [23, §6.2.5].

Corollary 3.7.3. *Any compatible choice of admissible smooth rational polyhedral cone decomposition data* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{\Phi_{\mathcal{H}}} = \{\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J_{\Phi_{\mathcal{H}}}}\}_{\Phi_{\mathcal{H}}}$ *for* $\mathsf{M}_{\mathcal{H}}$ *in the sense of* [23, Def. 6.3.3.2] *determines for each* $g \in \mathrm{G}(\mathbb{A}^\infty)$ *a* $\Gamma^{(g)}_{\mathcal{H}}$*-admissible collection* $\Sigma^{(g)} = \{\sigma_j\}_{j\in J^{(g)}}$ *for* X_0 *in the sense of* [2, p. 252]*. Consequently, the quasi-projective variety* $\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0$ *has a smooth toroidal compactification* $(\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0)^{\mathrm{tor}}_{\Sigma^{(g)}}$ *for every* $g \in \mathrm{G}(\mathbb{A}^\infty)$ *by the main results of* [2, Ch. III]*. Using the disjoint union* (2.5.2)*, we obtain a smooth toroidal compactification* $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *by forming the disjoint union of toroidal compactifications* $(\Gamma^{(g_i)}_{\mathcal{H}}\backslash\mathsf{X}_0)^{\mathrm{tor}}_{\Sigma^{(g_i)}}$*. We shall denote the corresponding algebraic space over* $\mathrm{Spec}(\mathbb{C})$ *by* $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$*.*

Remark 3.7.4. In the adelic setting, the collections of admissible cone decompositions accepted by [27] (in the pure case), by [18], and by [23], are slightly different from each other. Nevertheless, for each given collection of admissible cone decompositions (that is accepted by any one of these works), there exist a refinement that is accepted by all of these works.

Let us fix a choice of $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{\Phi_{\mathcal{H}}} = \{\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J_{\Phi_{\mathcal{H}}}}\}_{\Phi_{\mathcal{H}}}$ for $\mathsf{M}_{\mathcal{H}}$, which determines a $\Gamma^{(g)}_{\mathcal{H}}$-admissible collection $\Sigma^{(g)} = \{\sigma_j\}_{j\in J^{(g)}}$ for X_0 in Corollary 3.7.3. Let $\sigma \in \Sigma_{\Phi^{(g)}_{\mathcal{H}}}$ be a particular cone in $\mathbf{P}^+_{\Phi^{(g)}_{\mathcal{H}}}$. Then we have an identification

$$(\Gamma^{\mathsf{F}^{(g)},h}_{\mathcal{H}}\backslash(\Gamma^{\mathsf{F}^{(g)},\mathrm{U}}_{\mathcal{H}}\backslash\mathsf{X}^{\mathsf{F}^{(g)}}_2)) \overset{E_{\Phi^{(g)}_{\mathcal{H}}}(\mathbb{C})}{\times} E_{\Phi^{(g)}_{\mathcal{H}}}(\mathbb{C})(\sigma) \overset{\sim}{\to} \Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathrm{an}}(\sigma), \tag{3.7.5}$$

where $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathrm{an}}(\sigma)$ is the analytification of the pullback $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}(\sigma)$ of $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}(\sigma) \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}^{(g)}_{\mathcal{H}}}_{\mathcal{H}}$ under $\mathsf{Sh}^{\mathsf{F}^{(g)}}_{\mathcal{H},0,\mathrm{alg}} \hookrightarrow \mathsf{M}^{\mathsf{Z}^{(g)}_{\mathcal{H}}}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \to \mathsf{M}^{\mathsf{Z}^{(g)}_{\mathcal{H}}}_{\mathcal{H}}$. The stratifications by faces of σ on both sides are matched with each other, because they are defined by the same torus $E_{\Phi^{(g)}_{\mathcal{H}}}$. (This also justifies the notations with "(σ)" appearing after the subscripts "an" or "$\mathbb{C}$".) When $\mathcal{H}$ is neat, the choice of Σ (satisfying [23, Cond. 6.2.5.25 in the revision]) forces the stabilizer of $\Gamma^{\mathsf{F}^{(g)},l}_{\mathcal{H}}$ on the σ-stratum to act trivially. Therefore, quotient by $\Gamma^{\mathsf{F}^{(g)},l}_{\mathcal{H}}$ in the analytic theory induces an embedding of the σ-stratum $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathrm{an}}$ of $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathrm{an}}(\sigma)$ into $(\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0)^{\mathrm{tor}}_{\Sigma^{(g)}}$.

Let us summarize the setup as follows, each statement depending on the statements preceding it:

(1) The moduli problem $\mathsf{M}_{\mathcal{H}}$ has a smooth toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, which is an algebraic space over $\mathrm{Spec}(F_0)$, by the main results of [23, Ch. 6].

(2) Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C}}$ denote the pullback of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ under the structural morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(F_0)$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is smooth, the closure of $\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C}}$ defines a smooth toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ of $\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$.

(3) There is a stratification of $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ by locally closed sub-algebraic spaces labeled by the equivalence classes $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$, the same labels we have for $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. We do not need to know if the stratum labeled by a particular class $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ is empty or not.

(4) Every stratum of $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ is embedded as a union of connected components of the corresponding stratum of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ with the same label.

The issue is that we do not know if there is a morphism from the whole algebraic space $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ inducing the morphisms on the strata.

Remark 3.7.6. Any reader unwilling to work with algebraic spaces may assume that Σ is (not only smooth but also) *projective* (as in [23, Def. 7.3.1.3]). In this case, $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C}}$ is a smooth and projective variety over $\mathrm{Spec}(F_0)$ (see [23, Thm. 7.3.3.1]). On the other hand, the smooth toroidal compactification $(\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0)^{\mathrm{tor}}_{\Sigma^{(g)}}$ of $\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0$ is projective for every $g \in \mathrm{G}(\mathbb{A}^\infty)$ by the main results of [2, Ch. IV], and hence $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ is a projective variety over $\mathrm{Spec}(\mathbb{C})$.

4. Main comparison

Throughout the section, we retain the assumptions and notations in the previous sections. In particular, the open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\mathbb{A}^\infty)$ will always be assumed to be *neat*. (See Assumption 2.5.4.)

4.1. **Main Theorem.** Let us state our main theorem as follows:

Theorem 4.1.1. *Let $\mathcal{H}$ be a* **neat** *open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ (defining a level for our moduli problem), and let Σ be any compatible choice of admissible smooth rational polyhedral cone decomposition data $\Sigma = \{\Sigma_{\Phi_\mathcal{H}}\}_{\Phi_\mathcal{H}} = \{\Sigma_{\Phi_\mathcal{H}} = \{\sigma_j\}_{j \in J_{\Phi_\mathcal{H}}}\}_{\Phi_\mathcal{H}}$ for $\mathsf{M}_\mathcal{H}$ in the sense of* [23, Def. 6.3.3.2] *(cf. Corollary* 3.7.3*). Then there is a canonical strata-preserving isomorphism $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$ extending the canonical isomorphism $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H},\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$.*

The proof of Theorem 4.1.1 will be carried out in subsequent subsections.

Remark 4.1.2 (continuation of Remark 2.5.5). When $\mathcal{H}$ is not necessarily neat, by taking a *neat* normal subgroup $\mathcal{H}'$ of $\mathcal{H}$ such that the cone decomposition induced by Σ at level $\mathcal{H}'$ is also *smooth*, the obvious analogue of Theorem 4.1.1 at level $\mathcal{H}$ follows from the case at level $\mathcal{H}'$ by taking quotients by the finite group $\mathcal{H}/\mathcal{H}'$.

4.2. **Tautological degeneration data, Mumford families.** Let us fix a choice of $(\mathtt{V}, g, \varepsilon^{(g)})$ defining $\mathtt{F}^{(g)}$ and $(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$ as before. Let $\sigma \in \Sigma_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}}$ be a cone in $\mathbf{P}^+_{\Phi^{(g)}_\mathcal{H}}$. Then the identification (3.7.5) allows us to identify the formal completion $\mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}} := (\Xi_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \mathbb{C}}(\sigma))^\wedge_{\Xi_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}}}$ with the formal completion of $(\Gamma^{(g)}_\mathcal{H} \backslash \mathsf{X}_0)_{\Sigma^{(g)}}$ along its $[(\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma)]$-strata. Over $\mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}}$, we have the following tautological degeneration data:

1. A triple $(\mathtt{Z}^{(g)}_\mathcal{H},\ \Phi^{(g)}_\mathcal{H} = (X^{(g)}, Y^{(g)}, \phi^{(g)}, \varphi^{(g)}_{-2,\mathcal{H}}, \varphi^{(g)}_{0,\mathcal{H}}),\ \delta^{(g)}_\mathcal{H})$ representing a cusp label, defined by forming the equivalence class of the $\mathcal{H}$-orbit of $(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$.
2. The data $(A_\mathbb{C}, \lambda_{A_\mathbb{C}}, i_{A_\mathbb{C}}, \varphi_{-1,\mathcal{H},\mathbb{C}})$ on the abelian part, and the remaining data $(\varphi^{(g),\sim}_{-2,\mathcal{H}}, \varphi^{(g),\sim}_{0,\mathcal{H}})$ on the torus part inducing $(\varphi^{(g)}_{-2,\mathcal{H}}, \varphi^{(g)}_{0,\mathcal{H}})$.
3. The data $(c_{\mathcal{H},\mathbb{C}}, c^\vee_{\mathcal{H},\mathbb{C}})$.
4. The datum $\tau_{\mathcal{H},\mathbb{C}}$.

Putting these data together, we obtain by Mumford's construction (explained in [23, §6.2.5]) a degenerating family $(\,^\heartsuit G,\ ^\heartsuit\lambda,\ ^\heartsuit i,\ ^\heartsuit\alpha_\mathcal{H}) \to \mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}}$, called the *Mumford family*.

By construction of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$, the Mumford family $(\,^\heartsuit G,\ ^\heartsuit\lambda,\ ^\heartsuit i,\ ^\heartsuit\alpha_\mathcal{H}) \to \mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}}$ is the pullback of the degenerating family $(G, \lambda, i, \alpha_\mathcal{H}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ given by [23, Thm. 6.4.1.1] under the morphism $\mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}} \to \mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. On the other hand, there is the structural morphism $\mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}} \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ which identifies $\mathfrak{X}_{\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma, \mathbb{C}}$ with the completion of $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ along its $[(\Phi^{(g)}_\mathcal{H}, \delta^{(g)}_\mathcal{H}, \sigma)]$-stratum. For simplicity, let denote the previous composition by F_1, and denote the second composition by F_2. A priori, it is not clear how F_1 and F_2 should be compared.

Remark 4.2.1. Before moving on, let us clarify that this comparison is not related to the question of whether the algebraic construction of the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}$ has an analytic analogue. (This question is only related to the definition of F_1.) To relate F_2 to F_1 *at all*, we need to make use of the analytic toroidal boundary charts studied in §3, because it is the only way we can see how F_2 is defined.

