INTEGRAL MODELS OF TOROIDAL COMPACTIFICATIONS WITH PROJECTIVE CONE DECOMPOSITIONS

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Abstract. We construct integral models of toroidal compactifications of PEL-type Shimura varieties with projective cone decompositions as normalizations of certain explicit blowups of the corresponding minimal compactifications, generalizing works of Tai's, Chai's, Faltings and Chai's, and the author's in zero or good reduction characteristics. We show that such integral models still enjoy many features of the good reduction theory, regardless of the levels and ramifications involved.

Contents

1. Introduction

In the works of Tai's [1, Ch. IV, Sec. 2], Chai's [4, Ch. IV], Faltings and Chai's [5, Ch. V, Sec. 5], and the author's [15, Sec. 7.3] in zero or good reduction characteristics, it was shown that, with the notation in [15], when the level $\mathcal{H}$ of the relevant Shimura variety or PEL moduli problem $\mathsf{M}_{\mathcal{H}}$ is *neat*, the toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ defined by a compatible collection Σ of *projective* cone decompositions (satisfying certain other running assumptions in each of the works) is canonically isomorphic to the normalization $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}}$ of some explicit blowup of the minimal compactification $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$, where pol is a compatible collection of polarization functions for the corresponding Σ, and where $d_0 \geq 1$ is some integer depending on pol.

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In this article, we will show that, when the image $\mathcal{H}^p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ is *neat*, such normalizations of blowups provide p-integral models $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}}$ of toroidal compactifications of PEL-type Shimura varieties in all characteristics, which still enjoy many features of the good reduction theory, regardless of the levels and ramifications involved at p. For example, we will show that they admit stratifications by locally closed subschemes, with formal completions along the strata comparable with the completions of certain putative boundary charts parameterizing degeneration data of PEL structures, extending the familiar ones in zero and good reduction characteristics. We will also show that they carry semi-abelian schemes which are universal in a sense that can be made precise using the theory of degeneration of PEL structures developed in [5] and [15].

The idea will be to make use of the integral models $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ constructed by taking normalizations over good reduction auxiliary models as in [18, Sec. 7] (where the Σ' here is the Σ there), which were constructed only for those Σ' induced by certain auxiliary ones; and compare them with $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}}$ with the help of the putative boundary charts as in [18, Sec. 8] defined for some common projective smooth refinements Σ'' of Σ and Σ'. As a result, we can construct $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (with desired properties) not just for those Σ's induced by auxiliary ones as in [18, Sec. 7], but for all projective Σ's (satisfying the mild [15, Cond. 6.2.5.25]).

While for many applications the choices of cone decompositions hardly matter, such a construction still has the following advantages.

Firstly, we now have a uniform construction of integral models of toroidal compactifications in arbitrarily ramified characteristics, for a large and familiar class of cone decompositions which can be qualitatively described, without the need to even mention any auxiliary choices of good reduction models of toroidal compactifications. While it is still true that we need the auxiliary models in the proofs, the fact that the constructions and results can be formulated without them is not meaningless. By more practically knowing which cone decompositions are allowed in the constructions, we can more easily generalize arguments involving simultaneous refinements of cone decompositions (see, for example, [14, Prop. 3.19]). Hence, we consider the construction here a practical improvement over that in [18].

Secondly, we can write down invertible sheaves over the integral models of toroidal compactifications that are relatively ample over the corresponding integral models of minimal compactifications (see Corollary 6.7 below). Such relatively ample invertible sheaves have played crucial roles in many of our earlier works in good reduction characteristics, such as [20], [21], and [17]. (See, for example, the results in Section 8.) We believe that they should be provided in any sufficiently complete theory of toroidal and minimal compactifications.

Thirdly, even for $\mathcal{A}_g$, the Siegel moduli of principally polarized abelian schemes of relative dimension g, it is not clear whether one can construct its toroidal compactification, with the usual expected properties (other than smoothness), for all (possibly nonsmooth) cone decompositions (see [24, Rem. 4.1.10]). Although we have not addressed this issue either—indeed, our assumption that the level is neat trivially ruled out $\mathcal{A}_g$—at least at neat levels, the construction in this article will allow all projective cone decompositions satisfying the relatively mild [15, Cond. 6.2.5.25]. (In particular, even in good reduction characteristics, we will obtain integral models of toroidal compactifications not already constructed in [5] and [15].)

Fourthly, except at places where we consider semi-abelian schemes over the integral models of toroidal compactifications, the rest of the arguments will not just work for the PEL-type setting, but also for more general types of Shimura varieties, as soon as good integral models of the minimal compactifications and some (possibly rather restrictive) classes of good toroidal compactifications have been constructed.

Here is an outline of this article.

In Section 2, we introduce the PEL moduli problem $\mathsf{M}_{\mathcal{H}}$ at level $\mathcal{H}$ in characteristic zero, and review the notion of compatible collections of projective cone decompositions and their polarization functions, which we denote by the symbols Σ and pol, respectively, and summarize (after some minor modification or correction) certain known facts in the literature that will be used later. (We also take this opportunity to fix a minor error in the literature; see Remark 2.15 below.)

In Section 3, we construct certain integral models $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ of toroidal compactifications of $\mathsf{M}_{\mathcal{H}}$ as normalizations $\mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})$ of certain blowups of the integral models $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ of minimal compactifications of $\mathsf{M}_{\mathcal{H}}$ constructed in [18, Sec. 6], along some coherent $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}$-ideals $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ defined by multiples $d\mathsf{pol}$ of pol, for $d \in \mathbb{Z}_{\geq 1}$.

In Section 4, for each representative $(\Phi_{\mathcal{H}}, \mathsf{Z}_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, which define some stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, by studying the pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ to certain putative boundary chart $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ (based on the constructions in [15, Sec. 6.2] and [18, Sec. 8]), we show that there is a canonical morphism from $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ to $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}})]}}$, the formal completion of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ along the preimage of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}})]}$ in $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$, for all sufficiently divisible d.

In Section 5, we show that such canonical morphisms are isomorphisms for all sufficiently divisible d. Then we deduce from this and from general facts about blowups that, for all sufficiently divisible d, the schemes $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ are canonically isomorphic to each other. Henceforth, we may and we shall abusively write $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ instead of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$. (It will be justified later that $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ does not depend on pol at all.) Moreover, the above-mentioned isomorphisms allow us to stratify $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ by locally closed subschemes with familiar parameterizations and incidence relations (as in [15, Thm. 6.4.1.1(2)]), such that the formal completions of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ along the strata are canonically isomorphic to the corresponding formal completions of the putative boundary charts. Based on this, by a descent argument as in [18, Sec. 11], we show that the tautological objects over $\mathsf{M}_{\mathcal{H}}$ extend to semi-abelian degenerating families over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.

In Section 6, we summarize our main results in Theorem 6.1, in a format similar to those of [15, Thm. 6.4.1.1 and 7.2.4.3]. Moreover, in the same theorem, we also state and prove that $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is universal among base schemes carrying semi-abelian degenerations of certain patterns determined by Σ. In particular, up to canonical isomorphism, $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ depends only on Σ, but not on pol. (This, finally, justifies the notation of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.) The statements are admittedly very lengthy, but in practice we have found it more useful to have a place where almost all important information can be found. We also include the Corollary 6.7 concerning invertible sheaves over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ that are relatively ample over $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, and record some byproducts concerning local properties along the boundary (as in [18, Sec. 14]).

In Section 7, we explain how the functorial morphisms and Hecke actions in [18, Sec. 13] can be defined for the toroidal compactifications constructed here. We also record some important facts about higher direct images of structural sheaves and boundary ideals under the canonical morphisms between toroidal compactifications, and about the canonical extensions of relative first de Rham homology groups of the tautological abelian schemes over the integral models.

In Section 8, we show that, under mild assumptions on the coefficient modules, the analogue of the vanishing of higher direct images under the canonical morphisms from toroidal compactifications to minimal compactifications as in [17, Thm. 3.9] remain valid in the context of this article. (Such vanishing have played crucial roles in several recent developments in the constructions of p-adic automorphic forms and p-adic Galois representations. See the overviews in [10], [16, Sec. 8.2], [19], and [17]. The generalization here is not completely routine, because it allows nonordinary loci and arbitrary levels and ramifications.) When $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a simple $\mathbb{Q}$-algebra, we also show that the analogue of Koecher's principle as in [17, Thm. 2.3] holds here. (Both of these allow general coefficient rings not necessarily flat over $\mathbb{Z}_{(p)}$.)

We shall follow [15, Notation and Conventions] unless otherwise specified. While for practical reasons we cannot explain everything we need from [15], we recommend the reader to make use of the reasonably detailed index and table of contents there, when looking for the numerous definitions. We recommend that the reader go through the review materials in [14, Sec. 1; see also the errata] before reading Section 2 below. It is not necessary to have completely mastered the techniques in [18] before reading this article.

2. Projective cone decompositions

Suppose we have an integral PEL datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$, where $\mathcal{O}$ is an order in a semisimple algebra finite-dimensional over $\mathbb{Q}$, together with a positive involution $\star$, and where $(L, \langle \cdot\,, \cdot \rangle, h_0)$ is a PEL-type $\mathcal{O}$-lattice as in [15, Def. 1.2.1.3], which defines a group functor G over $\operatorname{Spec}(\mathbb{Z})$ as in [15, Def. 1.2.1.6], the *reflex field* F_0 (as a subfield of $\mathbb{C}$) as in [15, Def. 1.2.5.4], and a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 := \operatorname{Spec}(F_0)$ as in [15, Def. 1.4.1.4] (with $\Box = \emptyset$ there). Suppose that L satisfies [15, Cond. 1.4.3.10]. (This is harmless in practice, as explained in [15, Rem. 1.4.3.9].)

Definition 2.1 (cf. [15, Cond. 6.3.3.2 and Def. 6.3.3.4])**.** *A compatible collection of* **admissible rational polyhedral cone decomposition data** *for* $\mathsf{M}_{\mathcal{H}}$ *is a complete set* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *of compatible choices of* $\Sigma_{\Phi_{\mathcal{H}}}$ *(satisfying* [15, Cond. 6.2.5.25]*) such that, for every surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *of representatives of cusp labels, the cone decompositions* $\Sigma_{\Phi_{\mathcal{H}}}$ *and* $\Sigma_{\Phi'_{\mathcal{H}}}$ *define a surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *as in* [15, Def. 6.2.6.4].

Definition 2.2. *We shall say that a compatible collection* Σ *in Definition* 2.1 *is* **smooth** *if the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is smooth as in* [15, Def. 6.1.1.12]*, for each representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of cusp label for* $\mathsf{M}_{\mathcal{H}}$*.*

Remark 2.3. Every Σ induced by some auxiliary choices as in [15, Sec. 7] is a (possibly nonsmooth) compatible collection as in Definition 2.1.

Remark 2.4. We remind the reader that, for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp label, the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ is a decomposition (satisfying certain properties) of the set $\mathbf{P}_{\Phi_{\mathcal{H}}}$ of positive semi-definite $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-valued Hermitian pairings with rational radicals over some $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-module defined by $\Phi_{\mathcal{H}}$. (See the beginning of [15, Sec. 6.2.5], and the references from there to earlier sections in [15].) In addition to our recommendation of [14, Sec. 1] at the end of Section 1, the reader might also find the review materials in [16, Sec. 1.1.1, 1.1.2, 1.2.1, 1.2.2, and 1.3.1] helpful.

Definition 2.5 (see [15, Def. 7.3.1.1])**.** *Let* $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j \in J}$ *be any* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$*. An* (**invariant**) **polarization function** *on* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *for the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is a* $\Gamma_{\Phi_{\mathcal{H}}}$*-invariant continuous piecewise linear function* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *such that:*

(1) $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear* (*i.e., coincides with a linear function*) *on each cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}}$*.* (*Therefore,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(tx) = t\mathsf{pol}_{\Phi_{\mathcal{H}}}(x)$ *for all* $x \in \mathbf{P}_{\Phi_{\mathcal{H}}}$ *and* $t \in \mathbb{R}_{\geq 0}$*.*)
(2) $\mathsf{pol}_{\Phi_{\mathcal{H}}}((\mathbf{P}_{\Phi_{\mathcal{H}}} \cap \mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}) - \{0\}) \subset \mathbb{Z}_{>0}$*.* (*Therefore,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x) > 0$ *for all nonzero* x *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$*.*)
(3) $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear* (*in the above sense*) *on a rational polyhedral cone* σ *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *if and only if* σ *is contained in some cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}}$*.*
(4) *For all* $x, y \in \mathbf{P}_{\Phi_{\mathcal{H}}}$*, we have* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x+y) \geq \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) + \mathsf{pol}_{\Phi_{\mathcal{H}}}(y)$*. This is called the* **convexity** *of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$*.*

If such a polarization function exists, then we say that the $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is* **projective***.*

Proposition 2.6 (cf. [1, Ch. II], [5, p. 173], and [15, Prop. 7.3.1.2])**.** *Suppose* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *is any polarization function as in Definition* 2.5*.*

(1) $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} := \{x \in \mathbf{P}_{\Phi_{\mathcal{H}}} : \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) \geq 1\}$ *is a convex subset of* $\mathbf{P}_{\Phi_{\mathcal{H}}} - \{0\}$ *such that* $\mathbb{R}_{\geq 1} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *and* $\mathbb{R}_{\geq 0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \supset \mathbf{P}_{\Phi_{\mathcal{H}}}$*, whose closure* $\overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *in* $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ *is a* **cocore** *in the context of* [1, Ch. II, Sec. 5]*. For simplicity, we shall also call* $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *a cocore.*
(2) *The dual* $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y \rangle \geq 1 \ \forall y \in K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}$ *of* $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *is a convex subset in* $(\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}})^{\circ}$*, the interior of* $\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}}$*, such that* $\mathbb{R}_{\geq 1} \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *and* $\mathbb{R}_{>0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = (\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}})^{\circ}$*, which is a* **core** *in the context of* [1, Ch. II, Sec. 5]*.*
(3) *The top-dimensional cones* σ *in the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *correspond bijectively to the vertices* ℓ *of the core* $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$*, which are linear forms whose restrictions to each* σ *coincide with the restriction of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *to* σ*.*

Definition 2.7 (cf. [15, Def. 7.3.1.3])**.** *We say that a compatible collection* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *as in Definition* 2.1 *is* **projective** *if it satisfies the following condition: There is a compatible collection* $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *of polarization functions labeled by representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of cusp labels, each* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *being a polarization function of the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *in* Σ (*see Definition* 2.5)*, which are* **compatible** *in the following sense: For every surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *of representatives of cusp labels* (*see* [15, Def. 5.4.2.12]) *inducing an embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$*, we have* $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$*. In this case, because of condition* (3) *of Definition* 2.5*, we also say that* Σ *is* **induced** *by* pol*.*

Proposition 2.8 (cf. [15, Prop. 6.3.3.5 and 7.3.1.4] and [16, Prop. 1.2.2.17])**.**

(1) *A compatible choice Σ of admissible rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$, as in Definition* 2.1, **exists**. *Moreover, we may assume that Σ is smooth as in Definition* 2.2, *or projective as in Definition* 2.7, *or both.*
(2) *Given any Σ and Σ', we can find a common refinement for them, which we may require to be smooth, or projective, or both. The same is true if we allow varying levels or twists by Hecke actions* (*see* [15, Def. 6.4.2.8 and 6.4.3.2]). *We may assume that this common refinement is invariant under any choice of an open compact subgroup $\mathcal{H}'$ of $\mathrm{G}(\mathbb{A}^\infty)$ normalizing $\mathcal{H}$.*

Proof. See the proof of [16, Prop. 1.2.2.17] and the references made there (to [25, 5.21, 5.23, 5.24, 5.25] and [12, Ch. I, Sec. 2, Thm. 11 on pp. 33–35]). □

Remark 2.9. By the last statement of [9, Prop. 3.4], since we have assumed (for simplicity) that the auxiliary compatible collections of cone decompositions in [15, Sec. 7] are all projective, the induced Σ there is *projective* as in Definition 2.7.

Lemma 2.10 (cf. [5, Ch. V, Lem. 5.3] and [15, Lem. 7.3.1.7]). *Let Σ and pol be as in Definition* 2.7. *For each open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, there is an open compact subgroup $\mathcal{H}' \subset \mathcal{H}$* (*which can be taken to be normal*) *such that the compatible collections $\Sigma^{(\mathcal{H}')} = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\mathsf{pol}^{(\mathcal{H}')} = \{\mathsf{pol}_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ defined in* [15, Constr. 7.3.1.6] *for $\mathsf{M}_{\mathcal{H}'}$ satisfy the following condition: For each lifting $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ of $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ to level $\mathcal{H}'$, and for each vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}'}}}$ corresponding to a top-dimensional cone σ_0 in $\Sigma_{\Phi_{\mathcal{H}'}}$, we have*

$$\langle \ell_0, x\rangle < \langle \gamma\cdot\ell_0, x\rangle \tag{2.11}$$

for all $x \in \overline{\sigma}_0 \cap \mathbf{P}^+_{\Phi_{\mathcal{H}'}}$ and all $\gamma \in \Gamma_{\Phi_{\mathcal{H}'}}$ such that $\gamma \neq 1$.

