DUAL BGG COMPLEXES FOR AUTOMORPHIC BUNDLES

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Abstract. We generalize the construction of dual BGG complexes in Faltings–Chai and Mokrane–Tilouine–Polo (from the case of Siegel modular varieties) to all smooth integral models of PEL-type Shimura varieties.

1. Introduction

Shimura varieties are generalizations of modular curves, whose cohomology groups with coefficients in the so-called automorphic bundles set up natural stages for relating automorphic representations to Galois representations. In order to understand the Hodge structures on the de Rham version of such cohomology groups, Faltings introduced the dual BGG spectral sequence (over $\mathbb{C}$) in [9], following older ideas of Bernstein, I. M. Gelfand, and S. I. Gelfand [2], and verified its degeneration in [10, Ch. VI] using toroidal compactifications of fiber products of the universal abelian schemes over the Siegel modular varieties.

The geometric construction of compactifications in [10, Ch. VI] is actually carried out over $\mathbb{Z}$, and (parabolic) BGG complexes have been constructed over $\mathbb{Z}_{(p)}$ (under a *p-smallness* assumption on the highest weights) by Tilouine and the second author in [33]. Based on these inputs, Mokrane and Tilouine studied the de Rham cohomology of Siegel modular varieties with coefficients in vector bundles over $\mathbb{Z}_{(p)}$ in [31] by constructing analogues of Faltings's *dual BGG complexes*, and obtained several interesting applications to the cohomology of Siegel modular varieties. (In [8], Dimitrov applied similar ideas to the cohomology of Hilbert modular varieties.)

The aim of this article is to explain that the constructions of dual BGG complexes in [9, Sec. 3], [10, Ch. VI], and [31, Sec. 5] have analogues over all (smooth integral models of) PEL-type Shimura varieties, under a p-smallness assumption.

The main geometric input, generalizing the constructions of toroidal compactifications in [10], has been carried out by the first author in [22] and [25]. (We will refer to the published revision [26] instead of the original thesis [22]. When

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the Shimura variety we consider is compact, the shorter article [23] would suffice, because it explains that no compactification is needed.)

In [33], Theorems 2.8 and 4.3 were stated (and proved) for a connected, split reductive group G with a simply-connected derived group. In fact, these hypotheses can be relaxed, and we will show that a similar result still holds when G has a factor isomorphic to some orthogonal group O_{2r} (whose derived group is not simply-connected). (For readers familiar with the classification of PEL-type Shimura varieties, the point is that we allow all possibilities, including those with factors *of type* D.)

We will review the geometric setup in Section 2, review the representation theory we need in Section 3, explain the construction of differential operators in Section 4, and prove our main results in Section 5.

We shall follow [26, Notations and Conventions] unless otherwise specified. By symplectic isomorphisms between modules with symplectic pairings, we *always* mean isomorphisms between the modules matching the pairings up to an invertible scalar multiple. (These are often called symplectic similitudes, but our understanding is that the codomains of pairings are modules rather than rings, which ought to be matched as well.) Sheaves on schemes, algebraic spaces, or algebraic stacks are étale sheaves by default, although for coherent sheaves on schemes it would suffice to work in the Zariski topology.

Contents

2. Geometric setup

2.1. **Linear algebraic data.** Let $\mathcal{O}$ be an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra with a positive involution $\star$. Here, an *involution* means an *anti*-automorphism of order 2, and *positivity* of $\star$ means that for every $x \neq 0$ in $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, one has $\mathrm{Tr}_{(\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R})/\mathbb{R}}(xx^\star) > 0$. We assume that $\mathcal{O}$ is stable under $\star$. We shall denote the center of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ by F. (Then F is a product of number fields.)

Let $\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times) = (2\pi\sqrt{-1})\mathbb{Z}$, which is a free $\mathbb{Z}$-module of rank one. For any $\mathbb{Z}$-module M, we denote by $M(1)$ the module $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}(1)$, called the Tate twist of M, which is noncanonically isomorphic to M as $\mathbb{Z}$-modules.

By a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot\,, \cdot \rangle, h_0)$, we mean the following data:

1. An $\mathcal{O}$-lattice L, namely, a finite free $\mathbb{Z}$-module L with the structure of an $\mathcal{O}$-module.
2. An alternating pairing $\langle \cdot\,, \cdot \rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y \rangle = \langle x, b^\star y \rangle$ for all $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism
$$h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}),$$
satisfying:
   (a) For any $z \in \mathbb{C}$ and $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we have $\langle h_0(z)x, y \rangle = \langle x, h_0(z^c)y \rangle$, where $z \mapsto z^c$ is complex conjugation.
   (b) The $\mathbb{R}$-bilinear pairing $(2\pi\sqrt{-1})^{-1}\langle \cdot\,, h_0(\sqrt{-1})\,\cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is (symmetric and) positive definite. (See [26, Def. 1.2.1.3], where h_0 was denoted by h.)

The tuple $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ will be called an *integral* PEL datum. It is an integral version of the data $(B, \star, V, \langle \cdot\,, \cdot \rangle, h_0)$ in [21] and related works.

Definition 2.1 (cf. [26, Def. 1.2.1.6])**.** *Let $\mathcal{O}$ and $(L, \langle \cdot\,, \cdot \rangle)$ be given as above. We define for each $\mathbb{Z}$-algebra R*

$$\mathrm{G}(R) := \left\{ (g, r) \in \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \langle gx, gy \rangle = r\langle x, y \rangle,\ \forall x, y \in L \underset{\mathbb{Z}}{\otimes} R \right\}.$$

The assignment is functorial in R and defines a group functor G over $\mathbb{Z}$. The projection to the second factor $(g, r) \mapsto r$ defines a homomorphism $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$, which we call the **similitude character***. For simplicity, we shall often denote elements (g, r) in G by simply g, and denote by $\nu(g)$ the value of r when we need it.*

The homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ defines a Hodge structure of weight -1 on L, with Hodge decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c, \tag{2.2}$$

such that $h_0(z)$ acts by $1 \otimes z$ on V_0, and by $1 \otimes z^c$ on V_0^c. One can easily check that V_0 is (maximal) totally isotropic under the non-degenerate pairing $\langle \cdot\,, \cdot \rangle$, and hence (2.2) induces canonically an isomorphism

$$V_0^c \cong V_0^\vee(1) := \mathrm{Hom}_{\mathbb{C}}(V_0, \mathbb{C})(1). \tag{2.3}$$

Let F_0 be the *reflex field* of the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module V_0. Recall (see [21, p. 389] or [26, Def. 1.2.5.4]) that F_0 is the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by $\{\mathrm{Tr}_{\mathbb{C}}(b|V_0)\}_{b\in\mathcal{O}}$.

By abuse of notation, we shall denote the ring of integers in F (resp. F_0) by $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$). This is in conflict with the notation of the order $\mathcal{O}$ in the integral PEL datum, but the precise interpretation will be clear from the context.

We say that a rational prime number $p > 0$ is *good* if it satisfies the following conditions (cf. [21, Sec. 5] or [26, Def. 1.4.1.1]):

(1) p is unramified in $\mathcal{O}$ (as in [26, Def. 1.1.1.8]).
(2) $p \neq 2$ if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [26, Def. 1.2.1.15]).
(3) If we consider $L^{\#} := \{x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : \langle x, y\rangle \in \mathbb{Z}(1), \forall y \in L\}$, the dual lattice of L under the $\mathbb{Z}(1)$-valued pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, then $p \nmid [L^{\#} : L]$. Equivalently, after base change to $\mathbb{Z}_{(p)}$, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is perfect in the sense that it induces an isomorphism $L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)} \cong L^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}(1) := \mathrm{Hom}_{\mathbb{Z}_{(p)}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}(1))$.

Let us fix any choice of a good prime p.

By [26, Lem. 1.2.5.9], there exists a finite field extension F_0' of F_0 in $\mathbb{C}$, unramified at p, and an $\mathcal{O}_{F_0'}$-torsion-free $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0',(p)}$-module L_0, such that $L_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} \mathbb{C} \cong V_0$. Let us fix the choices of F_0' and L_0 from now on. (In practice, there might be optimal choices in each special case.)

Let us denote (cf. [26, Lem. 1.1.4.13]) by

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to \mathcal{O}_{F_0',(p)}(1)$$

the alternating pairing defined by

$$\langle (x_1, f_1), (x_2, f_2)\rangle_{\mathrm{can.}} := f_2(x_1) - f_1(x_2).$$

The natural right action of $\mathcal{O}$ on $L_0^\vee(1)$ defines a natural left action of $\mathcal{O}$ by composition with $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$. Then (2.3) induces canonically an isomorphism $L_0^\vee(1) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0^\vee(1) \cong V_0^c$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.

Definition 2.4. *For any $\mathcal{O}_{F_0',(p)}$-algebra R, set*

$$\mathrm{G}_0(R) := \left\{ \begin{array}{c} (g, r) \in \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}((L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy\rangle_{\mathrm{can.}} = r\langle x, y\rangle_{\mathrm{can.}},\ \forall x, y \in (L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R \end{array} \right\},$$

$$\mathrm{P}_0(R) := \left\{ (g, r) \in \mathrm{G}_0(R) : g(L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) = L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R \right\},$$

$$\mathrm{M}_0(R) := \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R),$$

We shall view $\mathrm{M}_0(R)$ *canonically as a quotient of* $\mathrm{P}_0(R)$ *by*

$$\mathrm{P}_0(R) \to \mathrm{M}_0(R) : (g, r) \mapsto (g|_{L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R}, r).$$

The assignments are functorial in R, and define group functors G_0, P_0, *and* M_0 *over* $\mathcal{O}_{F_0',(p)}$.

By [26, Prop. 1.1.1.21, Cor. 1.2.5.7, and Cor. 1.2.3.10], and by our choice that F_0' is unramified at p, there exists a discrete valuation ring R_1 over $\mathcal{O}_{F_0',(p)}$ satisfying the following:

Condition 2.5. (1) *The maximal ideal of R_1 is generated by p, and the residue field κ_1 of R_1 is a finite field of characteristic p. In this case, the p-adic completion of R_1 is isomorphic to the Witt vectors $W(\kappa_1)$ over κ_1.*

(2) *The $\mathbb{Z}$-algebra $\mathcal{O}_F$ is split over R_1, in the sense that*

$$\Upsilon := \mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$$

has cardinality $[F:\mathbb{Q}]$. Then there is a canonical isomorphism

$$\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1 \cong \prod_{\tau\in\Upsilon} \mathcal{O}_{F,\tau}, \tag{2.6}$$

where each $\mathcal{O}_{F,\tau}$ can be identified as the $\mathcal{O}_F$-algebra R_1 via τ.

(3) *There exists an isomorphism*

$$(L \underset{\mathbb{Z}}{\otimes} R_1, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (L_0 \oplus L_0^\vee(1), \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \tag{2.7}$$

inducing an isomorphism

$$\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1 \cong \mathrm{G}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \tag{2.8}$$

realizing $\mathrm{P}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$ as a subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$. (The existence of the isomorphism (2.7) *follows from* [26, Cor. 1.2.3.10].*)*

From now on, let us fix the choice of R_1 and the isomorphism (2.7), and set $\mathcal{O}_{F,1} := \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1$, $\mathcal{O}_1 := \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$, $L_1 := L \underset{\mathbb{Z}}{\otimes} R_1$, $L_{0,1} := L_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$, $\mathrm{G}_1 := \mathrm{G}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \cong \mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$, $\mathrm{P}_1 := \mathrm{P}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$, and $\mathrm{M}_1 := \mathrm{M}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$.

Remark 2.9. The group functors in Definitions 2.1 and 2.4 are representable because they are defined by closed conditions in general linear group schemes. By the same explicit classification as in the proof of [26, Prop. 1.2.3.11] (which works verbatim over the R_1 here instead of the R' there), $\mathrm{G}_1 = \mathrm{G}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \cong \mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$ is a split reductive group scheme over R_1, the group scheme P_1 is a parabolic subgroup scheme of G_1, and M_1 is canonically isomorphic to the Levi quotient of P_1.

2.2. **PEL-Type Shimura varieties and automorphic bundles.** Let $\mathcal{H}$ be a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$. (See [32, 0.6] or [26, Def. 1.4.1.8] for the definition of neatness.) By [26, Def. 1.4.1.4] (with $\Box = \{p\}$ there), the data of $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, parameterizing tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ over schemes S over S_0 of the following form:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization of degree prime to p.
(3) $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure as in [26, Def. 1.3.3.1].
(4) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-module structure given naturally by i satisfies the determinantal condition in [26, Def. 1.3.4.1] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$.

(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p, \langle \cdot\,,\,\cdot \rangle)$ as in [26, Def. 1.3.7.6].

(The definition can be identified with the one in [21, Sec. 5] by [26, Prop. 1.4.3.4].) By [26, Thm. 1.4.1.11 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over S_0 (under the assumption that $\mathcal{H}$ is neat).

Let $(A, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}$ be the universal tuple over $\mathsf{M}_{\mathcal{H}}$. Consider the *relative de Rham cohomology* $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$, with the dual

$$\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}), \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$$

defined to be the *relative de Rham homology*. Consider the canonical pairing

$$\langle \cdot\,,\,\cdot \rangle_\lambda : \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1) \tag{2.10}$$

defined by the pullback under $\mathrm{Id} \times \lambda_*$ of the canonical perfect pairing

$$\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(A^\vee/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$$

defined by the first Chern class of the Poincaré line bundle over $A \underset{\mathsf{M}_{\mathcal{H}}}{\times} A^\vee$. (See for example [7, 1.5].) Under the assumption that λ has degree prime to p, we know that λ is separable, that λ_* is an isomorphism, and hence that the pairing $\langle \cdot\,,\,\cdot \rangle_\lambda$ above is *perfect*. Let $\langle \cdot\,,\,\cdot \rangle_\lambda$ also denote the induced pairing on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$ by duality. By [3, Lem. 2.5.3], we have canonical short exact sequences

$$0 \to \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}(1) \to \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H}}} \to 0$$

and

$$0 \to \underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}}(1) \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H}}} \to 0.$$

The submodules $\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}(1)$ and $\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}}(1)$ are maximal totally isotropic with respect to $\langle \cdot\,,\,\cdot \rangle_\lambda$. (The Tate twists on $\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}(1)$ and $\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}}(1)$ were omitted in some of the first author's earlier writings, which we have reinstated for the sake of clarity.)

Let $\tilde{\mathsf{M}}^{(m)}_{\mathcal{H}}$ be the m-th infinitesimal neighborhood of the diagonal image of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{M}_{\mathcal{H}}$, and let $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{\mathsf{M}}^{(m)}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}$ be the two projections. Then we have by definition the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \to \mathscr{P}^m_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} := \mathrm{pr}_{1,*}\,\mathrm{pr}_2^*(\mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$. The isomorphism $s : \tilde{\mathsf{M}}^{(m)}_{\mathcal{H}} \to \tilde{\mathsf{M}}^{(m)}_{\mathcal{H}}$ over $\mathsf{M}_{\mathcal{H}}$ swapping two components of the fiber product then defines an automorphism s^* of $\mathscr{P}^m_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$. When $m = 1$, the kernel of the structural morphism $\mathrm{str}^* : \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$, canonically isomorphic to $\Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$ by definition, is spanned by the image of $s^* - \mathrm{Id}^*$ (induced by $\mathrm{pr}_1^* - \mathrm{pr}_2^*$).