Let s be any point of the σ-stratum of $\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}$, and let R_s denote the completion of the local ring of $\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}$ at s. The ring R_s is noetherian and normal because $\Xi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}(\sigma)$ is of finite type over $\mathbb{C}$ (and hence excellent). Let $K_s := \mathrm{Frac}(R_s)$. Then we obtain by pullback an object $({}^{\heartsuit}G_{K_s}, {}^{\heartsuit}\lambda_{K_s}, {}^{\heartsuit}i_{K_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(K_s)$ of $\mathsf{M}_{\mathcal{H}}(K_s)$ corresponding to a canonical morphism $\mathrm{Spec}(K_s) \to \mathsf{M}_{\mathcal{H}}$. Concretely, the pullback of the tautological degeneration data define an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ over $\mathrm{Spec}(R_s)$, and hence determines by [23, Thm. 5.3.1.17] a degenerating family $({}^{\heartsuit}G_{R_s}, {}^{\heartsuit}\lambda_{R_s}, {}^{\heartsuit}i_{R_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(R_s)$ in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ over $\mathrm{Spec}(R_s)$. Then the object $({}^{\heartsuit}G_{K_s}, {}^{\heartsuit}\lambda_{K_s}, {}^{\heartsuit}i_{K_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(K_s)$ of $\mathsf{M}_{\mathcal{H}}(K_s)$ above is isomorphic to the pullback of $({}^{\heartsuit}G_{R_s}, {}^{\heartsuit}\lambda_{R_s}, {}^{\heartsuit}i_{R_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(R_s)$ to $\mathrm{Spec}(K_s)$.

The morphism $F_1 : \mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to \mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ induces a morphism $f_1^{\mathrm{tor}} : \mathrm{Spec}(R_s) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, and hence a morphism $f_1 : \mathrm{Spec}(K_s) \to \mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$. The morphism $F_2 : \mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ induces a morphism $f_2^{\mathrm{tor}} : \mathrm{Spec}(R_s) \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$, and hence a morphism $f_2 : \mathrm{Spec}(K_s) \to \mathsf{M}_{\mathcal{H}}$ using the canonical morphism $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$.

Proposition 4.2.2. *Let* $({}^{\heartsuit}G_{K_s}, {}^{\heartsuit}\lambda_{K_s}, {}^{\heartsuit}i_{K_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(K_s)$ *be defined as above* (*which is isomorphic to the pullback under* f_1 *of the universal tuple over* $\mathsf{M}_{\mathcal{H}}$), *and let* $(G_{K_s}, \lambda_{K_s}, i_{K_s}, \alpha_{\mathcal{H},K_s}) \to \mathrm{Spec}(K_s)$ *be the pullback under* f_2 *of the universal tuple over* $\mathsf{M}_{\mathcal{H}}$. *Then there is a canonical isomorphism* $(G_{K_s}, \lambda_{K_s}, i_{K_s}, \alpha_{\mathcal{H},K_s}) \cong ({}^{\heartsuit}G_{K_s}, {}^{\heartsuit}\lambda_{K_s}, {}^{\heartsuit}i_{K_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s})$. *Consequently,* $f_1 = f_2$ *by the universal property of* $\mathsf{M}_{\mathcal{H}}$.

The proof of Proposition 4.2.2 will be completed in §4.6. Assuming Proposition 4.2.2 for now, we can prove our main theorem as follows:

Proof of Theorem 4.1.1. Let us consider the graph Δ of the isomorphism $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ as a locally closed sub-algebraic space of $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \times \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, and let Δ^{tor} denote its algebraic-space closure. We would like to show that Δ^{tor} defines an isomorphism from $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, i.e. Δ^{tor} is the graph of an isomorphism.

Let us take any cusp label $[(\Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}}, \sigma)]$ and any point s of the σ-stratum of $\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}$. Let R_s, K_s, f_1^{tor}, f_1, f_2^{tor}, and f_2 be defined as in the two paragraphs preceding Proposition 4.2.2. Under the flat morphism $f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}} : \mathrm{Spec}(R_s) \times \mathrm{Spec}(R_s) \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \times \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, the pullback $(f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}})^*(\Delta^{\mathrm{tor}})$ is the closure of $(f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}})^*(\Delta)$ in $\mathrm{Spec}(R_s) \times \mathrm{Spec}(R_s)$. Since $f_1 = f_2$ by Proposition 4.2.2, the generic point of $(f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}})^*(\Delta)$ is the image of the diagonal morphism $\mathrm{Spec}(K_s) \to \mathrm{Spec}(K_s) \times \mathrm{Spec}(K_s) \to \mathrm{Spec}(R_s) \times \mathrm{Spec}(R_s)$, and therefore $(f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}})^*(\Delta^{\mathrm{tor}})$ is the image of the diagonal morphism $\mathrm{Spec}(R_s) \to \mathrm{Spec}(R_s) \times \mathrm{Spec}(R_s)$. By flatness of $f_2^{\mathrm{tor}} \times f_1^{\mathrm{tor}}$, this shows that Δ^{tor} defines an isomorphism from an open neighborhood of the image of s in $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ to an open neighborhood of the image of s in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$. Thus the theorem follows because $[(\Phi^{(g)}_{\mathcal{H}}, \delta^{(g)}_{\mathcal{H}}, \sigma)]$ and s are arbitrary. □

To prepare for the proof of Proposition 4.2.2, let us summarize the important properties of ${}^\heartsuit G_{K_s}$ in terms of the *algebraic theta functions* given by sections of powers of the ample line bundle ${}^\heartsuit\mathcal{L}_{K_s}$. (These involve only the morphism f_1.) The reason to consider such theta functions is because they have obvious analytic analogues (which will allow us to involve the morphism f_2 as well).

Let ${}^\heartsuit\mathcal{L}_{K_s} := (\mathrm{Id}_{{}^\heartsuit G_{K_s}}, {}^\heartsuit\lambda_{K_s})^* \mathcal{P}_{{}^\heartsuit G_{K_s}}$, let $\mathcal{M}_{K_s} := (\mathrm{Id}_{A_{K_s}}, \lambda_{A_{K_s}})^* \mathcal{P}_{A_{K_s}}$, and let $\mathscr{O}_{\chi,K_s} := \mathcal{P}_{A_{K_s}}|_{A_{K_s} \times c_{\mathcal{H},K_s}(\chi)}$. According to [17, Ch. III, proof of Thm. 6.1] (see also [23, Cor. 4.5.4.24 and proof of Thm. 4.5.4.17 in the revision]), we have the following facts:

(1) There exists (non-canonically) an integer $m_s > 0$ depending on s (such that a relatively complete model for $(G_{K_s}, \mathcal{L}_{K_s}^{\otimes m_s})$ exists, and) such that, for any integer $m \geq 0$ divisible by m_s, $\Gamma({}^\heartsuit G_{K_s}, {}^\heartsuit\mathcal{L}_{K_s}^{\otimes m})$ is canonically isomorphic to the K_s-subspace V_m of $\prod\limits_{\chi \in X^{(g)}} \Gamma(A_{K_s}, \mathcal{M}_{K_s}^{\otimes m} \underset{\mathscr{O}_{A_{K_s}}}{\otimes} \mathscr{O}_{\chi,K_s})$ described by
$$V_m := \left\{ \begin{array}{l} (\theta_\chi)_{\chi \in X^{(g)}} : \theta_\chi \in \Gamma(A_{K_s}, \mathcal{M}_{K_s}^{\otimes m} \underset{\mathscr{O}_{A_{K_s}}}{\otimes} \mathscr{O}_{\chi,K_s}), \\ \forall y,\ \theta_{\chi + 2m\phi^{(g)}(y)} = \tau(y, \chi + m\phi^{(g)}(y))\, T^*_{c^\vee(y)}\, \theta_\chi \end{array} \right\}.$$
In what follows, let us fix the choice of such an m_s.

(2) The canonical morphisms
$$(4.2.3)\quad \Gamma(A_{K_s}, \mathcal{M}_{K_s}^{\otimes m} \underset{\mathscr{O}_{A_{K_s}}}{\otimes} \mathscr{O}_{\chi,K_s}) \otimes \Gamma(A_{K_s}, \mathcal{M}_{K_s}^{\otimes m'} \underset{\mathscr{O}_{A_{K_s}}}{\otimes} \mathscr{O}_{\chi',K_s}) \to \Gamma(A_{K_s}, \mathcal{M}_{K_s}^{\otimes (m+m')} \underset{\mathscr{O}_{A_{K_s}}}{\otimes} \mathscr{O}_{\chi+\chi',K_s})$$
induce K_s-module morphisms
$$(4.2.4)\quad V_m \otimes V_{m'} \to V_{m+m'} : \quad (\theta_\chi)_{\chi \in X^{(g)}} \otimes (\theta'_\chi)_{\chi \in X^{(g)}} \mapsto \Big(\sum_{\chi' \in X^{(g)}} \theta_{\chi+\chi'} \theta'_{\chi-\chi'} \Big)_{\chi \in X^{(g)}}$$
for integers $m, m' \geq 0$ divisible by m_s, where the infinite sum over $X^{(g)}$ makes sense because (by Mumford's construction) for each $(\theta_\chi)_{\chi \in X^{(g)}} \otimes (\theta'_\chi)_{\chi \in X^{(g)}}$ there is a finitely generated R_s-submodule of $V_{m+m'}$ containing all the entries $\theta_{\chi+\chi'}\theta'_{\chi-\chi'}$, in which $\sum\limits_{\chi' \in X^{(g)}} \theta_{\chi+\chi'}\theta'_{\chi-\chi'}$ form a convergent sequence with respect to the ideal of definition of R_s.

(3) The above isomorphisms are compatible and define a K_s-algebra isomorphism $\underset{m \geq 0, m_s | m}{\oplus} \Gamma({}^\heartsuit G_{K_s}, {}^\heartsuit\mathcal{L}_{K_s}^{\otimes m}) \overset{\sim}{\to} \underset{m \geq 0, m_s | m}{\oplus} V_m$.

(4) Since ${}^\heartsuit G_{K_s} \cong \mathrm{Proj}\Big(\underset{m \geq 0, m_s | m}{\oplus} \Gamma({}^\heartsuit G_{K_s}, {}^\heartsuit\mathcal{L}_{K_s}^{\otimes m}) \Big)$, the K_s-algebra $\underset{m \geq 0, m_s | m}{\oplus} V_m$ determines the isomorphism class of $({}^\heartsuit G_{K_s}, {}^\heartsuit\lambda_{K_s})$.

We will show (in Proposition 4.6.7 below) that the K_s-algebra $\underset{m\geq 0, m_s|m}{\oplus} V_m$ can be compared with the K_s-algebra $\underset{m\geq 0, m_s|m}{\oplus} \Gamma(G_{K_s}, \mathcal{L}_{K_s}^{\otimes m})$ (which involves only the morphism f_2). (This will allow us to show that $f_1 = f_2$.) As mentioned in Remark 4.2.1, this requires the detailed description of the analytic toroidal boundary charts in §3. It is irrelevant whether f_1 (or F_1, or the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}}^{(g)}, \delta_{\mathcal{H}}^{(g)}, \sigma, \mathbb{C}}$) has an analytic construction or not.