Lemma 2.12 (cf. [5, Ch. V, Lem. 5.4]). *Let σ be a top-dimensional cone in $\Sigma_{\Phi_{\mathcal{H}}}$, which corresponds to a vertex $\ell_{\sigma,0}$ of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. Then there exist elements $\ell_{\sigma,1}, \ldots, \ell_{\sigma,n_\sigma}$ of $\mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$* (*which are not necessarily vertices of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$*) *such that*

$$\mathbb{R}_{\geq 0}\cdot\sigma^\vee = \sum_{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}\cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}} \mathbb{R}_{\geq 0}\cdot(\ell-\ell_{\sigma,0}) = \sum_{1\leq i\leq n_\sigma} \mathbb{R}_{\geq 0}\cdot(\ell_{\sigma,i}-\ell_{\sigma,0}). \tag{2.13}$$

Proof. Let $\tau_1, \ldots, \tau_r$ be all the one-dimensional faces of σ. For each $1 \leq j \leq r$, consider the unique element y_j of τ_j such that $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}} \cap \tau_j = \mathbb{Z}_{\geq 1}\cdot y_j$, so that $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \cap \tau_j = \mathbb{R}_{\geq 1}\cdot(\mathsf{pol}_{\Phi_{\mathcal{H}}}(y_j)^{-1}y_j)$. For each j, let $L_j := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} \otimes_{\mathbb{Z}} \mathbb{R} : \langle x, y_j\rangle = \mathsf{pol}_{\Phi_{\mathcal{H}}}(y_j)\}$. Then $\ell_{\sigma,0}$ is the intersection of $L_1, \ldots, L_r$ by definition, and $L_j \cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ is a top-dimensional face of $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ for each j, whose vertices are in $\mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ because $y_j \in \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$ and $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ takes integral values on $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$. Consequently, the $\mathbb{R}_{\geq 0}$-span of $\ell - \ell_{\sigma,0}$ for all $\ell \in K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ can be identified with the two outside members of (2.13), for some finitely many $\ell_{\sigma,1}, \ldots, \ell_{\sigma,n_\sigma} \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, as desired. □

Remark 2.14 (cf. [5, Ch. V, Sec. 5, p. 175, Rem.]). The integral version of Lemma 2.12 is not true in general. We cannot replace $\mathbb{R}_{\geq 0}$ with $\mathbb{Z}_{\geq 0}$ in (2.13). This difference is immaterial because we are taking normalizations of the blowups we consider. (But this is one of the reasons that we have to take normalizations.)

Remark 2.15 (Erratum). The literal statements of [5, Ch. V, Lem. 5.4], which are stronger than those of Lemma 2.12, are unfortunately flawed. For example, if $\mathbf{P}^+_{\Phi_{\mathcal{H}}} = \mathbb{R}_{>0} = \sigma$, then there are no other top-dimensional cones at all, and hence [5, Ch. V, Lem. 5.4] asserts that $\sigma^\vee = \{0\}$—but $\sigma^\vee$ is certainly nonzero. This error was inherited from a similar error in [1, Ch. IV, Sec. 2, p. 330] (which was still present in the recent revision in [2, Ch. IV, Sec. 2, p. 211]), and was in turn inherited by [5, Ch. V, Lem. 5.5] and [15, Lem. 7.3.1.9]. Nevertheless, all of these can be fixed by slightly weakening their statements, by also allowing some of $\ell_{\sigma,1}, \ldots, \ell_{\sigma,n_\sigma}$ to be non-vertices as in Lemma 2.12, which still suffice for the proofs in [1, Ch. IV, Sec. 2], [5, Ch. V, Sec. 5], and [15, Sec. 7.3.3] (after relatively minor changes). (See the errata of [15] and the latest revision of [13], both available at the author's website.)

3. Main constructions

Construction 3.1 (cf. [15, Def. 7.3.3.1]). Let Σ be a compatible collection that is projective, with a compatible collection pol of polarization functions, and let Σ'' be a projective smooth refinement of Σ (which always exists by Proposition 2.8), as in Definitions 2.1, 2.2, and 2.7. (If Σ is already smooth, we may take Σ'' to be Σ itself.) Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ be as in [15, Thm. 6.4.1.1 and 7.2.4.3] (which is a scheme projective and smooth over $\mathsf{S}_0 = \operatorname{Spec}(F_0)$). By [15, Thm. 6.4.1.1(3)], the complement $\mathsf{D}_{\infty,\mathcal{H},\Sigma''}$ of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ (with its reduced structure) is a relative Cartier divisor with normal crossings, each of whose irreducible components is an irreducible component of some $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ that is the closure of some strata $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ labeled by the equivalence class $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ of some triple $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ with σ a one-dimensional cone in the cone decomposition $\Sigma''_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$. Let $\mathcal{I}_{\mathcal{H},\mathsf{pol},\Sigma''}$ be the invertible sheaf of ideals over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ supported on $\mathsf{D}_{\infty,\mathcal{H},\Sigma''}$ such that the order of $\mathcal{I}_{\mathcal{H},\mathsf{pol},\Sigma''}$ along each $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is the value of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ at the $\mathbb{Z}_{>0}$-generator of $\sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$ for some (and hence every) representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$. This is well defined because of the compatibility condition for $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ as in Definition 2.7. For each $d \in \mathbb{Z}_{\geq 1}$, let $d\mathsf{pol}$ denote the compatible collection of polarization functions defined by multiplying all polarization functions in the collection pol by d. Then we have a canonical isomorphism $\mathcal{I}_{\mathcal{H},d\mathsf{pol},\Sigma''} \cong \mathcal{I}^{\otimes d}_{\mathcal{H},\mathsf{pol},\Sigma''}$.

Lemma 3.2. *In Construction* 3.1*, suppose there exists a refinement Σ' of Σ such that Σ'' is a refinement of Σ', and such that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is also defined* (*either as in* [15, Thm. 6.4.1.1] *when Σ' is smooth, or as in* [18, (7.10)] *when Σ' is induced by some auxiliary choices*)*, with a canonical proper surjection $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$* (*as in* [15, Prop. 6.4.2.3] *or* [18, Lem. 9.8]). *Then the coherent $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-ideal*

$$\mathcal{I}_{\mathcal{H},d\mathsf{pol},\Sigma'} := (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})_* \mathcal{I}_{\mathcal{H},d\mathsf{pol},\Sigma''} \tag{3.3}$$

is invertible, and depends only on Σ'. Thus, if Σ' can be taken to be Σ itself, then

$$\mathcal{I}_{\mathcal{H},d\mathsf{pol}} := (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_* \mathcal{I}_{\mathcal{H},d\mathsf{pol},\Sigma''} \tag{3.4}$$

is invertible and independent of the choice of Σ''.

Proof. Since invertibility of coherent sheaves can be checked by pullback to completions by fpqc descent (cf. [7, VIII, 1.11]), it suffices to show that, for each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$ and for each $\sigma \in \Sigma'_{\Phi_{\mathcal{H}}}$ satisfying $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, the pullback of $\mathcal{I}_{\mathcal{H},d\mathsf{pol}}$ to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (by [15, Thm. 6.4.1.1(5)] or [18, Thm. 10.13]) is invertible. Also, since the canonical morphism $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is proper, by [8, III-1, 4.1.5], taking direct images commute with pulling back to formal completions of the target. Consider (as in [18, (10.5), (10.6), (10.7), (10.8)]) the formal completion $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)'' := \underset{\tau\in\Sigma''_{\Phi_{\mathcal{H}}},\tau\subset\sigma}{\cup} \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\tau)$ along the closed subscheme $\Xi''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} := \underset{\tau\in\Sigma''_{\Phi_{\mathcal{H}}},\tau\subset\sigma}{\cup} \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, which induces a canonical proper surjection $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ which can be identified (as explained in the proof of [18, Thm. 10.13]) with the pullback of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$.

By the same argument as in the second paragraph of [15, proof of Thm. 7.3.3.4(1)], the pullback of $\jmath_{\mathcal{H},d\mathsf{pol},\Sigma''}$ to the open formal subscheme $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} := \mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)''}{\times} \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\tau)$ of $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ corresponds to the sub-$\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-module $\underset{\langle\ell,y\rangle\geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y),\forall y\in\tau}{\hat{\oplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ of $\mathscr{O}_{\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \cong \underset{\ell\in\tau^\vee}{\hat{\oplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, and hence (by [8, III-1, 4.1.5]) the pullback of $\jmath_{\mathcal{H},d\mathsf{pol}}$ to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ corresponds to the sub-$\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-module $\underset{\langle\ell,y\rangle\geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y),\forall y\in\sigma}{\hat{\oplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ of $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \underset{\ell\in\sigma^\vee}{\hat{\oplus}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ (where the sums $\hat{\oplus}$ all denote the formal completions with respect to the topology induced by that of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$), which is invertible because the restriction of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ to any cone in $\Sigma_{\Phi_{\mathcal{H}}}$ is a linear function by definition (see (1) of Definition 2.5), and because $\Sigma'_{\Phi_{\mathcal{H}}}$ and $\Sigma''_{\Phi_{\mathcal{H}}}$ are refinements of $\Sigma_{\Phi_{\mathcal{H}}}$. $\square$

Definition 3.5 (see [15, Sec. 7.3]; cf. [5, Ch. V])**.** *For any $\mathcal{H}$, Σ, pol, and Σ'' as in Construction* 3.1*, and for each $d \in \mathbb{Z}_{\geq 1}$, let*

$$\mathscr{J}_{\mathcal{H},d\mathsf{pol}} := \mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}} := (\oint_{\mathcal{H},\Sigma''})_*(\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}) \cong (\oint_{\mathcal{H},\Sigma''})_*\jmath_{\mathcal{H},d\mathsf{pol}}, \tag{3.6}$$

where the canonical morphism $\oint_{\mathcal{H},\Sigma''} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ is as in [15, Thm. 7.2.4.1(3)]. *Then we also define*

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} := \mathrm{NBl}_{\mathscr{J}_{\mathcal{H},d\mathsf{pol}}}(\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}), \tag{3.7}$$

where NBl. $(\cdot)$ *denotes the normalization of the blowup* (*see* [15, Def. 7.3.2.1]).

Remark 3.8. We introduced the notation $\mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ for the sake of consistency with [5, Ch. V] and [15, Sec. 7.3]. Later we will mainly use $\mathscr{J}_{\mathcal{H},d\mathsf{pol}}$ and $\jmath_{\mathcal{H},d\mathsf{pol}}$ in our exposition. Note that $\mathscr{J}_{\mathcal{H},d\mathsf{pol}}$ is a coherent $\mathscr{O}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}}$-ideal because $\oint_{\mathcal{H},\Sigma''}$ is proper and because the canonical morphism $\mathscr{O}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}} \to (\oint_{\mathcal{H},\Sigma''})_*\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}}$ is an isomorphism. By Lemma 3.2, $\mathscr{J}_{\mathcal{H},d\mathsf{pol}}$ does not depend on the choice of Σ'', and coincides with the $\mathscr{J}_{\mathcal{H},d\mathsf{pol}} = \mathscr{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ in [15, Sec. 7.3] when Σ is (projective and) smooth.

Let us introduce the following condition for any Σ and pol (cf. [15, Lem. 7.3.1.7]) as in Definition 2.7:

Condition 3.9. (*See* [15, Cond. 7.3.3.3]*; cf.* [1, Ch. IV, Sec. 2, p. 329] *and* [5, Ch. V, Sec. 5, p. 178].) *For each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$ and each vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional cone σ_0, we have*

$$\langle \ell_0, x\rangle < \langle \gamma\cdot\ell_0, x\rangle$$

for all $x \in \overline{\sigma}_0 \cap \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ and all $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$ such that $\gamma \neq 1$.

Then we have the following prototype for the later constructions and results:

Theorem 3.10 (see [15, Thm. 7.3.3.4] and [16, Thm. 1.3.1.10]; cf. [1, Ch. IV, Sec. 2.1, Thm.] and [5, Ch. V, Thm. 5.8])**.** *Suppose $\mathcal{H}$ is neat, and suppose Σ is projective smooth with a compatible collection* pol *of polarization functions, as in Definitions* 2.1*,* 2.2*, and* 2.7*. For each $d \in \mathbb{Z}_{\geq 1}$, suppose $\jmath_{\mathcal{H},d\mathsf{pol}}$ is defined over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as in Construction* 3.1 (*with $\Sigma'' = \Sigma$ there*)*, or equivalently as in* (3.4) (*by Lemma* 3.2)*, and suppose $\mathcal{J}_{\mathcal{H},d\mathsf{pol}}$ is defined over $\mathsf{M}^{\min}_{\mathcal{H}}$ as in Definition* 3.5*. Then there exists $d_0 \in \mathbb{Z}_{\geq 1}$ such that the canonical morphism $\oint^{-1}_{\mathcal{H},\Sigma} \mathcal{J}_{\mathcal{H},d_0\mathsf{pol}} \cdot \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \to \jmath_{\mathcal{H},d_0\mathsf{pol}}$ of coherent $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-ideals is an isomorphism, and such that the canonical morphism $\mathrm{NBl}_{\mathcal{J}_{\mathcal{H},d_0\mathsf{pol}}}(\oint_{\mathcal{H},\Sigma}) : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}} = \mathrm{NBl}_{\mathcal{J}_{\mathcal{H},d_0\mathsf{pol}}}(\mathsf{M}^{\min}_{\mathcal{H}})$ over $\mathsf{S}_0 = \mathrm{Spec}(F_0)$, induced by the universal property of the normalization of blowup* (*see* [15, Def. 7.3.2.2])*, is an isomorphism. In particular, $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a scheme projective over S_0. If Condition* 3.9 *is satisfied, then the above are true for all $d_0 \in \mathbb{Z}_{\geq 3}$.*

Remark 3.11*.* Theorem 3.10 serves as a prototype, but will not be needed in the later constructions and proofs. The results we will obtain, however, have no explicit control on the possible d_0's even when Condition 3.9 is satisfied.

Construction 3.12*.* Let $p > 0$ be any rational prime number. Let $\mathcal{H}$, Σ, pol, Σ'', $\jmath_{\mathcal{H},d\mathsf{pol},\Sigma''}$, $\jmath_{\mathcal{H},d\mathsf{pol}}$, and $\mathcal{J}_{\mathcal{H},d\mathsf{pol}}$ be as in Construction 3.1, Lemma 3.2, and Definition 3.5, for each $d \in \mathbb{Z}_{\geq 1}$, with the additional running assumption that the image $\mathcal{H}^p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ is *neat* (which means, a fortiori, that $\mathcal{H}$ is also neat; the neatness of $\mathcal{H}^p$ was the running assumption in [18]). Let $\vec{\mathsf{M}}_{\mathcal{H}}$ and $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ be constructed over $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ as in [18, Prop. 6.1 and 6.4], with a fixed choice of some lattice collection $\{(g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ as in [18, Sec. 2]. (In what follows, all objects denoted with an arrow on the top will mean the p-integral versions over $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ of the analogous characteristic zero objects over $\mathsf{S}_0 = \mathrm{Spec}(F_0)$, often constructed using certain auxiliary choices.) For each $d \in \mathbb{Z}_{\geq 1}$, let $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ be the coherent $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}$-ideal defining the schematic closure in $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ of the closed subscheme of $\mathsf{M}^{\min}_{\mathcal{H}}$ defined by the coherent $\mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H}}}$-ideal $\mathcal{J}_{\mathcal{H},d\mathsf{pol}}$; and let (cf. (3.7) and [16, Prop. 2.2.2.1])

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} := \mathrm{NBl}_{\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\min}_{\mathcal{H}}). \tag{3.13}$$

By construction, $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ is a normal scheme projective and flat over $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$. When Σ is (projective and) smooth, by Theorem 3.10, there is some $d_0 \in \mathbb{Z}_{\geq 1}$ such that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \cong \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}} \cong \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d_0\mathsf{pol}} \underset{\vec{\mathsf{S}}_0}{\times} \mathsf{S}_0$ over $\mathsf{S}_0 = \mathrm{Spec}(F_0)$. Our goal is to show that there also exists some (possibly much larger) $d \in \mathbb{Z}_{\geq 1}$ such that $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ satisfies sufficiently many desired properties, extending as many as possible those in [15, Thm. 6.4.1.1] in the good reduction case, which will then force $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ to be canonical—i.e., depending only on Σ and the linear algebraic data involved in the construction of $\vec{\mathsf{M}}_{\mathcal{H}}$, but not on the choices of pol and d.

4. Formal local description of ideal sheaves

Lemma 4.1. *Suppose $\bar{x}$ is a geometric point over the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \cong \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$* (*see* [18, Thm. 12.1 and 12.6])*. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be any representative of the cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$. As in* [18, Prop. 12.14], *let* $(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ *denote the completion of the strict localization of* $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ *at* $\bar{x}$, *let* $(\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} := \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}{\times} (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$, *and let* $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}} := \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}{\times} (\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}}$. *For each* $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, *let* $(\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ *denote the pullback of* $\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} := (\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$ *under the canonical morphism* $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, *where* $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ *is as in* [18, Prop. 8.7]. *Then the pullback* $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$ *of the* $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}$*-ideal* $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ *to* $(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ *can be identified with the subsheaf of* $\mathscr{O}_{(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}} \cong \big(\prod_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}\big)^{\Gamma_{\Phi_{\mathcal{H}}}}$ *consisting of sections whose nonzero terms are supported on those* $\ell \in d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ *(see* (2) *of Proposition* 2.6*).*

Proof. By definition (see [8, I, 9.5.1 and 9.5.4]), since the canonical morphism $\mathsf{M}^{\min}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ is quasi-compact and separated, the $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}$-ideal $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ is the kernel of $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}} \to (\mathsf{M}^{\min}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}})_*\big(\mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H}}}/\mathcal{J}_{\mathcal{H},d\mathsf{pol}}\big)$, and the formation of such a kernel is compatible with flat base change. Therefore, by taking any geometric point $\bar{x}'$ of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ specializing to $\bar{x}$ such that $(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}'} \to (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ is defined, it follows that $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$ is the kernel of $\mathscr{O}_{(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}} \to ((\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}'} \to (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}})_*\big(\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}'}}/(\mathcal{J}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}'}\big)$, and it suffices to show that $(\mathcal{J}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}'}$ corresponds to the subsheaf of $\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}'}} \cong \big(\prod_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}'}\big)^{\Gamma_{\Phi_{\mathcal{H}}}}$ consisting of sections whose nonzero terms are supported on those $\ell \in d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. Then the assertion follows from Proposition 2.6 and from the same argument as in the proof of [15, Thm. 7.3.3.4(1)], by computing $(\mathcal{J}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}'}$ using the pullback of the proper morphism $\oint_{\mathcal{H},\Sigma''} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\min}_{\mathcal{H}}$ to $(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}'}$ (by [8, III-1, 4.1.5]), which shows that $(\mathcal{J}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}'}$ is (by abuse of language) the *common intersection* of the $\mathscr{O}_{(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}'}}$-modules $\hat{\bigoplus}_{\langle\ell,y\rangle\geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y),\forall y\in\sigma} (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^{\wedge}_{\bar{x}'}$ (see the proof of Lemma 3.2), for all $\sigma \in \Sigma''_{\Phi_{\mathcal{H}}} \in \Sigma''$ satisfying $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, which has the desired form. □

Let us fix once and for all a collection $\{\ell_{\sigma,i}\}_{\sigma,i}$ as in Lemma 2.12, where σ runs through all top-dimensional cones in $\Sigma_{\Phi_{\mathcal{H}}}$, and where i runs through integers from 0 to n_σ, such that $n_{\gamma\sigma} = n_\sigma$ and $\ell_{\gamma\sigma,i} = \ell_{\sigma,i}$, for all $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$ and $0 \leq i \leq n_\sigma$.