An important property of the relative de Rham cohomology of any smooth morphism like $A \to \mathsf{M}_{\mathcal{H}}$ is that, for any two smooth lifts $\tilde{A}_1 \to \tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}$ and $\tilde{A}_2 \to \tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}$ of $A \to \mathsf{M}_{\mathcal{H}}$, there is a canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{A}_2/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(\tilde{A}_1/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$. (See for example [26, Prop. 2.1.6.4].) If we consider $\tilde{A}_1 := \mathrm{pr}_1^* A$ and $\tilde{A}_2 := \mathrm{pr}_2^* A$, then we obtain a canonical isomorphism

$$\mathrm{pr}_2^*\,\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \cong \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_1^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \cong \mathrm{pr}_1^*\,\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}),$$

which we denote by Id^* by abuse of notation. On the other hand, pullback by the swapping automorphism $s : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H}} \xrightarrow{\sim} \tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}$ defines another canonical isomorphism

$$s^* : \mathrm{pr}_2^*\,\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \cong \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_1^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \cong \mathrm{pr}_1^*\,\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}).$$

Hence, we can define the Gauss–Manin connection as follows (cf. [26, Rem. 2.1.7.4]):

Definition 2.11. *The Gauss–Manin connection*

$$\nabla : \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.12}$$

on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$ is the composition

$$\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\mathrm{pr}_2^*} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H}}) \xrightarrow{s^*-\mathrm{Id}^*} \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}.$$

Definition 2.13. *The composition*

$$\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}}(1) \hookrightarrow \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\nabla} \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \twoheadrightarrow \underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$$

defines by duality a morphism

$$\mathrm{KS}_{A/\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}(1) \to \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}, \tag{2.14}$$

which we call the **Kodaira–Spencer morphism***.*

Definition 2.15 (cf. [26, Def. 2.3.5.1])**.** *The sheaf $\underline{\mathrm{KS}}_{A/\mathsf{M}_{\mathcal{H}}} := \underline{\mathrm{KS}}_{(A,\lambda,i,\alpha_{\mathcal{H}})/\mathsf{M}_{\mathcal{H}}}$ is the quotient*

$$(\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}) / \begin{pmatrix} \lambda^*(y)\otimes z - \lambda^*(z)\otimes y \\ i(b)^*(x)\otimes y - x\otimes(i(b)^\vee)^*(y) \end{pmatrix}_{\substack{x\in\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}},\\ y,z\in\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}},\\ b\in\mathcal{O}.}}$$

Proposition 2.16 (see [26, Prop. 2.3.5.2])**.** *The Kodaira–Spencer morphism* (2.14) *factors through the canonical quotient $\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}}(1) \twoheadrightarrow \underline{\mathrm{KS}}_{A/\mathsf{M}_{\mathcal{H}}}(1)$ and induces an isomorphism*

$$\underline{\mathrm{KS}}_{A/\mathsf{M}_{\mathcal{H}}}(1) \xrightarrow{\sim} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}, \tag{2.17}$$

which we call the **Kodaira–Spencer isomorphism***, and denote again* (*by abuse of notation*) *by* $\mathrm{KS}_{A/\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$*.*

Consider the set

$$\mathsf{X} := \mathrm{G}(\mathbb{R})h_0$$

of $\mathrm{G}(\mathbb{R})$-conjugates $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ of $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$, which is canonically a finite union of Hermitian symmetric spaces. Let $H_p := \mathrm{G}(\mathbb{Z}_p)$ (an open compact subgroup of $\mathrm{G}(\mathbb{Q}_p)$) and let H be the open compact subgroup $\mathcal{H}H_p$ of $\mathrm{G}(\hat{\mathbb{Z}})$. It is well known (see [21, Sec. 8] or [24, Sec. 2]) that there exists a quasi-projective variety Sh_H over F_0, together with a canonical open and closed immersion $\mathsf{Sh}_H \hookrightarrow \mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0$ (because $\mathcal{H}$ is neat), such that the analytification of $\mathsf{Sh}_H \underset{F_0}{\otimes} \mathbb{C}$ can be canonically identified with the double coset space $\mathrm{G}(\mathbb{Q})\backslash\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)/H$. (Note that $\mathsf{Sh}_H \hookrightarrow \mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0$ is not an isomorphism in general, due to the so-called failure of Hasse's principle. See [21, Sec. 8] and [26, Rem. 1.4.3.12].)

Let $\mathsf{M}_{\mathcal{H},0}$ denote the schematic closure of Sh_H in $\mathsf{M}_{\mathcal{H}}$. Then $\mathsf{M}_{\mathcal{H},0}$ is smooth over S_0. By [23], $\mathsf{M}_{\mathcal{H},0}$ is proper over S_0 if $\mathrm{G}(\mathbb{Q})\backslash\mathsf{X}\times\mathrm{G}(\mathbb{A}^\infty)/H$ is compact.

Let $\mathsf{S}_1 := \mathrm{Spec}(R_1)$, and let $\mathsf{M}_{\mathcal{H},1} := \mathsf{M}_{\mathcal{H},0} \underset{\mathsf{S}_0}{\times} \mathsf{S}_1$. By abuse of notation, we denote the pullback of the universal object over $\mathsf{M}_{\mathcal{H}}$ to $\mathsf{M}_{\mathcal{H},1}$ by $(A,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H},1}$.

As in [25, Sec. 6A] and [27, Sec. 1.3], let us define the *principal bundles*

$$\mathcal{E}_{\mathrm{G}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}\Big(\big(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)\big), \big((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)\big)\Big), \tag{2.18}$$

$$\mathcal{E}_{\mathrm{P}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}\Big(\big(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}(1)\big), \big((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}\big)\Big), \tag{2.19}$$

and

$$\mathcal{E}_{\mathrm{M}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}\Big(\big(\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}(1), \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)\big), \big(L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)\big)\Big), \tag{2.20}$$

which are étale torsors for G_1, P_1, and M_1, respectively, over $\mathsf{M}_{\mathcal{H},1}$. (The entries $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)$ in the tuples represent the values of the pairings, which are matched up to unit by the isomorphisms, by our convention. The groups G_1, P_1, and M_1 act as automorphisms on the respective second tuples in the $\underline{\mathrm{Isom}}$ functors, by definition.)

Definition 2.21. *For any R_1-algebra R, we denote by $\mathrm{Rep}_R(\mathrm{G}_1)$ (resp. $\mathrm{Rep}_R(\mathrm{P}_1)$, resp. $\mathrm{Rep}_R(\mathrm{M}_1)$) the category of R-modules of finite presentation with algebraic actions of $\mathrm{G}_1 \underset{R_1}{\otimes} R$ (resp. $\mathrm{P}_1 \underset{R_1}{\otimes} R$, resp. $\mathrm{M}_1 \underset{R_1}{\otimes} R$).*

Definition 2.22. *Let R be any R_1-algebra. For any $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, we define*

$$\mathcal{E}_{\mathrm{G}_1,R}(W) := (\mathcal{E}_{\mathrm{G}_1} \underset{R_1}{\otimes} R) \overset{\mathrm{G}_1 \underset{R_1}{\otimes} R}{\times} W, \tag{2.23}$$

called the **automorphic sheaf** *over $\mathsf{M}_{\mathcal{H},1} \underset{R_1}{\otimes} R$ associated with W. It is called an* **automorphic bundle** *if W is locally free as an R-module. We define similarly for $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ (resp. $W \in \mathrm{Rep}_R(\mathrm{M}_1)$) by replacing G_1 with P_1 (resp. with M_1) in (2.23). (These are coherent sheaves by fpqc descent. See [15, VIII, 1.1 and 1.10].)*

By [27, Lem. 1.18, 1.19, and 1.20, and Cor. 1.21], we have the following:

Lemma 2.24. *Let R be any R_1-algebra.*

1. *The assignment $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$ (resp. $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$, resp. $\mathcal{E}_{\mathrm{M}_1,R}(\,\cdot\,)$) defines an* **exact functor** *from $\mathrm{Rep}_R(\mathrm{G}_1)$ (resp. $\mathrm{Rep}_R(\mathrm{P}_1)$, resp. $\mathrm{Rep}_R(\mathrm{M}_1)$) to the category of coherent sheaves on $\mathsf{M}_{\mathcal{H},1}$.*
2. *If we consider an object $W \in \mathrm{Rep}_R(\mathrm{G}_1)$ as an object in $\mathrm{Rep}_R(\mathrm{P}_1)$ by restriction to P_1, then we have a canonical isomorphism $\mathcal{E}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}_{\mathrm{P}_1,R}(W)$.*
3. *If we view an object $W \in \mathrm{Rep}_R(\mathrm{M}_1)$ as an object in $\mathrm{Rep}_R(\mathrm{P}_1)$ in the canonical way (under the canonical surjection $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$), then we have a canonical isomorphism $\mathcal{E}_{\mathrm{P}_1,R}(W) \cong \mathcal{E}_{\mathrm{M}_1,R}(W)$.*
4. *Suppose $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ has a decreasing filtration by subobjects $\mathtt{F}^a(W) \subset W$ in $\mathrm{Rep}_R(\mathrm{P}_1)$ such that each graded piece $\mathrm{Gr}^a_{\mathtt{F}}(W) := \mathtt{F}^a(W)/\mathtt{F}^{a+1}(W)$ can be identified with an object of $\mathrm{Rep}_R(\mathrm{M}_1)$. Then $\mathcal{E}_{\mathrm{P}_1,R}(W)$ has a filtration $\mathcal{E}_{\mathrm{P}_1,R}(\mathtt{F}^a(W))$ with graded pieces $\mathcal{E}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$.*

2.3. **Toroidal compactifications and canonical extensions.** Under the assumption that $\mathcal{H}$ is neat, by [26, Thm. 6.4.1.1 and 7.3.3.4], $\mathsf{M}_{\mathcal{H}}$ admits a *toroidal compactification* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, a scheme projective and smooth over S_0, depending on a compatible collection Σ of cone decompositions that is *projective* and *smooth* in the sense of [26, Def. 6.3.3.4 and 7.3.1.3], with the following properties:

(1) The universal abelian scheme $A \to \mathsf{M}_{\mathcal{H}}$ extends to a semi-abelian scheme $A^{\mathrm{ext}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the polarization $\lambda : A \to A^\vee$ extends to a prime-to-p isogeny $\lambda^{\mathrm{ext}} : A^{\mathrm{ext}} \to (A^{\mathrm{ext}})^\vee$ between semi-abelian schemes, and the endomorphism structure $i : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H}}}(A)$ extends to an endomorphism structure $i^{\mathrm{ext}} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(A^{\mathrm{ext}})$. (These extensions are unique because the base is noetherian and normal. See [10, Ch. I, Prop. 2.7].)

(2) The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (with its reduced structure) is a relative Cartier divisor $\mathsf{D} = \mathsf{D}_{\infty,\mathcal{H}}$ with *simple* normal crossings. Here simpleness of the normal crossings uses [26, Cond. 6.2.5.25 and Lem. 6.2.5.27] (cf. [10, Ch. IV, Rem. 5.8(a)]) and the assumption that $\mathcal{H}$ is neat.

(3) Let
$$\underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \underline{\mathrm{KS}}_{(A^{\mathrm{ext}},\lambda^{\mathrm{ext}},i^{\mathrm{ext}})/\mathsf{M}_{\mathcal{H}}}$$
be the quotient of $\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ by the relations as in Definition 2.15. Let
$$\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} := \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}(\log \mathsf{D}) := \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log \mathsf{D}]$$
be the sheaf of modules of log 1-differentials on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ over S_0, with respect to the relative Cartier divisor D with normal crossings. Then the Kodaira–Spencer morphism (2.14) extends to a morphism
$$\mathrm{KS}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1) \to \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}, \tag{2.25}$$
called the *extended Kodaira–Spencer morphism*, which factors through the canonical quotient $\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1) \twoheadrightarrow \underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1)$ and induces a canonical *isomorphism*
$$\mathrm{KS}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1) \xrightarrow{\sim} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.26}$$
extending (2.17), called the *extended Kodaira–Spencer isomorphism.*

In what follows, we shall fix the choice of a (projective and smooth) Σ, and suppress Σ from the notation.

Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},0}$ denote the schematic closure of Sh_H in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, and let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$ denote the pullback of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},0}$ under $\mathsf{S}_1 \to \mathsf{S}_0$. Then $\mathsf{M}_{\mathcal{H},1}$ is smooth over S_1, and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1} \to \mathsf{S}_1$ is proper smooth with properties analogous to those of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{S}_0$. By abuse of notation, let us denote the pullback of D to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$ by the same notation D.

Proposition 2.27 (see [25, Prop. 6.9])**.** *The locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})$ extends to a locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$, which can be characterized by the following properties:*

(1) *The sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$, canonically identified as a subsheaf of the quasi-coherent sheaf $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1})_*(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}))$, is self-dual under*

the pairing $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1})_*(\langle \cdot , \cdot \rangle_\lambda)$. *We shall denote the induced pairing by* $\langle \cdot , \cdot \rangle^{\mathrm{can}}_\lambda$.

(2) $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ *contains* $\underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$ *as a subsheaf totally isotropic under* $\langle \cdot , \cdot \rangle^{\mathrm{can}}_\lambda$.

(3) *The quotient sheaf* $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}/\underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$ *can be canonically identified with the subsheaf* $\underline{\mathrm{Lie}}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$ *of* $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1})_*\underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H},1}}$.

(4) *The pairing* $\langle \cdot , \cdot \rangle^{\mathrm{can}}_\lambda$ *induces an isomorphism*

$$\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$$

which coincides with $d\lambda^{\mathrm{ext}}$.

(5) *Let*

$$\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}).$$

The Gauss–Manin connection (2.12) *extends to an integrable connection*

$$\nabla : \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1} \tag{2.28}$$

with log poles along D, *called the extended Gauss–Manin connection, such that the composition*

$$\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}(1) \hookrightarrow \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \xrightarrow{\nabla} \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1}$$
$$\twoheadrightarrow \underline{\mathrm{Lie}}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1}$$

induces by duality the extended Kodaira–Spencer morphism (2.25) (*and hence the extended Kodaira–Spencer isomorphism* (2.26)).

Remark 2.29. Any construction achieving the properties in Proposition 2.27 will serve the same purpose in what follows. Therefore, one can refer to [10, Ch. VI] and related works in special cases, without having to explain the consistency with [25]. (This is desirable because the methods in [10, Ch. VI] and [25] are different.)

As in [25, Sec. 6B] and [28, Sec. 4.2], by replacing

$$\big(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}), \langle \cdot , \cdot \rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}(1)\big)$$

(and its subtuples) with

$$\big(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \langle \cdot , \cdot \rangle^{\mathrm{can}}_\lambda, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}(1)\big)$$

(and the corresponding subtuples) in the definitions (2.18), (2.19), and (2.20), the principal bundles $\mathcal{E}_{\mathrm{G}_1}$, $\mathcal{E}_{\mathrm{P}_1}$, and $\mathcal{E}_{\mathrm{M}_1}$ over $\mathsf{M}_{\mathcal{H},1}$ extend canonically to the principal bundles $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1}$, and $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$, respectively,

Definition 2.30. *Let* R *be any* R_1*-algebra. For any* $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, *we define*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1} \underset{R_1}{\otimes} R) \overset{\mathrm{G}_1 \underset{R_1}{\otimes} R}{\times} W, \tag{2.31}$$

called the **canonical extension** *of* $\mathcal{E}_{\mathrm{G}_1,R}(W)$, *and define*

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \mathscr{I}_{\mathsf{D}},$$

called the **subcanonical extension** *of* $\mathscr{E}_{\mathrm{G}_1,R}(W)$*, where* $\mathscr{I}_{\mathsf{D}}$ *is the* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$*-ideal defining the relative Cartier divisor* D*. Also, we define similarly* $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$*,* $\mathscr{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$*,* $\mathscr{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(W)$*, and* $\mathscr{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(W)$ *with* G_1 *(and its principal bundle) replaced accordingly with* P_1 *and* M_1 *(and their respective principal bundles).*

Lemma 2.32 (cf. [28, Lem. 4.14])**.** *Lemma* 2.24 *remains true if we replace the automorphic sheaves with their canonical or subcanonical extensions.*

2.4. **De Rham complexes.** Let R be any R_1-algebra. For simplicity, we shall denote pullbacks of objects from R_1 to R by replacing the subscript "1" with "R", although we shall use the same notation D for its pullback.

First, let us explain how the Gauss–Manin connection (2.12) induces integrable connections on automorphic sheaves.

In Definition 2.11, the Gauss–Manin connection (2.12) is defined by the difference between the two isomorphisms $\mathrm{Id}^*, s^* : \mathrm{pr}_2^* \, \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\sim} \mathrm{pr}_1^* \, \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})$. Since s^* has a simple definition, we can interpret Id^* (whose definition as in [26, Prop. 2.1.6.4] is far from simple) as induced by the Gauss–Manin connection (2.12) (and s^*).

By construction of $\mathscr{E}_{\mathrm{G}_1,R}(\,\cdot\,)$ (cf. (2.23)), for each $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, the two isomorphisms above induce two isomorphisms $\mathrm{Id}^*, s^* : \mathrm{pr}_2^*(\mathscr{E}_{\mathrm{G}_1,R}(W)) \xrightarrow{\sim} \mathrm{pr}_1^*(\mathscr{E}_{\mathrm{G}_1,R}(W))$ lifting the identity morphism on $\mathscr{E}_{\mathrm{G}_1,R}(W)$. Hence, the difference $s^* - \mathrm{Id}^*$ induces an integrable connection

$$\nabla : \mathscr{E}_{\mathrm{G}_1,R}(W) \to \mathscr{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}. \tag{2.33}$$

Definition 2.34. *The integrable connection* ∇ *in* (2.33) *above is called the* **Gauss–Manin connection** *for* $\mathscr{E}_{\mathrm{G}_1,R}(W)$*.*

Next, let us explain how the extended Gauss–Manin connection (2.28) induces integrable connections on canonical and subcanonical extensions (extending the integrable connections induced by the Gauss–Manin connection (2.12)). Set

$$\overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1} := \Omega^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1}(\log \mathsf{D}) \cong \wedge^{\bullet}(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}/\mathsf{S}_1}[d\log \mathsf{D}]).$$

Let $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ be the subsheaf of $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_* \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$ corresponding to the subsheaf $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \oplus \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ of $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0})$ under the canonical splitting $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \cong \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, with the summand $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$ given by the image of $\mathrm{pr}_2^* : \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \to \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, and with the summand $\Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$ spanned by the image of $(\mathrm{pr}_1^* - \mathrm{pr}_2^*) = (s^* - \mathrm{Id}^*) \circ \mathrm{pr}_2^* : \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \to \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$. Then the morphisms $\mathrm{pr}_1^*, \mathrm{pr}_2^*, \mathrm{Id}^*, s^*$ induce respectively morphisms $\overline{\mathrm{pr}}_1^*, \overline{\mathrm{pr}}_2^* : \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ and $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \xrightarrow{\sim} \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ such that $\overline{s}^* - \overline{\mathrm{Id}}^*$ induces the universal log derivation $d : \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$. Since $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ is defined only axiomatically, it is convenient that the above objects are uniquely determined by their pullbacks to $\mathsf{M}_{\mathcal{H}}$, and that we can define them as induced objects, without having to resort to their interpretations in log geometry. (Certainly, any reasonable theory should be compatible with such extensions.)

