Local systems over Shimura varieties: a comparison of two constructions

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(joint work with Hansheng Diao, Ruochuan Liu, and Xinwen Zhu)

Let (G, X) be any Shimura datum, where G is a reductive algebraic group over $\mathbb{Q}$ and X is a $\mathrm{G}(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{C}^\times \to \mathrm{G}(\mathbb{R})$ satisfying certain axioms. Given any neat open compact subgroup K of $\mathrm{G}(\mathbb{A}^\infty)$, by results of Baily–Borel and Borel, the double coset space $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}} := \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)$ can be identified with the complex analytification of a canonical quasi-projective variety $\mathrm{Sh}_{K,\mathbb{C}}$ over $\mathbb{C}$. More precisely, the whole tower $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ with its right action by $\mathrm{G}(\mathbb{A}^\infty)$ has a canonical algebraic structure. Furthermore, by results of Shimura, Deligne, Borovoi, and Milne, among others, the whole tower $\{\mathrm{Sh}_{K,\mathbb{C}}\}_K$ with its canonical right action by $\mathrm{G}(\mathbb{A}^\infty)$ has a *canonical model* $\{\mathrm{Sh}_K\}_K$ over the *reflex field* E, which is a number field E depending only on (G, X) but not on K. We shall call any of these varieties the *Shimura varieties* associated with (G, X). For simplicity of exposition, we shall assume that $E = \mathbb{Q}$ in what follows.

Let G^c denote quotient of G by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus of the center of G. For any coefficient field F, we shall denote by $\mathrm{Rep}_F(\mathrm{G}^c)$ the category of algebraic representations of G^c over F.

Suppose $V \in \mathrm{Rep}_{\mathbb{Q}}(\mathrm{G}^c)$, with $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$. Then the local sections of $\mathrm{G}(\mathbb{Q})\backslash\big((\mathsf{X} \times V_{\mathbb{C}}) \times \mathrm{G}(\mathbb{A}^\infty)/K\big) \to \mathrm{G}(\mathbb{Q})\backslash\big(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/K\big)$ defines a canonical Betti *local system* ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}_{K,\mathbb{C}}^{\mathrm{an}}$. There is also a canonical (algebraic) filtered regular connection $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$ (satisfying Griffiths transversality) such that $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla)$ corresponds to ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ under Deligne's classical Riemann–Hilbert correspondence [5], and such that $\mathrm{Fil}^\bullet$ is induced by the Hodge cocharacters μ_h given by $h \in \mathsf{X}$. Such local systems and filtered connections are well-known complex analytically constructed objects over Shimura varieties.

On the other hand, for each prime number $p > 0$, consider $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, together with the canonical *p-adic étale local system* (i.e., lisse p-adic étale sheaf) ${}_{\text{ét}}\underline{V}_{\mathbb{Q}_p}$ over Sh_K defined using the tower of canonical models $\{\mathrm{Sh}_{K'}\}_{K' \subset K}$. By [11], this p-adic étale local system is *de Rham* in the sense that its geometric stalks over all classical points (defined by finite extensions of $\mathbb{Q}_p$) are de Rham as p-adic Galois representations. As in the case above over $\mathbb{C}$, but by using instead the *algebraic p-adic Riemann–Hilbert functor* (over $\mathbb{Q}_p$) we constructed (in [7, §6]), we also obtain a canonical (algebraic) filtered regular connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{Q}_p}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{Q}_p}$. By base change under any field homomorphism from $\mathbb{Q}_p$ to $\mathbb{C}$, we obtain a filtered regular connection $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ over $\mathrm{Sh}_{K,\mathbb{C}}$.

Note that the above base change from $\mathbb{Q}_p$ to $\mathbb{C}$ makes sense because we are working with *algebraic* filtered connections! The constructions over the analytification of $\mathrm{Sh}_{K,\mathbb{Q}_p}$ as in [16] and [11] are insufficient because canonical extensions and algebraizations generally do not exist in the rigid analytic world, unlike in the complex analytic world. Rather, we constructed (in [7, §5]) an analytic *logarithmic Riemann–Hilbert functor*, by working with pro-Kummer étale sites and log de

Rham period sheaves over suitable smooth compactifications, which provides the desired canonical extensions to which GAGA applies. Crucially, we showed that all the *eigenvalues of residues* are in $\mathbb{Q} \cap [0, 1)$, and we made essential uses of the finiteness of $[k : \mathbb{Q}_p]$ and the theory of decompletions.

By the classical Riemann–Hilbert correspondence again (in the easier direction), $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla)$ defines a Betti local system ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ over $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$. Such ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ are our new *p-adic analytically constructed* objects (with coefficient field $\mathbb{C}$!) over Shimura varieties. It is natural to ask how these objects compare with their complex analytically constructed counterparts.

Our main result is that ${}_{p\text{-B}}\underline{V}_{\mathbb{C}}$ and $({}_{p\text{-dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$ can be canonically identified with ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ and $({}_{\mathrm{dR}}\underline{V}_{\mathbb{C}}, \nabla, \mathrm{Fil}^\bullet)$, respectively, in a way compatible with the Hecke action of $\mathrm{G}(\mathbb{A}^\infty)$, with morphisms of Shimura varieties induced by morphisms of Shimura data, and with descent to canonical models of filtered connections (as in Harris's and Milne's works; see [10] and [14]). (See [7, §7], where we treated more general $V \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathrm{G}^c)$.)

Our proof uses several of the most general results and techniques available for Shimura varieties and their canonical models, from the (known) abelian case of Fontaine–Mazur conjecture [9] to Deligne's and Blasius's results [6, 1] that Hodge cycles on abelian varieties over number fields are *absolute Hodge* and *de Rham*, and then from Margulis's *superrigidity theorem* [12] and Borel's *density theorem* [2, 3] to a construction credited to Piatetski-Shapiro by Borovoi [4] and Milne [13].

Consequently, by the p-adic de Rham comparison results (for general smooth varieties over $\mathbb{Q}_p$) in [7, §6], we know that $H^i_{\text{ét}}(\mathrm{Sh}_{K,\overline{\mathbb{Q}}_p}, {}_{\text{ét}}\underline{V}_{\mathbb{Q}_p})$ is *de Rham* as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and that the Hodge–Tate weights of this representation is determined by the dimensions of certain coherent cohomology (and hence relative Lie algebra cohomology) given by Faltings's dual BGG complexes (see [8] and [10]). We also obtain a new proof of the degeneracy of the Hodge–de Rham spectral sequence for $H^i_{\mathrm{dR}}(\mathrm{Sh}_{K,\mathbb{C}}, {}_{\mathrm{dR}}\underline{V}_{\mathbb{C}})$ on the E_1 page, based on p-adic Hodge theory instead of complex Hodge theory. (In particular, we have not used Saito's theory of mixed Hodge modules [15].) We will extend these results and treat the compactly supported cohomology and interior cohomology in a forthcoming work.

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