4.3. **Classical theta functions.** By Lemmas 2.4.2, there exists a maximal totally isotropic submodule $L_{\mathrm{MI}}^{(g)}$ of $L^{(g)}$ containing $\mathtt{F}_{-2}^{(g)}$. Let $\{e_i\}_{1\leq i\leq r}$ be a $\mathbb{Z}$-basis of $\mathtt{F}_{-2}^{(g)}$. Let $\{e_i\}_{1\leq i\leq d}$ be a $\mathbb{Z}$-basis of $L_{\mathrm{MI}}^{(g)}$ extending the basis $\{e_i\}_{1\leq i\leq r}$ of $\mathtt{F}_{-2}^{(g)}$. Let $\{f_i\}_{1\leq i\leq d}$ be the elements of $L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ given by Lemma 2.4.3. Then the images of $\{f_i\}_{1\leq i\leq d}$ in $(L^{(g)}/L_{\mathrm{MI}}^{(g)}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ form a $\mathbb{Z}$-basis of $(L^{(g)})^{\#}/(L^{(g)})_{\mathrm{MI}}^{\#}$ dual to $\{e_i\}_{1\leq i\leq d}$, and the image of $\{f_i\}_{1\leq i\leq r}$ in $\mathrm{Gr}_{0,\mathbb{Q}}^{\mathtt{F}^{(g)}} = Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ form a $\mathbb{Z}$-basis of $X^{(g)}$ (identified as a submodule of $Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as usual) dual to $\{e_i\}_{1\leq i\leq r}$. Moreover, $(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}} := L_{\mathrm{MI}}/\mathtt{F}_{-2}^{(g)}$ is a maximal totally isotropic submodule of $\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}$, and the images $\{\bar{e}_i\}_{r<i\leq d}$ of $\{e_i\}_{r<i\leq d}$ in $\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}$ form a basis of $L_{\mathrm{MI}}/\mathtt{F}_{-2}^{(g)}$. The elements $\{f_i\}_{r<i\leq d}$ of $L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ lie in $\mathtt{F}_{-1,\mathbb{Q}}^{(g)} = (\mathtt{F}_{-2,\mathbb{Q}}^{(g)})^{\perp}$ because $\langle e_i, f_j\rangle = 0$ for $1 \leq i \leq r$ and $r < j \leq d$. The images $\{\bar{f}_i\}_{r<i\leq d}$ of $\{f_i\}_{r<i\leq d}$ in $\mathrm{Gr}_{-1,\mathbb{Q}}^{\mathtt{F}^{(g)}}$ satisfy $\langle \bar{e}_i, \bar{f}_j\rangle_{11}^{(g)} = 2\pi\sqrt{-1}\,\delta_{ij}$ and $\langle f_i, f_j\rangle_{11}^{(g)} = 0$ for $1 \leq i, j \leq d$. We shall fix the choices of these $\{e_i\}_{1\leq i\leq d}$ and $\{f_i\}_{1\leq i\leq d}$. As in §2.6, this allows us to study the line bundles with sections varying holomorphically with h.

At any $h \in \mathsf{X}_0$, $G_{h,g}$ (resp. $\mathcal{L}_{h,g}$) is realized canonically as a three-step quotient $((V_h/\mathtt{F}_{-2}^{(g)})/\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})/\mathrm{Gr}_0^{\mathtt{F}^{(g)}}$ (resp. $(((V_h \times \mathbb{C})/\mathtt{F}_{-2}^{(g)})/\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})/\mathrm{Gr}_0^{\mathtt{F}^{(g)}}$). For any integer $m \geq 0$, the line bundle $\mathcal{L}_{h,g}^{\otimes m}$ is canonically realized as the quotient of $V_h \times \mathbb{C}$ by the action of $L^{(g)}$ defined by sending $l \in L^{(g)}$ to the holomorphic map

$$V_h \times \mathbb{C} \to V_h \times \mathbb{C} :$$
$$(x, w) \mapsto (x + l,\ w\,\mathbf{e}(-\tfrac{1}{2}m(B_{h,g} - H_{h,g})(l,l) - m(B_{h,g} - H_{h,g})(l,x))).$$

Suppose we have a section of $\Gamma(G_{h,g}, \mathcal{L}_{h,g}^{\otimes m})$ represented by some function $f : V_h \to \mathbb{C}$. Then we have

$$f(x+l) = f(x)\,\mathbf{e}(-\tfrac{1}{2}m(B_{h,g} - H_{h,g})(l,l) - m(B_{h,g} - H_{h,g})(l,x)). \tag{4.3.1}$$

Since $\mathtt{F}_{-2}^{(g)} \subset L_{\mathrm{MI}}^{(g)}$, we have $(B_{h,g} - H_{h,g})(l,x) = 0$ for any $l \in \mathtt{F}_{-2}^{(g)}$ and $x \in V_h$ by definition. This shows that $f(x)$ is periodic in $\mathtt{F}_{-2}^{(g)}$.

Lemma 4.3.2. *For any $l \in L^{(g)}$ and any $x \in \mathtt{F}_{-2,h(\mathbb{C})}^{(g)}$, we have*

$$\mathbf{e}(\tfrac{1}{2}(B_{h,g} - H_{h,g})(l,x)) = \mathbf{e}(\langle x, \bar{l}\rangle_{20}),$$

where $\bar{l}$ is the image of l in $\mathrm{Gr}_0^{\mathtt{F}^{(g)}}$.

Proof. Since $(B_{h,g} - H_{h,g})(l,x)$ is $\mathbb{C}$-linear in x, we may assume that $x \in \mathtt{F}_{-2,\mathbb{R}}^{(g)}$. Let us write $l = l_1 + h(\sqrt{-1})l_2 + l_3$ for $l_1, l_2 \in \mathtt{F}_{-2,\mathbb{R}}^{(g)}$ and

$l_3 \in (\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp$. Then we have $H_{h,g}(l,x) = \langle l,x\rangle^{(g)} - \sqrt{-1}\,\langle l,h(\sqrt{-1})x\rangle^{(g)} = \langle h(\sqrt{-1})l_2,x\rangle^{(g)} - \sqrt{-1}\,\langle l_1,h(\sqrt{-1})x\rangle^{(g)}$ and $B_{h,g}(l,x) = H_{h,g}(l^c,x) = -\langle h(\sqrt{-1})l_2,x\rangle^{(g)} - \sqrt{-1}\,\langle l_1,h(\sqrt{-1})x\rangle^{(g)}$, so that $\frac{1}{2}(B_{h,g} - H_{h,g})(l,x) = \langle x,h(\sqrt{-1})l_2\rangle^{(g)} = \langle x,l\rangle^{(g)} = \langle x,\bar{l}\rangle^{(g)}_{20}$, as desired. □

Definition 4.3.3. *For each $y \in Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, we define the function*

$$\mathbf{e}_{h,g,y} : V_h \to \mathbb{C} : x \mapsto \mathbf{e}(\tfrac{1}{2}(B_{h,g} - H_{h,g})(\varepsilon^{(g)}(0,0,y),x)).$$

For each $\chi \in X^{(g)}$, we define $\mathbf{e}_{h,g,\chi}$ by viewing χ as an element of $Y^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ using the homomorphism $\phi^{(g)} : Y^{(g)} \hookrightarrow X^{(g)}$ with finite cokernel.

Corollary 4.3.4. *Every holomorphic function $f : V_h \to \mathbb{C}$ that is periodic in $\mathtt{F}^{(g)}_{-2}$ can be written uniquely as*

$$f(x) = \sum_{\chi \in X^{(g)}} \mathbf{e}_{h,g,\chi}(x)\, f_\chi(x) \tag{4.3.5}$$

for some holomorphic functions $f_\chi : V_h \to \mathbb{C}$. The value of the functions $f_\chi(x)$ depend only on the image x_{-1} of x in $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1} \cong V_h/\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}$. (Therefore we shall sometimes write $f_\chi(x_{-1})$ instead of $f_\chi(x)$.)

4.4. **Quasi-periodicity in $\varepsilon^{(g)}(\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1})$.** Suppose $l_{-1} \in \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}$. Let $l := \varepsilon^{(g)}(0,l_{-1},0)$. Then $l = l_1 + l_3$ for some $l_1 \in \mathtt{F}^{(g)}_{-2,\mathbb{R}}$ and $l_3 \in (\mathtt{F}^{(g)}_{-2,h(\mathbb{C})})^\perp$, which shows that $l^c = l_1 + l_3^c$ because $\mathtt{F}^{(g)}_{-2,\mathbb{R}} \subset L_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$. Therefore, $(B_{h,g} - H_{h,g})(l,x) = H_{h,g}(l^c - l,x) = H_{h_{-1},g}(l^c_{-1} - l_{-1},x_{-1}) = (B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},x_{-1})$, where x_{-1} is the image of x in $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1} \cong V_h/\mathtt{F}^{(g)}_{-2,h(\mathbb{C})}$. Thus, by comparing the coefficients of $\mathbf{e}_{h,g,\chi}(x)$ in (4.3.1), we obtain

$$\begin{aligned} f_\chi(x_{-1}+l_{-1})\,\mathbf{e}_{h,g,\chi}(l) &= f_\chi(x_{-1}) \\ &\quad \mathbf{e}(-\tfrac{1}{2}m(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},l_{-1}) - m(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},x_{-1})). \end{aligned} \tag{4.4.1}$$

The line bundle $\mathcal{M}_{h_{-1},g} := (\mathrm{Id}_{A_{h_{-1},g}}, \lambda_{A_{h_{-1},g}})^*\mathcal{P}_{A_{h_{-1},g}}$ is isomorphic to the quotient of $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C}$ by the action of $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1}$ defined by sending $l_{-1} \in \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{Q}}$ to the holomorphic map $\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C} \to \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \times \mathbb{C}$ defined by

$$\begin{aligned} (x_{-1},w) \mapsto (x_{-1}+l,\ & \\ & w\,\mathbf{e}(-\tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},l_{-1}) - (B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},x_{-1}))), \end{aligned}$$

where $B_{h_{-1},g} : \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \times \mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1,\mathbb{R}} \to \mathbb{C}$ is the symmetric $\mathbb{C}$-bilinear pairing such that $B_{h_{-1},g}(x_{-1},y_{-1}) = H_{h_{-1},g}(x_{-1},y_{-1})$ for any $x_{-1},y_{-1} \in (\mathrm{Gr}^{\mathtt{F}^{(g)}}_{-1})_{\mathrm{MI}}$. (We introduce $B_{h_{-1},g}$ so that the line bundle $\mathcal{M}_{h_{-1},g}$ and its sections vary holomorphically with h_{-1}.) Thus, the factor

$$\mathbf{e}(-\tfrac{1}{2}m(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},l_{-1}) - m(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},x_{-1}))$$

defines sections of the line bundle $\mathcal{M}^{\otimes m}_{h_{-1},g}$. However, this differs from the factor in (4.4.1) by the inverse of the factor $\mathbf{e}_{h,g,\chi}(l)$. Let us clarify this factor.

Lemma 4.4.2. *With settings as above, we have*

$$\mathbf{e}_{h,g,\chi}(l) = \mathbf{e}(\tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1}, \varepsilon_{10}^{(g)}(h)(\chi))).$$

Proof. Since $B_{h,g}$ is symmetric, we have

$$\begin{aligned}&\tfrac{1}{2}(B_{h,g} - H_{h,g})(\varepsilon^{(g)}(0,0,\chi), l)\\ &\quad= \tfrac{1}{2}(B_{h,g} - H_{h,g})(l,\ \varepsilon^{(g)}(0,0,\chi)) + \langle l, \varepsilon^{(g)}(0,0,\chi)\rangle^{(g)}\\ &\quad= \tfrac{1}{2}H_{h_{-1},g}(l_{-1}^{c} - l_{-1},\ \varepsilon_{10}^{(g)}(h)(\chi)) + \langle l_{-1}, \chi\rangle_{10}^{(g)}\\ &\quad= \tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1},\ \varepsilon_{10}^{(g)}(h)(\chi)) + \langle l_{-1}, \chi\rangle_{10}^{(g)}.\end{aligned}$$

Now the lemma follows because $\langle l_{-1}, \chi\rangle_{10}^{(g)} \in \mathbb{Z}(1)$. □

This shows that the inverse of the factor $\mathbf{e}_{h,g,\chi}(l)$ defines the line bundle corresponding to the point on $A_{h_{-1},g}^{\vee}$ represented by $\varepsilon_{10}^{(g)}(h)(\chi)$.