The property (5) in Proposition 2.27 states that the Gauss–Manin connection (2.12) induces the extended Gauss–Manin connection (2.28), which is equivalent to the statement that the extended Gauss–Manin connection (2.28) is defined by the difference between the two isomorphisms $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathrm{pr}}_2^*(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}) \xrightarrow{\sim} \overline{\mathrm{pr}}_1^*(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$. (Again, we can interpret $\overline{\mathrm{Id}}^*$ as induced by the extended Gauss–Manin connection (2.28) and $\overline{s}^*$.) Note that here $\overline{\mathrm{pr}}_1^*$ and $\overline{\mathrm{pr}}_2^*$ are morphisms with their targets tensored with $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$, but not $\mathscr{P}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ (which can be identified with the structural sheaf of the first infinitesimal neighborhood of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$). By construction of $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(\,\cdot\,)$ (cf. (2.31)), for each $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, the two isomorphisms above induce two isomorphisms $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathrm{pr}}_2^*(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)) \xrightarrow{\sim} \overline{\mathrm{pr}}_1^*(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W))$ lifting the identity morphism on $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$. Hence, the difference $\overline{s}^* - \overline{\mathrm{Id}}^*$ induces a morphism

$$\nabla : \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \to \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \tag{2.35}$$

of sheaves of R-modules. Since the connection ∇ in (2.35) is induced by the connection ∇ in (2.33), the conditions for being an integrable connection with log poles are tautologically verified. By applying $\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \mathscr{I}_{\mathsf{D}}$, we obtain an integrable connection

$$\nabla : \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \to \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \tag{2.36}$$

with log poles.

Definition 2.37. *The integrable connection* ∇ (*with log poles*) *in* (2.33) (*resp.* (2.36)) *is called the* **extended Gauss–Manin connection** *for* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$ (*resp.* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W)$).

Definition 2.38. *The connections* (2.33), (2.35), *and* (2.36) *define respectively the* **de Rham complex**

$$(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and the **log de Rham complexes**

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and

$$(\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla).$$

3. Representation theory

3.1. **Decomposition of reductive groups.** By [26, Prop. 1.1.1.21], and by the decomposition of the center $\mathcal{O}_{F,1} = \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1$ given in (2.6), $\mathcal{O}_1 = \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$ is canonically isomorphic to a direct product $\prod_{\tau \in \Upsilon} \mathcal{O}_\tau$, where for each $\tau \in \Upsilon = \mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$ we have $\mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ for some t_τ, whose center is $\mathcal{O}_{F,\tau} = R_1$, on which $\mathcal{O}_F$ acts via the homomorphism $\tau : \mathcal{O}_F \to R_1$.

By [26, Lem. 1.1.3.4], for each $\tau \in \Upsilon$, there is a unique (up to isomorphism) indecomposable projective $\mathcal{O}_\tau$-module, which we shall denote by V_τ. Concretely, since $\mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$, we can take V_τ to be $\mathcal{O}_{F,\tau}^{\oplus t_\tau}$, in which case $\mathrm{End}_{\mathcal{O}_\tau}(V_\tau) \cong$

$\mathcal{O}_{F,\tau} \cong R_1$. Moreover, every finitely generated projective $\mathcal{O}_1$-module decomposes (up to isomorphism) into a direct sum $\underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus m_\tau}$ for some integers m_τ. We call the tuple $(m_\tau)_{\tau\in\Upsilon}$ of integers the *multi-rank* of such an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$-module. (See [26, Def. 1.1.3.5].) Let $(p_\tau)_{\tau\in\Upsilon}$ (resp. $(q_\tau)_{\tau\in\Upsilon}$) be the multi-rank of $L_{0,1}$ (resp. $L_{0,1}^\vee(1)$). Then $q_\tau = p_{\tau\circ c}$, where $c : \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_F$ is the restriction of $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}$. The multi-rank of L_1 is $(p_\tau + q_\tau)_{\tau\in\Upsilon}$, because we have the isomorphism (2.7) over R_1.

Fix any choice of an isomorphism $L_{0,1} \cong \underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus p_\tau}$, and fix any choices of the (non-canonical) isomorphisms $V_{\tau\circ c}^\vee(1) := \mathrm{Hom}_{R_1}(V_{\tau\circ c}, R_1(1)) \cong V_\tau$ (for $\tau \in \Upsilon$). Then these choices induce canonically an isomorphism

$$L_1 \cong \Big(\underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus p_\tau}\Big) \oplus \Big(\underset{\tau\in\Upsilon}{\oplus} (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}\Big) \cong \underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus(p_\tau+q_\tau)} \tag{3.1}$$

by (2.7), matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing

$$(((x_{1,\tau}, f_{1,\tau\circ c}))_{\tau\in\Upsilon}, ((x_{2,\tau}, f_{2,\tau\circ c}))_{\tau\in\Upsilon}) \mapsto \sum_{\tau\in\Upsilon} (f_{2,\tau}(x_{1,\tau}) - f_{1,\tau}(x_{2,\tau})). \tag{3.2}$$

Lemma 3.3 (see [27, Lem. 2.4])**.** *There exists a cocharacter $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{G}_1$ splitting the similitude character $\nu : \mathrm{G}_1 \to \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$, which acts trivially on $L_{0,1}^\vee(1)$* (*under the identification* (2.7)).

For each $\tau \in \Upsilon$, set $L_\tau := V_\tau^{\oplus p_\tau} \oplus (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}$, and define the canonical pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_\tau : L_\tau \times L_{\tau\circ c} \to R_1(1)$$

by

$$((x_{1,\tau}, f_{1,\tau\circ c}), (x_{2,\tau\circ c}, f_{2,\tau})) \mapsto f_{2,\tau}(x_{1,\tau}) - f_{1,\tau\circ c}(x_{2,\tau\circ c}).$$

(The two factors L_τ and $L_{\tau\circ c}$ of the domain of $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$ are *not* the same when $\tau \neq \tau\circ c$.) Then we see that the pairing (3.2) is simply the sum of $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$ over $\tau \in \Upsilon$. Note that $\mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_\tau \underset{R_1}{\otimes} R) \cong \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_{\tau\circ c} \underset{R_1}{\otimes} R)$ for every R_1-algebra R. If we define for each R_1-algebra R

$$\mathrm{G}_\tau(R) := \left\{ \begin{array}{c} g \in \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_\tau \underset{R_1}{\otimes} R) : \\ \langle gx, gy\rangle_\tau = \langle x, y\rangle_\tau,\ \forall x \in L_\tau \underset{R_1}{\otimes} R, \forall y \in L_{\tau\circ c} \underset{R_1}{\otimes} R \end{array} \right\},$$

then we obtain a group functor G_τ over R_1 which falls into three possibilities (by the same explicit classification as in the proof of [26, Prop. 1.2.3.11] again, as in Remark 2.9):

1. (1) $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau = q_\tau$ and Sp_{2r_τ} is the (split) symplectic group of rank r_τ over $\mathbb{Z}$.
2. (2) $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau = q_\tau$ and O_{2r_τ} is the (split) even orthogonal group of rank r_τ over $\mathbb{Z}$.
3. (3) $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau + q_\tau$ and GL_{r_τ} is the general linear group of rank r_τ over $\mathbb{Z}$.

Since $\langle\,\cdot\,,\,\cdot\,\rangle_\tau = -\langle\,\cdot\,,\,\cdot\,\rangle_{\tau\circ c}$ as pairings between L_τ and $L_{\tau\circ c}$, the two group functors G_τ and $\mathrm{G}_{\tau\circ c}$ are canonically isomorphic.

Thus, we obtain a decomposition

$$\mathrm{G}_1 \cong \Big(\prod_{\tau \in \Upsilon / c} \mathrm{G}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \tag{3.4}$$

over $\mathsf{S}_1 = \mathrm{Spec}(R_1)$, where $\tau \in \Upsilon/c$ means (by abuse of language) we pick exactly one representative τ in its c-orbit in Υ, and where the last factor $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$ is given by the cocharacter given by Lemma 3.3 splitting the similitude character.

3.2. **Decomposition of parabolic subgroups.** Under the identification (2.7), the submodule $L_{0,1}^\vee(1)$ of L_1 is matched with the submodule $0 \oplus \Big(\underset{\tau \in \Upsilon}{\oplus} (V_{\tau \circ c}^\vee(1))^{\oplus q_\tau} \Big)$ of the second member in (3.1). For each $\tau \in \Upsilon$, define group functors P_τ and M_τ over R_1 by setting for each R_1-algebra R

$$\mathrm{P}_\tau(R) := \left\{ \begin{gathered} g \in \mathrm{G}_\tau(R) : g(0 \oplus (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R)) = (0 \oplus (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R)) \\ \text{in } L_\tau \underset{R_1}{\otimes} R = (V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus ((V_{\tau\circ c}^\vee(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R) \end{gathered} \right\} \tag{3.5}$$

and

$$\mathrm{M}_\tau(R) := \left\{ \begin{gathered} g \in \mathrm{P}_\tau(R) : g((V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus 0) = ((V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus 0) \\ \text{in } L_\tau \underset{R_1}{\otimes} R = (V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus ((V_{\tau\circ c}^\vee(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R) \end{gathered} \right\}. \tag{3.6}$$

Then the subgroup P_1 of G_1 can be identified with the subgroup

$$\Big(\prod_{\tau \in \Upsilon / c} \mathrm{P}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon / c} \mathrm{G}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1),$$

and the canonical surjection $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$ has a splitting $\mathrm{M}_1 \subset \mathrm{P}_1$ given by

$$\Big(\prod_{\tau \in \Upsilon / c} \mathrm{M}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon / c} \mathrm{P}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1).$$

For each $\tau \in \Upsilon$, since $\mathrm{End}_{\mathcal{O}_1}(V_\tau) \cong \mathrm{End}_{\mathcal{O}_1}(V_{\tau\circ c}^\vee(1)) \cong \mathcal{O}_{F,\tau} \cong R_1$, we have a diagonal action of $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau})(R)$ on $(V_\tau^{\oplus p_\tau} \oplus (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}) \underset{R_1}{\otimes} R$, which is functorial in R and hence defines a homomorphism $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{M}_\tau$.

3.3. **Hodge filtrations.** Let R be any R_1-algebra. Fix any choice of a cocharacter as in Lemma 3.3, and consider its reciprocal $H : \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{G}_1$. (By definition, H factors through P_1.)

Definition 3.7. *Given any object $W \in \mathrm{Rep}_R(\mathrm{P}_1)$, the induced action of $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$ decomposes W into weight spaces $W^{(a)}$ for $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$, indexed by integers. Then the Hodge filtration $\mathtt{F}$ on W is the decreasing filtration $\mathtt{F}(W) = \{\mathtt{F}^a(W)\}_{a \in \mathbb{Z}}$ defined by $\mathtt{F}^a(W) := \underset{b \geq a}{\oplus} W^{(b)}$. (Note that the choice of H is not unique in general, but the resulting filtration is independent of this choice.)*

Example 3.8. Since the cocharacter H acts with weight 0 on $L_{0,1}^\vee(1)$ (as a submodule of L_1) and with weight -1 on $L_{0,1}$ (as a quotient module of L_1), the Hodge filtration $\mathtt{F}$ on L_1 is given by $\mathtt{F}^{-1}(L_1) = L_1$, $\mathtt{F}^0(L_1) = L_{0,1}^\vee(1)$, and $\mathtt{F}^1(L_1) = \{0\}$. Then the only possibly nonzero graded pieces are $\mathrm{Gr}_{\mathtt{F}}^{-1}(L_1) = L_{0,1}$ and $\mathrm{Gr}_{\mathtt{F}}^0(L_1) = L_{0,1}^\vee(1)$.

Lemma 3.9 (see [27, Lem. 2.11])**.** *Let $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ and let $\{\mathtt{F}^a(W)\}_{a\in\mathbb{Z}}$ be its Hodge filtration defined in Definition 3.7. Then the unipotent radical U_1 of P_1 acts trivially on $\mathrm{Gr}^a_{\mathtt{F}}(W)$ for every $a \in \mathbb{Z}$. In other words, each graded piece $\mathrm{Gr}^a_{\mathtt{F}}(W)$ can be identified as an object in $\mathrm{Rep}_R(\mathrm{M}_1)$.*

By Lemmas 2.24 and 2.32, the Hodge filtration on W defines similar filtrations on $\mathcal{E}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$, which we shall denote by $\mathtt{F}^a(\mathcal{E}_{\mathrm{P}_1,R}(W))$, $\mathtt{F}^a(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W))$, and $\mathtt{F}^a(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W))$, for $a \in \mathbb{Z}$; and we have the canonical isomorphisms $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$, $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$, and $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$ between the graded pieces.

Definition 3.10. *The filtrations $\mathtt{F}(\mathcal{E}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$, $\mathtt{F}(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$, and $\mathtt{F}(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$ are called the Hodge filtrations on $\mathcal{E}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$, respectively.*

Definition 3.11. *Let $W \in \mathrm{Rep}_R(\mathrm{G}_1)$. By considering W as an object of $\mathrm{Rep}_R(\mathrm{P}_1)$ by restriction from G_1 to P_1, we can define the Hodge filtration on $\mathcal{E}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}_{\mathrm{P}_1,R}(W)$ (resp. $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$, resp. $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$) (see Lemmas* 2.24 *and* 2.32) *as in Definition* 3.10. *The Hodge filtration on the de Rham complex $\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ is defined by*

$$\mathtt{F}^a(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}) := \mathtt{F}^{a-\bullet}(\mathcal{E}_{\mathrm{G}_1,R}(W)) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

The Hodge filtrations on the log de Rham complexes $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ are defined similarly.

These are respectively subcomplexes of the full de Rham complexes for the Gauss–Manin connections, thanks to the Griffiths transversality. We shall postpone the explanation for the Griffiths transversality to the end of Section 4.3. (This is not ideal for the exposition, but we will not need Griffiths transversality before then.)

Lemma 3.12 (see [28, Lem. 4.21])**.** *Suppose W_1 and W_2 are two objects in $\mathrm{Rep}_R(\mathrm{G}_1)$ such that the induced actions of P_1 and $\mathrm{Lie}(\mathrm{G}_1)$ on them satisfy $W_1|_{\mathrm{P}_1} \cong W_2|_{\mathrm{P}_1}$ and $W_1|_{\mathrm{Lie}(\mathrm{G}_1)} \cong W_2|_{\mathrm{Lie}(\mathrm{G}_1)}$. Then we have a canonical isomorphism*

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W_1) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla) \cong (\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W_2) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla) \tag{3.13}$$

respecting the Hodge filtrations on both sides. (*Consequently, the same is true with $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(\,\cdot\,)$ replaced with $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(\,\cdot\,)$ and $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$.*)

Remark 3.14. Lemma 3.12 will be needed only when G_1 is not connected, i.e., when $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [26, Def. 1.2.1.15]). (This happens exactly when $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ for some $\tau \in \Upsilon$.)

3.4. **Roots and weights.** We shall choose a maximal torus T_τ of M_τ by choosing a subgroup of $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ that embeds into M_τ under the natural homomorphism $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{M}_\tau$ defined at the end of Section 3.2. There are two cases:

(1) If $\tau = \tau \circ c$, then $p_\tau = q_\tau$ and we take $\mathrm{T}_\tau = \{t_\tau = (t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}\}$, embedded in $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ by $t_\tau \mapsto (t_\tau^{-1}, t_\tau)$.

(2) If $\tau \neq \tau \circ c$, then we take $\mathrm{T}_\tau = \{t_\tau = (t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}\}$, identified with $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ by $(t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau} \mapsto ((t_{\tau,q_\tau+i_\tau}^{-1})_{1 \leq i_\tau \leq p_\tau}, (t_{\tau,i_\tau})_{1 \leq i_\tau \leq q_\tau})\}$.

We take $\mathrm{T}_1 \subset \mathrm{M}_1$ to be the subgroup corresponding to

$$\Big(\prod_{\tau \in \Upsilon/c} \mathrm{T}_\tau \Big) \times (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon/c} \mathrm{M}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \tag{3.15}$$

(where the products are over S_0). Then the split torus T_1 is a maximal torus in both M_1 and G_1 (by comparing the ranks).