Lemma 4.2. *With the above choice of the collection* $\{\ell_{\sigma,i}\}_{\sigma,i}$, *there exists* $d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \in \mathbb{Z}_{\geq 1}$ *such that, for each top-dimensional cone* σ *in* $\Sigma_{\Phi_{\mathcal{H}}}$, *for each integer* i *such that* $0 \leq i \leq n_\sigma$, *for each geometric point* $\bar{x}$ *over the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *of* $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ *as in Lemma* 4.1, *and for each* $d \in \mathbb{Z}_{\geq 1}\cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, *the invertible sheaf* $(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_{\sigma,i}))^{\wedge}_{\bar{x}} \cong (\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\sigma,i})^{\otimes d})^{\wedge}_{\bar{x}}$ *over* $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ *is very ample, and hence is generated by its global sections, which can be canonically identified with the sections of* $(\underline{\vec{\mathrm{FJ}}}^{(d\cdot\ell_{\sigma,i})}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ *over* $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$.

Proof. Since $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ and its strata (as in [18, Thm. 12.1 and 12.6]) are quasi-compact and separated over $\vec{\mathsf{S}}_0$, and since there are only finitely many $\Gamma_{\Phi_{\mathcal{H}}}$-orbits of cones in $\Sigma_{\Phi_{\mathcal{H}}}$ (by its admissibility), it suffices to show that, for each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, the invertible sheaf $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ over $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is relatively ample over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. By

Proposition 2.6, each such ℓ lies in the interior $(\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}})^\circ$ of $\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}}$. Therefore, there exists some $N \in \mathbb{Z}_{\geq 1}$ such that $N \cdot \ell$ is the image of some $(\ell_{\mathrm{j,aux}})_{\mathrm{j} \in \mathrm{J}} \in \prod_{\mathrm{j} \in \mathrm{J}} \mathbf{S}_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}}}$ under the morphism $(\mathbf{S}_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}}})_{\mathbb{Q}} \twoheadrightarrow (\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{Q}}$ in [18, (5.21)], so that $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)^{\otimes N} \cong \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell)$ is isomorphic to the pullback of $\boxtimes_{\mathrm{j} \in \mathrm{J}} \Psi_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}(\ell_{\mathrm{j,aux}})$ under the finite morphism $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \prod_{\mathrm{j} \in \mathrm{J}} C_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}$ in [18, (8.6)], where $\ell_{\mathrm{j,aux}}$ lies in the interior of $\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}}}$ for each $\mathrm{j} \in \mathrm{J}$. Since each such $\Psi_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}(\ell_{\mathrm{j,aux}})$ over $C_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}$ is relatively ample over $\mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}^{\mathsf{Z}_{\mathcal{H}_{\mathrm{j,aux}}}}$ by the same argument as in the proof of [15, Thm. 7.3.3.4(1)], the pullback $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N \cdot \ell)$ of $\boxtimes_{\mathrm{j} \in \mathrm{J}} \Psi_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}(\ell_{\mathrm{j,aux}})$ to $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is also relatively ample over $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$; and so is $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, as desired. $\square$

Remark 4.3. If $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}} \to (\vec{\mathsf{M}}_{\mathcal{H}}^{\Phi_{\mathcal{H}}})^\wedge_{\bar{x}}$ is an abelian scheme torsor, then it suffices to take $d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = 3$, by Lefschetz's theorem (see, for example, [23, Sec. 17, Thm., p. 163] for the very ampleness of the pullback of $(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d \cdot \ell_0))^\wedge_{\bar{x}}$ to the fiber over $\bar{x}$, and see [23, Sec. 5] and [8, III-2, 7.7.5 and 7.7.10] for the base change argument). However, $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}} \to (\vec{\mathsf{M}}_{\mathcal{H}}^{\Phi_{\mathcal{H}}})^\wedge_{\bar{x}}$ is not an abelian scheme torsor in general.

Definition 4.4. *We define d_{pol} to be the smallest $d \in \mathbb{Z}_{\geq 1}$ such that, for every representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, there exists some $d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \in \mathbb{Z}_{\geq 1}$ as in Lemma 4.2 such that $d \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. (Note that each $d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ can be chosen to depend only on the cusp label represented by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, and there are only finitely many cusp labels for $\mathsf{M}_{\mathcal{H}}$.)*

Construction 4.5. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be any representative of cusp label for $\mathsf{M}_{\mathcal{H}}$. For each $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$, as in [15, Sec. 6.2.5] and [18, Sec. 8], we define

$$\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) := \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}} \Big(\bigoplus_{\ell \in \sigma^\vee} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \Big), \tag{4.6}$$

$$\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}} \Big(\bigoplus_{\ell \in \sigma^\perp} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \Big), \tag{4.7}$$

and

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} := (\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma))^\wedge_{\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}}. \tag{4.8}$$

(These constructions do not require σ to be either smooth or induced by some auxiliary choices.) As in [15, Sec. 6.2.5], let us also define the toroidal embedding

$$\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \hookrightarrow \vec{\overline{\Xi}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} = \bigcup_{\sigma \in \Sigma_{\Phi_{\mathcal{H}}}} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \tag{4.9}$$

using the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ (cf. [15, Thm. 6.1.2.8]), and define $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ to be the formal completion of $\vec{\overline{\Xi}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ along the union of $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ for all $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ satisfying $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$. For each such σ, we also define

$$\vec{\mathfrak{X}}^\circ_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} := \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \underset{\vec{\overline{\Xi}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}}{\times} \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}. \tag{4.10}$$

By their constructions, by [18, Prop. 8.14] (for the case of $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$), and by [8, IV-2, 7.8.3], these schemes and formal schemes are all normal (i.e., all the local rings are normal). By definition, we have canonical morphisms $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$, where the second one is an open immersion of formal schemes. We shall tacitly extend such definitions to the cases with other cone decompositions.

Lemma 4.11. *Suppose Σ' and Σ'' are two compatible collections as in Definition* 2.1 *such that Σ'' is a refinement of Σ'. Then there is a canonical proper morphism $\vec{\overline{\Xi}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} \to \vec{\overline{\Xi}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}$ inducing a canonical proper surjection*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}. \tag{4.12}$$

Proof. These follow from the definitions of the toroidal embeddings. □

Lemma 4.13. *For any compatible collection Σ' for which $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is constructed as in* [18, Sec. 7]*, we have canonical morphisms*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to (\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}, \tag{4.14}$$

where $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ is the formal completion of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ along the preimage of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ (*see* [18, Thm. 12.1])*, and where the third morphism* (4.14) *is the proper surjection induced by the canonical* (*necessarily proper and surjective*) *morphism $\vec{\oint}_{\mathcal{H},\Sigma'} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ as in* [18, Thm. 7.11]*.*

Proof. Let us construct the canonical isomorphism in (4.14). By construction, the formal scheme $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}$ is covered by its open formal subschemes $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\rho}$, for $\rho \in \Sigma'_{\Phi_{\mathcal{H}}}$ satisfying $\rho \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, each of which carries tautological tuples as in [18, (8.25)], so that, by Mumford's construction as in [15, Sec. 6.2.5], we have the corresponding Mumford families over $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\rho}$ (cf. [15, Sec. 6.2.5] and [18, (8.29)]), which induces compatible morphisms $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\rho} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ by the universal property of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ as in [18, Thm. 7.14 and 11.4], which patch together and define a canonical morphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$. On the other hand, the pullbacks of the tautological tuples over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ (as in [18, Thm. 11.4]) to $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ define degeneration data parameterized by $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}}$, which induce a canonical morphism $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}}$. These two canonical morphisms are inverses of each other by applying [15, Thm. 5.3.1.19] (to degenerating families of types $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, for all $\mathrm{j} \in \mathrm{J}$), which only require the affine open formal subschemes to have good generic characteristics. Hence they are both isomorphisms as desired. □

Lemma 4.15. *For each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, there exist canonical morphisms*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}, \tag{4.16}$$

where $(\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ is the formal completion of $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ along the locally closed subscheme $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (*cf. the explanation in* [15, Thm. 6.4.1.1(5)])*, and where the second morphism is proper and surjective, satisfying the following characterizing property: Suppose Σ' is any compatible collection for which $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is constructed* *as in* [18, Sec. 7], *and suppose* Σ'' *is any common refinement of* Σ *and* Σ' (*which exists by Proposition* 2.8). *Then the composition of the canonical proper surjection*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \tag{4.17}$$

(*cf. Lemma* 4.11) *with* (4.16) *coincides with the composition of canonical morphisms*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma'_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \tag{4.18}$$

(*cf. Lemmas* 4.11 *and* 4.13).

Proof. Let Σ' and Σ'' be as in the statement of the lemma, so that (4.17) and (4.18) are defined. Then the proper surjection $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ induced by (4.18) factors as a composition of the proper surjection $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}}$ induced by (4.17) with an induced proper surjection $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$, which is the desired second morphism in (4.16), because by [18, Prop. 12.14] so does its pullback to the completions of strict local rings of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ at geometric points over $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. $\square$

Proposition 4.19. *For each representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *of cusp label, there exists some* $d_0 \in \mathbb{Z}_{\geq 1}$, *which can be taken to be any* $d_0 \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (*see Lemma* 4.2) *when Condition* 3.9 *holds, such that, for any* $d \in \mathbb{Z}_{\geq 1} \cdot d_0$, *the pullback of* $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ *under the composition of* (4.16) *with the canonical morphism* $(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ *is the invertible sheaf over* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ *whose restriction to each open formal subscheme* $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *corresponds to the sub-*$\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$*-module* $\hat{\bigoplus}_{\langle \ell, y\rangle \geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y), \forall y\in\sigma} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ *of* $\mathscr{O}_{\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \hat{\bigoplus}_{\ell\in\sigma^{\vee}} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. *Hence, by the universal property of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} = \mathrm{NBl}_{\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})$ (*see* (3.13)), *the canonical morphisms* (4.16) *lift to canonical morphisms*

$$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}, \tag{4.20}$$

where $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ *denotes* (*as usual*) *the formal completion of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ *along the* (*locally closed*) *preimage of* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *in* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$, *and where the second morphism is proper and surjective.*

Proof. By Lemma 2.10, there exists a normal open compact subgroup $\mathcal{H}'$ of $\mathcal{H}$ such that Condition 3.9 is satisfied by the induced $\Sigma^{(\mathcal{H}')} = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and

$\mathsf{pol}^{(\mathcal{H}')} = \{\mathsf{pol}_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ as in [15, Constr. 7.3.1.6]. Then we have a commutative diagram

$$\begin{tikzcd} \coprod\limits_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})] \text{ lifts } [(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} (\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}}}/\Gamma_{\Phi_{\mathcal{H}'}}) \arrow[r] \arrow[d] & \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \arrow[d] \\ \coprod\limits_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})] \text{ lifts } [(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} (\vec{\mathsf{M}}^{\min}_{\mathcal{H}'})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}} \arrow[r] \arrow[d] & (\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \arrow[d] \\ \vec{\mathsf{M}}^{\min}_{\mathcal{H}'} \arrow[r] & \vec{\mathsf{M}}^{\min}_{\mathcal{H}} \end{tikzcd} \tag{4.21}$$

of canonical morphisms, in which the horizontal arrows induce an isomorphism from the quotients of the objects at level $\mathcal{H}'$ (at the left-hand sides) by $\mathcal{H}/\mathcal{H}'$ to the objects at level $\mathcal{H}$ (at the right-hand sides). If the assertions of Proposition 4.19 are true at level $\mathcal{H}'$ (with all notation accordingly denoted with a prime) when $d' \in \mathbb{Z}_{\geq 1} \cdot d'_0$ for some $d'_0 \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}'}}}$, by taking norms of local generators with respect to the action of $\mathcal{H}/\mathcal{H}'$, the corresponding assertions are also true at level $\mathcal{H}$ when $d \in \mathbb{Z}_{\geq 1} \cdot d_0$ for $d_0 := \#(\mathcal{H}/\mathcal{H}') \cdot d'_0$. Hence, we may and we shall assume that Condition 3.9 holds, and that $d \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

For each open formal subscheme $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$, for some $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ satisfying $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, since $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is an open formal subscheme of $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ for any top-dimensional cone τ in $\Sigma_{\Phi_{\mathcal{H}}}$ having σ as a face, we may and we shall assume that σ is top-dimensional, which corresponds to some vertex ℓ_0 of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (see (3) of Proposition 2.6) in the sense that $\mathsf{pol}_{\Phi_{\mathcal{H}}}(y) = \langle \ell_0, y\rangle$ for all $y \in \sigma$.

Let $\bar{x}$ be any geometric point of $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \cong \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. (We shall adopt the same notation system as in Lemma 4.1.) Since

$$(\mathbf{S}_{\Phi_{\mathcal{H}}} \cap (d \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})) + \sigma^{\vee} \subset d \cdot \ell_0 + \sigma^{\vee} = \{\langle \ell, y\rangle \geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y), \forall y \in \sigma\} \tag{4.22}$$

by Lemma 2.12, we can write each section f of $(\vec{\mathcal{J}}^{(d)}_{\mathcal{H},\mathsf{pol}})^{\wedge}_{\bar{x}} \subset \big(\prod\limits_{\ell \in \mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}\big)^{\Gamma_{\Phi_{\mathcal{H}}}}$ (see [18, Prop. 12.14] and Lemma 4.1) as a formal sum $f = \sum\limits_{\ell \in d\cdot\ell_0+\sigma^{\vee}} f^{(\ell)}$, where each $f^{(\ell)}$ is a section of $(\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$. Since f is $\Gamma_{\Phi_{\mathcal{H}}}$-invariant, it decomposes as a formal sum $f = \sum\limits_{[\ell] \in (\Gamma_{\Phi_{\mathcal{H}}} \cdot (d\cdot\ell_0+\sigma^{\vee}))/\Gamma_{\Phi_{\mathcal{H}}}} f^{[\ell]}$ of subseries $f^{[\ell]} = \sum\limits_{\ell \in [\ell]} f^{(\ell)}$, where each $[\ell]$ is by definition the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of some $\ell \in d \cdot \ell_0 + \sigma^{\vee}$. Since the largest ideal of definition of $\mathscr{O}_{\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \hat{\bigoplus\limits_{\ell\in\sigma^{\vee}}} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ consists of sections whose nonzero terms are supported on those ℓ's in $\sigma^{\vee}_{0+} := \bigcap\limits_{\tau \text{ a face of } \sigma,\, \tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}} \tau^{\vee}_0$, and since $\sigma^{\vee}_{0+}$ contains $\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}} - \{0\}$ (because each $\tau^{\vee}_0$ as above does), we see that $f^{[d\cdot\ell_0]} = \sum\limits_{\gamma \in \Gamma_{\Phi_{\mathcal{H}}}} f^{(\gamma\cdot(d\cdot\ell_0))}$ is a *leading subseries* of f in the sense that $f - f^{[d\cdot\ell_0]}$ has a higher degree than $f^{[d\cdot\ell_0]}$ in the natural grading of $\hat{\bigoplus\limits_{\ell\in\sigma^{\vee}}} (\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^{\wedge}_{\bar{x}}$ defined by the above ideal of definition of $\hat{\bigoplus\limits_{\ell\in\sigma^{\vee}}} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. Since Condition 3.9 holds by assumption, $f^{(d\cdot\ell_0)}$ is a *leading* *term* of $f^{[d\cdot\ell_0]}$ in the sense that $f^{[d\cdot\ell_0]} - f^{(d\cdot\ell_0)}$ (or equivalently $f - f^{(d\cdot\ell_0)}$) has a higher degree than $f^{(d\cdot\ell_0)}$ in the natural grading of $\hat{\bigoplus}_{\ell\in\sigma^\vee}(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$. (These are abused terminologies because the leading subseries or terms might be zero.)