Corollary 4.4.3. *For any function $\zeta : \mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}} \to \mathbb{C}$ and any point $a \in \mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}}$, define a function $T_a^*\,\zeta : \mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}} \to \mathbb{C}$ by $(T_a^*\,\zeta)(x) := \zeta(x+a)$. Then f_χ satisfies the relation* (4.4.1) *if and only if $T^*_{\frac{1}{m}\varepsilon_{10}^{(g)}(h)(\chi)}\, f_\chi$ represents a section of $\Gamma(A_{h_{-1},g}, \mathcal{M}_{h_{-1},g}^{\otimes m})$.*

Lemma 4.4.4. $\varepsilon_{10}^{(g)}(h)(\chi) + {}^t(\varepsilon_{21}^{(g)}(h))(\chi) \in (\mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}})^{\#}$.

Proof. By Lemma 3.5.8, we have for any $l_{-1} \in \mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}}$

$$\begin{aligned}\langle l_{-1},\ \varepsilon_{10}^{(g)}(h)(\chi) + {}^t(\varepsilon_{21}^{(g)}(h))(\chi)\rangle_{11}^{(g)} &= \langle l_{-1}, \varepsilon_{10}^{(g)}(h)(\chi)\rangle_{11}^{(g)} + \langle \varepsilon_{21}^{(g)}(h)(l_{-1}), \chi\rangle_{11}^{(g)}\\ &= \langle l_{-1}, \chi\rangle_{10}^{(g)} \in \mathbb{Z}(1).\end{aligned}$$ □

Corollary 4.4.5. *The points $\varepsilon_{10}^{(g)}(h)(\chi)$ and $-{}^t(\varepsilon_{21}^{(g)}(h))(\chi)$ of $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}}$ define the same point in $A_{h_{-1},g}^{\vee} = \mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}}/(\mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}})^{\#}$.*

Corollary 4.4.6. *Let $\mathscr{O}_{\chi,h_{-1},g} := \mathcal{P}_{A_{h_{-1},g}}|_{A_{h_{-1},g}\times c_{h,g}(\chi)}$. Let $f_\chi : \mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{F}^{(g)}} \to \mathbb{C}$ be a function satisfying the relation* (4.4.1)*. Then f_χ represents a section of $\Gamma(A_{h_{-1},g}, \mathcal{M}_{h_{-1},g}^{\otimes m} \underset{\mathscr{O}_{A_{h_{-1},g}}}{\otimes} \mathscr{O}_{\chi,h_{-1},g})$ if we represent the point $c_{h,g}(\chi)$ of $A_{h_{-1},g}^{\vee}$ by $\varepsilon_{10}^{(g)}(h)(\chi)$ (instead of $-{}^t(\varepsilon_{21}^{(g)}(h))(\chi)$, which also defines the correct datum $c_{h,g,\mathcal{H}}$).*

Remark 4.4.7. If we represent the point $c_{h,g}(\chi)$ of $A_{h_{-1},g}^{\vee}$ by $-{}^t(\varepsilon_{21}^{(g)}(h))(\chi)$, then we need to modify $f_\chi(x_{-1})$ by the factor

$$\mathbf{e}(-\tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(\varepsilon_{10}^{(g)}(h)(\chi) + {}^t(\varepsilon_{21}^{(g)}(h))(\chi), x_{-1})).$$

We shall refrain from doing so, because later we will need to work with $\varepsilon_{10}^{(g)}(h)(\chi)$ anyway when we study the action of $\mathrm{Gr}_0^{\mathrm{F}^{(g)}}$.

For any $\bar{l}_{-1} \in (\mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}})^{\#}/(\mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}})^{\#}_{\mathrm{MI}}$, consider the group homomorphism

$$\xi_{\bar{l}_{-1}} : (\mathrm{Gr}_{-1}^{\mathrm{F}^{(g)}})_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{C} \to \mathbb{C}^{\times} : x_{-1} \mapsto \mathbf{e}(\tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(l_{-1}, x_{-1})),$$

where $l_{-1} \in (\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}$ is some representative of $\bar{l}_{-1}$. Then we have

$$\xi_{\bar{l}_{-1}}(x_{-1}) = \mathbf{e}(\langle x_{-1}, l_{-1}\rangle_{11}^{(g)}) = \mathbf{e}(-\langle l_{-1}, x_{-1}\rangle_{11}^{(g)})$$

by Lemma 2.6.1.

Let $(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}}^{\#} := (\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#} \cap ((\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$. For any integer $m \geq 1$, let $\{\bar{l}_{-1}^{(j)}\}_{j\in J_m}$ be a complete set of representatives of

$$[(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})^{\#}/(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}}^{\#}]\,/\,[2m(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}/(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}})].$$

Let $\Omega_{h_{-1}}$ be determined by h_{-1}, $\{e_i\}_{r<i\leq d}$, and $\{f_i\}_{r<i\leq d}$ as in §2.4. Then, by Corollary 2.6.5, for each $j \in J_m$, the infinite sums

$$\theta_{m,h_{-1},g}^{(j)}(x_{-1}) := \sum_{\bar{l}_{-1}\in \mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}}/(\mathrm{Gr}_{-1}^{\mathtt{F}^{(g)}})_{\mathrm{MI}}} \xi_{\bar{l}_{-1}^{(j)}+m\bar{l}_{-1}}(\Omega_{h_{-1}}(\bar{l}_{-1}))\,\xi_{\bar{l}_{-1}^{(j)}+2m\bar{l}_{-1}}(x_{-1})$$

converges absolutely and uniformly over compact subsets of $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}}$, and the collection $\{\theta_{m,h_{-1},g}^{(j)}\}_{j\in J_m}$ define holomorphic functions over $\mathrm{Gr}_{-1,h}^{\mathtt{F}^{(g)}}$ representing a $\mathbb{C}$-basis of $\Gamma(A_{h_{-1},g}, \mathcal{M}_{h_{-1},g}^{\otimes m})$. Combining this with Corollary 4.4.3, we obtain:

Lemma 4.4.8. *Set* $\theta_{m,h_{-1},g}^{(j),\chi} := T^{*}_{-\frac{1}{m}\varepsilon_{10}^{(g)}(h)(\chi)}\,\theta_{m,h_{-1},g}^{(j)}$ *for any* $j \in J_m$ *and* $\chi \in X^{(g)}$. *Then the collection* $\{\theta_{m,h_{-1},g}^{(j),\chi}\}_{j\in J_m}$ *over* $\mathrm{Gr}_{-1,h}^{\mathtt{F}^{(g)}}$ *represents a* $\mathbb{C}$*-basis of* $\Gamma(A_{h_{-1},g}, \mathcal{M}_{h_{-1},g}^{\otimes m} \underset{\mathscr{O}_{A_{h_{-1},g}}}{\otimes} \mathscr{O}_{\chi,h_{-1},g})$.

4.5. **Quasi-periodicity in** $\varepsilon^{(g)}(\mathrm{Gr}_0^{\mathtt{F}^{(g)}})$**.** Suppose $y \in Y^{(g)} = \mathrm{Gr}_0^{\mathtt{F}^{(g)}}$. Let $l := \varepsilon^{(g)}(0,0,y)$.

Lemma 4.5.1. *For any* $\chi \in X^{(g)}$*, we have* (*cf. Definition* 4.3.3)

$$\mathbf{e}_{h,g,\chi}(l) = \mathbf{e}(\langle \varepsilon_{20}^{(g)}(y), \chi\rangle_{20}^{(g)})\,\mathbf{e}(\tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(\varepsilon_{10}^{(g)}(h)(\chi), \varepsilon_{10}^{(g)}(h)(y))),$$

which represents the section $\tau_{h,g}(y,\chi)$ *of* $(c_{h,g}^{\vee}(y), c_{h,g}(\chi))^{*}\,\mathcal{P}_{A_{h_{-1},g}}^{\otimes -1}$ *if we represent the point* $(c_{h,g}^{\vee}(y), c_{h,g}(\chi))$ *of* $A_{h_{-1},g} \times A_{h_{-1},g}^{\vee}$ *by the point* $(\varepsilon_{10}^{(g)}(h)(\chi), \varepsilon_{10}^{(g)}(h)(y))$ *of* $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}} \times \mathrm{Gr}_{-1,\mathbb{R}}^{\mathtt{F}^{(g)}}$.

Proof. Let us write $\varepsilon^{(g)}(0,0,\chi) = \chi_1 + h(\sqrt{-1})\chi_2 + \chi_3$ and $l = \varepsilon^{(g)}(0,0,y) = y_1 + h(\sqrt{-1})y_2 + y_3$ with $\chi_1, \chi_2, y_1, y_2 \in \mathrm{Gr}_{-2,\mathbb{R}}^{\mathtt{F}^{(g)}}$ and $\chi_3, y_3 \in (\mathrm{Gr}_{-2,h(\mathbb{C})}^{\mathtt{F}^{(g)}})^{\perp}$. Then

$$\begin{aligned}
&\tfrac{1}{2}(B_{h,g} - H_{h,g})(\varepsilon^{(g)}(0,0,\chi), l)\\
&= \tfrac{1}{2}H_{h,g}(-2h(\sqrt{-1})\chi_2,\ y_1 + h(\sqrt{-1})y_2) + \tfrac{1}{2}H_{h,g}(\chi_3^{c} - \chi_3,\ y_3)\\
&= -\langle h(\sqrt{-1})\chi_2, y_1\rangle^{(g)} - \sqrt{-1}\,\langle h(\sqrt{-1})\chi_2, y_2\rangle^{(g)} + \tfrac{1}{2}H_{h_{-1},g}(\varepsilon_{10}^{(g)}(\chi)^{c} - \varepsilon_{10}^{(g)}(\chi),\ y_3)\\
&= \langle \varepsilon_{20}^{(g)}(h)(y), \chi\rangle_{20}^{(g)} + \tfrac{1}{2}(B_{h_{-1},g} - H_{h_{-1},g})(\varepsilon_{10}^{(g)}(h)(\chi),\ \varepsilon_{10}^{(g)}(h)(y)),
\end{aligned}$$

and the lemma follows by evaluating $\mathbf{e}(\,\cdot\,)$. □

Remark 4.5.2. In the proof of Lemma 3.6.6, the expression of $\tau_{h,g}(y,\chi)$ was written as $\mathbf{e}(\langle \varepsilon_{20}^{(g)}(h)(y), \chi\rangle_{20}^{(g)})$ $\mathbf{e}(-\frac{1}{2}H_{h_{-1},g}(-{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi),\ \varepsilon_{10}^{(g)}(h)(y)))$ over $(\varepsilon_{10}^{(g)}(h)(y), -{}^{t}(\varepsilon_{21}^{(g)}(h))(\chi))$. Since we have twisted our coordinates by introducing

$B_{h_{-1},g}$, and since we have used the representative $(\varepsilon_{10}^{(g)}(h)(y), \varepsilon_{10}^{(g)}(h)(\chi))$ instead of $(\varepsilon_{10}^{(g)}(h)(y), -{}^t(\varepsilon_{21}^{(g)}(h))(\chi))$, the expressions are consistent.