Elements in T_1 can be written as $t = ((t_\tau)_{\tau \in \Upsilon/c}; t_0) = (((t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau})_{\tau \in \Upsilon/c}; t_0)$, and therefore elements in the character group $\mathrm{X} := \mathrm{Hom}_{R_1}(\mathrm{T}_1, \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1)$ of T_1 are of the from $\mu = ((\mu_\tau)_{\tau \in \Upsilon/c}; \mu_0) = (((\mu_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau})_{\tau \in \Upsilon/c}; \mu_0)$, given concretely by

$$t \mapsto (\prod_{\tau \in \Upsilon/c} \mu_\tau(t_\tau)) \, \mu_0(t_0) = (\prod_{\tau \in \Upsilon/c} \prod_{1 \leq i_\tau \leq r_\tau} t_{\tau,i_\tau}^{\mu_{\tau,i_\tau}}) \, t_0^{\mu_0}.$$

Let $\mathrm{X}^\vee := \mathrm{Hom}_{R_1}(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1, \mathrm{T}_1)$ be the cocharacter group of T_1, and let $(\,\cdot\,,\,\cdot\,) : \mathrm{X} \times \mathrm{X}^\vee \to \mathbb{Z}$ be the canonical pairing between X and $\mathrm{X}^\vee$ defined by sending $(\mu, \nu^\vee) \in \mathrm{X} \times \mathrm{X}^\vee$ to $\mu \circ \nu^\vee \in \mathrm{End}_{R_1}(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \cong \mathbb{Z}$. Let $\Phi_{\mathrm{G}_1} \subset \mathrm{X}$ (resp. $\Phi_{\mathrm{G}_1}^\vee \subset \mathrm{X}^\vee$) be the roots (resp. coroots) of the split reductive group scheme G_1 over R_1. The choice of $\Phi_{\mathrm{G}_1}^+$ corresponds to the choice of a Borel subgroup B_1 in G_1. Using the explicit identifications in (3.4), (3.5), (3.6), and (3.15), we can choose B_1 to contain the unipotent radical U_1 of P_1, and accordingly the positive roots $\Phi_{\mathrm{G}_1}^+$ in Φ_{G_1}, such that the set $\mathrm{X}_{\mathrm{G}_1}^+$ of dominant weights of G_1 consists of those $\mu \in \mathrm{X}$ as above with $\mu_{\tau,i_\tau} \geq \mu_{\tau,i_\tau+1}$ for every $\tau \in \Upsilon/c$ and for every $1 \leq i_\tau < r_\tau$, satisfying in addition:

(1) If $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $\mu_{\tau,r_\tau} \geq 0$.

(2) If $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $\mu_{\tau,r_\tau-1} \geq |\mu_{\tau,r_\tau}|$.

(If $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then there is no other condition on μ_τ.)

Remark 3.16. When G_1 is not connected (i.e., $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ for some $\tau \in \Upsilon$), it is isomorphic to a semi-direct product $\mathrm{G}_1^\circ \rtimes \Gamma$, where G_1° is the identity component of G_1, and where Γ is an elementary abelian 2-group normalizing B_1 and T_1; and the irreducible representations V of G_1 over $R_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ are parameterized not exactly by a single dominant weight $\mu \in \mathrm{X}_{\mathrm{G}_1}^+$, but instead by the Γ-orbit $[\mu]$ of μ in $\mathrm{X}_{\mathrm{G}_1}^+$ plus the action on V of the stabilizer of μ in Γ. Suppose any V as above has $\mu \in \mathrm{X}_{\mathrm{G}_1}^+$ as a highest weight. Then $V|_{\mathrm{G}_1^\circ} \cong \underset{\mu' \in [\mu]}{\oplus} V_{\mu'}$, where each $V_{\mu'}$ is an irreducible representation of G_1° over $R_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ of highest weight μ', and where $[\mu]$ is the Γ-orbit of μ in $\mathrm{X}^+_{\mathrm{G}_1}$. (Since there is a central isogeny $\Big(\prod_{\tau\in\Upsilon/c}\mathrm{G}_\tau\Big)\times(\mathbf{G}_{\mathrm{m}}\otimes_{\mathbb{Z}} R_1)\to\mathrm{G}_1$, it suffices to verify the analogous statements for the factors G_τ, or rather just those such that $\mathrm{G}_\tau\cong\mathrm{O}_{2r_\tau}\otimes_{\mathbb{Z}} R_1$, by following the same argument as in [14, Sec. 5.5.5].) By Lemma 3.12, two representations of G_1 will serve the same purpose for us if their restrictions to G_1° are isomorphic. Hence, we shall abusively denote by $V_{[\mu]}$ any irreducible representation V (as above) having μ as a highest weight.

Let Φ_{M_1} be the roots of the split reductive group scheme M_1 over R_1. Then intersection of the above-chosen B_1 with M_1 (realized as a subgroup in P_1 as above) determines the choice of positive roots $\Phi^+_{\mathrm{M}_1}$ in Φ_{M_1}, so that $\Phi^+_{\mathrm{M}_1}=\Phi_{\mathrm{M}_1}\cap\Phi^+_{\mathrm{G}_1}$. Then the set $\mathrm{X}^+_{\mathrm{M}_1}$ of dominant weights of M_1 consists of those $\mu\in\mathrm{X}$ as above with $\mu_{\tau,i_\tau}\geq\mu_{\tau,i_\tau+1}$ for every $\tau\in\Upsilon/c$ and for every $1\leq i_\tau<q_\tau$ or $q_\tau<i_\tau<r_\tau$. (When $\mathrm{G}_\tau\cong\mathrm{Sp}_{2r_\tau}\otimes_{\mathbb{Z}} R_1$ or $\mathrm{G}_\tau\cong\mathrm{O}_{2r_\tau}\otimes_{\mathbb{Z}} R_1$, this means we drop the conditions (1) and (2) above. When $\mathrm{G}_\tau\cong\mathrm{GL}_{r_\tau}\otimes_{\mathbb{Z}} R_1$, this means we drop the condition $\mu_{\tau,q_\tau}\geq\mu_{\tau,q_\tau+1}$.)

It is conventional to say that a root $\alpha\in\Phi_{\mathrm{G}_1}$ is *compact* if it is an element of Φ_{M_1}, and that α is *non-compact* if otherwise. We denote the non-compact roots of Φ_{G_1} by Φ^{M_1}, and denote the collection of positive non-compact roots by $\Phi^{\mathrm{M}_1,+}$. For negative roots, we replace $+$ with $-$ in the above notation.

Let $\mathrm{W}_{\mathrm{G}_1}$ (resp. $\mathrm{W}_{\mathrm{M}_1}$) be the Weyl group of G_1 (resp. M_1). The realization of M_1 as a subgroup of G_1 containing T_1 identifies $\mathrm{W}_{\mathrm{M}_1}$ as a subgroup of $\mathrm{W}_{\mathrm{G}_1}$. We define

$$\mathrm{W}^{\mathrm{M}_1}:=\{w\in\mathrm{W}_{\mathrm{G}_1}:w(\mathrm{X}^+_{\mathrm{G}_1})\subset\mathrm{X}^+_{\mathrm{M}_1}\}.$$

Then any element w in $\mathrm{W}_{\mathrm{G}_1}$ has a unique expression as $w=w_1w_2$ with $w_1\in\mathrm{W}_{\mathrm{M}_1}$ and $w_2\in\mathrm{W}^{\mathrm{M}_1}$. (The elements of $\mathrm{W}^{\mathrm{M}_1}$ are the minimal length representatives of $\mathrm{W}_{\mathrm{M}_1}\setminus\mathrm{W}_{\mathrm{G}_1}$.)

For any root $\alpha\in\Phi_{\mathrm{G}_1}$, we shall denote by $\alpha^\vee\in\Phi^\vee_{\mathrm{G}_1}$ the associated coroot. Let $\rho:=\frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathrm{G}_1}}\alpha$ be the half-sum of positive roots in Φ_{G_1}. The *dot action* of $\mathrm{W}_{\mathrm{G}_1}$ (and its subset $\mathrm{W}^{\mathrm{M}_1}$) is defined by setting $w\cdot\mu:=w(\mu+\rho)-\rho$ for each $w\in\mathrm{W}_{\mathrm{G}_1}$.

3.5. p-small weights and Weyl modules.

Definition 3.17. *Let $\mu\in\mathrm{X}$. We say μ is* p**-small for** G_1 (*resp.* **for** M_1) *if $(\mu+\rho,\alpha^\vee)\leq p$ for every $\alpha\in\Phi_{\mathrm{G}_1}$ (resp. $\alpha\in\Phi_{\mathrm{M}_1}$). We denote by $\mathrm{X}^{<p}_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{<p}_{\mathrm{M}_1}$) the subset of X consisting of $\mu\in\mathrm{X}$ that are p-small for G_1 (resp. M_1), and we set $\mathrm{X}^{+,<p}_{\mathrm{G}_1}:=\mathrm{X}^+_{\mathrm{G}_1}\cap\mathrm{X}^{<p}_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{+,<p}_{\mathrm{M}_1}:=\mathrm{X}^+_{\mathrm{M}_1}\cap\mathrm{X}^{<p}_{\mathrm{M}_1}$).*

Remark 3.18. Note that $\mathrm{X}^{<p}_{\mathrm{M}_1}$ is stable under the dot action of $\mathrm{W}_{\mathrm{G}_1}$, and that $w\cdot\mu$ belongs to $\mathrm{X}^{+,<p}_{\mathrm{M}_1}$ for any $w\in\mathrm{W}^{\mathrm{M}_1}$ and $\mu\in\mathrm{X}^{+,<p}_{\mathrm{G}_1}$.

Remark 3.19 (cf. [33, 1.9]). Since $\rho_{\mathrm{M}_1}:=\frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathrm{M}_1}}\alpha$ satisfies $(\rho,\alpha^\vee)=(\rho_{\mathrm{M}_1},\alpha^\vee)$ for every $\alpha\in\Phi_{\mathrm{M}_1}$, the given definition of p-smallness for the Levi subgroup M_1 is the same as the one when M_1 is regarded as a reductive group on its own.

Since G_1 (resp. M_1) is split reductive over R_1, there exists a split reductive group scheme $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) over $\mathbb{Z}_{(p)}$ such that $\mathrm{G}_1\cong\mathrm{G}_{\mathrm{split},R_1}$ (resp. $\mathrm{M}_1\cong\mathrm{M}_{\mathrm{split},R_1}$). Note that $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) has the same roots and weights as G_1

(resp. M_1), and is a semi-direct product of $\mathbf{G}_{\mathrm{m}}$ with the split symplectic, (even) orthogonal, and general linear groups over $\mathbb{Z}_{(p)}$. For $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ (resp. $\mu \in \mathrm{X}^+_{\mathrm{M}_1}$), let $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) be any irreducible $\mathbb{Q}$-representation of $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) having μ as a highest weight (see Remark 3.16). As in [33, 1.5], a $\mathbb{Z}_{(p)}$-lattice in a $\mathbb{Q}$-representation of a group scheme over $\mathbb{Z}_{(p)}$ is called *admissible* if it is stable under the group scheme action. Let $V_{[\mu],\mathbb{Z}_{(p)}} \subset V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Z}_{(p)}} \subset W_{\mu,\mathbb{Q}}$) be the span of a highest weight vector under the action of the group scheme over $\mathbb{Z}_{(p)}$, which is (by construction) minimal among admissible lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) that contain the same highest weight vector. If we denote by $\mathrm{G}^\circ_{\mathrm{split}}$ the identity component of $\mathrm{G}_{\mathrm{split}}$, then $\mathrm{G}^\circ_{\mathrm{split},R_1} \cong \mathrm{G}^\circ_1$ and $V_{[\mu],\mathbb{Z}_{(p)}}|_{\mathrm{G}^\circ_{\mathrm{split}}} \cong \underset{\mu' \in [\mu]}{\oplus} V_{\mu',\mathbb{Z}_{(p)}}$ (see Remark 3.16), where each $V_{\mu',\mathbb{Z}_{(p)}}$ is the span of some highest weight vector under the action of $\mathrm{G}^\circ_{\mathrm{split}}$ in an irreducible $\mathbb{Q}$-representation of highest μ'. (Then $V_{\mu',\mathbb{Z}_{(p)}}$ and $W_{\mu,\mathbb{Z}_{(p)}}$ are *Weyl modules* of $\mathrm{G}^\circ_{\mathrm{split}}$ and $\mathrm{M}_{\mathrm{split}}$, respectively; cf. [33, 1.3].)

According to [33, Cor. 1.9] (cf. [33, Cor. 5]), if $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ (resp. $\mu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$), then all admissible $\mathbb{Z}_{(p)}$-lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$), including ones constructed by plethysm as in [11] or [14], are isomorphic to $V_{[\mu],\mathbb{Z}_{(p)}}$ (resp. $W_{\mu,\mathbb{Z}_{(p)}}$). Then we set $V_{[\mu]} := V_{[\mu],\mathbb{Z}_{(p)}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$ (resp. $W_\mu := W_{\mu,\mathbb{Z}_{(p)}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$), and set $V_{[\mu],R} := V_{[\mu]} \underset{R_1}{\otimes} R$ (resp. $W_{\mu,R} := W_\mu \underset{R_1}{\otimes} R$) for each R_1-algebra R.

4. Differential operators

4.1. **Verma modules.** Let U_1 be the unipotent radical of the parabolic subgroup P_1 of G_1. Then $\mathfrak{u}_1 := \mathrm{Lie}(\mathrm{U}_1)$ is the unipotent radical of the parabolic subalgebra $\mathfrak{p}_1 := \mathrm{Lie}(\mathrm{P}_1)$ of $\mathfrak{g}_1 := \mathrm{Lie}(\mathrm{G}_1)$. Let $\mathfrak{p}_1^-$ be the parabolic subalgebra of $\mathfrak{g}_1$ *opposite* to $\mathfrak{p}_1$, and let $\mathfrak{u}_1^-$ be the unipotent radical of $\mathfrak{p}_1^-$.

Our convention in Section 3.4 is that the weights of $\mathfrak{u}_1$ are in $\Phi^{\mathrm{M}_1,+}$, and so that the weights of $\mathfrak{u}_1^-$ are in $\Phi^{\mathrm{M}_1,-}$. Let $\mathrm{U}(\mathfrak{g}_1)$ (resp. $\mathrm{U}(\mathfrak{p}_1)$, resp. $\mathrm{U}(\mathfrak{u}_1^-)$)) denote the universal enveloping algebra of $\mathfrak{g}_1$ (resp. $\mathfrak{p}_1$, resp. $\mathfrak{u}_1^-$). As always, for each R_1-algebra R, we denote the pullbacks of objects from R_1 to R by replacing the subscript "1" with "R".

Now let us fix the choice of an R_1-algebra R. We view $\mathfrak{g}_R$, $\mathfrak{p}_R$, and $\mathfrak{u}_R$ as objects in $\mathrm{Rep}_R(\mathrm{P}_1)$ canonically, and we view $\mathfrak{u}_R^-$ as an object in $\mathrm{Rep}_R(\mathrm{P}_1)$ by $\mathfrak{u}_R^- \cong \mathfrak{g}_R/\mathfrak{p}_R$. We also view $\mathfrak{u}_R$ and $\mathfrak{u}_R^-$ as objects in $\mathrm{Rep}_R(\mathrm{M}_1)$ because U_1 acts trivially on them.

Definition 4.1. *By a* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-**module**, *we mean a module with actions of* $\mathrm{U}(\mathfrak{g}_R)$ *and* P_1 *that induce the same action of* $\mathfrak{p}_R$. *By a morphism between* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-*modules, we mean a morphism of* $\mathrm{U}(\mathfrak{g}_R)$-*modules that induces a morphism of* $\mathrm{U}(\mathfrak{p}_R)$-*modules coming from an algebraic morphism between* P_1-*modules. We shall use the notation* $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}_R)\text{-}\mathrm{P}_1}(\,\cdot\,,\,\cdot\,)$ *to mean the group of morphisms between* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-*modules.*

Lemma 4.2. *Let* $W \in \mathrm{Rep}_R(\mathrm{P}_1)$. *Then the module*

$$\mathrm{Verm}(W) := \mathrm{U}(\mathfrak{g}_R) \underset{\mathrm{U}(\mathfrak{p}_R)}{\otimes} W \tag{4.3}$$

with canonical action of $\mathrm{U}(\mathfrak{g}_R)$ *on the first component, and with canonical diagonal action of* P_1 *on both components, is a* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-*module.*

Proof. We need to show that the two induced actions of $\mathfrak{p}_R$ agree. Let "ad" denote the adjoint action of $\mathfrak{p}_R$ on $\mathfrak{g}_R$, induced by the canonical adjoint action of P_1 on $\mathfrak{g}_1$. Then the lemma follows from the identity

$$(pu)\otimes v = (pu - up)\otimes v + (up)\otimes v = (\mathrm{ad}(p)(u))\otimes v + u\otimes(pv),$$

for all $p \in \mathfrak{p}_R$, $u \in \mathrm{U}(\mathfrak{g}_R)$, and $v \in W$. □

Definition 4.4. *Let $W \in \mathrm{Rep}_R(\mathrm{P}_1)$. We define the (generalized)* **Verma module** *for W to be the $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-module $\mathrm{Verm}(W)$ defined as in (4.3). (Elements in $\mathrm{U}(\mathfrak{g}_R)$ but not in $\mathrm{U}(\mathfrak{p}_R)$ do not act on the second component even when W comes from an object in $\mathrm{Rep}_R(\mathrm{G}_1)$.)*

Remark 4.5. Such modules are more often called *generalized Verma modules* because $\mathfrak{p}_1$ is seldom the Borel subalgebra of $\mathfrak{g}_1$. Since the choice of the parabolic subalgebra $\mathfrak{p}_1$ is fixed in what follows, we shall drop the modifier *generalized* from all terminologies for simplicity.