Suppose $f_1,\ldots,f_k$ are sections of $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^\wedge_{\bar{x}}$ generating $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^\wedge_{\bar{x}}$ such that their respective leading terms $f_1^{(d\cdot\ell_0)},\ldots,f_k^{(d\cdot\ell_0)}$ generate $(\underline{\vec{\mathrm{FJ}}}^{(d\cdot\ell_0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$. Since $d\in\mathbb{Z}_{\geq 1}\cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ by assumption, by Lemma 4.2, the pullbacks of $f_1^{(d\cdot\ell_0)},\ldots,f_k^{(d\cdot\ell_0)}$ to $(\vec{\mathfrak{X}}^\circ_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}$ generates the pullback of $(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_0))^\wedge_{\bar{x}}$, while the pullbacks of $f_1,\ldots,f_k$ generate the (coherent ideal) pullback of $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^\wedge_{\bar{x}}$. By (4.22), this last pullback is invertible and corresponds to the sub-$\mathscr{O}_{(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}}$-module $\hat{\bigoplus}_{\langle\ell,y\rangle\geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y),\forall y\in\sigma}(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$ of $\mathscr{O}_{(\vec{\mathfrak{X}}^\circ_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}}\cong\hat{\bigoplus}_{\ell\in\sigma^\vee}(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$. Since $\bar{x}$ is arbitrary, the proposition follows, by fpqc descent (cf. [7, VIII, 1.11]). $\square$

Corollary 4.23. *Suppose $d\in\mathbb{Z}_{\geq 1}$ such that, for each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ for $\mathsf{M}_{\mathcal{H}}$, it is divisible by some integer d_0 as in Proposition* 4.19*. Suppose Σ'' is a projective smooth refinement of the Σ induced by* pol*. Then there is a canonical proper surjective morphism*

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}\to\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} \tag{4.24}$$

such that, for each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} \arrow[r, "\text{can.}"] \arrow[d, "\text{can.}"'] & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \arrow[r, "\text{can.}"] \arrow[d, "(4.20)\underset{\mathbb{Z}}{\otimes}\mathbb{Q}"] & \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \arrow[d, "(4.20)"] \\
\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \arrow[r, "(4.24)"'] & \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} \arrow[r, "\text{can.}"'] & \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}
\end{tikzcd} \tag{4.25}$$

(*where $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} := \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ and $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}} := \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$—or they can be more directly constructed using toroidal embeddings of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*).

Proof. For any representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, since d is divisible by some d_0 as in Proposition 4.19, the pullback of $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ to $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ is invertible, and so its further pullback to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}$ is also invertible. By the definition of $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ (see Construction 3.12), and by the characterizing property of (4.16) in Lemma 4.15, this pullback to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}$ is canonically isomorphic to the pullback of $\mathcal{J}_{\mathcal{H},d\mathsf{pol}}$ under the composition of the canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma''_{\Phi_{\mathcal{H}}}}\to\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ (given by the universal property [15, Thm. 6.4.1.1(6)], by the same argument of the proof of [15, Thm. 6.4.1.1(5)]) with the canonical morphism $\oint_{\mathcal{H},\Sigma''}:\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}\to\mathsf{M}^{\min}_{\mathcal{H}}$ (as in [15, Prop. 7.2.4.1(3)]). By fpqc descent (cf. [7, VIII, 1.11]), this shows that the pullback of $\mathcal{J}_{\mathcal{H},d\mathsf{pol}}$ under $\oint_{\mathcal{H},\Sigma''}$ is invertible, and hence $\oint_{\mathcal{H},\Sigma''}$ lifts to the desired canonical morphism (4.24), which is necessarily proper surjective and makes the diagram (4.25) commutative. $\square$

5. Stratification and completions

Proposition 5.1. *In Proposition* 4.19*, up to replacing d_0 with a multiple, we can assert additionally that the second morphism in* (4.20) *is an isomorphism.*

It is convenient to first introduce the following consequence of Proposition 5.1:

Corollary 5.2. *Suppose $d, d' \in \mathbb{Z}_{\geq 1}$ such that, for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp label for $\mathsf{M}_{\mathcal{H}}$, they are both divisible by some integer d_0 as in Proposition* 5.1. *Then the canonical morphisms*

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},dd'\mathsf{pol}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} \tag{5.3}$$

and

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},dd'\mathsf{pol}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d'\mathsf{pol}} \tag{5.4}$$

induced by the canonical morphisms $\vec{\mathscr{J}}^{d'}_{\mathcal{H},d\mathsf{pol}} \to \vec{\mathscr{J}}_{\mathcal{H},dd'\mathsf{pol}}$ and $\vec{\mathscr{J}}^{d}_{\mathcal{H},d'\mathsf{pol}} \to \vec{\mathscr{J}}_{\mathcal{H},dd'\mathsf{pol}}$ between coherent $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}}$-ideals are both isomorphisms. Consequently, up to canonical isomorphism, $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ does not depend on the precise choice of $d \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Proof. This is because, by Proposition 5.1, for each stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, (5.3) and (5.4) induce isomorphisms $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},dd'\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ and $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},dd'\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d'\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ over the formal completion $(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$. □

Now we begin with some reduction step:

Lemma 5.5. *It suffices to prove Proposition* 5.1 *under the additional assumption that Condition* 3.9 *holds, and that $d \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.*

Proof. Suppose that we are in the context of the first paragraph of the proof of Proposition 4.19, that the assertions of Propositions 4.19 and 5.1 and hence of Corollary 5.2 are true at level $\mathcal{H}'$ (with all notation accordingly denoted with a prime) when $d' \in \mathbb{Z}_{\geq 1} \cdot d'_0$ for some $d'_0 \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}'}}}$, and that the assertions of Proposition 4.19 are true at level $\mathcal{H}$ when $d \in \mathbb{Z}_{\geq 1} \cdot d_0$ for some $d_0 \in \mathbb{Z}_{\geq 1} \cdot d'_0$. Then the commutative diagram (4.21) induces a similar commutative diagram

$$\begin{array}{ccc} \coprod\limits_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})] \text{ lifts } [(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} (\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}}}/\Gamma_{\Phi_{\mathcal{H}'}}) & \longrightarrow & \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \\ \downarrow \wr & & \downarrow \\ \coprod\limits_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})] \text{ lifts } [(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}^{(\mathcal{H}')}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}} & \longrightarrow & (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \\ \downarrow & & \downarrow \\ \vec{\mathsf{M}}^{\min}_{\mathcal{H}'} & \longrightarrow & \vec{\mathsf{M}}^{\min}_{\mathcal{H}} \end{array} \tag{5.6}$$

of canonical morphisms, in which the top and bottom horizontal arrows induce an isomorphism from the quotients of the objects at level $\mathcal{H}'$ (at the left-hand sides) by $\mathcal{H}/\mathcal{H}'$ to the objects at level $\mathcal{H}$ (at the right-hand sides), and in which the middle horizontal arrow is defined and $\mathcal{H}/\mathcal{H}'$-equivariant for the following reason: Let $\vec{\jmath}_d$ (resp. $\vec{\jmath}'_d$) be the invertible ideal pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ (resp. $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}^{(\mathcal{H}')}}$) under $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ (resp. $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}^{(\mathcal{H}')}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}'}$). Then the ideal pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ under the composition $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}^{(\mathcal{H}')}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}'} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ of canonical morphisms coincides with $\vec{\jmath}'_d$, because its further pullback under each isomorphism

$\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}}}/\Gamma_{\Phi_{\mathcal{H}'}} \xrightarrow{\sim} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}}$ coincides with the pullback of $\vec{\mathcal{J}}_d$ under the composition $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}}}/\Gamma_{\Phi_{\mathcal{H}'}} \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ of canonical morphisms, whose restriction to each open formal subscheme $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\tau}$ (for $\tau \in \Sigma_{\Phi_{\mathcal{H}'}}$ satisfying $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}'}}$) corresponds to the sub-$\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}}}$-module $\hat{\bigoplus}_{\langle \ell,y\rangle \geq d\mathsf{pol}_{\Phi_{\mathcal{H}'}}(y), \forall y\in\sigma'} \vec{\Psi}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}}(\ell)$ of $\mathscr{O}_{\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\tau}} \cong \hat{\bigoplus}_{\ell\in\tau^{\vee}} \vec{\Psi}_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}}(\ell)$. Hence, by the universal property of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} = \mathrm{NBl}_{\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})$ as a normalization of blowup, we have a canonical morphism

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} \tag{5.7}$$

under which the ideal pullback of $\vec{\mathcal{J}}_d$ is $\vec{\mathcal{J}}'_d$, which is $\mathcal{H}/\mathcal{H}'$-equivariant (because all objects involved are) and induces the middle horizontal arrow in (5.6).

The ideal pullback of $\vec{\mathcal{J}}_d$ to $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')}/(\mathcal{H}/\mathcal{H}')$ under the canonical morphism induced by (5.7) coincides with $\Big(\big(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')}/(\mathcal{H}/\mathcal{H}')\big)_{*}\vec{\mathcal{J}}'_d\Big)^{\mathcal{H}/\mathcal{H}'}$, because their pullbacks under each canonical isomorphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} \big(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')}/(\mathcal{H}/\mathcal{H}')\big)^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ induced by (5.6) correspond to the same sub-$\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-module $\hat{\bigoplus}_{\langle \ell,y\rangle \geq d\mathsf{pol}_{\Phi_{\mathcal{H}}}(y), \forall y\in\sigma} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ of $\mathscr{O}_{\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \hat{\bigoplus}_{\ell\in\sigma^{\vee}} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ over each open formal subscheme $\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (for $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ satisfying $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$). Therefore, by Corollary 5.2, and by [5, Ch. V, Lem. 5.9 and 5.10, and Prop. 5.13] (cf. [1, Ch. IV, Sec. 2, p. 327, Lem.] or [2, Ch. IV, Sec. 2, Lem. 2.14], and [15, Prop. 7.3.2.3]), up to replacing d_0 with some multiple, we may and we shall assume that (5.7) identifies $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ with the quotient of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',d\mathsf{pol}(\mathcal{H}')}$ by $\mathcal{H}/\mathcal{H}'$, so that (5.6) also induces the isomorphisms $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$, as desired. $\square$

Proof of Proposition 5.1. By Lemma 5.5, we may and we shall assume that Condition 3.9 holds, and that $d \in \mathbb{Z}_{\geq 1} \cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Let $\bar{x}$ be an arbitrary geometric point of $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \cong \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. (We shall adopt the same notation system as in Lemma 4.1.) Consider the proper morphism

$$(\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}})^{\wedge}_{\bar{x}}/\Gamma_{\Phi_{\mathcal{H}}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}. \tag{5.8}$$

induced by (4.20). It suffices to show that (5.8) is an isomorphism.

By the definition of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}}$ as the normalization of a blowup, since the formation of normalizations is compatible with pullbacks to the formal completions of strict local rings for excellent schemes, $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$ has an open covering by affine open formal subschemes $\{\mathfrak{U}_f\}_f$ labeled by nonzero sections f of $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$, where each $\mathfrak{U}_f$ is tautological for the pullback of $\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}}$ to be an invertible ideal generated by f, which only depends on sufficiently high powers of f. Concretely, let R denote the ring of global sections of $\mathscr{O}_{(\vec{\mathsf{M}}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}} \cong \big(\prod_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}\big)^{\Gamma_{\Phi_{\mathcal{H}}}}$, which is a noetherian normal domain, and let J denote the R-ideal of global sections of $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$; then $\mathfrak{U}_f \cong \mathrm{Spf}(R_{(f)})$, where $R_{(f)}$ is the integral closure in $\mathrm{Frac}(R)$ of the subring of

$\mathrm{Frac}(R)$ generated by R and by fractions of the form $f^{-k}\prod_{0\leq j<k} g_j$, where $g_j\in J$, for all $0\leq j<k$.

For each such f, let us write it as a formal sum $f=\sum_{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}\cap(d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})} f^{(\ell)}$, where each $f^{(\ell)}$ is a section of $(\underline{\vec{\mathrm{FJ}}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$, which can be further decomposed as a formal sum $f=\sum_{[\ell]\in(\mathbf{S}_{\Phi_{\mathcal{H}}}\cap(d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}))/\Gamma_{\Phi_{\mathcal{H}}}} f^{[\ell]}$ of subseries $f^{[\ell]}=\sum_{\ell\in[\ell]} f^{(\ell)}$, where each $[\ell]$ is by definition the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of some $\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}\cap(d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})$. Note that $\Gamma_{\Phi_{\mathcal{H}}}$ acts freely on $d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, because (by Proposition 2.6) any element of $d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ can be identified with some positive-definite pairing over some $Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ (as in [15, Sec. 6.2.5]), whose stabilizer in $\Gamma_{\Phi_{\mathcal{H}}}$ can be identified with a discrete subgroup of a compact orthogonal subgroup of $\mathrm{GL}_{\mathbb{R}}(Y\underset{\mathbb{Z}}{\otimes}\mathbb{R})$, which must be finite and hence trivial, by the neatness of $\mathcal{H}$.

In order to give an open covering of $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$, we only need a collection of f's such that every section g of $(\vec{\mathcal{J}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$ has a sufficiently high power which lies in the subideal generated by the collection. Consequently, we only need those f's such that $f^{[\ell]}\neq 0$ only for one $\Gamma_{\Phi_{\mathcal{H}}}$-orbit $[\ell]=[\ell_0]$, represented by some ℓ_0, and we may and we shall assume furthermore that ℓ_0 is a vertex of $d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, which is equal to $d\cdot\ell_{\tau,0}$ for some vertex $\ell_{\tau,0}$ of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional τ in $\Sigma_{\Phi_{\mathcal{H}}}$. Let $\mathfrak{V}_{f^{(\ell_0)}}$ denote the maximal open formal subscheme of $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ over which $f^{(\ell_0)}$ is a generator of the pullback of $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$, and let $\mathfrak{W}_{f^{(\ell_0)}}$ denote the preimage of $\mathfrak{V}_{f^{(\ell_0)}}$ under the canonical morphism $(\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau})^{\wedge}_{\bar{x}}\to(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$. By the proof of Proposition 4.19, $\mathfrak{W}_{f^{(\ell_0)}}$ is the preimage of $\mathfrak{U}_f$ under the canonical morphism $(\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau})^{\wedge}_{\bar{x}}\to(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}})^{\wedge}_{\bar{x}}$ induced by (5.8), so that we have a canonical morphism

$$\mathfrak{W}_{f^{(\ell_0)}}\to\mathfrak{U}_f. \tag{5.9}$$

Since f is $\Gamma_{\Phi_{\mathcal{H}}}$-invariant, for each $\gamma\in\Gamma_{\Phi_{\mathcal{H}}}$, the similarly defined canonical morphism $\mathfrak{W}_{f^{(\gamma\ell_0)}}\to\mathfrak{U}_f$ is compatible with (5.9) and with the canonical isomorphism $\gamma:\mathfrak{W}_{f^{(\ell_0)}}\xrightarrow{\sim}\mathfrak{W}_{f^{(\gamma\ell_0)}}$ induced by the isomorphism $\gamma:\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}\xrightarrow{\sim}\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\gamma\tau}$.

With the collection $\{\ell_{\sigma,i}\}_{\sigma,i}$ chosen in the paragraph preceding Lemma 4.2, there are $\ell_{\tau,1},\ldots,\ell_{\tau,n_\tau}$ in $\mathbf{S}_{\Phi_{\mathcal{H}}}\cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ such that $\mathbb{R}_{\geq 0}\cdot\tau^{\vee}=\sum_{1\leq i\leq n_\tau}\mathbb{R}_{\geq 0}\cdot(\ell_{\tau,i}-\ell_{\tau,0})$ (cf. (2.13)). Since $d\in\mathbb{Z}_{\geq 1}\cdot d_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, by Lemma 4.2, the sections of $(\underline{\vec{\mathrm{FJ}}}^{(d\cdot\ell_{\tau,i})}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ generate $(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d\cdot\ell_{\tau,i}))^{\wedge}_{\bar{x}}$ over $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$, for each integer i such that $0\leq i\leq n_\tau$. For each section $g^{(d\cdot\ell_{\tau,i})}$ of $(\underline{\vec{\mathrm{FJ}}}^{(d\cdot\ell_{\tau,i})}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$, the formal sum $g_i=g^{[d\cdot\ell_{\tau,i}]}=\sum_{\gamma\in\Gamma_{\Phi_{\mathcal{H}}}}\gamma g^{(d\cdot\ell_{\tau,i})}$, where $\gamma g^{(d\cdot\ell_{\tau,i})}$ is a section of $(\underline{\vec{\mathrm{FJ}}}^{(d\cdot(\gamma\ell_{\tau,i}))}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ for each $\gamma\in\Gamma_{\Phi_{\mathcal{H}}}$, defines an element of J, and so $f^{-1}g_i$ defines an element of $R_{(f)}$. (Since $\Gamma_{\Phi_{\mathcal{H}}}$ acts freely on $d\cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, there is no cancellation among different terms in the formal sum $g_i=\sum_{\gamma\in\Gamma_{\Phi_{\mathcal{H}}}}\gamma g^{(d\cdot\ell_{\tau,i})}$.)