By expanding $f(x+l)$ using (4.3.5), we obtain

$$\begin{aligned} f(x+l) &= \sum_{\chi\in X^{(g)}} \mathbf{e}_{h,g,\chi}(x+l)\, f_\chi(x_{-1}+\varepsilon_{10}^{(g)}(h)(y)) \\ &= \sum_{\chi\in X^{(g)}} \mathbf{e}_{h,g,\chi}(x)\, \mathbf{e}_{h,g,\chi}(l)\, f_\chi(x_{-1}+\varepsilon_{10}^{(g)}(h)(y)). \end{aligned}$$

On the other hand, we have (by shifting summation indices by $2m\phi^{(g)}(y)$)

$$\begin{aligned} &f(x)\,\mathbf{e}(-\tfrac{1}{2}m(B_{h,g}-H_{h,g})(l,l) - m(B_{h,g}-H_{h,g})(l,x)) \\ &= \sum_{\chi\in X^{(g)}} \mathbf{e}_{h,g,\chi}(x) f_\chi(x_{-1})\, \mathbf{e}_{h,g,-my}(l)\, \mathbf{e}_{h,g,-2my}(x) \\ &= \sum_{\chi\in X^{(g)}} \mathbf{e}_{h,g,\chi}(x)\, f_{\chi+2m\phi^{(g)}(y)}(x_{-1})\, \mathbf{e}_{h,g,-my}(l). \end{aligned}$$

By comparing the coefficients of $\mathbf{e}_{h,g,\chi}(x)$, the relation (4.3.1) implies

$$f_\chi(x_{-1}+\varepsilon_{10}^{(g)}(h)(y))\, \mathbf{e}_{h,g,\chi}(l) = f_{\chi+2m\phi^{(g)}(y)}(x_{-1})\, \mathbf{e}_{h,g,-my}(l),$$

or equivalently

$$f_{\chi+2m\phi^{(g)}(y)} = \mathbf{e}_{h,g,\chi+m\phi^{(g)}(y)}(l)\, T^*_{\varepsilon_{10}^{(g)}(h)(y)}\, f_\chi. \tag{4.5.3}$$

Written symbolically using Lemma 4.5.1, this is

$$f_{\chi+2m\phi^{(g)}(y)} = \tau(y, \chi+m\phi^{(g)}(y))\, T^*_{c^\vee_{h,g}(y)}\, f_\chi,$$

which is exactly the same relation appeared in the definition of V_m.

Lemma 4.5.4. *If $\{\chi^{(j')}\}_{j'\in J'_m}$ is a complete set of representatives of $X^{(g)}/2\phi^{(g)}(Y^{(g)})$, then any family $(f_\chi)_{\chi\in X}$ that satisfies* (4.5.3) *for every $y\in Y^{(g)}$ and every $\chi\in X^{(g)}$ is uniquely determined by the finite subfamily $(f_{\chi^{(j')}})_{j'\in J'_m}$. Moreover, the sum* (4.3.5) *converges for such a family and defines an element of $\Gamma(G_{h,g}, \mathcal{L}_{h,g}^{\otimes m})$.*

Proof. The first statement is obvious. Since the $\mathbb{C}$-dimension of $\Gamma(G_{h,g}, \mathcal{L}_{h,g}^{\otimes m})$ is explicitly known (by Corollary 2.6.5), the convergence of the sum (4.3.5) is forced. □

Corollary 4.5.5. *Let us denote by $\theta_{m,h,g}^{(j,j')}$ the family (f_χ) satisfying* (4.5.3) *(for every $y\in Y^{(g)}$ and every $\chi\in X^{(g)}$) such that $f_\chi = 0$ for $\chi\notin\chi^{(j')}+2m\phi^{(g)}(Y^{(g)})$ and $f_{\chi^{(j')}} = \theta_{m,h_{-1},g}^{(j)}$. Then the collection $\{\theta_{m,h,g}^{(j,j')}\}_{j\in J_m, j'\in J'_m}$ defines a $\mathbb{C}$-basis of $\Gamma(G_{h,g}, \mathcal{L}_{h,g}^{\otimes m})$.*

4.6. **Explicit bases in analytic families.** Let K_0 be the function field of the irreducible component of $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$ containing s. Let U be any (connected) complex analytic neighborhood of s in $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, and let K_U denote the ring of meromorphic functions on U. As in [3, p. 117, (10.1)] (with an obvious analogue for algebraic spaces), there are natural inclusions

$$K_0 \hookrightarrow K_U \to K_s.$$

Let $U_0 := U \cap \mathsf{Sh}_{\mathcal{H}}$. Let $(G_{U_0}, \lambda_{U_0}, i_{U_0}, \alpha_{\mathcal{H},U_0}) \to U_0$ denote the pullback of $(G_{\mathrm{hol}}, \lambda_{\mathrm{hol}}, i_{\mathrm{hol}}, \alpha_{\mathcal{H},\mathrm{hol}}) \to \mathsf{Sh}_{\mathcal{H}}$ under $U_0 \hookrightarrow \mathsf{Sh}_{\mathcal{H}}$, or rather the holomorphic family descended from $(G_{\mathrm{hol}}, \lambda_{\mathrm{hol}}, i_{\mathrm{hol}}, \alpha_{\mathcal{H},\mathrm{hol}}) \to \mathsf{X}$. By shrinking U if necessary, we may assume that $U_0 \hookrightarrow U$ is given by

$$\Delta_0^a \times \Delta^b \hookrightarrow \Delta^{a+b}, \tag{4.6.1}$$

where Δ is the unit disk, Δ_0 is the unit punctured disk, and $a, b \geq 0$ are integers. By the theorem of [2, p. 279], we see that $\pi_1(U_0)$ (as a subgroup of $\Gamma_{\mathcal{H}}^{(g)}$) lies in $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2}$. Therefore, by choosing a point $\tilde{s}$ of

$$(\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2} \backslash \mathsf{X}_2^{\mathsf{F}^{(g)}}) \overset{E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})}{\times} E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})(\sigma)$$

that is mapped to s, (which is necessarily in the interior of the closure of $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2} \backslash \mathsf{X}$,) we may assume that U is the isomorphic image of some complex analytic polydisk $\tilde{U}$ containing $\tilde{s}$. Let $\tilde{U}_0$ be the preimage of U_0 in $\tilde{U}$. Then we may realize $(G_{U_0}, \lambda_{U_0}, i_{U_0}, \alpha_{\mathcal{H},U_0}) \to U_0$ as the holomorphic family $(G_{\tilde{U}_0}, \lambda_{\tilde{U}_0}, i_{\tilde{U}_0}, \alpha_{\mathcal{H},\tilde{U}_0}) \to \tilde{U}_0$ descended from X.

Lemma 4.6.2. *The topology of R_s is finer than the topology defined by the functions vanishing on the σ-stratum $\Xi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\sigma,\mathrm{an}}$ of $\Xi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(\sigma)$, which is therefore finer than the topology defined by the collection of functions $\{\mathbf{e}_{h,g,y}(\varepsilon^{(g)}(0,0,y))\}_{y \in Y^{(g)}, y \neq 0}$ in complex coordinates. (Here, by abuse of notation, we consider $\mathbf{e}_{h,g,y}(\varepsilon^{(g)}(0,0,y))$ as functions varying holomorphically with h for each g and y.) Therefore, a formal power series convergent for the topology defined by $\{\mathbf{e}_{h,g,y}(\varepsilon^{(g)}(0,0,y))\}_{y \in Y^{(g)}, y \neq 0}$ is also convergent for the topology of R_s.*

Proof. The first half of the first statement is true because s is a point of the σ-stratum. The second half of the first statement is true because the definition of the σ-stratum uses the sub-semigroup $\sigma_0^\vee$ of $\mathbf{S}_{\Phi_n^{(g)}}$, which contains all elements of the form $[y \otimes \phi(y)]$ for some $y \in Y^{(g)}$ because $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}^{(g)}}$. □

Lemma 4.6.3. *The $\mathbb{C}$-basis $\{\theta_{m,h,g}^{(j,j')}\}_{j \in J_m, j' \in J'_m}$ of $\Gamma(G_{h,g}, \mathcal{L}_{h,g}^{\otimes m})$ varies holomorphically with $h \in \tilde{U}_0$, and it descends to U_0 because $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2}$ acts trivially on all the data involved. Moreover, the element $\theta_{m,\mathrm{hol}}^{(j,j')}$ of $\Gamma(G_{U_0}, \mathcal{L}_{U_0}^{\otimes m})$ defined by $\theta_{m,h,g}^{(j,j')}$ stays bounded when the point of U_0 defined by h approaches the boundary $U - U_0$.*

Proof. Both statements are clear from the explicit expression of each $\theta_{m,h,g}^{(j,j')}$. □

Corollary 4.6.4. *The collection $\{\theta_{m,\mathrm{hol}}^{(j,j')}\}_{j \in J_m, j' \in J'_m}$ represents a K_s-basis of $\Gamma(G_{K_s}, \mathcal{L}_{K_s}^{\otimes m})$.*

Lemma 4.6.5. *The sections $\{\theta_{m,h_{-1},g}^{(j),\chi}\}_{j \in J_m}$ of $\Gamma(A_{h_{-1},g}, \mathcal{M}_{h_{-1},g}^{\otimes m} \underset{\mathscr{O}_{A_{h_{-1},g}}}{\otimes} \mathscr{O}_{\chi,h_{-1},g})$ vary holomorphically with $h \in \tilde{U}_0$ (or rather with the h_{-1} associated with h), descend to sections $\{\theta_{m,\mathrm{hol}}^{(j),\chi}\}_{j \in J_m}$ of $\Gamma(A_{U_0}, \mathcal{M}_{U_0}^{\otimes m} \underset{\mathscr{O}_{A_{U_0}}}{\otimes} \mathscr{O}_{\chi,U_0})$, and extend canonically to U.*

Proof. The identification of $U_0 \hookrightarrow U$ in (4.6.1) implies implicitly that U_0 and U parameterize the same abelian parts. □

Corollary 4.6.6. *The collection* $\{\theta^{(j),\chi}_{m,\mathrm{hol}}\}_{j\in J_m}$ *over* U_0 *defines a* K_s*-basis of* $\Gamma(A_{K_s},\mathcal{M}^{\otimes m}_{K_s}\underset{\mathscr{O}_{A_{K_s}}}{\otimes}\mathscr{O}_{\chi,K_s})$.

Completely analogous to the case of $\{\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ above, the collection $\{\theta^{(j),\chi}_{m,\mathrm{hol}}\}_{j\in J_m}$ and the same representatives $\{\chi^{(j')}\}_{j'\in J'_m}$ determine a K_s-basis $\{{}^\heartsuit\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ of V_m.

Now Proposition 4.2.2 follows (as outlined in §4.2) from:

Proposition 4.6.7. *The assignments* $\theta^{(j,j')}_{m,\mathrm{hol}}\mapsto {}^\heartsuit\theta^{(j,j')}_{m,\mathrm{hol}}$ *(for* $m\geq 0$ *divisible by* m_s*) define a canonical (*K_s*-algebra) isomorphism*

$$\underset{m\geq 0,m_s|m}{\oplus}\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})\overset{\sim}{\to}\underset{m\geq 0,m_s|m}{\oplus}V_m\cong\underset{m\geq 0,m_s|m}{\oplus}\Gamma({}^\heartsuit G_{K_s},{}^\heartsuit\mathcal{L}^{\otimes m}_{K_s})$$

inducing an isomorphism $(G_{K_s},\lambda_{K_s},i_{K_s},\alpha_{\mathcal{H},K_s})\cong({}^\heartsuit G_{K_s},{}^\heartsuit\lambda_{K_s},{}^\heartsuit i_{K_s},{}^\heartsuit\alpha_{\mathcal{H},K_s})$.

Proof. Since the periods $\tau(y,\chi+m\phi^{(g)}(y))$ involved are identical over K_s, the explicit formula (4.5.3) allows us to identify $\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})$ canonically with V_m for any $m\geq 0$ (divisible by m_s).