According to the Poincaré–Birkhoff–Witt theorem over $\mathbb{Z}$ (and hence over $\mathbb{Z}_{(p)}$) for the split forms of the Lie algebras, and hence over R_1 and over R by base change, we have a canonical isomorphism

$$\mathrm{Verm}(W) = \mathrm{U}(\mathfrak{g}_R) \underset{\mathrm{U}(\mathfrak{p}_R)}{\otimes} W \cong \mathrm{U}(\mathfrak{u}_R^-) \underset{R}{\otimes} W$$

of P_1-modules. Since $\mathfrak{u}_R^-$ is abelian, we have a canonical isomorphism $\mathrm{U}(\mathfrak{u}_R^-) \cong \mathrm{Sym}(\mathfrak{u}_R^-)$, in which $\mathrm{Sym}(\mathfrak{u}_R^-)$ can be identified with a polynomial algebra over R with variables given by any free R_1-basis of $\mathfrak{u}_R^-$. For any integer $m \geq 0$, we denote by $\mathrm{U}(\mathfrak{u}_R^-)^m$ (resp. $\mathrm{U}(\mathfrak{u}_R^-)^{\leq m}$, resp. $\mathrm{U}(\mathfrak{u}_R^-)^{<m}$) the elements of degree m (resp. at most m, resp. strictly less than m) in $\mathrm{U}(\mathfrak{u}_R^-)$. We use similar notation for any algebra with a natural grading.

Note that $\mathrm{U}(\mathfrak{u}_R^-)^{\leq m}$ is naturally a filtered P_1-module with $\mathrm{U}(\mathfrak{u}_R^-)^m$ as the top graded piece. In general the canonical morphism $\mathrm{U}(\mathfrak{u}_R^-)^{\leq m} \twoheadrightarrow \mathrm{U}(\mathfrak{u}_R^-)^m$ does not split as a morphism of P_1-modules.

Definition 4.6. *We say that a P_1-submodule of $\mathrm{Verm}(W)$ is* **of bounded degree** *if it is contained in $\mathrm{U}(\mathfrak{u}_R^-)^{\leq m} \underset{R}{\otimes} W$ for some $m \geq 0$. We say it is* **of degree** *m if it is contained in $\mathrm{U}(\mathfrak{u}_R^-)^{\leq m} \underset{R}{\otimes} W$ but not in $\mathrm{U}(\mathfrak{u}_R^-)^{<m} \underset{R}{\otimes} W$.*

Let $W_1, W_2 \in \mathrm{Rep}_R(\mathrm{P}_1)$. By finiteness of W_1 as an R-module, we know that any morphism in $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}_R)\text{-}\mathrm{P}_1}(\mathrm{Verm}(W_1), \mathrm{Verm}(W_2)) \cong \mathrm{Hom}_{\mathrm{P}_1}(W_1, \mathrm{Verm}(W_2))$ sends W_1 to a P_1-submodule of $\mathrm{Verm}(W_2)$ of bounded degree.

Definition 4.7. *We say that a morphism in $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}_R)\text{-}\mathrm{P}_1}(\mathrm{Verm}(W_1), \mathrm{Verm}(W_2))$ is* **of degree** *m if the image of the induced morphism in $\mathrm{Hom}_{\mathrm{P}_1}(W_1, \mathrm{Verm}(W_2))$ has the same property.*

4.2. **Construction of differential operators.** Let $W_1, W_2 \in \mathrm{Rep}_R(\mathrm{P}_1)$, and let ϕ be a morphism in $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}_R)\text{-}\mathrm{P}_1}(\mathrm{Verm}(W_1), \mathrm{Verm}(W_2))$ of degree m, induced by a morphism in $\mathrm{Hom}_{\mathrm{P}_1}(W_1, \mathrm{U}(\mathfrak{u}_R^-)^{\leq m} \underset{R}{\otimes} W_2)$ which we denote again by ϕ. Suppose W_1 and W_2 are locally free as R-modules.

By local freeness of $\mathfrak{u}_R^-$ over R_1, we have a canonical perfect pairing

$$\mathrm{Sym}(\mathfrak{u}_R^-) \times \Gamma((\mathfrak{u}_R^-)^\vee) \to R, \tag{4.8}$$

compatible with the canonical P_1-actions, matching elements of the same degree, where $\Gamma(\,\cdot\,)$ is the divided power analogue of $\mathrm{Sym}(\,\cdot\,)$. (See [4, Appendix A, especially Prop. A.10] for the precise definition of $\Gamma(\,\cdot\,)$ and for the perfectness of (4.8). For simplicity, we have omitted the subscript "R" of $\mathrm{Sym}(\,\cdot\,)$ and $\Gamma(\,\cdot\,)$ indicating that the constructions are over R.)

Let us abusively denote $(\mathfrak{u}_R^-)^\vee$ as $\mathfrak{u}_R^\#$, which can be identified with $\mathfrak{u}_R$ when $p > 2$, or when $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau, R_1}$ for some $\tau \in \Upsilon$. Then (4.8) induces an isomorphism

$$(\mathrm{U}(\mathfrak{u}_R^-)^{\leq m})^\vee \cong \Gamma(\mathfrak{u}_R^\#)/\Gamma^{>m}(\mathfrak{u}_R^\#) =: \Gamma_{\leq m}(\mathfrak{u}_R^\#),$$

and the morphism $\phi : W_1 \to \mathrm{U}(\mathfrak{u}_R^-)^{\leq m} \underset{R}{\otimes} W_2$ is canonically dual to a morphism

$$\phi^\vee : W_2^\vee \underset{R}{\otimes} \Gamma_{\leq m}(\mathfrak{u}_R^\#) \to W_1^\vee, \tag{4.9}$$

all considered as morphisms in $\mathrm{Rep}_R(\mathrm{P}_1)$.

There is a degree-preserving canonical morphism

$$\mathrm{Sym}(\mathfrak{u}_R^\#) \to \Gamma(\mathfrak{u}_R^\#)$$

(of P_1-modules), which induces a canonical morphism

$$\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^\#) := \mathrm{Sym}(\mathfrak{u}_R^\#)/\,\mathrm{Sym}^{>m}(\mathfrak{u}_R^\#) \to \Gamma_{\leq m}(\mathfrak{u}_R^\#) \tag{4.10}$$

in $\mathrm{Rep}_R(\mathrm{P}_1)$. This morphism is an isomorphism either when the residue characteristics of R are all zero, or when ϕ is p-small, namely when $m < p$, because $m!$ is invertible in R_1, and hence in R.

The above morphisms (4.9) and (4.10) induce another morphism

$$\phi^\vee : W_2^\vee \underset{R}{\otimes} \mathrm{Sym}_{\leq m}(\mathfrak{u}_R^\#) \to W_1^\vee \tag{4.11}$$

in $\mathrm{Rep}_R(\mathrm{P}_1)$.

Lemma 4.12. *For any isomorphism $\iota : R_1(1) \xrightarrow{\sim} R_1$ inducing an isomorphism $L_{0,1}^\vee(1) \to L_{0,1}^\vee$ which we also denote by ι, the R_1-module $\mathfrak{u}_1^\# = \mathfrak{u}_{R_1}^\#$ is isomorphic to*

$$(L_{0,1}^\vee \underset{R_1}{\otimes} L_{0,1}^\vee(1))/\begin{pmatrix} \iota(y)\otimes z - \iota(z)\otimes y \\ (b^\star x)\otimes y - x\otimes(by) \end{pmatrix}_{x\in L_{0,1}^\vee,\, y,z\in L_{0,1}^\vee(1),\, b\in\mathcal{O}_1}.$$

Proof. This follows from the definition of P_1 in Definition 2.4. $\square$

Corollary 4.13. *There are canonical isomorphisms $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^\#) \cong \underline{\mathrm{KS}}_{A_R/\mathsf{M}_{\mathcal{H},R}}(1)$ and $\mathcal{E}_{\mathrm{P}_1,R}^{\mathrm{can}}(\mathfrak{u}_R^\#) \cong \underline{\mathrm{KS}}_{A_R^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H},R}^{\mathrm{tor}}}(1)$ (cf. Definition 2.15 and the definition of $\underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ in Section 2.3).*

Proof. By definition (cf. (2.10), (2.19), and (2.23)), we can identify $\mathcal{E}_{\mathrm{P}_1,R}(L_{0,1}^\vee)$ with $\mathcal{E}_{\mathrm{P}_1,R}(L_{0,1}^\vee)$ under the isomorphism $\lambda^* : \underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H},1}}^\vee \xrightarrow{\sim} \underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H},1}}^\vee$. Hence, by Lemma 4.12 and by functoriality, we obtain $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^\#) \cong \underline{\mathrm{KS}}_{A_R/\mathsf{M}_{\mathcal{H},R}}(1)$. The case for $\mathcal{E}_{\mathrm{P}_1,R}^{\mathrm{can}}(\mathfrak{u}_R^\#) \cong \underline{\mathrm{KS}}_{A_R^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H},R}^{\mathrm{tor}}}(1)$ is similar. $\square$

We shall always identify $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^\#)$ with $\Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ using Corollary 4.13 and the Kodaira–Spencer isomorphism (2.17).

Lemma 4.14. *Under the above identification, consider the morphism*

$$(4.15)\quad (\mathcal{E}_{\mathrm{P}_1} \underset{R_1}{\otimes} R) \times \mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} : (\xi, (r, u)) \mapsto (r, [\xi^{-1} u \xi]),$$

where ξ is any section of $\mathcal{E}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$, where $(r, u) \in \mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#})$, with r in degree zero and u in degree one, and where $[\xi^{-1} u \xi]$ is defined as follows: (For simplicity of notation, let us treat only sections defined globally, although the argument works also locally.) Any section ξ of $\mathcal{E}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$ induces by definition (cf. (2.19)) an isomorphism

$$\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}) \to (L_{0,1} \oplus L_{0,1}^{\vee}(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$$

(which we again denote by ξ) matching the natural filtrations, and hence also induces a splitting

$$\underline{H}_1^{\mathrm{dR}}(A_R/\mathsf{M}_{\mathcal{H},R}) \cong \underline{\mathrm{Lie}}_{A_R/\mathsf{M}_{\mathcal{H},R}} \oplus \underline{\mathrm{Lie}}^{\vee}_{A_R^{\vee}/\mathsf{M}_{\mathcal{H},R}}(1)$$

(corresponding to the canonical splitting of $L_{0,1} \oplus L_{0,1}^{\vee}(1)$). Then $\xi^{-1} u \xi$ induces a morphism $\underline{\mathrm{Lie}}_{A_R/\mathsf{M}_{\mathcal{H},R}} \to \underline{\mathrm{Lie}}^{\vee}_{A_R^{\vee}/\mathsf{M}_{\mathcal{H},R}}(1)$, which in turn induces a section of $\Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ under the Kodaira–Spencer morphism (2.14), which we denote by $[\xi^{-1} u \xi]$.

For any section η of $\mathrm{P}_1 \underset{R_1}{\otimes} R$, both $\eta(\xi, (r, u)) = (\eta\xi, (r, \eta u \eta^{-1}))$ and $(\xi, (r, u))$ have the same image $(r, [(\eta\xi)^{-1}(\eta u \eta^{-1})(\eta\xi)]) = (r, [\xi^{-1} u \xi])$, and hence the morphism (4.15) induces a morphism (see Definition 2.22)

$$(4.16)\quad \mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#})) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

This morphism is an isomorphism of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$-modules.

Similarly, we have an isomorphism

$$(4.17)\quad \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#})) \xrightarrow{\sim} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}} \oplus \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}.$$

Proof. By trivializing $\mathcal{E}_{\mathrm{P}_1,R}$ étale locally, we see that the morphism (4.16) is indeed a morphism of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$-modules. By definition, it sends the submodule $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^{\#})$ of $\mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#}))$ (induced by the canonical submodule $\mathfrak{u}_R$ of $\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#})$ embedded in degree one) to the submodule $\Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, and the induced morphism $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R) \to \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ is an isomorphism by Corollary 4.13 and by the extended Kodaira–Spencer isomorphism (2.26). On the other hand, the induced morphism $\mathcal{E}_{\mathrm{P}_1,R}(R) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$ between quotient modules is clearly an isomorphism. Therefore, (4.16) is an isomorphism, as desired.

The proof for (4.17) is similar. □

Lemma 4.18. *For any integer $m \geq 0$, we have a canonical filtered isomorphism*

$$\mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#})) \xrightarrow{\sim} \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R},$$

where $\mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ is the sheaf of principal parts of order m over $\mathsf{M}_{\mathcal{H},R}$ (see [16, IV-4, 16.3]).

Proof. Let $\tilde{\mathsf{M}}^{(m)}_{\mathcal{H},R}$ be defined as in the paragraph preceding Definition 2.11, with the canonical splitting $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \cong \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ when $m = 1$. Then (4.16) can be rewritten as a filtered isomorphism $\mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#})) \xrightarrow{\sim} \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$. Since the functor $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ is functorial and exact, and since $\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#}) \cong \mathrm{Sym}_m(\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^{\#}))$ as $\mathrm{P}_{1,R}$-modules, we obtain a canonical filtered isomorphism

$$\mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#})) \cong \mathrm{Sym}_{\leq m}(\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^{\#})) \cong \mathrm{Sym}_{\leq m}(\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}).$$

Since $\mathsf{M}_{\mathcal{H},R} \to \mathsf{S}_R$ is smooth (and hence differential smooth), the canonical filtered morphism $\mathrm{Sym}_{\leq m}(\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}) \to \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ is an isomorphism. (It suffices to compare the graded pieces. See [16, IV-4, 17.12.4].) Then the lemma follows by composing all these filtered isomorphisms. $\square$

Proposition 4.19. *For any integer $m \geq 0$, the morphism* (4.11) *corresponds under the functor $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ to a morphism*

$$\mathcal{E}_{\mathrm{P}_1,R}(W_2^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \mathcal{E}_{\mathrm{P}_1,R}(W_1^\vee)$$

between locally free $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$-modules. The pre-composition of this morphism with the canonical morphism $\mathcal{E}_{\mathrm{P}_1,R}(W_2^\vee) \to \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \mathcal{E}_{\mathrm{P}_1,R}(W_2^\vee)$ gives a differential operator

$$d_\phi : \mathcal{E}_{\mathrm{P}_1,R}(W_2^\vee) \to \mathcal{E}_{\mathrm{P}_1,R}(W_1^\vee)$$

of order m. (*See* [16, IV-4, 16.8.1].) *Moreover, this construction is compatible with composition of morphisms.*

Proof. The first statement follows immediately from Lemma 4.18. The construction is compatible with composition of morphisms because the identification

$$\mathcal{E}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#})) \cong \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

in Lemma 4.18 is compatible with the canonical morphism

$$\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#}) \underset{R}{\otimes} \mathrm{Sym}_{\leq m'}(\mathfrak{u}_R^{\#}) \to \mathrm{Sym}_{\leq m''}(\mathfrak{u}_R^{\#})$$

for all $0 \leq m, m', m''$ with $m'' \leq m + m'$. $\square$

For each $m \geq 0$, let us define $\overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ to be the canonical extension of $\mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, namely $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#}))$. (This is consistent with the construction of $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ in Section 2.4. For our purpose, we do not need to know any interpretation of $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ and $\overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ in log geometry.)

Lemma 4.20. *The canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \to \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ over $\mathsf{M}_{\mathcal{H},R}$ admits a unique extension $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}} \to \overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}$.*

Proof. Note that $\overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} = \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq m}(\mathfrak{u}_R^{\#}))$ is locally free, and there is a canonical decomposition $\overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \cong \underset{0 \leq a \leq m}{\oplus} \mathrm{Sym}^a(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R})$ as $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$-modules. (This decomposition is compatible with restriction to $\mathsf{M}_{\mathcal{H},R}$.) For any $0 \leq a \leq m$, since the composition of the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \to \mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ with the canonical projection $\mathscr{P}^m_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \mathrm{Sym}^a(\Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R})$ is nothing but the a-th symmetric power of the universal derivation $d : \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}} \to \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, the lemma follows from the fact that, by the very definition of $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$, the morphism $(\mathsf{M}_{\mathcal{H},R} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R})_*(d)$ sends the subsheaf $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$ of $(\mathsf{M}_{\mathcal{H},R} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R})_*(\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}})$ to the subsheaf $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ of $(\mathsf{M}_{\mathcal{H},R} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R})_*(\Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R})$. $\square$

Proposition 4.21. *For any integer $m \geq 0$, the morphism* (4.11) *corresponds under the functor $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\,\cdot\,)$ to a morphism*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_2^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \to \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_1^\vee)$$

between locally free $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules. The pre-composition of this morphism with the canonical morphism $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_2^\vee) \to \overline{\mathscr{P}}^m_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_2^\vee)$ (*induced by the extended canonical morphism in Lemma* 4.20) *gives a* **log differential operator**

$$d_\phi : \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_2^\vee) \to \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W_1^\vee)$$

of order m. Moreover, this construction is compatible with composition of morphisms.