Let V be any complete discrete valuation ring with valuation $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ and with an algebraically closed residue field k, and let $z : \mathrm{Spf}(V) \to \mathfrak{U}_f$ be any morphism. Without loss of generality, up to replacing ℓ_0 with another representative in its $\Gamma_{\Phi_\mathcal{H}}$-orbit, we have $\upsilon(f^{(\ell_0)}) \leq \upsilon(g^{(\ell)})$ for all $g = \sum\limits_{\ell \in \mathbf{S}_{\Phi_\mathcal{H}} \cap (d \cdot K^\vee_{\mathrm{pol}_{\Phi_\mathcal{H}}})} g^{(\ell)} \in J$ and all $\ell \in \mathbf{S}_{\Phi_\mathcal{H}} \cap (d \cdot K^\vee_{\mathrm{pol}_{\Phi_\mathcal{H}}})$. By applying this to formal sums $g_i = \sum\limits_{\gamma \in \Gamma_{\Phi_\mathcal{H}}} \gamma g^{(d \cdot \ell_{\tau,i})}$ as in the previous paragraph, for all $0 \leq i \leq n_\tau$, we have $\upsilon(f^{(\ell_0)}) \leq \upsilon(g^{(d \cdot \ell_{\tau,i})})$ for every section $g^{(d \cdot \ell_{\tau,i})}$ of $(\underline{\vec{\mathrm{FJ}}}^{(d \cdot \ell_{\tau,i})}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}$. By Lemma 4.2 again, the sections $(f^{(\ell_0)})^{-1} g^{(d \cdot \ell_{\tau,i})}$ generate $(\vec{\Psi}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(d \cdot (\ell_{\tau,i} - \ell_{\tau,0})))^\wedge_{\bar{x}}$ over $\mathfrak{V}_{f^{(\ell_0)}}$. Since $\mathbb{R}_{\geq 0} \cdot \tau^\vee = \sum\limits_{1 \leq i \leq n_\tau} \mathbb{R}_{\geq 0} \cdot (\ell_{\tau,i} - \ell_{\tau,0})$, it follows that any $y : \mathrm{Spf}(V) \to \mathfrak{U}_f$ as above necessarily uniquely lifts to a morphism $y : \mathrm{Spf}(V) \to \mathfrak{W}_{f^{(\ell_0)}}$ via the canonical morphism (5.9). Since V and y are arbitrary, this shows, in particular, that $\mathfrak{W}_{f^{(\ell_0)}}$ is the preimage of $\mathfrak{U}_f$ under the proper morphism (5.8). Since both $\mathfrak{W}_{f^{(\ell_0)}}$ and $\mathfrak{U}_f$ are affine, this forces the morphism (5.9) to be finite, which is induced by some finite homomorphism

$$R_{(f)} = \Gamma(\mathfrak{U}_f, \mathscr{O}_{\mathfrak{U}_f}) \to \Gamma(\mathfrak{W}_{f^{(\ell_0)}}, \mathscr{O}_{\mathfrak{W}_{f^{(\ell_0)}}}) \cong \hat{\bigoplus_{\ell \in \sigma^\vee}} \Gamma(\mathfrak{V}_{f^{(\ell_0)}}, (\vec{\Psi}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell))^\wedge_{\bar{x}}) \tag{5.10}$$

of R-algebras. Since $R_{(f)}$ is noetherian normal by construction, the above unique liftability (for arbitrary V and y) also shows that (5.10) induces an isomorphism between the total rings of fractions, and so (5.10) and (5.9) are isomorphisms.

Thus, the inverse of (5.9) defines a local inverse of (5.8) over $\mathfrak{U}_f$, which is (up to canonical isomorphism) independent of the choice of ℓ_0 in its $\Gamma_{\Phi_\mathcal{H}}$-orbit $[\ell_0]$. Since $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathrm{pol}})^\wedge_{\bar{x}}$ is covered by such $\mathfrak{U}_f$'s, (5.8) is an isomorphism, as desired. $\square$

Corollary 5.11. *Suppose $d \in \mathbb{Z}_{\geq 1}$ such that, for each representative $(\Phi_\mathcal{H}, \delta_\mathcal{H})$ for $\mathsf{M}_\mathcal{H}$, it is divisible by some integer d_0 as in Proposition* 5.1. *Suppose that $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is constructed as in* [18, Sec. 7] (*which means that the Σ induced by* pol *is also induced by some auxiliary choices of cone decompositions there*). *Then the canonical morphism $\vec{\oint}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{min}}_\mathcal{H}$ lifts to a canonical isomorphism*

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \xrightarrow{\sim} \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathrm{pol}}. \tag{5.12}$$

Proof. By Lemma 4.13, we have $\vec{\mathfrak{X}}_{\Phi_\mathcal{H},\delta_\mathcal{H},\Sigma_{\Phi_\mathcal{H}}} / \Gamma_{\Phi_\mathcal{H}} \cong (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}}$, for each representative $(\Phi_\mathcal{H}, \delta_\mathcal{H})$ of cusp label for $\mathsf{M}_\mathcal{H}$. By Proposition 4.19, and by the same argument as in the proof of Corollary 4.23, the pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathrm{pol}}$ to $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ under $\vec{\oint}_{\mathcal{H},\Sigma}$ is invertible. Hence, by the universal property of the normalization of blowup (see [15, Def. 7.3.2.1]), $\vec{\oint}_{\mathcal{H},\Sigma}$ induces a canonical morphism $\mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathrm{pol}}}(\vec{\oint}_{\mathcal{H},\Sigma}) : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathrm{pol}} = \mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathrm{pol}}}(\vec{\mathsf{M}}^{\mathrm{min}}_\mathcal{H})$, which is an isomorphism because, for each representative $(\Phi_\mathcal{H}, \delta_\mathcal{H})$ of cusp label for $\mathsf{M}_\mathcal{H}$, its pullback $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^\wedge_{\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathrm{pol}})^\wedge_{\vec{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}}$ is, by Proposition 5.1. $\square$

Definition 5.13. *By abuse of notation, we shall henceforth denote any $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathrm{pol}}$ in Proposition* 5.1 *and Corollary* 5.2 *as $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, and adjust all related notation accordingly. We shall denote the canonical morphism $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{min}}_\mathcal{H}$ by $\vec{\oint}_{\mathcal{H},\Sigma}$, or simply* *by* $\vec{\oint}_{\mathcal{H}}$ *when the context is clear.* (*To fully justify such notation, we will show in* (6) *of Theorem* 6.1 *below that* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *does not depend on the choice of* pol.)

Remark 5.14. By Corollary 5.11, this is justified even when $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ has already been constructed in [18, Sec. 7] for the Σ induced by pol.

Proposition 5.15. *Let* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *be as in Definition* 5.13*, and let* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *denote the isomorphic image of the locally closed subscheme* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *of* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}/\Gamma_{\Phi_{\mathcal{H}}}$ *under the composition of the second morphism in* (4.20) (*which is an isomorphism by Proposition* 5.1) *with the canonical morphism* $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$*, for any representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*. Then* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is well defined and locally closed in* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (*with admits the canonical structure of a reduced locally closed subscheme of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$)*, and we have a stratification*

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \tag{5.16}$$

in which the $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ *lies in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *if and only if* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *is a face of* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ *as in* [15, Thm. 6.3.2.14 and Rem. 6.3.2.15].

By construction, we have a canonical isomorphism

$$(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \tag{5.17}$$

inducing a canonical isomorphism

$$\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \cong \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}. \tag{5.18}$$

Moreover, the canonical morphism $\vec{\oint}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ *maps* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *to* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$*, and induces a surjection* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \twoheadrightarrow \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *which can be canonically identified with the canonical surjection* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \twoheadrightarrow \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (*via* (5.18) *and the canonical isomorphism* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \cong \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ *in* [18, Thm. 12.16]).

If Σ'' *is as in Corollary* 4.23*, then the morphism* (4.24) *maps the stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ (*defined as in* [15, Thm. 6.4.1.1(2)]) *to the stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} := \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} := \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (*where* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *is defined as in Definition* 5.13) *whenever* $\tau \in \Sigma''_{\Phi_{\mathcal{H}}} \in \Sigma''$ *is contained in* $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$ *in* $\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$*. Moreover, each* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is the union of the images of all such* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$.

Proof. By the stratification by locally closed subschemes $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (see [18, Thm. 12.1]), and by Proposition 5.1, each $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is locally closed in $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, and we have a disjoint union as in (5.16). Moreover, the assertions in the second paragraph (of the proposition) also follow from Proposition 5.1, and the assertions in the last paragraph follow from Corollary 4.23. Thanks to the isomorphism (5.18), the characteristic zero fiber $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is dense in $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, because the analogous assertion for $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is true (cf. [18, Cor. 10.15]). Consequently, by the assertions in the last paragraph, the locally closed subschemes in (5.16) satisfy the desired incidence relation, because the (finer) strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ satisfy the analogous incidence relation as in [15, Thm. 6.4.1.1(2)], for any projective smooth refinement Σ'' of Σ. $\square$

Lemma 5.19 (cf. [18, Lem. 11.1]). *Suppose $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$, where $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$, is as in* [15, Def. 6.2.6.1]*, and suppose $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)] \neq [(0, 0, \{0\})]$. Let U be any open subscheme of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ that is a union of strata and contains $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ as a closed subscheme; and let U' be the complement of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in U, which necessarily contains $\vec{\mathsf{Z}}_{[(0,0,\{0\})]} = \vec{\mathsf{M}}_{\mathcal{H}}$ because $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)] \neq [(0, 0, \{0\})]$ (see Proposition* 5.15*). By definition, the formal completion $\mathfrak{U}$ of U along $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ can be canonically identified with $\big(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\big)^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, so that we have a canonical isomorphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \xrightarrow{\sim} \mathfrak{U}$ given by* (5.17)*. Suppose $\mathrm{j} \in \mathrm{J}$, and suppose the tautological object $(A_{\mathrm{j}}, \lambda_{\mathrm{j}}, i_{\mathrm{j}}, \alpha_{\mathcal{H}_{\mathrm{j}}})$ over $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}} \cong \mathsf{M}_{\mathcal{H}}$ (see* [18, (2.1)]*) extends to a degenerating family $(G_{\mathrm{j},U'}, \lambda_{\mathrm{j},U'}, i_{\mathrm{j},U'}, \alpha_{\mathcal{H}_{\mathrm{j}},U'})$ of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ over U' (see* [15, Def. 5.3.2.1]*), where $\alpha_{\mathcal{H}_{\mathrm{j}},U'}$ is only required to be defined over $\mathsf{M}_{\mathcal{H}}$. Then this degenerating family further extends to a degenerating family $(G_{\mathrm{j},U}, \lambda_{\mathrm{j},U}, i_{\mathrm{j},U}, \alpha_{\mathcal{H}_{\mathrm{j}},U})$ of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ over U.*

Proof. Let Σ'' be any projective smooth refinement of Σ. Consider (as in [18, (10.5), (10.6), (10.7), (10.8)]; see also the proof of Lemma 3.2) the formal completion $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)'' := \bigcup\limits_{\tau \in \Sigma''_{\Phi_{\mathcal{H}}}, \tau \subset \sigma} \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\tau)$ along the closed subscheme $\Xi''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} := \bigcup\limits_{\tau \in \Sigma''_{\Phi_{\mathcal{H}}}, \tau \subset \sigma} \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, which induces a canonical proper morphism $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Then $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is also the formal completion of $\bigcup\limits_{\tau \in \Sigma''_{\Phi_{\mathcal{H}}}, \tau \subset \sigma} \mathfrak{X}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ along $\Xi''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. By the same argument as in the proof of [15, Thm. 6.4.1.1(5)], by [15, Thm. 6.4.1.1(6)], the Mumford families carried by $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ induces a canonical isomorphism $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''})^{\wedge}_{\mathsf{Z}''_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, where $\mathsf{Z}''_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} := \bigcup\limits_{\tau \in \Sigma''_{\Phi_{\mathcal{H}}}, \tau \subset \sigma} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$. Since this isomorphism is induced by the universal property in [15, Thm. 6.4.1.1(6)], for each affine open formal subscheme $\mathrm{Spf}(R'')$ of $\mathfrak{X}''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, the pullback of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ under the induced morphism $\mathrm{Spec}(R'') \to \Xi''_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ coincides with the pullback of $\mathsf{M}_{\mathcal{H}}$ under the induced morphism $\mathrm{Spec}(R'') \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$. Consequently, by Corollary 4.23, for each affine open subscheme $\mathrm{Spec}(R)$ of U inducing an affine open subscheme $\mathrm{Spf}(R^{\wedge}, I)$ of $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathfrak{U}$, with canonical morphisms $\mathrm{Spec}(R^{\wedge}) \to \mathrm{Spec}(R) \to U$, there is a canonical isomorphism over the preimage of $\mathsf{M}_{\mathcal{H}}$ in $\mathrm{Spec}(R^{\wedge})$ between the pullbacks of the tautological object $(A_{\mathrm{j}}, \lambda_{\mathrm{j}}, i_{\mathrm{j}}, \alpha_{\mathcal{H}_{\mathrm{j}}})$ over $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}} \cong \mathsf{M}_{\mathcal{H}}$ and of the Mumford family $({}^{\heartsuit}\vec{G}_{\mathrm{j}}, {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}}, {}^{\heartsuit}\vec{i}_{\mathrm{j}}, {}^{\heartsuit}\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ over $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (cf. [15, Sec. 6.2.5] and [18, (8.29)]). Then the lemma follows from the same descent argument as in the proof of [18, Lem. 11.1] (which was based on [7, VIII, 7.8] and [22, Thm. 1.1]). $\square$

Proposition 5.20 (cf. [18, Thm. 11.2]). *For each $\mathrm{j} \in \mathrm{J}$, there exists a degenerating family $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (see* [15, Def. 5.3.2.1]*), whose pullback to $\mathsf{M}_{\mathcal{H}} \cong \mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ (see* [18, (2.1)]*) is isomorphic to the tautological object $(A_{\mathrm{j}}, \lambda_{\mathrm{j}}, i_{\mathrm{j}}, \alpha_{\mathcal{H}_{\mathrm{j}}})$ over $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, and whose pullback to $\vec{\mathsf{M}}_{\mathcal{H}}$ is isomorphic to the degenerating family of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ over $\vec{\mathsf{M}}_{\mathcal{H}}$ which was denoted $(\vec{A}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ in* [18, Prop. 6.1]*.* (*The notation of $\vec{\lambda}_{\mathrm{j}}$, $\vec{i}_{\mathrm{j}}$, and $\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}}$ is abusive and dependent on the context.*) *For each $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$, the pullback of $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ to $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ via the canonical morphism*

(5.17) *is canonically isomorphic to the Mumford family* $({}^{\heartsuit}\vec{G}_{\mathrm{j}}, {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}}, {}^{\heartsuit}\vec{i}_{\mathrm{j}}, {}^{\heartsuit}\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (*cf.* [15, Sec. 6.2.5] *and* [18, (8.29)]).

Proof. The same argument as in the proof of [18, Thm. 11.2] works here, with the stratification in [18, Thm. 9.13] there replaced with the stratification (5.18) here, and with [18, Lem. 11.1] there replaced with Lemma 5.19 here. □

6. Main results

Theorem 6.1 (cf. [15, Thm. 6.4.1.1 and 7.2.4.1(3)–(5)]). *For each open compact subgroup* $\mathcal{H}$ *of* $\mathrm{G}(\hat{\mathbb{Z}})$ *whose image* $\mathcal{H}^p$ *under the canonical homomorphism* $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ *is* **neat**, *for each choice of lattice collection* $\{(g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ *as in* [18, Sec. 2], *and for each* **projective** *compatible choice* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *of admissible rational polyhedral cone decomposition data as in Definitions* 2.1 *and* 2.7, *there is a normal scheme* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *projective and flat over* $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, *containing the scheme* $\vec{\mathsf{M}}_{\mathcal{H}}$ *as in* [18, Prop. 6.1] *as an open fiberwise dense subscheme, together with a tautological degenerating family* $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *of type* $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ *over* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (*see* [15, Def. 5.3.2.1]), *for each* $\mathrm{j} \in \mathrm{J}$, *where* $\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}}$ *is defined only over the open dense subscheme* $\mathsf{M}_{\mathcal{H}} \cong \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, *such that we have the following:*

1. *For each* $\mathrm{j} \in \mathrm{J}$, *the pullback of* $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *to* $\vec{\mathsf{M}}_{\mathcal{H}}$ *is the tautological tuple* $(\vec{A}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathsf{M}}_{\mathcal{H}}$ *as in* [18, Prop. 6.1].
2. (*Compare with* [18, Prop. 7.11].) *There exists a canonical proper surjection* $\vec{\oint}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ (*over* $\vec{\mathsf{S}}_0$), *where* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ *is as in* [18, Prop. 6.4]. *If* Σ' *is a refinement of* Σ, *then there exists a canonical morphism* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *compatible with* $\vec{\oint}_{\mathcal{H},\Sigma'}$ *and* $\vec{\oint}_{\mathcal{H},\Sigma}$. *If we denote by* $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ *the pullback of the ample invertible sheaf* $\omega_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{J}}$ *over* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ *as in* [18, Prop. 6.4], *then it is canonically isomorphic to* $\underset{\mathrm{j}\in\mathrm{J}}{\otimes}\, \omega^{\otimes a_{\mathrm{j}}}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$, *where* $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}} := \wedge^{\mathrm{top}}\, \underline{\mathrm{Lie}}^{\vee}_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ *and where* a_{j} *is as in* [18, Lem. 5.30, and Prop. 6.1 and 6.4], *for each* $\mathrm{j} \in \mathrm{J}$, *and we have* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus}\, \Gamma(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}})\Big)$. *Moreover,* $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$ *descends to an invertible sheaf* $\omega_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{j}}$ *over* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$, *for each* $\mathrm{j} \in \mathrm{J}$, *and the above canonical isomorphism* $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}} \cong \underset{\mathrm{j}\in\mathrm{J}}{\otimes}\, \omega^{\otimes a_{\mathrm{j}}}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$ *over* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *descends to a canonical isomorphism* $\omega_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{J}} \cong \underset{\mathrm{j}\in\mathrm{J}}{\otimes}\, \omega^{\otimes a_{\mathrm{j}}}_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{j}}$ *over* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$.
3. (*Compare with* [18, Thm. 9.13, and Cor. 10.18 and 11.9].) $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *has a stratification by locally closed subschemes*

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}, \tag{6.2}$$

with $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *running through a complete set of equivalence classes of* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ (*as in* [15, Def. 6.2.6.1]) *with* $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ *and* $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$. (*Here* $\mathsf{Z}_{\mathcal{H}}$ *is suppressed in the notation by* [15, Conv. 5.4.2.5].)