By Lemma 4.6.2, the infinite sums (4.3.5) for elements in $\{\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ correspond to infinite sums convergent in the topology defined by R_s. This allows us to identify the canonical morphisms $\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})\otimes\Gamma(G_{K_s},\mathcal{L}^{\otimes m'}_{K_s})\to\Gamma(G_{K_s},\mathcal{L}^{\otimes(m+m')}_{K_s})$ with the morphisms (4.2.4) using the canonical morphisms (4.2.3). As a result, the graded K_s-algebra $\underset{m\geq 0,m_s|m}{\oplus}\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})$ is canonically isomorphic to $\underset{m\geq 0,m_s|m}{\oplus}V_m$, and hence to $\underset{m\geq 0,m_s|m}{\oplus}\Gamma({}^\heartsuit G_{K_s},{}^\heartsuit\mathcal{L}^{\otimes m}_{K_s})$. This gives the canonical isomorphism $G_{K_s}\cong{}^\heartsuit G_{K_s}$ matching λ_{K_s} and ${}^\heartsuit\lambda_{K_s}$.

Take any integer $n\geq 3$ such that $\mathcal{U}(n)\subset\mathcal{H}$. Since $\mathcal{L}_{K_s}$ is ample over G_{K_s}, giving a point of $G_{K_s}[n]$ is equivalent to giving (compatibly) for each $m\geq 0$ divisible by m_s a K_s-linear morphism $\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})\to K_s$. (The same is true with m_s replaced with any positive integer.) For each $\bar{l}\in\frac{1}{n}L/L$, the evaluation of $\{\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ at the point $\alpha_{h,g,n}(\bar{l})$ of $G_{h,g}$ varies holomorphically with respect to $h\in\tilde{U}_0$, and hence defines a morphism $\Gamma(G_{K_s},\mathcal{L}^{\otimes m}_{K_s})\to K_s$ giving a point of $G_{K_s}[n]$. This defines a tautological level-n structure $\alpha_{n,K_s}:\frac{1}{n}L^{(g)}/L^{(g)}\overset{\sim}{\to}G_{K_s}[n]$, whose $(\mathcal{H}/\mathcal{U}(n))$-orbit is $\alpha_{\mathcal{H},K_s}$. By Corollary 4.5.5, the evaluation defining α_{n,K_s} can be determined by a similar evaluation of the collection $\{\theta^{(j),\chi}_{m,h_{-1}}\}_{j\in J_m}$ at points of $A_{h_{-1},g}[n]$ (without having to evaluate at the points of $G_{h,g}[n]$). Since the two collections $\{\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ and $\{{}^\heartsuit\theta^{(j,j')}_{m,\mathrm{hol}}\}_{j\in J_m,j'\in J'_m}$ are defined using the collection $\{\theta^{(j),\chi}_{m,\mathrm{hol}}\}_{j\in J_m}$ by the same formulae (for each $m\geq 0$ divisible by m_s), and since the level structures are built from their graded pieces by the same tautological relations (in particular, the relations (3.5.10) and (3.6.7) for α_{n,K_s} correspond to the same tautological relations for ${}^\heartsuit\alpha_{n,K_s}$; cf. [23, Thm. 5.2.3.13 and §6.2.3]), the isomorphism $G_{K_s}\cong{}^\heartsuit G_{K_s}$ matches α_{n,K_s} and ${}^\heartsuit\alpha_{n,K_s}$, the latter being the tautological level-n structure determined by the $\mathcal{U}(n)$-orbit of $(\mathsf{Z}^{(g)},\Phi^{(g)},\delta^{(g)})$. By taking $(\mathcal{H}/\mathcal{U}(n))$-orbits, we see that the isomorphism $G_{K_s}\cong{}^\heartsuit G_{K_s}$ matches $\alpha_{\mathcal{H},K_s}$ and ${}^\heartsuit\alpha_{\mathcal{H},K_s}$ as well.

By [24, §21, Thm. 5 and its proof], the isomorphism $G_{K_s} \cong {}^{\heartsuit}G_{K_s}$ matches i_{K_s} and ${}^{\heartsuit}i_{K_s}$ because it matches α_{n,K_s} and ${}^{\heartsuit}\alpha_{n,K_s}$ with $n \geq 3$. Thus we have the desired isomorphism $(G_{K_s}, \lambda_{K_s}, i_{K_s}, \alpha_{\mathcal{H},K_s}) \cong ({}^{\heartsuit}G_{K_s}, {}^{\heartsuit}\lambda_{K_s}, {}^{\heartsuit}i_{K_s}, {}^{\heartsuit}\alpha_{\mathcal{H},K_s})$. □

5. Applications

5.1. **Minimal compactifications.**

Theorem 5.1.1. *Let* $\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}}$ *denote the minimal compactification of* $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ *construction by* [4, 10.11]*, and let* $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ *denote the closure of* $\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ *in* $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C}}$*, the pullback of* $\mathsf{M}^{\min}_{\mathcal{H}}$ *under* $F_0 \hookrightarrow \mathbb{C}$*. Then there is a canonical strata-preserving isomorphism* $\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ *extending the canonical isomorphism* $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$.

Proof. Let $\omega := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} = \wedge^{\mathrm{top}} e^{*}_{G}\, \Omega^{1}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$, where $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is the degenerating family given by [23, Thm. 6.4.1.1]. By [23, Thm. 7.2.4.1], we have $\mathsf{M}^{\min}_{\mathcal{H}} \cong \mathrm{Proj}\left(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes k})\right)$. By pullback to closures of $\mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ (in the normal ambient varieties), we obtain

$$\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \cong \mathrm{Proj}\left(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}, \omega^{\otimes k})\right). \tag{5.1.2}$$

On the other hand, according to [4, 10.11] and [27, 8.2], and by Theorem 4.1.1, we have

$$\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}} \cong \mathrm{Proj}\left(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}, \overline{\Omega}^{\otimes k})\right), \tag{5.1.3}$$

where $\overline{\Omega} := \wedge^{\mathrm{top}}\ (\Omega^{1}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}/\mathbb{C}}[d\log\infty])$ (with $d\log\infty$ defined as in [23, Thm. 6.4.1.1]).

Let us denote by $\mathsf{M}^{1}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ the open subscheme of $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ formed by the union of the strata of $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ of codimension at most one. By the description of the fibers of the canonical morphism $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \to \mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ in [23, Thm. 7.2.4.1], its restriction to $\mathsf{M}^{1}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ is an isomorphism. Hence we may view $\mathsf{M}^{1}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ as an open subscheme of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ as well.

Since the complement of $\mathsf{M}^{1}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ in $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ has codimension at least two, and since the sheaves on the right-hand sides of (5.1.2) and (5.1.3) descend to $\mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ (because $\mathcal{H}$ is neat; cf. [23, Thm. 7.2.4.1] and [4, 10.14]), to compare the right-hand sides of (5.1.2) and (5.1.3), it suffices to compare them over connected components of $\mathsf{M}^{1}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$. Since each of the connected components decomposes (up to subspaces of codimension at least two) according to the decomposition of $\mathrm{G}^{\mathrm{ad}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ into $\mathbb{Q}$-simple factors, we may as well assume that $\mathrm{G}^{\mathrm{ad}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is $\mathbb{Q}$-simple and nontrivial. By using the extended Kodaira–Spencer isomorphism (see [23, Thm. 6.4.1.1]), and by tensoring with the analogues isomorphism with F-action twists by *, we know that some positive tensor power $\omega|_{\mathsf{M}^1_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}}$ is isomorphic to some positive tensor power of $\overline{\Omega}|_{\mathsf{M}^1_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}}$. Thus, we obtain a canonical isomorphism $\mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ extending the canonical isomorphism $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$, as desired.

The statement that this isomorphism is strata-preserving follows from Theorem 4.1.1, from the description of the morphism $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \to \mathsf{Sh}^{\min}_{\mathcal{H},\mathrm{alg}}$ in [2, pp. 254–256], and from the description of the morphism $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\min}_{\mathcal{H}}$ in [23, Thm. 7.2.4.1]. □

Remark 5.1.4. The statement that the restriction of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \to \mathsf{M}^{\min}_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ to $\mathsf{M}^1_{\mathcal{H},\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}}$ is an isomorphism, together with Theorems 4.1.1 and 5.1.1, explains that growth conditions commonly imposed on sections of coherent sheaves can be understood as the Riemann extension theorem applied to the analytic boundary, essentially (products of) the one-dimensional case.

5.2. **Automorphic bundles.** Let us denote the universal object over $\mathsf{M}_{\mathcal{H}}$ by $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}$, interpreted as the pullback of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}_{\mathcal{H}}$. Consider the relative de Rham cohomology $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) := R^1(G_{\mathsf{M}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}})_*(\Omega^\bullet_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$ with its self-dual pairing structure $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ induced by $\lambda : G_{\mathsf{M}_{\mathcal{H}}} \to G^\vee_{\mathsf{M}_{\mathcal{H}}}$. (See [15, 1.5] for the definition of $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$.) Let

$$\underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}(\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}),$$

and denote the induced pairing on $\underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ by the same notation $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$. By [5, Lem. 2.5.3], we have canonical short exact sequences $0 \to \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to \underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to 0$ and $0 \to \underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to 0$. The submodules $\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$ and $\underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$ are maximal totally isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$.

On the other hand, consider the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-module morphism

$$L\underset{\mathbb{Z}}{\otimes}\mathbb{C} \to V_0 := (L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}_{h_0}, \tag{5.2.1}$$

where $\mathsf{P}_{h_0} := \{\sqrt{-1}\,x - h_0(\sqrt{-1})x : x \in L\underset{\mathbb{Z}}{\otimes}\mathbb{R}\} \subset L\underset{\mathbb{Z}}{\otimes}\mathbb{C}$ is defined as in Lemma 2.1.2. Let F'_0 be any field extension of F_0 in $\mathbb{C}$ over which there exists an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}F'_0$-module L_0 such that $L_0\underset{F'_0}{\otimes}\mathbb{C} \cong V_0$ as $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules. (The choice does not matter for us here, but in practice there might be an optimal choice in each special case. The reader can take for example $F'_0 = \mathbb{C}$ in what follows.) Let us fix the choice of L_0 and denote by

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to F'_0(1)$$

the alternating pairing defined by $\langle(x_1, f_1), (x_2, f_2)\rangle_{\mathrm{can.}} := f_2(x_1) - f_1(x_2)$.

Definition 5.2.2. *For any F_0'-algebra R, set*

$$\mathrm{G}_0(R) := \left\{ \begin{array}{c} (g,r) \in \mathrm{GL}_{\mathcal{O} \underset{F_0'}{\otimes} R}((L_0 \oplus L_0^\vee(1)) \underset{F_0'}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy \rangle = r\langle x,y\rangle, \forall x,y \in (L_0 \oplus L_0^\vee(1)) \underset{F_0'}{\otimes} R \end{array} \right\},$$

$$\mathrm{P}_0(R) := \{(g,r) \in \mathrm{G}_0(R) : g(L_0^\vee(1) \underset{F_0'}{\otimes} R) = L_0^\vee(1) \underset{F_0'}{\otimes} R\},$$

$$\mathrm{M}_0(R) := \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_0^\vee(1) \underset{F_0'}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R),$$

where we view $\mathrm{M}_0(R)$ canonically as a quotient of $\mathrm{P}_0(R)$ by $\mathrm{P}_0(R) \to \mathrm{M}_0(R) : (g,r) \mapsto (g|_{L_0^\vee(1) \underset{F_0'}{\otimes} R}, r)$. The assignments are functorial in R and define group functors G_0, P_0, and M_0 over $\mathrm{Spec}(F_0')$.

Lemma 5.2.3. *The choice of h_0 defines by Lemma 2.1.2 a canonical isomorphism*

$$(L \underset{\mathbb{Z}}{\otimes} \mathbb{C}, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (L_0 \oplus L_0^\vee(1), \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \underset{F_0'}{\otimes} \mathbb{C},$$

and hence a canonical isomorphism $\mathrm{G}(\mathbb{C}) \cong \mathrm{G}_0(\mathbb{C})$. Consequently, the choice of h_0 identifies $\mathrm{P}_0 \underset{F_0'}{\otimes} \mathbb{C}$ canonically with a parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$.