The analogous statements for the functor $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(\,\cdot\,)$ are also true.

Proof. The case of $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(\,\cdot\,)$ is similar to the case of $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$. (See the proof of Proposition 4.19.) The case of $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(\,\cdot\,)$ then follows by applying $\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \mathscr{I}_{\mathsf{D}}$ to all $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$-modules. □

Remark 4.22. While the restriction (4.11) gives rise to (log) differential operators, the original morphism (4.9) gives rise to *(log) HPD differential operators*. The attachment of the restriction (4.11) to (4.9) then corresponds to the attachment of a (log) differential operator to a (log) HPD (or rather PD) differential operator, as in [4, paragraph following Def. 4.4].

For later reference, let us record the following observation:

Lemma 4.23. *If there exists integers a_0 and m_0 such that $\mathrm{Gr}^{a_1}_{\mathsf{F}}(W_2^\vee) \neq 0$ only when $a_1 \leq a_0$, but $\mathrm{Gr}^{a_2}_{\mathsf{F}}(W_1^\vee) = 0$ for all $a_2 > a_0 + m_0$, then there is no nonzero morphism as in* (4.9) *with $m > m_0$. Therefore, the construction of Proposition* 4.19 (*resp. Proposition* 4.21) *gives no nonzero differential operator* (*resp. log differential operator*) *of order greater than m_0 from $W_2^\vee$ to $W_1^\vee$.*

Proof. This is because all elements in $\mathfrak{u}_R^-$ have H-weight -1. □

4.3. **Standard complexes and de Rham complexes.** Let $W \in \mathrm{Rep}_R(\mathrm{G}_1)$ be locally free as an R-module, which we also consider as an element of $\mathrm{Rep}_R(\mathrm{P}_1)$ by restriction to P_1.

Let us identify $\mathfrak{u}_R^-$ with $\mathfrak{g}_R/\mathfrak{p}_R$ as algebraic representations of P_1 as usual. Let n denote the relative dimension of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ over S_0, which is also the rank of $\mathfrak{u}_R^-$ as a free R-module. Consider the complex of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules

$$\begin{aligned} 0 \to \mathrm{Verm}(\wedge^n(\mathfrak{u}_R^-) \underset{R}{\otimes} W) &\xrightarrow{d_n} \mathrm{Verm}(\wedge^{n-1}(\mathfrak{u}_R^-) \underset{R}{\otimes} W) \\ &\xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathrm{Verm}((\mathfrak{u}_R^-) \underset{R}{\otimes} W) \xrightarrow{d_1} \mathrm{Verm}(W) \end{aligned} \tag{4.24}$$

with differentials given by morphisms

$$\begin{aligned} d_a : \mathrm{Verm}(\wedge^a(\mathfrak{u}_R^-) \underset{R}{\otimes} W) &= \mathrm{U}(\mathfrak{g}_R) \underset{\mathrm{U}(\mathfrak{p}_R)}{\otimes} (\wedge^a(\mathfrak{u}_R^-) \underset{R}{\otimes} W) \\ \to \mathrm{Verm}(\wedge^{a-1}(\mathfrak{u}_R^-) \underset{R}{\otimes} W) &= \mathrm{U}(\mathfrak{g}_R) \underset{\mathrm{U}(\mathfrak{p}_R)}{\otimes} (\wedge^{a-1}(\mathfrak{u}_R^-) \underset{R}{\otimes} W) \end{aligned}$$

of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules, for $1 \leq a \leq n$, defined by

$$\begin{aligned} & d_a(u \otimes ((x_1 \wedge x_2 \wedge \cdots \wedge x_a) \otimes v) \\ (4.25) \qquad & := \sum_{1 \leq i \leq a} (-1)^{i-1} (u x_i) \otimes ((x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_a) \otimes v) \\ & + \sum_{1 \leq i \leq a} (-1)^i u \otimes ((x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_a) \otimes (x_i v)) \end{aligned}$$

for all $u \in \mathrm{U}(\mathfrak{g}_R)$, $x_1, \ldots, x_a \in \mathfrak{u}_R^-$, and $v \in W$. We omitted the usual terms involving $[x_i, x_j]$ (see [33, 2.2]) because $\mathfrak{u}_R^-$ is abelian. (When $W = R$ is the trivial representation, this is induced by the standard Koszul complex resolving the trivial $\mathrm{U}(\mathfrak{g}_R)$-module R.) One can easily check (with any free R-basis of $\mathfrak{u}_R^-$) that this complex is exact.

Definition 4.26. *The complex* (4.24) *with differentials given by* (4.25) *is called the* **standard complex** (*of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules*) *for the module* W *in* $\mathrm{Rep}_R(\mathrm{G}_1)$*, which we shall denote as* $\mathrm{Std}_\bullet(W)$.

Proposition 4.27. *Under the functor* $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ (*as in Proposition* 4.19)*, the canonical morphism*

$$(4.28) \quad W^\vee \underset{R}{\otimes} \mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^\#) \to W^\vee \underset{R}{\otimes} \mathfrak{u}_R^\# : w \otimes (c + e) \mapsto w \otimes e + \sum_{1 \leq j \leq n} (y_j w) \otimes (c f_j),$$

for all $c \in R_1$, $e \in \mathfrak{u}_R^-$*, and* $w \in W^\vee$*, and for any free* R*-basis* $y_1, \ldots, y_n$ *of* $\mathfrak{u}_R^-$ *with dual free* R*-basis* $f_1, \ldots, f_n$ *of* $\mathfrak{u}_R^\#$*, is associated with the canonical morphism*

$$(4.29) \qquad \mathcal{E}_{\mathrm{P}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \mathcal{E}_{\mathrm{P}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

inducing the Gauss–Manin connection defined as in Definition 2.34.

Proof. As explained in the proof of Lemma 4.18, the canonical morphism $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ corresponds to $\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^\#) \to \mathfrak{u}_R^\# : c + e \mapsto e$, for all $c \in R$ and $e \in \mathfrak{u}_R^-$. The canonical morphism (4.29) inducing the Gauss–Manin connection is defined by (the restriction to $\mathrm{pr}_2^*(\mathcal{E}_{\mathrm{P}_1,R}(W^\vee))$ of) $s^* - \mathrm{Id}^*$ on $\mathcal{E}_{\mathrm{P}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, satisfying

$$(s^* - \mathrm{Id}^*)(z \otimes x) = ((s^* - \mathrm{Id}^*)(z \otimes 1))x + z \otimes ((s^* - \mathrm{Id}^*)x)$$

for all sections z of $\mathcal{E}_{\mathrm{P}_1,R}(W^\vee)$ and sections x of $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, because

$$((s^* - \mathrm{Id}^*)(z \otimes 1)) \otimes ((s^* - \mathrm{Id}^*)x) = 0.$$

Since $(s^* - \mathrm{Id}^*)x$ is known to agree with the image of the canonical morphism $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ when restricted to sections x in $\mathrm{pr}_2^*(\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}})$, it remains to study $(s^* - \mathrm{Id}^*)(z \otimes 1)$.

Let us adopt the notation in the proof of Lemma 4.18, with two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},R} \to \mathsf{M}_{\mathcal{H},R}$. Then $\mathrm{pr}_i^* \, \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}) \cong \underline{H}_1^{\mathrm{dR}}(\mathrm{pr}_i^* \, A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},R})$, and we obtain a morphism $(s^* - \mathrm{Id}^*) : \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}) \to \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$.

For any section v of $\underline{\mathrm{Der}}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, we obtain a morphism $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}) \to \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R})$ respecting $\langle \cdot, \cdot \rangle_\lambda$, and inducing a trivial morphism on the top

Hodge graded piece (after taking quotient by the bottom Hodge graded pieces). If we identify

$$(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},R}}(1))$$

with

$$((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}(1), L_R^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}))$$

by any section of $\mathcal{E}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$, this morphism induced by v defines a section u_v of the pullback of $\mathfrak{u}_R^-$ to $\mathsf{M}_{\mathcal{H},R}$. (This is compatible with the identification $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}_R^\#) \cong \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ based on Corollary 4.13 and the Kodaira–Spencer isomorphism (2.17).) Hence,

$$v(s^* - \mathrm{Id}^*)(z \otimes 1) = u_v z,$$

and so

$$(s^* - \mathrm{Id}^*)(z \otimes 1) = \sum_{1 \leq j \leq n} (y_j z) \otimes f_j$$

by duality, as desired. □

Corollary 4.30. *The complex associated with* $\mathrm{Std}_\bullet(W)$ *under the functor* $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ (*as in Proposition* 4.19) *is canonically isomorphic to the de Rham complex*

$$(\mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla).$$

Proof. For each $1 \leq a \leq n$, the morphism d_a corresponds to the morphism

$$d_a^\vee : W^\vee \underset{R}{\otimes} \wedge^{a-1}(\mathfrak{u}_R^\#) \underset{R}{\otimes} \mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^\#) \to W^\vee \underset{R}{\otimes} \wedge^a(\mathfrak{u}_R^\#)$$

defined by

$$\begin{aligned} &d_a^\vee(w \otimes ((e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1}) \otimes (c + e_a)) \\ &:= w \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1} \wedge e_a) + \sum_{1 \leq j \leq n} (y_j w) \otimes (e_1 \wedge e_2 \wedge \cdots \wedge e_{a-1} \wedge (c f_j)), \end{aligned}$$

for all $w \in W^\vee$, $e_1, \ldots, e_{a-1}, e_a \in \mathfrak{u}_R^-$, and $c \in R$, and for any free R-basis $y_1, \ldots, y_n$ of $\mathfrak{u}_R^-$ dual to a free R-basis $f_1, \ldots, f_n$ of $\mathfrak{u}_R^\#$. (This can be checked using the explicit bases we have chosen.)

By Proposition 4.27, the morphism associated with $d_a^\vee$ under the functor $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ is the composition of the canonical morphisms

$$\begin{aligned} &\mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{a-1}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \\ &\to \mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{a-1}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \\ &\to \mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^a_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \end{aligned}$$

inducing the Gauss–Manin connection, as desired. □

Proposition 4.31. *The analogues of Proposition* 4.27 *for the canonical and subcanonical extensions are true, and the complexes associated with* $\mathrm{Std}_\bullet(W)$ *under the functors* $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\,\cdot\,)$ *and* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(\,\cdot\,)$ (*as in Proposition* 4.21) *are canonically isomorphic to the log de Rham complexes*

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and

$$(\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla),$$

respectively.

Proof. By functoriality and compatibility between $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$ and $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\,\cdot\,)$, the identification $\mathcal{E}_{\mathrm{P}_1,R}(\mathfrak{u}^\#_R) \cong \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ based on Corollary 4.13 and the Kodaira–Spencer isomorphism (2.17) extends to the identification

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathfrak{u}^\#_R) \cong \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$$

based on the extended Kodaira–Spencer isomorphism (2.26). Then, by (5) in Proposition 2.27, we have the canonical morphism $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \to \overline{\Omega}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ extending $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \to \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, inducing the log de Rham complex $(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ extending $(\mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ (by its verify construction in Section 2.4). Then the proofs of Proposition 4.27 and Corollary 4.30 also work for the canonical extensions, and show that $(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ is associated with $\mathrm{Std}_\bullet(W)$ under the functor $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\,\cdot\,)$. (The case of $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(\,\cdot\,)$ then follows by applying $\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \mathscr{I}_{\mathsf{D}}$ to all $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$-modules, as usual.) □

Corollary 4.32. *The de Rham complex*

$$(\mathcal{E}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and the log de Rham complexes

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and

$$(\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

all satisfy the Griffiths transversality. (*The remark following Definition* 3.11 *is now justified.*)

Proof. By Corollary 4.30 and Proposition 4.31, it suffices to note that in (4.25) the action of $\mathfrak{u}^-_R$ on W increases the H-weights by 1 (cf. the proof of Lemma 3.9). □

5. Main results

5.1. **Notation.** Let R be any R_1-algebra. Let $\mathrm{X}^{+,<p}_{\mathrm{G}_1,R} := \mathrm{X}^+_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{+,<p}_{\mathrm{M}_1,R} := \mathrm{X}^+_{\mathrm{M}_1}$) if the residue characteristics of R are all zero, and let $\mathrm{X}^{+,<p}_{\mathrm{G}_1,R} := \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{+,<p}_{\mathrm{M}_1,R} := \mathrm{X}^{+,<p}_{\mathrm{M}_1}$) as in Definition 3.17 if otherwise.

For each $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1,R}$, let $V_{[\mu],R} \in \mathrm{Rep}_R(\mathrm{G}_1)$ be defined as in the last paragraph of Section 3.5. Since the underlying R-module of $V_{[\mu],R}$ is locally free, we can consider the contragredient representation $V^\vee_{[\mu],R} \in \mathrm{Rep}_R(\mathrm{G}_1)$. Then we have the associated automorphic bundle

$$\underline{V}^\vee_{[\mu],R} := \mathcal{E}_{\mathrm{G}_1,R}(V^\vee_{[\mu],R})$$

over $\mathsf{M}_{\mathcal{H},R}$, and its canonical and subcanonical extensions

$$(\underline{V}^{\vee}_{[\mu],R})^{\mathrm{can}} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(V^{\vee}_{[\mu],R})$$

and

$$(\underline{V}^{\vee}_{[\mu],R})^{\mathrm{sub}} := \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(V^{\vee}_{[\mu],R}),$$

respectively, to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}$. Similarly, for each $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1,R}$, let $W_{\nu,R}$ be defined as in the last paragraph of Section 3.5. For any $w \in \mathrm{W}^{\mathrm{M}_1}$, we define

$$W_{w\cdot[\mu],R} := \underset{\nu\in w\cdot[\mu]}{\oplus} W_{\nu,R}.$$

(Although $V_{[\mu],R_1} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is irreducible by definition, $W_{w\cdot[\mu],R_1} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is not necessarily irreducible when G_1 is not connected.) Then we define $W^{\vee}_{w\cdot[\mu],R}$, $\underline{W}^{\vee}_{w\cdot[\mu],R}$, $(\underline{W}^{\vee}_{w\cdot[\mu],R})^{\mathrm{can}}$, and $(\underline{W}^{\vee}_{w\cdot[\mu],R})^{\mathrm{sub}}$ in the obvious way.

The connections (2.33), (2.35), and (2.36) define respectively the de Rham complex

$$(\underline{V}^{\vee}_{[\mu],R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and the log de Rham complexes

$$((\underline{V}^{\vee}_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

and

$$((\underline{V}^{\vee}_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla).$$

5.2. **BGG complexes.**

Definition 5.1. *A complex of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules formed by direct sums of Verma modules* (*see Definition* 4.4) *is called a summand* **of degree zero** *of another complex of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules formed by direct sums of Verma modules if both the embedding and the splitting morphisms defining the summand are defined by direct sums of morphisms of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules of degree zero* (*see Definition* 4.7).

For any integer $a \geq 0$, we denote by $\mathrm{W}^{\mathrm{M}_1}(a)$ the elements w in $\mathrm{W}^{\mathrm{M}_1}$ with length $l(w) = a$.