In this stratification, the $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ *lies in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *if and only if* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$

is a face of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ *as in* [15, Thm. 6.3.2.14 and Rem. 6.3.2.15]. *The analogous assertion holds after pullback to fibers over* $\vec{\mathsf{S}}_0$.

The $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *is flat over* $\vec{\mathsf{S}}_0$ *and normal, and is isomorphic to the support of the formal scheme* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ *for any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. *The formal scheme* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ *admits a canonical structure as the completion of an affine toroidal embedding* $\vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)$ (*along its* σ*-stratum* $\vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$) *of a torus torsor* $\vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}$ *over a scheme* $\vec{C}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}$ *flat over* $\vec{\mathsf{S}}_0$ *and normal. The scheme* $\vec{C}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}$ *is proper* (*and surjective*) *over a finite cover* $\vec{\mathsf{M}}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ *of the boundary version* $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *of* $\vec{\mathsf{M}}_{\mathcal{H}}$ (*cf.* [18, Prop. 7.4]). (*Note that* $\mathsf{Z}_{\mathcal{H}}$ *and the isomorphism class of* $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ *depend only on the class* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, *but not on the choice of the representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.)

In particular, $\vec{\mathsf{M}}_{\mathcal{H}} = \vec{\mathsf{Z}}_{[(0,0,\{0\})]}$ *is an open fiberwise dense stratum in this stratification.*

(4) (*Compare with* [18, Thm. 10.13 and 11.12, and Cor. 10.16].) *The formal completion* $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *along its* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *is canonically isomorphic to the formal scheme* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ *for any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$.

For any open immersion $\mathrm{Spf}(R, I) \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ *inducing morphisms* $\mathrm{Spec}(R) \to \vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)$ *and* $\mathrm{Spec}(R) \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (*via the above-mentioned isomorphism*), *the preimage of* $\vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}$ *under* $\mathrm{Spec}(R) \to \vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}}(\sigma)$ *coincides with the preimage of* $\vec{\mathsf{M}}_{\mathcal{H}}$ *under* $\mathrm{Spec}(R) \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.

For each $\mathrm{j} \in \mathrm{J}$, *the pullback to* $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}}$ *of the degenerating family* $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *is canonically isomorphic to the Mumford family* $({}^{\heartsuit}\vec{G}_{\mathrm{j}}, {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}}, {}^{\heartsuit}\vec{i}_{\mathrm{j}}, {}^{\heartsuit}\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ (*cf.* [15, Def. 6.2.5.28] *and* [18, (8.29)]), *after we identify the bases using the above-mentioned isomorphism.*

(5) (*Compare with* [18, Cor. 10.15, and Thm. 12.1 and 12.16].) *The stratification* (6.2) *is compatible with the stratification of* $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ *as in* [18, Thm. 12.1 and 12.16], *in the sense that the restriction of the proper surjection* $\vec{\oint}_{\mathcal{H},\Sigma}$ *in* (2) *to the stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *induces a surjection to the stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *of* $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, *which can be identified with the composition of the canonical isomorphism* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]} \xrightarrow{\sim} \vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma}$ (*induced by the canonical isomorphism in* (4)), *the structural morphism* $\vec{\Xi}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma} \to \vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, *and the isomorphism* $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \xrightarrow{\sim} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *given by* [18, Thm. 12.16]. *In particular, it is proper and surjective if* σ *is top-dimensional in* $\mathbf{P}^{+}_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$.

(6) (*Compare with* [18, Thm. 7.14 and 11.4].) *Let* S *be an irreducible noetherian normal scheme over* $\vec{\mathsf{S}}_0$, *with generic point* η, *which is equipped with a morphism*

$$\eta \to \mathsf{M}_{\mathcal{H}}. \tag{6.3}$$

Let $(A_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H},\eta})$ *denote the pullback of the tautological object of* $\mathsf{M}_{\mathcal{H}}$ *to* η *under* (6.3). *Suppose that, for each* $\mathrm{j} \in \mathrm{J}$, *we have a degenerating family* $(G^{\dagger}_{\mathrm{j}}, \lambda^{\dagger}_{\mathrm{j}}, i^{\dagger}_{\mathrm{j}}, \alpha^{\dagger}_{\mathcal{H}_{\mathrm{j}}})$ *of type* $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ *over* S, *whose pullback* $(G_{\mathrm{j},\eta}, \lambda_{\mathrm{j},\eta}, i_{\mathrm{j},\eta}, \alpha_{\mathcal{H}_{\mathrm{j}},\eta})$

to η defines a morphism

$$\eta \to \mathsf{M}_{\mathcal{H}_{\mathrm{j}}} \tag{6.4}$$

by the universal property of $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, which we assume to coincide with the composition of (6.3) *with the canonical isomorphism $\mathsf{M}_{\mathcal{H}} \cong \mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ given by* [18, (2.1)].

Then (6.3) (*necessarily uniquely*) *extends to a morphism*

$$S \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \tag{6.5}$$

(*over $\vec{\mathsf{S}}_0$*) *if the following condition is satisfied at each geometric point $\bar{s}$ of S:*

Consider any dominant morphism $\mathrm{Spec}(V) \to S$ centered at $\bar{s}$, where V is a complete discrete valuation ring with fraction field K, algebraically closed residue field k, and discrete valuation υ. By the semistable reduction theorem (*see, for example,* [15, Thm. 3.3.2.4]), *up to replacing K with a finite extension field and replacing V accordingly, we may assume that the pullback of A_η to $\mathrm{Spec}(K)$ extends to a semi-abelian scheme $G^\ddagger$ over $\mathrm{Spec}(V)$. By the theory of Néron models* (*see* [3]*; cf.* [26, IX, 1.4], [5, Ch. I, Prop. 2.7], *or* [15, Prop. 3.3.1.5]), *the pullback of $(A_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H},\eta})$ to $\mathrm{Spec}(K)$ extends to a degenerating family $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}})$ of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(V)$, where $\alpha^\ddagger_{\mathcal{H}}$ is defined only over $\mathrm{Spec}(K)$, which defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$ corresponding to a tuple $(B^\ddagger, \lambda_{B^\ddagger}, i_{B^\ddagger}, \underline{X}^\ddagger, \underline{Y}^\ddagger, \phi^\ddagger, c^\ddagger, c^{\vee,\ddagger}, \tau^\ddagger, [\alpha^{\natural,\ddagger}_{\mathcal{H}}])$ in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$ under* [15, Thm. 5.3.1.19]. *Then we have a fully symplectic-liftable admissible filtration $\mathsf{Z}^\ddagger_{\mathcal{H}}$ determined by $[\alpha^{\natural,\ddagger}_{\mathcal{H}}]$. Moreover, the étale sheaves $\underline{X}^\ddagger$ and $\underline{Y}^\ddagger$ are necessarily constant, because the base ring V is strict local. Hence it makes sense to say we also have a uniquely determined torus argument $\Phi^\ddagger_{\mathcal{H}}$ at level $\mathcal{H}$ for $\mathsf{Z}^\ddagger_{\mathcal{H}}$. On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^\ddagger)$, $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G^\ddagger)}$, and $\underline{B}(G^\ddagger)$* (*see* [15, Constr. 6.3.1.1]), *which define objects $\Phi^\ddagger_{\mathcal{H}}$, $\mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}}$, and in particular $B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathrm{Inv}(V)$ over the special fiber. Then $\upsilon \circ B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathbb{Z}$ defines an element of $\mathbf{S}^\vee_{\Phi^\ddagger_{\mathcal{H}}}$, where $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ is the homomorphism induced by the discrete valuation of V.*

Then the condition is that, for each $\mathrm{Spec}(V) \to S$ as above (*centered at $\bar{s}$*), *and for some* (*and hence every*) *choice of $\delta^\ddagger_{\mathcal{H}}$, there is a cone $\sigma^\ddagger$ in the cone decomposition $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ such that $\overline{\sigma}^\ddagger$ contains all $\upsilon \circ B^\ddagger$ obtained in this way.* (*As explained in the proof of* [15, Prop. 6.3.3.11], *we may assume that $\sigma^\ddagger$ is minimal among such choices; also, it follows from the positivity of $\tau^\ddagger$ that $\sigma^\ddagger \subset \mathbf{P}^+_{\Phi^\ddagger_{\mathcal{H}}}$. Then the extended morphism* (6.5) *maps $\bar{s}$ to a geometric point over $\vec{\mathsf{Z}}_{[(\Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}}, \sigma^\ddagger)]}$; conversely, this property also characterizes the stratum $\vec{\mathsf{Z}}_{[(\Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}}, \sigma^\ddagger)]}$ of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.*)

In particular, since this condition involves only Σ, the scheme $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ depends (*up to canonical isomorphism*) *only on the compatible collection Σ induced by* pol *and on the linear algebraic data in* [18, Sec. 2], *but not on the choice of* pol *or on any auxiliary choices made in* [18, Sec. 7].

Proof. Let $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ be as in Definition 5.13, which is projective by construction (see Construction 3.12). By Proposition 5.20, it carries the tautological families $(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, which satisfy the assertion (1).

As for the assertion (2), the existence of the morphism $\vec{\oint}_{\mathcal{H},\Sigma}$ and the canonical isomorphism $\vec{\mathsf{M}}^{\min}_{\mathcal{H}} \cong \operatorname{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}})\Big)$ follow from the fact that (by abuse of language) the canonical proper surjective morphism $\vec{\oint}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ is its own Stein factorization (see [8, III-1, 4.3.3 and 4.3.4]), because $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ are normal, and because $\vec{\oint}_{\mathcal{H},\Sigma}$ is generically an isomorphism (over $\mathsf{M}_{\mathcal{H}}$).

The assertions (3), (4), and (5) follow from Propositions 5.15 and 5.20, and from the same argument as in the proof of [18, Cor. 10.16], except for the statement in assertion (3) that the stratification satisfies the same incidence relations after pullback to fibers over $\vec{\mathsf{S}}_0$ (which will be proved two paragraphs below).

Let us prove the assertion (6). Since the desired extensibility is a local question (because $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is separated over $\vec{\mathsf{S}}_0$), we may replace S with the spectrum of the completion R of the strict local ring of S at an arbitrary geometric point $\bar{s}$, and assume that it is local with $\bar{s}$ as a closed point. Let $K := \mathrm{Frac}(R)$, so that $\eta \cong \mathrm{Spec}(K)$. By applying [15, Thm. 5.3.1.19] (to degenerating families of types $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, for all $\mathrm{j} \in \mathrm{J}$), and by the same argument as in the proof of [15, Prop. 6.2.5.11], there exists a canonical morphism $\eta \to \Xi_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$, which extends to a canonical morphism $S \to \vec{\Xi}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}(\sigma)$ mapping $\bar{s}$ to $\vec{\Xi}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\sigma}$ and lifts to a canonical morphism $S \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\sigma}$ exactly when the condition in the second last paragraph of the assertion (6) (cf. the second last paragraphs of [15, Thm. 6.4.1.1(6)] and [18, Thm. 7.14]) is satisfied (at $\bar{s}$), under which the degenerating families $(G^{\dagger}_{\mathrm{j}}, \lambda^{\dagger}_{\mathrm{j}}, i^{\dagger}_{\mathrm{j}}, \alpha^{\dagger}_{\mathcal{H}_{\mathrm{j}}})$ over S are the pullbacks of the Mumford families over $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\sigma}$, for all $\mathrm{j} \in \mathrm{J}$. By Proposition 5.20, and by [5, Ch. I, Prop. 2.7] or [15, Prop. 3.3.1.5], the composition of any such morphism $S \to \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\sigma}$ with the canonical morphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (induced by (5.17)) gives the desired extension $S \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ of (6.3).

Then the statement in the assertion (3) that the stratification satisfies the same incidence relations after pullback to fibers over $\vec{\mathsf{S}}_0$ follows from same argument as in the proof of [18, Cor. 11.9].

Finally, let us complete the proof of the assertion (2). If Σ' is a refinement of Σ, then the existence of the canonical morphism $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (which is necessarily compatible with $\vec{\oint}_{\mathcal{H},\Sigma'}$ and $\vec{\oint}_{\mathcal{H},\Sigma}$ because all of them extend the identity morphism over the dense subscheme $\vec{\mathsf{M}}_{\mathcal{H}}$) follows from the assertion (6). In order to show that $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}} \cong \underset{\mathrm{j}\in\mathrm{J}}{\otimes} \omega^{\otimes a_{\mathrm{j}}}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$, since the canonical morphisms $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as above are their own Stein factorizations, or equivalently since the canonical morphisms $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}}$ are isomorphisms, by [15, Lem. 7.2.2.1], we may replace Σ (up to a common projective refinement) with one that is induced by some auxiliary choices as in [15, Sec. 7] (see Remarks 2.3 and 2.9), in which case the desired isomorphism follows from [18, Prop. 7.11]. Since the proper morphism $\vec{\oint}_{\mathcal{H},\Sigma}$ is its own Stein factorization (by what we have shown in the second paragraph of this proof), or equivalently since the canonical morphism $\mathscr{O}_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}}} \to (\vec{\oint}_{\mathcal{H},\Sigma})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$

is an isomorphism, by [15, Lem. 7.2.2.1] again, it remains to show that $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$ descends to an invertible sheaf $\omega_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{j}}$ over $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$, for each $\mathrm{j} \in \mathrm{J}$; or, rather, that the pushforward $(\vec{\oint}_{\mathcal{H},\Sigma})_* \omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$ is an invertible sheaf. By [8, III-1, 4.1.5] (since $\vec{\oint}_{\mathcal{H},\Sigma}$ is proper) and by fpqc descent (cf. [7, VIII, 1.11]), as in the proof of [15, Thm. 7.2.4.1], it suffices to note that, for each $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ defining a stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ as in the assertion (3), the pullback of $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{j}}$ to $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ descends to the invertible sheaf $(\wedge^{\mathrm{top}}_{\mathbb{Z}} X_{\mathrm{j}}) \underset{\mathbb{Z}}{\otimes} (\wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{\vec{B}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}})$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (cf. [15, Lem. 7.1.2.1] and its proof), where X_{j} is part of the torus argument $\Phi_{\mathcal{H}_{\mathrm{j}}}$ associated with $\Phi_{\mathcal{H}}$ as in [18, (3.3)], and where $\vec{B}_{\mathrm{j}}$ is part of the tautological family $(\vec{B}_{\mathrm{j}}, \lambda_{\vec{B}_{\mathrm{j}}}, i_{\vec{B}_{\mathrm{j}}}, \vec{\varphi}_{-1,\mathcal{H}_{\mathrm{j}}})$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ as in [18, Prop. 7.4]. □

Corollary 6.6. *In Corollary* 4.23*, if the* Σ *induced by* pol *is already smooth* (*and satisfies* [15, Cond. 6.3.3.2]) *as in Definition* 2.2*, and if we take* $\Sigma'' = \Sigma$ *there, then the canonical morphism* (4.24) *is an isomorphism, and the stratification of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *in* [15, Thm. 6.4.1.1(2)] *coincides with the one induced by* (5.16)*.*

Proof. This is because, by [18, Lem. 3.21], the universal properties of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ in [15, Thm. 6.4.1.1(6)] and in (6) of Theorem 6.1 imply each other. □

Corollary 6.7. *Let* $\mathcal{H}$ *and* Σ *be as in Theorem* 6.1*. There exists an effective Carter divisor* D' *over* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$*, with* $\mathsf{D}'_{\mathrm{red}} = \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} - \vec{\mathsf{M}}_{\mathcal{H}}$ (*with its canonical reduced closed subscheme structure*) *such that* $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(-\mathsf{D}')$ *is relatively ample over* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$*, with respect to the canonical morphism* $\vec{\oint}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$*.*

Proof. This follows from the definition of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},d\mathsf{pol}} = \mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}})$ (see Definition 5.13), because the pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ to $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is of the form $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(-\mathsf{D}')$ as in the statement of the corollary, by Propositions 4.19 and 5.1. □

By the same arguments as in the proofs of [18, Prop. 14.1 and 14.2, and Cor. 14.4], we obtain the following:

Proposition 6.8. *Suppose* Σ *is smooth as in Definition* 2.2*. Then* $\vec{\mathsf{M}}_{\mathcal{H}}$ *is regular if and only if* $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}$ *is.*

Proposition 6.9. *Let* P *be the property of being one of the following: reduced, geometrically reduced, normal, geometrically normal, Cohen–Macaulay,* (R_0)*, geometric* (R_0)*,* (R_1)*, geometric* (R_1)*, and* (S_i)*, one property for each* $i \geq 0$ (*see* [8, IV-2, 5.7.2 and 5.8.2])*. Then the fiber of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{S}}_0$ *over some point* s *of* $\vec{\mathsf{S}}_0$ *satisfies property* P *if and only if the corresponding fiber of the open subscheme* $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$ *over* s *does. If* Σ *is smooth as in* [15, Def. 6.3.3.4]*, then* P *can also be the property of being one of the following: regular, geometrically regular,* (R_i)*, and geometrically* (R_i)*, one property for each* $i \geq 0$*.*

Corollary 6.10. *Suppose that* $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$ *has geometrically normal fibers. Then all geometric fibers of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{S}}_0$ *have the same number of connected components, and the same is true for* $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$*.* (*The analogous statements are true if we consider irreducible components instead of connected components.*)

7. Functorial properties and Hecke twists

For most of this section (except for Proposition 7.15 and its proof), as in [18, Sec. 13], for the sake of clarity, we shall abusively denote all objects constructed using $\{(g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ by an additional subscript J.