Proof. It suffices to take any isomorphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0 \oplus V_0^\vee(1)$ matching V_0 (resp. $V_0^\vee(1)$) with the submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ on which $h_0(z)$ acts by $1 \otimes z$ (resp. $1 \otimes z^c$). □

In what follows, by abuse of notation, we shall replace $\mathsf{M}_{\mathcal{H}}$ etc with their base extensions from $\mathrm{Spec}(F_0)$ to $\mathrm{Spec}(F_0')$, and replace $\mathsf{M}_0 = \mathrm{Spec}(F_0)$ with $\mathrm{Spec}(F_0')$ accordingly.

Definition 5.2.4. *The* **principal P_0-bundle** *over $\mathsf{M}_{\mathcal{H}}$ is the P_0-torsor*

$$\mathcal{E}_{\mathrm{P}_0} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}),$$
$$((L_0 \oplus L_0^\vee(1)) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^\vee(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$-sheaves of symplectic $\mathcal{O}$-modules with maximal totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$-submodules. (The group P_0 acts as automorphisms on $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^\vee(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$ by definition. The third entries $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ in the tuples represent the values of the pairings. We allow isomorphisms of symplectic $\mathcal{O}$-modules to modify the pairings up to units.)

Here $\mathcal{E}_{\mathrm{P}_0}$ is an étale P_0-torsor because, by the condition on Lie algebra (giving sections pointwise), by the theory of infinitesimal deformations (giving sections over complete local bases; cf. for example [23, Ch. 2]), and by the theory of Artin's approximations (cf. [1, Thm. 1.10 and Cor. 2.5]), it has sections étale locally.

Definition 5.2.5. *The* **principal M_0-bundle** *over $\mathsf{M}_{\mathcal{H}}$ is the M_0-torsor*

$$\mathcal{E}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)), (L_0^\vee(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1))),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$-sheaves of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$-modules. (We view the second entries $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ as an additional structure, inherited from the corresponding *objects for* P_0. *The group* M_0 *acts as automorphisms on* $(L_0^\vee(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1))$ *by definition.)*

Definition 5.2.6. *For any* F_0'*-algebra* E, *we denote by* $\mathrm{Rep}_E(\mathrm{P}_0)$ (*resp.* $\mathrm{Rep}_E(\mathrm{M}_0)$) *the category of* E*-modules with algebraic actions of* $\mathrm{P}_0 \underset{F_0'}{\otimes} E$ (*resp.* $\mathrm{M}_0 \underset{F_0'}{\otimes} E$).

Definition 5.2.7. *Let* E *be any* F_0'*-algebra. For any* $W \in \mathrm{Rep}_E(\mathrm{P}_0)$, *we define*

$$\mathcal{E}_{\mathrm{P}_0,E}(W) := (\mathcal{E}_{\mathrm{P}_0} \underset{F_0'}{\otimes} E) \overset{\mathrm{P}_0 \underset{F_0'}{\otimes} E}{\times} W,$$

called the **automorphic bundle** *over* $\mathsf{M}_{\mathcal{H}} \underset{F_0'}{\otimes} E$ *associated with* W.

Lemma 5.2.8. *Let* E *be any* F_0'*-algebra. If we view an element in* $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ *as an element in* $\mathrm{Rep}_E(\mathrm{P}_0)$ *in the canonical way, then we have a canonical isomorphism*

$$\mathcal{E}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}_{\mathrm{M}_0,E}(W) := (\mathcal{E}_{\mathrm{M}_0} \underset{F_0'}{\otimes} E) \overset{\mathrm{M}_0 \underset{F_0'}{\otimes} E}{\times} W.$$

We call $\mathcal{E}_{\mathrm{M}_0,E}(W)$ *the* **automorphic bundle** *over* $\mathsf{M}_{\mathcal{H}} \underset{F_0'}{\otimes} E$ *associated with* W.

To define the canonical extensions of automorphic bundles, let us formulate axiomatically the input we need as follows:

Assumption 5.2.9. *The sheaf* $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ *extends to a locally free sheaf* $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))^{\mathrm{can}}$ *over* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$, *which is characterized by the following properties:*

1. *The sheaf* $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))^{\mathrm{can}}$, *canonically identified as a subsheaf of the quasi-coherent sheaf* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_*(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))$, *is self-dual under the pairing* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_*(\langle\,\cdot\,,\,\cdot\,\rangle_\lambda)$. *We shall denote the induced pairing by* $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda^{\mathrm{can}}$.
2. $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))^{\mathrm{can}}$ *contains* $\underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *as a subsheaf totally isotropic under* $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda^{\mathrm{can}}$.
3. *The quotient sheaf* $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))^{\mathrm{can}}/\underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *can be canonically identified with the subsheaf* $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *of* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_*\,\underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$.
4. *The pairing* $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda^{\mathrm{can}}$ *induces an isomorphism* $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \overset{\sim}{\to} \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *which coincides with* $d\lambda$.

The construction of $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))^{\mathrm{can}}$ with properties in Assumption 5.2.9 can be found in the forthcoming article [22]. (In the Siegel case, it suffices to refer to [17, Ch. VI, §§1–2].) We stated Assumption 5.2.9 to clarify that any construction achieving these properties would serve the same purpose.

Admitting Assumption 5.2.9 from now, the principle bundle $\mathcal{E}_{\mathrm{P}_0}$ extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ by setting

$$\begin{aligned}\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}}&((\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \langle\,\cdot\,,\,\cdot\,\rangle_\lambda^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}),\\ &((L_0 \oplus L_0^\vee)(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(1), L_0^\vee(1) \underset{F_0'}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}})),\end{aligned}$$

and the principle bundle $\mathcal{E}_{\mathrm{M}_0}$ extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ by setting

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}}((\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(1)), (L_0^{\vee}(1)\underset{F_0'}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(1))).$$

Definition 5.2.10. *Let E be any F_0'-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{P}_0)$, we define*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}\underset{F_0'}{\otimes}E) \overset{\mathrm{P}_0\underset{F_0'}{\otimes}E}{\times} W,$$

called the **canonical extension** *of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, and define*

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) := \mathcal{E}^{\mathrm{can}}(W)\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}}{\otimes}\mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}},$$

called the **subcanonical extension** *of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, where $\mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}}$ is the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-ideal defining the relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}} := \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} - \mathsf{M}_{\mathcal{H}}$ (with its reduced structure).*

Lemma 5.2.11. *Let E be any F_0'-algebra. If we view an element in $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as an element in $\mathrm{Rep}_E(\mathrm{P}_0)$ via the canonical morphism $\mathrm{P}_0 \to \mathrm{M}_0$, then we have canonical isomorphisms*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}\underset{F_0'}{\otimes}E) \overset{\mathrm{M}_0\underset{F_0'}{\otimes}E}{\times} W.$$

and

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}}{\otimes}\mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}}.$$

We call $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)$ (resp. $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W)$) the **canonical extension** *(resp.* **subcanonical extension***) of $\mathcal{E}_{\mathrm{M}_0,E}(W)$ over $\mathsf{M}_{\mathcal{H}}\underset{F_0'}{\otimes}E$ associated with W.*

By abuse of notation, we shall also denote by the subscript "$\mathbb{C}$" the pullbacks of various objects under the morphism $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \cong \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{F_0'}{\otimes}\mathbb{C}$. Moreover, we denote the analytifications of the various objects by replacing the subscripts "$\mathbb{C}$" with "an".

Theorem 5.2.12. *Suppose $\mathrm{G}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$, X, and Σ are chosen such that $\mathsf{Sh}_{\mathcal{H}}$ and $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ make sense in* [18], *in* [19, §2], *and in our theory. (In this case, Σ is smooth and satisfies* [23, Cond. 6.2.5.25 in the revision].) *For any $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{P}_0)$, the bundle $\mathcal{E}_{\mathrm{P}_0,\mathrm{an}}(W)$ defines an automorphic bundle over $\mathsf{Sh}_{\mathcal{H}}$, and the bundles $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,\mathrm{an}}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,\mathrm{an}}(W)$ define respectively the canonical and subcanonical extensions of $\mathcal{E}_{\mathrm{P}_0,\mathrm{an}}(W)$ over $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, in the senses of* [18, §4] *and* [19, §2]. *(The analogous statement for $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$ follows consequently.)*

Proof. Given a triple $(\mathtt{V}, g, \varepsilon^{(g)})$ inducing a rational boundary component of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^{\infty})$, let $\mathtt{F}^{(g)}$ be associated with $(\mathtt{V}, g)$ as in §3.1, and let other related objects be defined accordingly.

At the point of $\Gamma^{(g)}_{\mathcal{H}}\backslash\mathsf{X}_0$ represented by some $h \in \mathsf{X}_0$, the fiber of the analytification of the pullback of $(\underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$ to $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}$

can be canonically identified with

$$(L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{C}, \langle \cdot\,, \cdot \rangle^{(g)}, \mathbb{C}(1), \mathsf{P}_h). \tag{5.2.13}$$

When we vary h holomorphically in X_0, it is exactly the maximal totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-submodule P_h of $L^{(g)} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ that varies accordingly. Therefore, our construction of $\mathcal{E}_{\mathrm{P}_0}(W)$ in Definition 5.2.7 using $\mathcal{E}_{\mathrm{P}_0}$ implies that $\mathcal{E}_{\mathrm{P}_0,\mathrm{an}}(W)$ coincides with the construction in [18, §1] using the Borel compact dual of X_0.

Suppose no longer that h lies in X_0, but that h lies in $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})\,\mathsf{X}_0 \cong \mathsf{X}_2^{\mathtt{F}^{(g)}}$. Then P_h still represents a maximal totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, although it might not satisfy the positivity described in (3) of Lemma 2.1.2. Since the action of $\mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})$ does not modify the exact sequence (3.5.1), the identification $\mathsf{P}_h \cong \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{h,g}/\mathbb{C}}(1)$ is valid for all $h \in \mathrm{U}_{2,\mathtt{F}^{(g)}}(\mathbb{C})\,\mathsf{X}_0$, extending the identification for $h \in \mathsf{X}_0$. This allows us to descend the family of tuples (5.2.13) to $\mathsf{X}_1^{\mathtt{F}^{(g)}}$, or rather to $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h}\backslash(\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_1}\backslash\mathsf{X}_1^{\mathtt{F}^{(g)}})$. Over the interior of the closure of $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2}\backslash\mathsf{X}_0$ in $(\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_2}\backslash\mathsf{X}_2^{\mathtt{F}^{(g)}}) \overset{E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})}{\times} E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})(\sigma)$, the pullback of the descended family of tuples over $\mathsf{X}_1^{\mathtt{F}^{(g)}}$ agrees with the pullback of the analytification of $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \langle \cdot\,, \cdot \rangle_{\lambda}^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(1), \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}})$.

Therefore, our construction of $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}(W)$ in Definition 5.2.10 using $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}$ implies that $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,\mathrm{an}}(W)$ coincides with the construction in [18, §4] using pullbacks of descents of automorphic bundles to $\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},h}\backslash(\Gamma_{\mathcal{H}}^{\mathtt{F}^{(g)},\mathrm{U}_1}\backslash\mathsf{X}_1^{\mathtt{F}^{(g)}})$.