Theorem 5.2. *Let* $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1,R}$, *and let* $V_{[\mu],R} \in \mathrm{Rep}_R(\mathrm{G}_1)$ *be defined as in Section* 3.5. *Then there exists an* $\mathtt{F}$*-filtered complex of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules*

(5.3)

$$0 \to \mathrm{BGG}_n(V_{[\mu],R}) \overset{d_n}{\to} \mathrm{BGG}_{n-1}(V_{[\mu],R}) \to \cdots \to \mathrm{BGG}_1(V_{[\mu],R}) \overset{d_1}{\to} \mathrm{BGG}_0(V_{[\mu],R}),$$

canonically $\mathtt{F}$*-filtered quasi-isomorphically embedded as a summand of* $\mathrm{Std}_{\bullet}(V_{[\mu],R})$ (*see Definition* 4.26) *in the category of* $\mathtt{F}$*-filtered complexes of* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-modules, where*

$$\mathrm{BGG}_a(V_{[\mu],R}) \cong \underset{w\in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} \mathrm{Verm}(W_{w\cdot[\mu],R}) \tag{5.4}$$

(*as* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-submodules*) *for each* $0 \leq a \leq n$. *Moreover, the induced complex*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_{\bullet}(V_{[\mu],R}))$$

is a canonical (*quasi-isomorphic*) *summand of degree zero of* $\mathrm{Gr}_{\mathtt{F}}(\mathrm{Std}_{\bullet}(V_{[\mu],R}))$ *with trivial differentials.*

Proof. By the argument in [33, 4.4] (using [20, Sec. 8.2] and [33, Cor. 1.11 b)]), since U_1 and hence $\mathfrak{u}_R$ are abelian, we have $\wedge^a(\mathfrak{u}_R^-) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} W_{w\cdot[0],R}$ as P_1-modules, and hence

$$\mathrm{Std}_a(V_{[0],R}) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} \mathrm{Verm}(W_{w\cdot[0],R}) \tag{5.5}$$

as $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules. (When some residue characteristic of R is $p > 0$, the argument in [33, 4.4] requires that $p \geq h - 1$, where h is the Coxeter number of G_1. As pointed out in [33, Rem. 2.1], this is automatic if there is any $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<p}$ to begin with.) As in [33, 4.5], since $\mathrm{Std}_a(V_{[\mu],R}) \cong \mathrm{Std}_a(V_{[0],R}) \underset{R}{\otimes} V_{[\mu],R}$ (as complexes of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules), we deduce from (5.5) (and the tensor identity [12, Prop. 1.7]) that

$$\mathrm{Std}_\bullet(V_{[\mu],R}) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}}{\oplus} \mathrm{Std}_w(V_{[\mu],R}), \tag{5.6}$$

where

$$\mathrm{Std}_w(V_{[\mu],R}) := \mathrm{Verm}(W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R})$$

($W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R}$ being a tensor product in $\mathrm{Rep}_R(\mathrm{P}_1)$) appears in degree $l(w)$, and where the differentials are inherited from those of $\mathrm{Std}_\bullet(V_{[\mu],R})$. (In particular, (5.6) is not a decomposition into subcomplexes.)

By the same argument as in [33, 2.7] (using also [18]), the complex $\mathrm{Std}_\bullet(V_{[\mu],R})$ admits a canonical direct sum decomposition

$$\mathrm{Std}_\bullet(V_{[\mu],R}) \cong \underset{j \in J}{\oplus} \mathrm{Std}_\bullet(V_{[\mu],R})_{\bar{\chi}_j} \cong \underset{j \in J}{\oplus} \left(\underset{w \in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} \mathrm{Std}_w(V_{[\mu],R})_{\bar{\chi}_j} \right),$$

indexed by some finite set J, such that the center of $\mathrm{U}(\mathfrak{g}_R)$ acts on the reduction mod p of each direct summand $\mathrm{Std}_\bullet(V_{[\mu],R})_{\bar{\chi}_j}$ by a distinct character $\bar{\chi}_j$. The direct sum with respect to $j \in J$ is a decomposition into subcomplexes of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules, because the action of the center of $\mathrm{U}(\mathfrak{g}_R)$ commutes with the action of P_1. Take the unique index $j_0 \in J$ such that $\bar{\chi}_{j_0} = \bar{\chi}_{[\mu],p}$, where the latter is the unique character of the center of $\mathrm{U}(\mathfrak{g}_R)$ that acts nontrivially on the reduction mod p of $V_{[\mu],R}$. Then we define

$$\mathrm{BGG}_\bullet(V_{[\mu],R}) := \mathrm{Std}_\bullet(V_{[\mu],R})_{\bar{\chi}_{j_0}} = \mathrm{Std}_\bullet(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}}.$$

Thus, (5.6) has a refinement

$$\mathrm{BGG}_\bullet(V_{[\mu],R}) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}}{\oplus} \mathrm{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}}. \tag{5.7}$$

Since $\mu \in \mathrm{X}_{\mathrm{G}_1,R}^{+,<p}$, by [33, Lem. 2.3], all weights of $\wedge^\bullet(\mathfrak{u}_R^-) \underset{R}{\otimes} V_{[\mu],R}$ are p-small. Then, for each $w \in \mathrm{W}^{\mathrm{M}_1}$, there exists a finite filtration on $W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R}$ such that the graded pieces are of the form $W_{\nu,R}$ for some $\nu \in \mathrm{X}_{\mathrm{M}_1,R}^{+,<p}$. Since the functor $\mathrm{Verm}(\,\cdot\,)$ (see (4.3)) is exact (because $\mathrm{U}(\mathfrak{g}_R)$ is free over $\mathrm{U}(\mathfrak{p}_R)$), there is a corresponding finite filtration on $\mathrm{Std}_w(V_{[\mu],R}) = \mathrm{Verm}(W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R})$ by $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules, whose graded pieces are of the form $\mathrm{Verm}(W_{\nu,R})$ with $W_{\nu,R}$ appearing as a graded piece on the finite filtration on $W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R}$. We have a similar finite filtration for the direct summand $\mathrm{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}}$ of $\mathrm{Std}_w(V_{[\mu],R})$. As in [33, 2.7, 2.8] (using also [18]), for $\nu \in \mathrm{X}_{\mathrm{M}_1,R}^{+,<p}$, the $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-module $\mathrm{Verm}(W_{\nu,R})$

appears as a graded piece of such a filtration on $\mathrm{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}}$ if and only if the following two conditions hold:

(1) $W_{\nu,R}$ appears as a graded piece on the finite filtration on $W_{w\cdot[0],R} \underset{R}{\otimes} V_{[\mu],R}$.

(2) $\bar{\chi}_{\nu,p} = \bar{\chi}_{[\mu],p}$, or equivalently when $\nu = w' \cdot \mu'$ for some $w' \in \mathrm{W}^{\mathrm{M}_1}$ and some $\mu' \in [\mu]$.

But then, as explained in [33, proof of Lem. 4.5], this happens only when $w' = w$, and the multiplicity is exactly one for each μ'. (It suffices to check this in the universal case $R = R_1$, and hence after base change to the characteristic zero field $R_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.) This shows that

$$\mathrm{Std}_w(V_{[\mu],R})_{\bar{\chi}_{[\mu],p}} \cong \mathrm{Verm}(W_{w\cdot[\mu],R}). \tag{5.8}$$

Thus, (5.4) follows from the combination of (5.7) and (5.8).

As for last statement, first note that the morphisms of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules defining $\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V_{[\mu],R}))$ as a summand of $\mathrm{Gr}_{\mathtt{F}}(\mathrm{Std}_\bullet(V_{[\mu],R}))$ are of degree zero because U_R and hence $\mathfrak{u}_R$ act trivially on $\mathtt{F}$-graded pieces. Since there is no nonzero P_1-morphism from any P_1-summand of $W^\vee_{w_2\cdot[\mu],R}$ to $W^\vee_{w_1\cdot[\mu],R}$ when $w_1, w_2 \in \mathrm{W}^{\mathrm{M}_1}$ satisfy $w_1 \neq w_2$ (which is the case when $l(w_1) \neq l(w_2)$), the differentials of $\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V_{[\mu],R}))$ (which are sums of morphisms between Verma modules) are sums of morphisms that are either zero or of positive degree (in the sense of Definition 4.7). By Lemma 4.23 (with $m_0 = 0$), this shows that the differentials of $\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V_{[\mu],R}))$ are all zero, as desired. □

5.3. **Dual BGG complexes.**

Theorem 5.9. *For any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_{1,R}}$, there is a canonical $\mathtt{F}$-filtered complex*

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}),$$

with trivial differentials on its $\mathtt{F}$-graded pieces, such that

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})) \cong \underset{w\in\mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}}$$

as $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules, together with a canonical quasi-isomorphic embedding

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}) \tag{5.10}$$

in the category of complexes of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules, realizing the left-hand side as a summand of the right-hand side.

The embedding (5.10) *is induced by taking $\mathtt{F}$-graded pieces of a canonical $\mathtt{F}$-filtered morphism*

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}) \to (\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \tag{5.11}$$

in the categories of complexes of sheaves of R-modules, with morphisms in each degree given by differential operators (*rather than morphisms of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules*)*.*

Proof. The existence of the complex $\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})$ (with required properties) follows from Theorem 5.2 and Proposition 4.21. (Implicit in the condition of being $\mathtt{F}$-filtered is that its differential and its Hodge filtration satisfy the Griffiths transversality, which is true because the action of $\mathfrak{u}^-_R$ on any P_1-module increases the H-weights by 1, as in the proof of Corollary 4.32.)

The existence of the $\mathtt{F}$-filtered morphisms (5.10) and (5.11) in the categories of complexes of sheaves of R-modules, with morphisms in each degree given by differential operators, also follows from Theorem 5.2 and Proposition 4.21, and from Proposition 4.31. The fact that (5.10) is an embedding in the category of *complexes of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules* follows from the fact that $\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V_{[\mu],R}))$ is a quasi-isomorphic summand of *degree zero* of $\mathrm{Gr}_{\mathtt{F}}(\mathrm{Std}_\bullet(V_{[\mu],R}))$ (see the last statement in Theorem 5.2). $\square$

Corollary 5.12. *With the setting in Theorem* 5.9*, by setting*

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}) := \mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \mathscr{I}_{\mathsf{D}},$$

we obtain an $\mathtt{F}$*-filtered complex*

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}),$$

with trivial differentials on $\mathtt{F}$*-graded pieces, such that*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}})) \cong \underset{w\in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}}$$

as $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$*-modules, together with a canonical quasi-isomorphic embedding*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}})) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}) \tag{5.13}$$

in the category of complexes of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$*-modules, realizing the left-hand side as a summand of the right-hand side.*

The embedding (5.13) *is induced by taking* $\mathtt{F}$*-graded pieces of a canonical* $\mathtt{F}$*-filtered morphism*

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}) \hookrightarrow (\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R} \tag{5.14}$$

in the categories of complexes of sheaves of R*-modules, with morphisms in each degree given by differential operators* (*rather than morphisms of* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$*-modules*)*.*

Proof. Simply apply $\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \mathscr{I}_{\mathsf{D}}$ to all $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules in Theorem 5.9. $\square$

Remark 5.15. The canonical objects and morphisms in Theorem 5.9 and Corollary 5.12 are all functorial in R. In fact, by the smoothness of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1} \to \mathsf{S}_1$, in order to prove Theorem 5.9 and Corollary 5.12, it suffices to treat the universal cases $R = R_1$ (for $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$) and $R = R_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (for $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1,R}$ but $\mu \notin \mathrm{X}^{+,<p}_{\mathrm{G}_1}$, which can happen only when the residue characteristics of R are all zero).

Remark 5.16. If we are in the modular curve case, and if $[\mu]$ is chosen such that $\underline{V}^\vee_{[\mu],R} \cong \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R})$, then the degree zero term of (5.11) can be identified with a morphism

$$\underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H},R}} \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},R})$$

of sheaves of R-modules. If the residue characteristics of R are all zero, then we know that this is an embedding which splits the (relative) Hodge filtration *globally.* This is a differential operator but not a morphism of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules.

5.4. **Characteristic zero bases.**

Proposition 5.17. *If the residue characteristics of R are all zero, then the canonical morphisms* (5.11) *and* (5.14) *are* (F*-filtered*) *quasi-isomorphic embeddings* (*cf.* [9, Sec. 3] *and* [10, Ch. VI, Sec. 5]).

Proof. Since all objects involved can be defined over a field extension of F_0 that can be embedded into $\mathbb{C}$, we may assume that $R = \mathbb{C}$. Then we can verify the statements of quasi-isomorphisms of sheaves after analytification. We shall denote the analytifications of objects over $\mathbb{C}$ by the subscript "an".

By the comparisons given in [24, Sec. 4 and 5.2], the algebraic and analytic constructions are compatible with each other for the Shimura varieties and their toroidal compactifications, and for the automorphic bundles and their canonical extensions. Based on these objects, the algebraic and analytic constructions of differential operators from morphisms between Verma modules are also compatible with each other.

Then the proposition is known thanks to the same arguments in [9, Sec. 3 and 7] and [10, Ch. VI, Sec. 5]:

Over $\mathsf{M}_{\mathcal{H},\mathrm{an}}$, which is a finite union of arithmetic quotients of Hermitian symmetric spaces, we know (by [9, Sec. 3 and 7], by verifying the statements over each connected components) that the canonical morphism

$$\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],\mathrm{an}}) \to \underline{V}^\vee_{[\mu],\mathrm{an}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}$$

is a quasi-isomorphic embedding realizing the left-hand side as a summand of the right-hand side, and both sides give resolutions of the same local system $\underline{V}^\vee_{[\mu],\mathrm{Betti}}$ attached to the dual of the irreducible representation of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ containing the highest weight μ. Moreover (by [9, Sec. 7]) the analytifications of (5.11) and (5.14) realize the canonical embeddings

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{can}}) \hookrightarrow (\mathsf{M}_{\mathcal{H},\mathrm{an}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}})_*(\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],\mathrm{an}}))$$

and

$$\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{sub}}) \hookrightarrow (\mathsf{M}_{\mathcal{H},\mathrm{an}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}})_!(\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],\mathrm{an}})),$$

(term-by-term, compatibly) as summands of the canonical embeddings

$$(\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}} \hookrightarrow (\mathsf{M}_{\mathcal{H},\mathrm{an}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}})_*(\underline{V}^\vee_{[\mu],\mathrm{an}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}) \tag{5.18}$$

and

$$(\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}} \hookrightarrow (\mathsf{M}_{\mathcal{H},\mathrm{an}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}})_!(\underline{V}^\vee_{[\mu],\mathrm{an}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}), \tag{5.19}$$

respectively. Thus, to show that (the analytifications of) (5.11) and (5.14) are quasi-isomorphisms, it suffices to show that the canonical embeddings (5.18) and (5.19) are quasi-isomorphisms. (Note that, unlike their restrictions to $\mathsf{M}_{\mathcal{H},\mathrm{an}}$, the log de Rham complexes $\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{can}})$, $\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{sub}})$, $(\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}$, and $(\underline{V}^\vee_{[\mu],\mathrm{an}})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}}$ are *not* resolutions in general.)

According to [1, Ch. III, Sec. 5, Main Thm. I and its proof], the connected local charts of the toroidal compactifications about a boundary divisor can be given by partial toroidal embeddings of punctured polydisk bundles with fundamental groups canonically identified with a discrete subgroup of the unipotent radical of some maximal parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. (The unipotent radical depends on the boundary divisor in question.) Since the local system $\underline{V}^\vee_{[\mu],\mathrm{Betti}}$ is defined over $\mathsf{M}_{\mathcal{H},\mathrm{an}}$ using the action of the same group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, the monodromy transformations along irreducible components of the boundary divisor D are all unipotent. (The automorphic bundles $\underline{V}^\vee_{[\mu],\mathrm{an}}$ can be realized as summands of the relative de Rham cohomology of the log smooth morphisms from toroidal compactifications of Kuga families to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\mathrm{an}}$. By [19, VII], this implies that the eigenvalues of the residue maps of the Gauss–Manin connections along irreducible components of D are non-negative rational numbers strictly less than one. Therefore, by [19, VI], this shows that the condition that the monodromy transformations of $\underline{V}^\vee_{[\mu],\mathrm{Betti}}$ are unipotent are equivalent to the condition that the Gauss–Manin connections of $\underline{V}^\vee_{[\mu],\mathrm{an}}$ are nilpotent.)

Hence, since $\underline{V}^\vee_{[\mu],\mathrm{an}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\mathrm{an}}}$ is a resolution of $\underline{V}^\vee_{[\mu],\mathrm{Betti}}$, as explained in [10, Ch. VI, Prop. 5.4], by local calculations and by reducing to the one variable case using Künneth, we know that (5.18) and (5.19) are indeed quasi-isomorphisms, as desired. (The unipotence of the monodromy is used in the standard argument reducing these statements to the trivial coefficient case; see for example [6, II, Lem. 6.9].) □

Remark 5.20. Suppose any residue characteristic of R is $p > 0$. Then differential operators behave pathologically in general, and the left-hand sides of the morphisms (5.11) and (5.14) might fail to be summands of the right-hand sides for trivial reasons (because the differential operators may annihilate too many elements). This can be salvaged by introducing the language of divided powers, which are quite natural because log crystals (realized as coherent sheaves with log HPD stratifications) can be canonically attached to the generalized Verma modules. This has been studied in the Siegel case in [31, Sec. 4–5]. However, since this has not been needed in the applications we have in mind, we shall not carry this out in this article.

5.5. **Decomposition of (log) Hodge cohomology.** Variants of the following consequence of the canonical quasi-isomorphic embeddings (5.10) and (5.13) on the $\mathtt{F}$-graded pieces (without any reference to (5.11) and (5.14)) suffice in most applications we know (including those in [31] and subsequent works using patterns of Hodge–Tate weights):

Corollary 5.21. *For any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1,R}$, we have a canonical isomorphism*

$$\begin{aligned} &H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R})) \\ &\cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{a+b-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})). \end{aligned} \tag{5.22}$$

The same is true if we replace $(\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}$ with $(\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}$.

The upshot is that the left-hand side of (5.22) is a hypercohomology of complexes of sheaves, while the right-hand side is a direct sum of cohomology of sheaves. In practice, the right-hand side can be much easier to study.