Proposition 7.1 (cf. [18, Prop. 13.7 and 13.9])**.** *With the setting as in* [18, Prop. 13.1]*, suppose moreover that* Σ *and* Σ' *are compatible choices of admissible rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$ *and* $\mathsf{M}_{\mathcal{H}'}$*, respectively, which are projective as in Definitions* 2.1 *and* 2.7*, such that* Σ *is a* 1*-refinement of* Σ' *as in* [15, Def. 6.4.3.3]*.* (*The definition there naturally generalizes to the case of nonsmooth cone decompositions.*) *Then there is a canonical projective morphism*

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'} \tag{7.2}$$

extending the canonical proper morphism [18, (13.2)] *and is compatible with the canonical morphism* [18, (13.5)] *under the canonical projective morphisms* $\vec{\oint}_{\mathcal{H},\Sigma,\mathrm{J}} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H},\mathrm{J}}$ *and* $\vec{\oint}_{\mathcal{H}',\Sigma',\mathrm{J}'} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}',\mathrm{J}'}$*, which maps the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],\mathrm{J}}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}$ *to the* $[(\Phi'_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma')]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma')],\mathrm{J}'}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')]$*, respectively, such that* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a* 1*-refinement of* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *as in* [15, Def. 6.4.3.1]*.*

Proof. By the universal property of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}$ as in (6) of Theorem 6.1, the canonical morphism [18, (13.2)] extends to a canonical morphism (7.2), under which the subcollection $\{(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}'_{\mathrm{j}}})\}_{\mathrm{j}\in\mathrm{J}'}$ of $\{(\vec{G}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}}, \vec{\alpha}_{\mathcal{H}'_{\mathrm{j}}})\}_{\mathrm{j}\in\mathrm{J}}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}$ is the pullback of the corresponding collection over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}$, which maps $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],\mathrm{J}}$ to $\vec{\mathsf{Z}}_{[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma')],\mathrm{J}'}$ if and only if the condition as in the proposition holds. It is then compatible with the canonical morphism [18, (13.5)], by (2) of Theorem 6.1. $\square$

Proposition 7.3 (cf. [18, Prop. 13.15])**.** *Given any collection* $\{(g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ *satisfying the conditions imposed by an open compact subgroup* $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}})$ *as in* [18, Sec. 2]*, suppose that* $\mathcal{H}' \subset \mathrm{G}(\hat{\mathbb{Z}})$ *contains both* $\mathcal{H}$ *and* $g^{-1}\mathcal{H}g$*, and that* $g_{\mathrm{j}}^{-1}\mathcal{H}g_{\mathrm{j}}$ *stabilizes* $L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}$ *for all* $\mathrm{j} \in \mathrm{J}$*, so that* $\{(g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ *also satisfies the conditions imposed by* $\mathcal{H}'$*. Then the collection* $\{(g^{\epsilon}g_{\mathrm{j}}, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{(\epsilon,\mathrm{j})\in\{0,1\}\times\mathrm{J}}$ *satisfies the condition imposed by* $\mathcal{H}$ *as well, and we have two canonical projective morphisms as in* [18, (13.16) and (13.17)]*. Given any projective* Σ' *as in Definitions* 2.1 *and* 2.7*, there exist some refinement* Σ *of* Σ' *such that the two canonical projective morphisms* [18, (13.16)] *extend to two canonical projective morphisms*

$$\vec{[1]}^{\mathrm{tor}}, \vec{[g]}^{\mathrm{tor}} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\{0,1\}\times\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}} \tag{7.4}$$

compatible with the two canonical projective morphisms [18, (13.18)]*.*

The morphism $\vec{[1]}^{\mathrm{tor}}$ (*resp.* $\vec{[g]}^{\mathrm{tor}}$) *in* (7.4) *maps the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],\{0,1\}\times\mathrm{J}}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\{0,1\}\times\mathrm{J}}$ *to the* $[(\Phi'_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma')]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma')],\mathrm{J}}$ *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')]$*, respectively, such that* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a* 1*-refinement* (*resp. g-refinement*) *of* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *as in* [15, Def. 6.4.3.1]*.*

Proof. This follows from Proposition 7.1 and the same argument as in [18, Ex. 13.14] (with $(\mathrm{J}, \mathrm{J}_0)$ there replaced with $(\{0,1\} \times \mathrm{J}, \mathrm{J})$ here). $\square$

Proposition 7.5. *The morphism* (7.2) *induces canonical morphisms*

$$\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}} \tag{7.6}$$

and

$$\mathscr{I}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}} \to (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'})_* \mathscr{I}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}}, \tag{7.7}$$

where $\mathscr{I}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}}$ (*resp.* $\mathscr{I}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}}$) *denotes the coherent* $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}}$*-ideal* (*resp.* $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}}$*-ideal*) *defining the boundary* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} - \vec{\mathsf{M}}_{\mathcal{H},\mathrm{J}}$ (*resp.* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'} - \vec{\mathsf{M}}_{\mathcal{H}',\mathrm{J}'}$) *with its canonical reduced subscheme structure. When* $\mathcal{H}'$ *is a normal subgroup of* $\mathcal{H}$*, the finite group* $\mathcal{H}/\mathcal{H}'$ *acts on the right-hand side of* (7.6) (*resp.* (7.7)) *and identifies the left-hand side with the* $\mathcal{H}/\mathcal{H}'$*-invariants in the right-hand side. Moreover, we have*

$$R^i(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}} = 0 \tag{7.8}$$

and

$$R^i(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'})_* \mathscr{I}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}} = 0, \tag{7.9}$$

for all $i > 0$*. The analogous statements for the two morphisms* (7.4) *are also true.*

Proof. By the same argument as in the proof of Lemma 4.13, and by (6) of Theorem 6.1, $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}},\mathrm{J}}/\Gamma_{\Phi_{\mathcal{H}}}$ carries Mumford families (cf. [15, Sec. 6.2.5] and [18, (8.29)]) which are isomorphic to the pullback of the tautological objects over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}$ under the composition of the canonical isomorphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}},\mathrm{J}}/\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],\mathrm{J}}}$ (see Proposition 5.1) with the canonical morphism $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],\mathrm{J}}} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma,\mathrm{J}}$. The similar statement for $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}$ is also true. Hence, since the morphism (7.2) is induced by the universal property of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma',\mathrm{J}'}$ as in (6) of Theorem 6.1, it induces the canonical proper morphism

$$\coprod_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}}) \text{ lifts } (\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})} (\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}},\mathrm{J}}/\Gamma_{\Phi_{\mathcal{H}}}) \to \vec{\mathfrak{X}}_{\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'},\Sigma'_{\Phi'_{\mathcal{H}'}},\mathrm{J}'}/\Gamma_{\Phi'_{\mathcal{H}'}}, \tag{7.10}$$

which is the formal completion of the canonical proper morphism

$$\coprod_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}}) \text{ lifts } (\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})} (\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}},\mathrm{J}}/\Gamma_{\Phi_{\mathcal{H}}}) \to \vec{\Xi}_{\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'},\Sigma'_{\Phi'_{\mathcal{H}'}},\mathrm{J}'}/\Gamma_{\Phi'_{\mathcal{H}'}}. \tag{7.11}$$

Since the morphisms (7.2), (7.10), and (7.11) are all proper, by [8, III-1, 4.1.5], it suffices to prove the obvious analogues of the statements of the proposition for (7.11). Since the canonical morphism

$$\mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}'}} \to (\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}} \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}'})_* \mathscr{O}_{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}}} \tag{7.12}$$

is an isomorphism by Zariski's main theorem (see [8, III-1, 4.4.3, 4.4.11]) and by noetherian normality of $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}}$ and $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}'}$ (note the change from J to J′ in the subscripts), it suffices to prove the analogues of the statements for the morphism

$$\coprod_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}}) \text{ lifts } (\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}},\mathrm{J}'}/\Gamma_{\Phi_{\mathcal{H}}} \to \vec{\Xi}_{\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'},\Sigma'_{\Phi_{\mathcal{H}'}},\mathrm{J}'}/\Gamma_{\Phi'_{\mathcal{H}'}}, \tag{7.13}$$

now with the same J'; or rather the analogues for the morphism

$$\bigcup_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}}) \text{ lifts } (\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'}),\ \sigma\in\Sigma_{\Phi_{\mathcal{H}}},\ \sigma\subset\sigma'} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\mathrm{J}'}(\sigma) \to \vec{\Xi}_{\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'},\mathrm{J}'}(\sigma') \tag{7.14}$$

for each $\sigma' \in \Sigma_{\Phi'_{\mathcal{H}'}}$, which then follow from the arguments in [12, Ch. I, Sec. 3, especially p. 44, Cor. 2] (cf. the proof of [15, Lem. 7.1.1.4]), as usual. □

(From now on, for simplicity, we shall again drop J from the subscripts.)

Proposition 7.15. *For each* $\mathrm{j} \in \mathrm{J}$*, the locally free sheaves* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ *and* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})$ *over* $\vec{\mathsf{M}}_{\mathcal{H}}$*, where* $\vec{A}_{\mathrm{j}}$ *is as in* [18, Prop. 6.1]*, extends to locally free sheaves* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$ *and* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$ *over* $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$*, with a canonical pairing*

$$\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \times \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \to \mathscr{O}_{\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \tag{7.16}$$

(*necessarily uniquely*) *extending the canonical pairing*

$$\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}}) \to \mathscr{O}_{\vec{\mathsf{M}}_{\mathcal{H}}}. \tag{7.17}$$

(*Here we have ignored the Tate twists for simplicity, which can be compatibly reinstated when needed in applications.*) *Moreover, the canonical exact sequences*

$$0 \to \underline{\mathrm{Lie}}^\vee_{\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}}} \to \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}}} \to 0 \tag{7.18}$$

and

$$0 \to \underline{\mathrm{Lie}}^\vee_{\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}}} \to \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}}} \to 0 \tag{7.19}$$

over $\vec{\mathsf{M}}_{\mathcal{H}}$ *extend to canonical short exact sequences*

$$0 \to \underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \to \underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to 0 \tag{7.20}$$

and

$$0 \to \underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to \underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \to \underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to 0 \tag{7.21}$$

over $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$*, where* $\vec{G}_{\mathrm{j}}$ *and* $\vec{G}_{\mathrm{j}}^\vee$ *are the semi-abelian schemes as in Theorem* 6.1*, which are compatible with each other in the sense that the sheaves* $\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}}$ *and* $\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}}$ *in* (7.18) *and* (7.19) (*viewed as submodules of the middle terms*) *are annihilators of each other under the pairing* (7.16)*, and the canonically induced morphisms* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}/\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to (\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}})^\vee$ *and* $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}/\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to (\underline{\mathrm{Lie}}^\vee_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}})^\vee$ *can be identified with the identity morphisms on* $\underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}}$ *and* $\underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}}$*, respectively.*

Proof. Let us first show that $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ and $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})$ extend to locally free sheaves $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$ and $\underline{H}_1^{\mathrm{dR}}(\vec{A}_{\mathrm{j}}^\vee/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$ over $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, with short exact sequences (7.20) and (7.21) extending (7.18) and (7.19), respectively.

Let Σ' be any compatible collection for which $\vec{\mathsf{M}}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}$ is constructed as in [18, Sec. 7], and let Σ'' be any common projective smooth refinement of Σ and Σ' (which exists by Proposition 2.8), so that (by (2) of Theorem 6.1) we have canonical morphisms $\vec{\oint}_{\Sigma'',\Sigma} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\vec{\oint}_{\Sigma'',\Sigma'} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. By Proposition 7.5 (with $\mathcal{H} = \mathcal{H}'$ and $\mathrm{J} = \mathrm{J}'$ there), we have $(\vec{\oint}_{\Sigma'',\Sigma})_*\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}} \cong \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$, $(\vec{\oint}_{\Sigma'',\Sigma'})_*\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}} \cong \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}}$, $R^i(\vec{\oint}_{\Sigma'',\Sigma})_*\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}} = 0$, and $R^i(\vec{\oint}_{\Sigma'',\Sigma'})_*\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}} = 0$, for all $i > 0$. Therefore, if $\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ extends to a locally free sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, which we abusively also denote by $\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$, with a short exact sequence

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{\vec{G}^{\vee}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}} \to \underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \to \underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}} \to 0$$

extending (7.18) over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, where the same symbols $\vec{G}_{\mathrm{j}}$ and $\vec{G}^{\vee}_{\mathrm{j}}$ abusively also denote the semi-abelian schemes extending $\vec{A}_{\mathrm{j}}$ and $\vec{A}^{\vee}_{\mathrm{j}}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, then we have a similar short exact sequence

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{\vec{G}^{\vee}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \to (\vec{\oint}_{\Sigma'',\Sigma})_*(\vec{\oint}_{\Sigma'',\Sigma'})^*\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}} \to \underline{\mathrm{Lie}}_{\vec{G}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \to 0,$$

which shows that $(\vec{\oint}_{\Sigma'',\Sigma})_*(\vec{\oint}_{\Sigma'',\Sigma'})^*\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})^{\mathrm{can}}$ is the desired locally free extension of $\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Similarly, the corresponding assertion for $\underline{H}^{\mathrm{dR}}_1(\vec{A}^{\vee}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ is also true. Hence, it suffices to construct the locally free extensions over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. (If $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is already constructed in [18, Sec. 7], then we can take $\Sigma' = \Sigma$, in which case the reference to Theorem 6.1 is not really necessary.)

By [18, Lem. 9.8] and [14, Prop. 6.9], the pullback $\underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})$ of $\underline{H}^{\mathrm{dR}}_1(\vec{A}_{\mathrm{j}}/\vec{\mathsf{M}}_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ extends to a locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$, together with a short exact sequence

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathrm{j}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}} \to \underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \to \underline{\mathrm{Lie}}_{G_{\mathrm{j}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}} \to 0$$

over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$, extending the canonical short exact sequence

$$0 \to \underline{\mathrm{Lie}}^{\vee}_{A^{\vee}_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}}} \to \underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}}} \to 0$$

over $\mathsf{M}_{\mathcal{H}}$, where G_{j} and $G^{\vee}_{\mathrm{j}}$ are semi-abelian schemes over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$ extending A_{j} and $A^{\vee}_{\mathrm{j}}$, respectively. Let $\oint_{\Sigma'',\Sigma'} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ denote the canonical morphism as in [18, (9.9)]. Then the same argument as in the previous paragraph shows that $(\oint_{\Sigma'',\Sigma'})_*\underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ is locally free. Thus, $\underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})$ extends to a locally free sheaf $\mathscr{E}$ over the open subscheme $\vec{\mathsf{M}}_{\mathcal{H}} \cup \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, whose complement is a closed subscheme of codimension at least two (because $\vec{\mathsf{M}}_{\mathcal{H}}$ is open fiberwise dense in $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, by [18, Cor. 10.18]; cf. Theorem 6.1). Similarly, $\underline{H}^{\mathrm{dR}}_1(A^{\vee}_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})$ also extends to a locally free sheaf over $\vec{\mathsf{M}}_{\mathcal{H}} \cup \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. Let us denote the canonical open immersion $\vec{\mathsf{M}}_{\mathcal{H}} \cup \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ by j. Since $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is noetherian and normal by construction, it is (S_2) by Serre's criterion (see [8, IV-2, 5.8.6]). Therefore, by [6, VIII, Prop. 3.2], $j_*\mathscr{E}$ is a coherent sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$; and, by [11, Prop. 1.11 and Thm. 3.8], $j_*\mathscr{E} \cong \mathscr{F}$ if there exists any locally free extension $\mathscr{F}$ of $\mathscr{E}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. Consequently, it suffices to show that

$$\underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}^{\times a_{\mathrm{j},1}} \underset{\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}}{\times} (A^{\vee}_{\mathrm{j}})^{\times a_{\mathrm{j},2}}/\mathsf{M}_{\mathcal{H}}) \cong \underline{H}^{\mathrm{dR}}_1(A_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})^{\oplus a_{\mathrm{j},1}} \oplus \underline{H}^{\mathrm{dR}}_1(A^{\vee}_{\mathrm{j}}/\mathsf{M}_{\mathcal{H}})^{\oplus a_{\mathrm{j},2}}$$

extends to a locally free sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ for some integers $a_{\mathrm{j},1} > 0$ and $a_{\mathrm{j},2} \geq 0$ as in [18, Lem. 4.1]. By [18, Prop. 4.12], under the morphism $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}$ induced by [18, (6.2)], $\underline{H}_1^{\mathrm{dR}}(A_{\mathrm{j}}^{\times a_{\mathrm{j},1}} \underset{\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}}{\times} (A_{\mathrm{j}}^{\vee})^{\times a_{\mathrm{j},2}}/\mathsf{M}_{\mathcal{H}})$ is canonically isomorphic to the pullback of the sheaf $\underline{H}_1^{\mathrm{dR}}(A_{\mathrm{j,aux}}/\mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}})$ over $\mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}$; and, under any morphism $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{j,aux}},\Sigma_{\mathrm{j,aux}}}$ as in [18, (7.8)], any locally free extension of the latter sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{j,aux}},\Sigma_{\mathrm{j,aux}}}$ pulls back to a locally free extension of the former over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. Thus, it suffices to note that, by [14, Prop. 6.9] again, the latter extends to the locally free sheaf $\underline{H}_1^{\mathrm{dR}}(A_{\mathrm{j,aux}}/\mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}})^{\mathrm{can}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{\mathrm{j,aux}},\Sigma_{\mathrm{j,aux}}}$. Since the auxiliary polarization $\lambda_{\mathrm{j,aux}} : A_{\mathrm{j,aux}} \to A^{\vee}_{\mathrm{j,aux}}$ is prime-to-p (by assumption), the corresponding assertion for the dual abelian schemes is also true.