Since the isomorphism $\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}} \overset{\sim}{\to} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathbb{C},L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}}$ in Theorem 4.1.1 preserves the stratifications, the case of $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,\mathrm{an}}(W)$ follows from the case of $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,\mathrm{an}}(W)$. □

Remark 5.2.14. According to [19, §2], the coherent cohomology of the canonical (resp. subcanonical) extensions of automorphic bundles can be represented by differential forms that are *slowly increasing* (resp. *rapidly decreasing*). Moreover, according to [6, §3], nondegenerate limits of discrete series (not necessarily holomorphic ones) can be realized in such coherent cohomology spaces. Therefore, by taking integral models of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}$ (using [23] and [22]), Theorem 5.2.12 allows us to define (at least abstractly) the notion of integral structures on these spaces (at the good primes).

From now on, let us fix the choice of a triple $(\mathtt{V}, g, \varepsilon^{(g)})$ inducing a rational boundary component of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^{\infty})$, and let $\mathtt{F}^{(g)}$ and $(\mathtt{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$ be associated with $(\mathtt{V}, g, \varepsilon^{(g)})$ as in §3.1.

Definition 5.2.15. *The* **principal** M_0**-bundle** *over* $C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}$ *is the* M_0*-torsor*

$$\mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}}}((\underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}}, \mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}}(1)), (L_0^{\vee}(1) \underset{F_0'}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}}, \mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}}(1))),$$

with conventions as in Definition 5.2.5.

Then we define $\mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,E}(W)$ for any F'_0-algebra E and any $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as in Lemma 5.2.8, and we define $\mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W)$ and $\mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathrm{an}}(W)$ for $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$ with abuse of notation as above.

Lemma 5.2.16. *For any $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$, there is a canonical isomorphism*

$$(\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}})^* \, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathbb{C}}(W) \cong (\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}})^* \, \mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W).$$

Proof. This is because of the canonical isomorphism

$$(\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}})^* \, \underline{\mathrm{Lie}}^{\vee}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \cong (\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}})^* \, \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}}. \quad \square$$

The proof of Theorem 5.2.12 implies:

Corollary 5.2.17. *With settings as in Theorem 5.2.12, suppose $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$. Then the sections of $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathrm{an}}(W)$ invariant under the action of $\mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{R})$ descends to sections of $\mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathrm{an}}(W)$. This identification is consistent with the analytification of the canonical isomorphism in Lemma 5.2.16.*

5.3. **Fourier–Jacobi expansions.** We shall focus on the category $\mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$ in this section, and the global sections of the associated automorphic bundles.

First let us explain the analytic Fourier–Jacobi expansions.

Lemma 5.3.1. *The group $\mathcal{U}_\infty = \mathrm{Cent}_{\mathrm{G}(\mathbb{R})}(h_0)$, defined in §2.3 as the stabilizer of h_0 under the conjugation action of $\mathrm{G}(\mathbb{R})$, can be identified canonically as a subgroup of $\mathrm{P}_0(\mathbb{C})$. The composition $\mathcal{U}_\infty \hookrightarrow \mathrm{P}_0(\mathbb{C}) \twoheadrightarrow \mathrm{M}_0(\mathbb{C})$ is injective and identifies $\mathrm{M}_0(\mathbb{C})$ with the complexification of $\mathcal{U}_\infty$. Consequently, two objects in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$ are isomorphic if and only if their restrictions to $\mathcal{U}_\infty$ are isomorphic.*

Proof. If we consider the Hodge decomposition $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong \mathsf{P}_{h_0} \oplus V_0$ (splitting (5.2.1)) defined by mapping V_0 to be the subset of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ on which $h_0(z)$ acts by $1 \otimes z$, then the elements in $\mathrm{G}(\mathbb{C})$ stabilizing the Hodge decomposition gives a well-known splitting of $\mathrm{P}_0(\mathbb{C}) \twoheadrightarrow \mathrm{M}_0(\mathbb{C})$. Now it suffices to notice that an element in $\mathrm{G}(\mathbb{C})$ stabilizes the Hodge decomposition if and only if it lies in $\mathrm{Cent}_{\mathrm{G}(\mathbb{C})}(h_0)$. $\square$

Lemma 5.3.2. *Let $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$. Then $\mathcal{E}_{\mathrm{M}_0,\mathrm{an}}(W) \to \mathsf{Sh}_{\mathcal{H}}$ is canonically isomorphic to*

$$\mathrm{G}(\mathbb{Q})\backslash(\mathrm{G}(\mathbb{R}) \overset{\mathcal{U}_\infty}{\times} W) \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H} \to \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathcal{U}_\infty\mathcal{H}$$

Therefore, for $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M}_0)$, the sections of $\mathcal{E}_{\mathrm{M}_0,\mathrm{an}}(W) \to \mathsf{Sh}_{\mathcal{H}}$ can be represented by functions $f : \mathrm{G}(\mathbb{A}) \to W$ satisfying $f(\gamma g u_\infty u) = u_\infty^{-1} f(g)$ for $\gamma \in \mathrm{G}(\mathbb{Q})$, $g \in \mathrm{G}(\mathbb{A})$, $u_\infty \in \mathcal{U}_\infty$, and $u \in \mathcal{H}$.

By Corollary 3.6.10, elements ℓ in $\mathbf{S}_{\Phi^{(g)}_1} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ correspond bijectively to smooth functions $q_\ell : \mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{A}) \to \mathbb{C}^\times$ satisfying $q_\ell(\gamma g) = q_\ell(g)$ for any $\gamma \in \mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{Q})$ and $g \in \mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{A})$. (Here smoothness on the factor $\mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{A}^\infty)$ of $\mathrm{U}_{2,\mathbf{F}^{(g)}}(\mathbb{A})$ means right invariance by some sufficiently small open compact subgroup.) If $\ell \in \mathbf{S}_{\Phi^{(g)}_{\mathcal{H}}}$, then the corresponding function q_ℓ is right-invariant under $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{A}^\infty)\cap\mathcal{H}$, and vice versa. Such functions q_ℓ correspond canonically to the algebraic characters of $\Gamma_{\mathcal{H}}^{\mathsf{F}^{(g)},\mathrm{U}_2}\backslash\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{C}) \overset{\text{can.}}{\overset{\sim}{\to}} E_{\Phi_{\mathcal{H}}^{(g)}}(\mathbb{C})$. If we denote by $\Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}(\ell)$ the pullback of the line bundle $\Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(\ell)$ over $C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}$ (as in [23, §6.2.4]) under $C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}} \to C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}$, then we can describe sections of the line bundle $\Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}(\ell)$ over $C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}$ by multiplying q_ℓ by a function on $\mathsf{X}_1^{\mathsf{F}^{(g)}}$. (See Lemma 3.5.11 and Corollary 3.5.12.)

Suppose f is invariant under $\mathcal{H}$. Then the integral

$$\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f) := \int_{\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{Q})\backslash\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{A})} f(ng)\, q_\ell(n)^{-1}\, dn$$

is nonzero only when $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}$. This allows us to write f as an infinite sum

$$f = \sum_{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}} \mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f)\, q_\ell.$$

Here each coefficient $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f)$ represents a section of $\mathcal{E}_{\mathrm{M}_0,\mathrm{an}}(W)$ invariant under the action of $\mathrm{U}_{2,\mathsf{F}^{(g)}}(\mathbb{R})$ on X_0. By Corollary 5.2.17, each function $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f)$ represents a section of $\mathcal{E}_{\mathrm{M}_0,\mathrm{an}}^{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(W)$ over $\mathsf{X}_1^{\mathsf{F}^{(g)}}$. Therefore, each function $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f)\, q_\ell$ represents a section over $\mathsf{X}_2^{\mathsf{F}^{(g)}}$ of the vector bundle $\Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}}}{\otimes} \mathcal{E}_{\mathrm{M}_0,\mathrm{an}}^{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(W)$ over $C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}$.

Definition 5.3.3. *The* **ℓ-th analytic Fourier–Jacobi morphism**

$$\begin{aligned}\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}} : \Gamma(\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathrm{an}}(W)) \\ \to \Gamma(C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}, \Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}}}{\otimes} \mathcal{E}_{\mathrm{M}_0,\mathrm{an}}^{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(W))\end{aligned}$$

(along $(\mathsf{Z}^{(g)},\Phi^{(g)},\delta^{(g)})$*) is defined by sending a section represented by some function* $f : \mathrm{G}(\mathbb{A}) \to W$ *to the section represented by* $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathrm{an}}(f)\, q_\ell$.

On the other hand, algebraic Fourier–Jacobi expansions are defined simply using the geometric structure of the boundary. According to the construction of $\mathfrak{X}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\sigma,\mathbb{C}} \to C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}$ as a formal completion, we have a natural homomorphism $(\mathfrak{X}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\sigma,\mathbb{C}} \to C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}})_* \mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\sigma,\mathbb{C}}} \to \prod_{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}^{(g)}}} \Psi_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)}}(\ell)$ of $\mathscr{O}_{C_{\Phi_{\mathcal{H}}^{(g)},\delta_{\mathcal{H}}^{(g)},\mathbb{C}}}$-algebras. By Lemma 5.2.16, we have the composition of canonical morphisms

$$\begin{aligned}
&\Gamma(\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathbb{C}}(W)) \\
&\quad \to \Gamma(\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}, (\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to \mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}})^* \, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathbb{C}}(W)) \\
&\quad \cong \Gamma(\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}}, (\mathfrak{X}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\sigma,\mathbb{C}} \to C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}})^* \, \mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W)) \\
&\quad \to \prod_{\ell \in \mathbf{S}_{\Phi^{(g)}_{\mathcal{H}}}} \Gamma(C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}, \Psi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}}}{\otimes} \mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W)),
\end{aligned}$$

denoted by $\mathrm{FJ}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}$, which we call the morphism of *algebraic Fourier–Jacobi expansions.*

Definition 5.3.4. *The* **ℓ-th algebraic Fourier–Jacobi morphism**

$$\begin{aligned}
\mathrm{FJ}^{(\ell)}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}} :\ & \Gamma(\mathsf{Sh}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{alg}}, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,\mathbb{C}}(W)) \\
& \to \Gamma(C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}, \Psi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}}}{\otimes} \mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W))
\end{aligned}$$

(along $(\mathsf{Z}^{(g)}, \Phi^{(g)}, \delta^{(g)})$*) is the ℓ-th factor of the morphism* $\mathrm{FJ}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}$ *of algebraic Fourier–Jacobi expansions.*

Theorem 5.3.5. *For any* $\ell \in \mathbf{S}_{\Phi^{(g)}_{\mathcal{H}}}$*, the morphism* $\mathrm{FJ}^{(\ell)}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathrm{an}}$ *can be canonically identified with the analytification of* $\mathrm{FJ}^{(\ell)}_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}$.

Proof. This follows from the above constructions and the second statement of Corollary 5.2.17. □

Remark 5.3.6. Suppose $\mathrm{Gr}^{\mathsf{V}}_{-1} = \{0\}$, or equivalently $\mathrm{Gr}^{\mathsf{F}^{(g)}}_{-1} = \{0\}$ or $\mathrm{Gr}^{\mathsf{Z}^{(g)}}_{-1} = \{0\}$. Then $C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}$ is zero-dimensional (and reduced), and sections of $\Gamma(C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}, \Psi_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}},\mathbb{C}}}}{\otimes} \mathcal{E}^{\Phi^{(g)}_{\mathcal{H}},\delta^{(g)}_{\mathcal{H}}}_{\mathrm{M}_0,\mathbb{C}}(W))$ are represented by W-valued functions on a finite set. In such cases, the Fourier–Jacobi expansions are often called q-expansions (because no "Jacobi theta functions" are involved), and Theorem 5.3.5 says that the analytic and algebraic q-expansions agree under the canonical identifications.

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