Proof of Corollary 5.21. This is because of the quasi-isomorphisms (5.10) and (5.13), and because the left-hand sides of them have trivial differentials. □

Remark 5.23. Applications of (5.10) and (5.13) to the study of torsion in the cohomology of PEL-type Shimura varieties can be found in the joint work of the first author and Junecue Suh. (See [27] and [28].)

Remark 5.24. If the residue characteristics of R are all zero, then it is known that the Hodge spectral sequences

$$\begin{aligned} E_1^{a,b} &:= H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R})) \\ &\Rightarrow H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, (\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}) \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} E_1^{a,b} &:= H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R})) \\ &\Rightarrow H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, (\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}) \end{aligned} \tag{5.26}$$

degenerate at their E_1 terms. As a result, being defined by $\mathtt{F}$-filtered quasi-isomorphic summands, the dual BGG versions of the Hodge spectral sequences

$$\begin{aligned} E_1^{a,b} &:= H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}(\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}))) \\ &\cong \underset{w\in \mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{a+b-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})) \\ &\Rightarrow H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})) \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} E_1^{a,b} &:= H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}(\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}))) \\ &\cong \underset{w\in \mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{a+b-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}})) \\ &\Rightarrow H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, \mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}})) \end{aligned} \tag{5.28}$$

also degenerate at their E_1 terms. These can be proved by first reducing to the case $R = \mathbb{C}$, and by realizing both sides of these spectral sequences as summands of the corresponding Hodge spectral sequences of the toroidal compactifications of Kuga families (or certain mixed Shimura varieties) with trivial coefficients. (See [10, Ch. IV, Sec. 1–2] and [25] for the algebraic construction of toroidal compactifications of PEL-type Kuga families, and see [32] for the analytic construction of toroidal compactifications of mixed Shimura varieties.) Alternatively, one may resort to Saito's theory of mixed Hodge modules (see [34]). (See [10, Ch. VI, p. 234] and [17, Cor. 4.2.3] for the methods of proving the degeneration.)

Remark 5.29. Suppose $R = \mathbb{C}$. As explained in the proof of Proposition 5.17, the right-hand sides of (5.25) and (5.27) (resp. (5.26) and (5.28)) are canonically isomorphic to $H^{a+b}(\mathsf{M}_{\mathcal{H},\mathrm{an}}, \underline{V}^\vee_{[\mu],\mathrm{Betti}})$ (resp. $H^{a+b}_{\mathrm{c}}(\mathsf{M}_{\mathcal{H},\mathrm{an}}, \underline{V}^\vee_{[\mu],\mathrm{Betti}})$). As a result, the left-hand side of (5.22) (resp. the analogue for $(\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}$) gives the Hodge graded pieces of the cohomology of $H^{a+b}(\mathsf{M}_{\mathcal{H},\mathrm{an}}, \underline{V}^\vee_{[\mu],\mathrm{Betti}})$ (resp. $H^{a+b}_{\mathrm{c}}(\mathsf{M}_{\mathcal{H},\mathrm{an}}, \underline{V}^\vee_{[\mu],\mathrm{Betti}})$).

5.6. **Descending the BGG and dual BGG complexes.** Up to modifying the choices of $\mathcal{O}_{F_0',(p)}$ and R_1 in Section 2.1, consider any $\mathcal{O}_{F_0,(p)}$-algebra R' and any R_1-algebra R satisfying the following:

Assumption 5.30. *There exist a faithfully flat homomorphism $R' \to R$, and an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R'$-module L_0', locally free over R', such that $L_0' \underset{R'}{\otimes} R \cong L_{0,1} \underset{R_1}{\otimes} R$.*

Remark 5.31. The upshot is that R' does not have to satisfy Condition 2.5 as R_1 does. (It does not even have to be an $\mathcal{O}_{F_0',(p)}$-algebra for some F_0'.)

Remark 5.32. If $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0$ is a product of matrix algebras over fields (e.g., if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a field), then, up to modifying the choices of $\mathcal{O}_{F_0',(p)}$ and R_1 in Section 2.1, R' can be taken to be any $\mathcal{O}_{F_0,(p)}$-algebra.

Let us denote by $\iota_1, \iota_2 : R \to R^{(2)} := R \underset{R'}{\otimes} R$ the two natural homomorphisms. By the theory of descent, the category of R'-modules is equivalent to the category of R-modules with descent data. Namely, the datum of an R'-module M' is equivalent to the datum of a pair (M, δ_M), where M is an R-module, and where $\delta_M : M \underset{R,\iota_1}{\otimes} R^{(2)} \xrightarrow{\sim} M \underset{R,\iota_2}{\otimes} R^{(2)}$ is an isomorphism of $R \underset{R'}{\otimes} R$-modules such that the three pullbacks of δ_M to $R^{(3)} := R \underset{R'}{\otimes} R \underset{R'}{\otimes} R$ satisfy the usual cocycle condition.

By Assumption 5.30, and by imitating Definition 2.4, with $\mathcal{O}_{F_0',(p)}$ and L_0 replaced with R' and L_0', we obtain group functors G', P', and M', together with the canonical morphisms among them, such that their base changes from R' to R are compatibly isomorphic to $\mathrm{G}_1 \underset{R_1}{\otimes} R$, $\mathrm{P}_1 \underset{R_1}{\otimes} R$, and $\mathrm{M}_1 \underset{R_1}{\otimes} R$. We will fix compatible choices of the isomorphisms $\mathrm{G}' \underset{R'}{\otimes} R \cong \mathrm{G}_1 \underset{R_1}{\otimes} R$, $\mathrm{P}' \underset{R'}{\otimes} R \cong \mathrm{P}_1 \underset{R_1}{\otimes} R$, and $\mathrm{M}' \underset{R'}{\otimes} R \cong \mathrm{M}_1 \underset{R_1}{\otimes} R$, and suppress them in what follows. We shall consider objects and morphisms canonical if they canonically depend on these fixed choices.

Then we can define the categories $\mathrm{Rep}_{R'}(\mathrm{G}')$, $\mathrm{Rep}_{R'}(\mathrm{P}')$, and $\mathrm{Rep}_{R'}(\mathrm{M}')$ as in Definition 2.21. Moreover, we can define the analogues over R' of all the objects in Sections 2.2–2.4, with the superscript "1" replaced with a prime in the notation system. For example, for $W' \in \mathrm{Rep}_{R'}(\mathrm{G}')$, we can define the de Rham complex

$$(\mathcal{E}_{\mathrm{G}',R'}(W') \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R'}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R'}/\mathsf{S}_{R'}}, \nabla)$$

and the log de Rham complexes

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}',R'}(W') \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}/\mathsf{S}_{R'}}, \nabla)$$

and

$$(\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}',R'}(W') \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}/\mathsf{S}_{R'}}, \nabla)$$

as in Definition 2.38. We can also define Hodge filtrations as in Section 3.3. Apart from the remark in the second paragraph of Section 4.1, the construction of differential operators in Section 4 works verbatim when $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules are replaced with $\mathrm{U}(\mathfrak{g}')$-$\mathrm{P}'$-modules. The pullbacks of these objects from R' to R all carry descent data in the same way as R'-modules (because they are all given by complexes of sheaves of R'-modules).

Suppose $V' \in \mathrm{Rep}_{R'}(\mathrm{G}')$, and suppose $V := V' \underset{R'}{\otimes} R$ decomposes as

$$V \cong \underset{i \in I_V}{\oplus} V_{[\mu_i],R}, \tag{5.33}$$

where I_V is an index set, and where $\mu_i \in \mathrm{X}^{+,<p}_{\mathrm{G}_1,R}$ for each $i \in I_V$ and each $\mu_i \in [\mu_i]$. By Definition 4.26, we have the canonical $\mathtt{F}$-filtered complex of $\mathrm{U}(\mathfrak{g}')$-$\mathrm{P}'$-modules $\mathrm{Std}_\bullet(V')$ (resp. of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules $\mathrm{Std}_\bullet(V)$), and $\mathrm{Std}_\bullet(V)$ carries the descent isomorphism $\delta_{\mathrm{Std}_\bullet(V)} : \mathrm{Std}_\bullet(V) \underset{R,\iota_1}{\otimes} R^{(2)} \xrightarrow{\sim} \mathrm{Std}_\bullet(V) \underset{R,\iota_2}{\otimes} R^{(2)}$ canonically identifying $\mathrm{Std}_\bullet(V)$ as the pullback of $\mathrm{Std}_\bullet(V')$ from R' to R.

On the other hand, we define (according to (5.33)) the $\mathtt{F}$-filtered complex of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules

$$\mathrm{BGG}_\bullet(V) := \underset{i \in I_V}{\oplus} \mathrm{BGG}_\bullet(V_{[\mu_i],R}). \tag{5.34}$$

By applying Theorem 5.2 to $[\mu_i]$ for each $i \in I_V$, we see that $\mathrm{BGG}_\bullet(V)$ is canonically $\mathtt{F}$-filtered quasi-isomorphically embedded as a summand of $\mathrm{Std}_\bullet(V)$ in the category of $\mathtt{F}$-filtered complexes of $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules. Moreover, the induced complex $\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V))$ is a canonical (quasi-isomorphic) summand of degree zero of $\mathrm{Gr}_{\mathtt{F}}(\mathrm{Std}_\bullet(V))$ with trivial differentials. (The embedding of $\mathrm{BGG}_\bullet(V)$ as an $\mathtt{F}$-filtered summand of $\mathrm{Std}_\bullet(V)$ is canonically determined by the actions of $\mathrm{U}(\mathfrak{g}_R)$ and P_1.) Analogues of these statements remain true when we replace R with any R-algebra.

Theorem 5.35. *With the setting as above, there exists an* $\mathtt{F}$*-filtered complex of* $\mathrm{U}(\mathfrak{g}')$-$\mathrm{P}'$*-modules*

$$0 \to \mathrm{BGG}_n(V') \xrightarrow{d_n} \mathrm{BGG}_{n-1}(V') \to \cdots \to \mathrm{BGG}_1(V') \xrightarrow{d_1} \mathrm{BGG}_0(V'), \tag{5.36}$$

canonically $\mathtt{F}$*-filtered quasi-isomorphically embedded as a summand of* $\mathrm{Std}_\bullet(V')$ (*see Definition* 4.26) *in the category of* $\mathtt{F}$*-filtered complexes of* $\mathrm{U}(\mathfrak{g}')$-$\mathrm{P}'$*-modules, such that*

$$\mathrm{BGG}_a(V') \underset{R'}{\otimes} R \cong \mathrm{BGG}_a(V) \cong \underset{i \in I_V}{\oplus} \Big(\underset{w \in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} \mathrm{Verm}(W_{w \cdot [\mu_i],R}) \Big) \tag{5.37}$$

(*as* $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$*-submodules*) *for each* $0 \leq a \leq n$. *Moreover, the induced complex*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}_\bullet(V'))$$

is a canonical (*quasi-isomorphic*) *summand of degree zero of* $\mathrm{Gr}_{\mathtt{F}}(\mathrm{Std}_\bullet(V'))$ (*see Definition* 5.1) *with trivial differentials.*

Proof. Since $\mathrm{U}(\mathfrak{g}_R)$ and P_1 have models over R', the descent isomorphism

$$\delta_{\mathrm{Std}_\bullet(V)} : \mathrm{Std}_\bullet(V) \underset{R,\iota_1}{\otimes} R^{(2)} \xrightarrow{\sim} \mathrm{Std}_\bullet(V) \underset{R,\iota_2}{\otimes} R^{(2)}$$

is compatible with the pullbacks of the actions of $\mathrm{U}(\mathfrak{g}_R)$ and P_1 under ι_1 and ι_2 (on the two sides). Since these actions determine the canonical summand $\mathrm{BGG}_\bullet(V) \underset{R,\iota_1}{\otimes} R^{(2)}$ of $\mathrm{Std}_\bullet(V) \underset{R,\iota_1}{\otimes} R^{(2)}$ and the corresponding canonical summand $\mathrm{BGG}_\bullet(V) \underset{R,\iota_2}{\otimes} R^{(2)}$ of $\mathrm{Std}_\bullet(V) \underset{R,\iota_2}{\otimes} R^{(2)}$, the isomorphism $\delta_{\mathrm{Std}_\bullet(V)}$ induces a descent isomorphism

$$\delta_{\mathrm{BGG}_\bullet(V)} : \mathrm{BGG}_\bullet(V) \underset{R,\iota_1}{\otimes} R^{(2)} \xrightarrow{\sim} \mathrm{BGG}_\bullet(V) \underset{R,\iota_2}{\otimes} R^{(2)},$$

whose three pullbacks to $R^{(3)}$ satisfy the usual cocycle condition. Thus, the $\mathtt{F}$-filtered summand $\mathrm{BGG}_\bullet(V)$ of $\mathrm{Std}_\bullet(V)$ descends to an $\mathtt{F}$-filtered summand of $\mathrm{Std}_\bullet(V')$, which is the desired complex $\mathrm{BGG}_\bullet(V')$. $\square$

Let $\underline{V}' := \mathscr{E}_{\mathrm{G}',R'}(V')$ and $\underline{V}'^\vee := \mathscr{E}_{\mathrm{G}',R'}(V'^\vee)$. (The notation makes sense because they are canonically dual to each other.) Let $(\underline{V}')^{\mathrm{can}} := \mathscr{E}^{\mathrm{can}}_{\mathrm{G}',R'}(V')$, $(\underline{V}')^{\mathrm{sub}} := \mathscr{E}^{\mathrm{sub}}_{\mathrm{G}',R'}(V')$, $(\underline{V}'^\vee)^{\mathrm{can}} := \mathscr{E}^{\mathrm{can}}_{\mathrm{G}',R'}(V'^\vee)$, and $(\underline{V}'^\vee)^{\mathrm{sub}} := \mathscr{E}^{\mathrm{sub}}_{\mathrm{G}',R'}(V'^\vee)$. (Then $(\underline{V}')^{\mathrm{can}}$ and $(\underline{V}'^\vee)^{\mathrm{can}}$ are canonically dual to each other.)

By Theorem 5.35, since the construction of differential operators in Section 4 works verbatim with $\mathrm{U}(\mathfrak{g}_R)$-$\mathrm{P}_1$-modules replaced with $\mathrm{U}(\mathfrak{g}')$-$\mathrm{P}'$-modules, the same proofs of Theorem 5.9 and Corollary 5.12 give the following:

Theorem 5.38 (cf. Theorem 5.9 and Corollary 5.12)**.** *With the setting as above, there is a canonical* $\mathtt{F}$*-filtered complex*

$$\mathrm{BGG}^\bullet((\underline{V}'^\vee)^{\mathrm{can}}),$$

with trivial differentials on its $\mathtt{F}$*-graded pieces, such that*

$$\begin{aligned}\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a((\underline{V}'^\vee)^{\mathrm{can}})) \underset{R'}{\otimes} R &\cong \mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a((\underline{V}^\vee)^{\mathrm{can}})) \\ &\cong \underset{i\in I_V}{\oplus}\left(\underset{w\in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus}(\underline{W}^\vee_{w\cdot[\mu_i],R})^{\mathrm{can}}\right)\end{aligned} \tag{5.39}$$

as $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$*-modules, together with a canonical quasi-isomorphic embedding*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^\bullet(\underline{V}'^\vee)^{\mathrm{can}})) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}((\underline{V}'^\vee)^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}/\mathsf{S}_{R'}}) \tag{5.40}$$

in the category of complexes of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}$*-modules, realizing the left-hand side as a summand of the right-hand side.*

The embedding (5.40) *is induced by taking* $\mathtt{F}$*-graded pieces of a canonical* $\mathtt{F}$*-filtered morphism*

$$\mathrm{BGG}^\bullet((\underline{V}'^\vee)^{\mathrm{can}}) \to (\underline{V}'^\vee)^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}/\mathsf{S}_{R'}} \tag{5.41}$$

in the categories of complexes of sheaves of R'*-modules, with morphisms in each degree given by differential operators* (*rather than morphisms of* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}$*-modules*)*.*

By setting

$$\mathrm{BGG}^\bullet((\underline{V}'^\vee)^{\mathrm{sub}}) := \mathrm{BGG}^\bullet((\underline{V}'^\vee)^{\mathrm{can}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R'}}}{\otimes} \mathscr{I}_{\mathsf{D}},$$

we obtain an $\mathtt{F}$*-filtered complex*

$$\mathrm{BGG}^\bullet((\underline{V}'^\vee)^{\mathrm{sub}}),$$

with trivial differentials on $\mathtt{F}$*-graded pieces, and with other properties similar to the above* (*with canonical extensions replaced with subcanonical extensions*)*.*

Since Proposition 5.17 was proved by working over $\mathbb{C}$, we also have:

Proposition 5.42 (cf. Proposition 5.17)**.** *If the residue characteristics of* R' *are all zero, then the canonical morphisms* (5.41) *and its subcanonical analogue are* ($\mathtt{F}$*-filtered*) *quasi-isomorphic embeddings.*

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