Thus, we have constructed the desired locally free extensions over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, and hence also over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, with short exact sequences as in (7.20) and (7.21), extending (7.18) and (7.19), respectively. Since $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is noetherian normal, to construct the canonical pairing (7.16) extending (7.17), with the desired compatibility with (7.20) and (7.21) as in the statement of the proposition, it suffices to construct it over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \cup \vec{\mathsf{M}}_{\mathcal{H}}$; or, rather, over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''} \cup \vec{\mathsf{M}}_{\mathcal{H}}$, by the same pushforward argument as above; or, rather, just over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma''}$. This, again, follows from [14, Prop. 6.9]. □

8. Vanishing of higher direct images, and Koecher's principle

As in [15, Sec. 7.1.2], let $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}} : \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ denote the structural morphism. As in [17, Sec. 6], let $\mathbf{P}^{\vee,+}_{\Phi_{\mathcal{H}}} := \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle > 0, \forall y \in \mathbf{P}_{\Phi_{\mathcal{H}}} - \{0\}\}$.

Lemma 8.1. *There exist infinitely many integers n prime to p such that, for each such n, there exists a finite étale commutative group scheme H_n of order prime to p over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ acting on $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ via morphisms compatible with $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$, inducing canonical morphisms $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/H_n \xrightarrow{\sim} \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, whose composition we denote as $[n]$, such that $[n]^*\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \cong \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \cong \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)^{\otimes n^2}$, for each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$. Moreover, for any $\mathcal{O}_{F_0,(p)}$-algebra R, the canonical morphism*

$$\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R \to [n]_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) \tag{8.2}$$

defined by adjunction identifies the left-hand side with a direct summand of the right-hand side, consisting of H_n-invariants (*cf.* [23, p. 72, Cor.]).

Proof. Let $m \geq 1$ be any integer such that $\ker(\mathrm{G}_{\mathrm{j}}(\hat{\mathbb{Z}}) \to \mathrm{G}_{\mathrm{j}}(\mathbb{Z}/m\mathbb{Z})) \subset \mathcal{H}_{\mathrm{j}}$ and such that multiplication by m annihilates the analogues for $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ (see [18, Sec. 3]) of the finite étale group schemes $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ as in [15, Prop. 6.2.2.4], for all $\mathrm{j} \in \mathrm{J}$. By the construction of $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \cong C_{\Phi_{\mathcal{H}_{\mathrm{j}}},\delta_{\mathcal{H}_{\mathrm{j}}}}$ (see [15, Sec. 6.2.3–6.2.4]), for each of the infinitely many integers n that are prime to p and congruent to 1 moduli m^2, and for each $\mathrm{j} \in \mathrm{J}$, the multiplication by n on the tautological tuple $(c_{\mathcal{H}_{\mathrm{j}}}, c^{\vee}_{\mathcal{H}_{\mathrm{j}}})$ (which is an orbit of objects at level m satisfying certain liftability and pairing conditions that are unaffected by multiplication by n, because of the choice of m) induces a canonical morphism $[n] : C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}} \cong \mathsf{M}^{\Phi_{\mathcal{H}_{\mathrm{j}}}}_{\mathcal{H}_{\mathrm{j}}}$ (which is then also a morphism over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \cong \mathsf{M}^{\mathsf{Z}_{\mathcal{H}_{\mathrm{j}}}}_{\mathcal{H}_{\mathrm{j}}}$), which can be realized as a quotient of $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by a finite étale subgroup scheme H_n of order prime to p. Up to replacing m with a (positive) multiple, the canonical morphisms

$[n] : C_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}} \to C_{\Phi_{\mathcal{H}_{\mathrm{j,aux}}},\delta_{\mathcal{H}_{\mathrm{j,aux}}}}$ are similarly defined by multiplication by n on the tautological tuples $(c_{\mathcal{H}_{\mathrm{j,aux}}}, c^\vee_{\mathcal{H}_{\mathrm{j,aux}}})$, for all $\mathrm{j} \in \mathrm{J}$, which are naturally compatible with the above morphism $[n] : C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ via the canonical morphism [18, (8.5)]. Therefore, by the construction of $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ (see [18, Prop. 8.4]), the action of H_n on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to an action of H_n on $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, which also induces the multiplication by n on the tautological tuple $(\vec{c}_{\mathcal{H}_{\mathrm{j}}}, \vec{c}^\vee_{\mathcal{H}_{\mathrm{j}}})$, for all J; and the induced finite morphism $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/H_n \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ between noetherian normal schemes is necessarily an isomorphism, because it is so in characteristic zero, by Zariski's main theorem (see [8, III-1, 4.4.3, 4.4.11]). As for the canonical isomorphisms $[n]^*\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \cong \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \cong \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)^{\otimes n^2}$, they exist over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by the construction in [15, Sec. 6.2.4], and they extend over $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by the argument in the proof of [18, Prop. 8.7] (for extending the $E_{\Phi_{\mathcal{H}}}$-torsor $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ to the $E_{\Phi_{\mathcal{H}}}$-torsor $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$). Finally, since the order of H_n is invertible in the base ring $\mathcal{O}_{F_0,(p)}$, the canonical morphism (8.2) admits a splitting (by descent and) by taking averages under H_n-action, as in the proof of [23, p. 72, Cor.]. Consequently, its left-hand side can be identified with a direct summand of its right-hand side, consisting of H_n-invariants, as desired. □

Proposition 8.3. *Suppose $\ell \in \mathbf{P}^{\vee,+}_{\Phi_{\mathcal{H}}}$. Then $R^i(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$ for all $i > 0$ and all $\mathcal{O}_{F_0,(p)}$-algebra R.*

Proof. Since $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$ is proper and since $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is quasi-projective over $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, by the usual limit argument (cf. [15, Thm. 1.3.1.3] and the references made there), we may and we shall assume that R is noetherian. For any such ℓ, as explained in the proof of Lemma 4.2, the invertible sheaf $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ over $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is relatively ample over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. Since $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$ is proper and since $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is of finite type over the noetherian ring $\mathcal{O}_{F_0,(p)}$, there exists some integer $N_0 \geq 1$ (depending on R) such that $R^i(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$ for all $i > 0$ and $N \geq N_0$. Let n be any integer considered in Lemma 8.1 such that $n^2 \geq N_0$. Then $R^i(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$ for all $i > 0$, because it is a direct summand of $R^i(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$, by Lemma 8.1. □

Proposition 8.4. *Suppose that $\mathbf{S}_{\Phi_{\mathcal{H}}} \cong \mathbb{Z}$, that $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$ is negative, and that the morphism $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$ has positive-dimensional fibers. Then $(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$ for all $\mathcal{O}_{F_0,(p)}$-algebra R.*

Proof. As in the proof of Proposition 8.3, we may and we shall assume that R is noetherian. Suppose that $(\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) \neq 0$. Since $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$ is proper, by Grothendieck's fundamental theorem [8, III-1, 4.1.5], there exists some morphism $U = \mathrm{Spec}(R_0) \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$, where R_0 is an Artinian local ring whose residue field we denote by k_0, and some nonzero f in $\Gamma(\tilde{U}, \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)|_{\tilde{U}})$, where $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)|_{\tilde{U}}$ denotes the pullback of $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ under the canonical morphism $\tilde{U} := \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}{\times} U \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By Lemma 8.1, for each integer n considered there, f has nonzero image f_n in $\Gamma(\tilde{U}, \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell)|_{\tilde{U}})$ under the canonical morphism induced by (8.2), where $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell)|_{\tilde{U}}$ denotes the similar pullback of $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell)$.

Since $\tilde{U}$ is noetherian, by using primary decompositions of zero ideals in noetherian rings of sections over affine open subschemes, there exists an integer $N \geq 1$ such that, for each open subscheme V of $\tilde{U}$ and each nonzero h in $\mathscr{O}_{\tilde{U}}(V)$, there exists some $x \in V$ such that h has a nonzero pullback to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},x}/\mathfrak{m}_x^N)$, where $\mathfrak{m}_x$ denotes the maximal ideal of the local ring $\mathscr{O}_{\tilde{U},x}$. (We will use similar notation without further explanation.) Moreover, for each x as above, there is some associated point y of V (i.e., $y \in \mathrm{Ass}(\mathscr{O}_V)$; see [8, IV-2, 3.1.1]) whose closure $\overline{\{y\}}$ in V contains x such that h also has a nonzero pullback to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},y}/\mathfrak{m}_y^N)$. Since invertible sheaves are locally trivial, the analogous statements are true for their sections. (The same N works for all invertible sheaves over $\tilde{U}$.)

Since $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathrm{Z}_{\mathcal{H}}}$ has positive-dimensional fibers by assumption, for any integer n considered in Lemma 8.1, each geometric fiber of the morphism $[n]$ there has cardinality increasing with n^2, because (under the H_n-action) the number of values of the tautological tuple $(\vec{c}_{\mathcal{H}_{\mathrm{j}}}, \vec{c}^{\vee}_{\mathcal{H}_{\mathrm{j}}})$ over each geometric fiber also does. If f has nonzero pullbacks to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},x}/\mathfrak{m}_x^N)$ only at some closed points x, which are necessarily associated points of $\tilde{U}$, then f_n has nonzero pullbacks to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},x}/\mathfrak{m}_x^N)$ only when x is in the preimage of these closed points under $[n]$. But this is impossible because, for all sufficiently large n, such a preimage cannot be supported on the finitely many closed associated points of $\tilde{U}$. Since there are only finitely many associated points of $\tilde{U}$, there exists one of them with positive-dimensional closure $\overline{\{y\}}$ in $\tilde{U}$ and with an infinite sequence $n_1 < n_2 < \cdots$ of integers considered in Lemma 8.1 such that f_{n_i} has nonzero pullback to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},y}/\mathfrak{m}_y^N)$ for all $i \geq 1$. Up to replacing R_0 with a flat extension, we may and we shall assume that its residue field k_0 is algebraically closed with uncountable cardinality. Then the closed points in $\overline{\{y\}}$, which are all k_0-points, cannot be a countable union of its proper closed subsets, and hence there exist mutually distinct closed k_0-points x_j in $\overline{\{y\}}$, indexed by integers $1 \leq j \leq r$ for some integer $r > \mathrm{length}_{R_0}(\Gamma(\tilde{U}, \mathscr{O}_{\tilde{U}}))$ (which is possible because $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathrm{Z}_{\mathcal{H}}}$ is proper and R_0 is Artinian), such that f_{n_i} has nonzero pullback to $\mathrm{Spec}(\mathscr{O}_{\tilde{U},x_j}/\mathfrak{m}_{x_j}^N)$ for all $i \geq 1$ and all $1 \leq j \leq r$.

Consider the (coherent) $\mathscr{O}_{\tilde{U}}$-ideal $\mathscr{I} := \ker\big(\mathscr{O}_{\tilde{U}} \twoheadrightarrow \bigoplus_{1\leq j\leq r}(\mathscr{O}_{\tilde{U},x_j}/\mathfrak{m}_{x_j}^N)\big)$. Since ℓ is negative, $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-\ell)$ is relatively ample over $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathrm{Z}_{\mathcal{H}}}$, as explained in the proof of Proposition 8.3. Hence, by Serre vanishing [8, III-1, 2.2.1], there exists some sufficiently large $i_0 \geq 1$ such that $H^1(\tilde{U}, \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}} \underset{\mathscr{O}_{\tilde{U}}}{\otimes} \mathscr{I}) = 0$, and so that $\Gamma(\tilde{U}, \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}}) \to \bigoplus_{1\leq j\leq r}\big(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}} \underset{\mathscr{O}_{\tilde{U}}}{\otimes}(\mathscr{O}_{\tilde{U},x_j}/\mathfrak{m}_{x_j}^N)\big)$ is surjective. For each $1 \leq j \leq r$, let g_j be any element of $\Gamma(\tilde{U}, \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}})$ such that its image in $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}} \underset{\mathscr{O}_{\tilde{U}}}{\otimes}(\mathscr{O}_{\tilde{U},x_{j'}}/\mathfrak{m}_{x_{j'}}^N)$ is 1 (resp. 0) when $j = j'$ (resp. $j \neq j'$), for all $1 \leq j' \leq r$. If $g = \sum_{1\leq j\leq r} c_j g_j$ for some $c_j \in R_0$, then $f_{n_{i_0}} g = \sum_{1\leq j\leq r} c_j f_{n_{i_0}} g_j$ is a global section of $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n_{i_0}^2\ell)|_{\tilde{U}} \underset{\mathscr{O}_{\tilde{U}}}{\otimes} \vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(-n_{i_0}^2\ell)|_{\tilde{U}} \cong \mathscr{O}_{\tilde{U}}$ whose image in $\mathscr{O}_{\tilde{U},x_j}/\mathfrak{m}_{x_j}^N$ is $c_j f_{n_{i_0}}$, which is nonzero when $c_j \in R_0^\times$, for each $1 \leq j \leq r$. This shows that $\mathrm{length}_{R_0}(\Gamma(\tilde{U}, \mathscr{O}_{\tilde{U}})) \geq r$, contradicting the choice of r, as desired. $\square$

Definition 8.5 (cf. [17, Cor. 5.8])**.** *Let R be an $\mathcal{O}_{F_0,(p)}$-algebra. We say that a quasi-coherent sheaf $\mathscr{E}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is* **formally canonical** (*resp.* **formally subcanonical**) (*over R*) *if it satisfies the following condition: Suppose $\bar{x}$ is a geometric point over the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ as in* [18, Thm. 12.1], *which is canonically isomorphic to $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ by* [18, Thm. 12.16]. *Then there exists a quasi-coherent sheaf $\mathscr{E}_{0,\bar{x}}$ over $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ satisfying the following properties:*

(1) *For each $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ satisfying $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$, the pullback $\mathscr{E}^{\wedge}$ of $\mathscr{E}$ to the affine formal subscheme $(\vec{\mathfrak{X}}^{\circ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^{\wedge}_{\bar{x}}$ of $(\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}})^{\wedge}_{\bar{x}}$ (via the canonical morphisms induced by* (4.20)*; see Proposition* 5.1) *is of the form $\hat{\bigoplus}_{\ell\in ?}((\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^{\wedge}_{\bar{x}} \underset{\mathscr{O}_{(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}}}{\otimes} \mathscr{E}_{0,\bar{x}})$ (as an $\mathscr{O}_{(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}}$-module), where $? = \sigma^{\vee}$ (resp. $? = \sigma^{\vee}_{+}$), where $\sigma^{\vee}_{+}$ is the intersection of $\tau^{\vee}_{0}$ (in $\mathbf{S}_{\Phi_{\mathcal{H}}}$) for τ running through faces of σ in $\Sigma_{\Phi_{\mathcal{H}}}$ (including σ itself).*

(2) *There is a finite exhaustive filtration on $\mathscr{E}_{0,\bar{x}}$ whose graded pieces are isomorphic to pullbacks of quasi-coherent sheaves over $\vec{\mathsf{S}}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ associated with finite R-modules, under the structural morphism $(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}} \to \vec{\mathsf{S}}_0$.*

Theorem 8.6 (vanishing of higher direct images; cf. [17, Thm. 3.9])**.** *Suppose R is an $\mathcal{O}_{F_0,(p)}$-algebra, and suppose that $\mathscr{E}$ is a quasi-coherent sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ that is formally canonical* (*resp. formally subcanonical*) *over R, as in Definition* 8.5. *Let D' be as in Corollary* 6.7, *and let $\mathscr{E}(-n\mathsf{D}') := \mathscr{E} \underset{\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(-n\mathsf{D}')$, for each integer n. Then $R^i(\vec{\oint}_{\mathcal{H},\Sigma})_*\mathscr{E}(-n\mathsf{D}') = 0$ for all $i > 0$ and $n > 0$* (*resp. $n \geq 0$*).

Proof. Thanks to Theorem 6.1, which provides almost the same axiomatic setup in [17, Sec. 4], except that $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is in general not an abelian scheme torsor over a finite cover of $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$; and thanks to Proposition 8.3, which implies the analogue of [17, Lem. 6.1] for the context here; the same argument as in the proof of [17, Thm. 3.9] also works here (see Remark 8.9 below). ☐

Theorem 8.7 (Koecher's principle; cf. [17, Thm. 2.3])**.** *Suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a simple algebra over $\mathbb{Q}$. Suppose R is an $\mathcal{O}_{F_0,(p)}$-algebra, and suppose that $\mathscr{E}$ is a quasi-coherent sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ that is formally canonical over R, as in Definition* 8.5. *For each open subset $U^{\min}$ of $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$, consider its preimage U^{tor} in $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ under the canonical morphisms $\vec{\oint}_{\mathcal{H},\Sigma}$, and its preimage U in $\vec{\mathsf{M}}_{\mathcal{H}}$ under the canonical morphism $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\min}_{\mathcal{H}}$. Then the canonical restriction map*

$$\Gamma(U^{\mathrm{tor}}, \mathscr{E}|_{U^{\mathrm{tor}}}) \to \Gamma(U, \mathscr{E}|_U) \tag{8.8}$$

is a bijection, except when both $\dim(\mathsf{M}_{\mathcal{H}}) = 1$ *and* $U^{\min} - U \neq \emptyset$ *hold.*

Proof. As in the proof of Theorem 8.6, thanks to Theorem 6.1, and thanks to Proposition 8.4, which implies the analogue of [17, Lem. 6.2] for the context here (under the assumption that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a simple algebra over $\mathbb{Q}$), the same argument as in the proof of [17, Thm. 2.3] also works here (see Remark 8.9 below). ☐

Remark 8.9. While we assumed in [17] that Σ is not only projective but also smooth, the arguments there were carried out up to replacing Σ with its smooth refinements (see [17, Rem. 4.17]). Hence they also work for possibly nonsmooth Σ's in the contexts of Theorems 8.6 and 8.7. Note that [17] was written for the smooth integral models in [5] and [15], where Σ was always assumed to be smooth.

Remark 8.10. However, since the proof of [17, Thm. 2.5] made use of Serre duality, we cannot easily generalize the higher Koecher's principle to the context here. In general, we do not yet know whether it is still true in ramified characteristics.

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