VANISHING THEOREMS FOR TORSION AUTOMORPHIC SHEAVES ON GENERAL PEL-TYPE SHIMURA VARIETIES

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Abstract. We consider the interior cohomology (and the Hodge graded pieces in the case of the de Rham realization) of general — not necessarily compact — PEL-type Shimura varieties with coefficients in the local systems corresponding to sufficiently regular algebraic representations of the associated reductive group. For primes p bigger than an effective bound, we prove that the $\mathbb{F}_p$- and $\mathbb{Z}_p$-cohomology groups are concentrated in the middle degree, that the $\mathbb{Z}_p$-cohomology groups are free of p-torsion, and that every $\mathbb{F}_p$-cohomology class lifts to a $\mathbb{Z}_p$-cohomology class.

Introduction

In the introduction to [42], we asked the following question on the cohomology of a (general) Shimura variety with coefficients in algebraic representations of the associated reductive group:

Question. *Let p be a prime number. When is the* (*Betti*) *cohomology of the Shimura variety with* (*possibly nontrivial*) *integral coefficients p-torsion free?*

We provided a partial answer in [42], by proving the torsion freeness under the following conditions: that the Shimura variety is *compact*, is *of PEL type*, and has a neat and prime-to-p level; that the weight of the algebraic representation is *sufficiently regular*; and that p is greater than an *explicit* bound determined by the linear algebraic data and the weight (and independent of the neat and prime-to-p level). More precisely, we showed that the cohomology groups with p-torsion coefficients are concentrated in the middle degree, and this last fact implied the desired p-torsion freeness statement. The key ingredients of the proof included Illusie's vanishing theorem in characteristic $p > 0$, Faltings's dual BGG construction, and an observation relating the (geometric) "Kodaira type" conditions on the coefficient systems to the (representation-theoretic) "sufficient regularity" conditions.

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One naturally asks for analogous statements for noncompact PEL-type Shimura varieties. Of course, unlike in the compact case, here we have a plethora of cohomology theories, for some of which torsion and integral coefficients make sense. Among them are the ordinary cohomology, the cohomology with compact support, the interior cohomology (Harder's "*innere Kohomologie*" [20, p. 41]; cf. Remark 7.31), and the intersection cohomology (with respect to the minimal (Satake–Baily–Borel) compactification), which in general are all different.

The techniques of vanishing theorems in positive characteristics can be most readily applied to the first two, and hence to the third as a result. This is due to the fact that the automorphic vector bundles with integrable connections extend naturally to ones with log poles (along the boundaries) over smooth toroidal compactifications (see [50], [18, Ch. VI], [21], [22], and [46]), and that we have good integral models of both the compactifications and the bundles (see [18, Ch. VI], [48], and [38]) that we can then reduce modulo good primes. We prove that the ordinary cohomology groups (resp. the compactly supported cohomology groups) vanish below the middle degree (resp. above the middle degree) for *sufficiently regular* weights and for neat and prime-to-p levels as before (in [42]). These vanishing statements imply, in the middle degree, the p-torsion freeness of the ordinary cohomology and the liftability of the compactly supported cohomology classes with $\mathbb{F}_p$-coefficients. We also deduce that the interior cohomology is concentrated in the middle degree and enjoys both the p-torsion freeness and the liftability.

In addition to the representation-theoretic techniques (such as the dual BGG construction and procuring suitable line bundles) employed already in [42], we will need a new vanishing theorem in characteristic $p > 0$. While the ample automorphic line bundles (that we used in the compact case [42]) over the Shimura variety extend canonically to ample line bundles over the (generally singular) minimal compactification, their pullbacks to the (projective smooth) toroidal compactifications are only *nef* and *big*, but *not ample* in general, except in the very short list of special cases where the toroidal compactifications agree with the minimal compactification. (There are certainly ample line bundles on the projective toroidal compactifications, but we need the *canonical* extensions of *automorphic* line bundles.)

Dealing with this qualitative difference falls naturally in the realm of birational geometry, and is done in Section 3. It is supported by the first two sections, and these three sections can be read independently of the rest of the article.

Another complication arises from the fact that we cannot expect, in general, to have compactifications of the self-fiber products of the universal family of abelian schemes, that are *semistable* over toroidal compactifications; what we manage to get are morphisms that are log smooth and log integral. This leads to two difficulties. The first is that it makes the literal form of Illusie's decomposition theorem, let alone the Kodaira type vanishing that results from it, inapplicable in the general case. The second is that for general log smooth and log integral morphisms, the residue maps on the relative cohomology need not be nilpotent, and this interferes with the vanishing theorem.

It turns out that these two difficulties can be overcome for our families over Shimura varieties. The decomposition can be obtained by using certain properties of the families and then applying a theorem of Ogus, and the residues can be shown to be nilpotent in various ways. (As to the former, we note that Illusie had already outlined how one could obtain the decomposition in a setting more general than just the semistable case.) After these considerations, the variant in Section 3.3 will give the vanishing that is necessary for our PEL-type Shimura varieties.

Here is an outline of the article. In Section 1, we recall Illusie's decomposition theorem for the relative log de Rham cohomology of semistable morphisms and Ogus's (partial) generalization for certain F-spans; the latter will apply to the case of PEL-type Shimura varieties. In Section 2, we record some facts in logarithmic geometry that we will need for our later proofs. By adapting Esnault and Viehweg's method of cyclic coverings, we prove a vanishing statement in Section 3 that applies to the kinds of morphisms considered in the preceding sections. In Section 4, we review the notion of PEL-type Shimura varieties and automorphic bundles, as in [42], and introduce the so-called canonical extensions of automorphic bundles over toroidal compactifications. In Section 5, we explain how to geometrically realize the canonical extensions of the automorphic bundles, using results from [38]. In Section 6, we give two proofs of the nilpotence of the residues on the canonical extensions of automorphic vector bundles on toroidal compactifications of PEL-type Shimura varieties; the first uses the comparison in [37] and the analytic local charts in Ash–Mumford–Rapoport–Tai [2], and the second is purely algebraic and uses only the definition. In Sections 7–8, we apply the vanishing statement in Section 3 to obtain our main results on the (log) de Rham and Hodge cohomology of automorphic bundles. In Sections 9–10, we translate these into results on the étale and Betti cohomology, using crystalline comparison isomorphisms. (Some materials in Sections 2, 4–6, and 9 are unsurprising and can be considered reviews for experts working in this area; but since they are not readily available in the literature, we develop the theories for the convenience of the reader.)

The broad line of proof in this article is parallel to that in [42], and we will indicate the corresponding steps along the way. We repeat, as we emphasized in [42, Introduction], that all the conditions in our results are explicit and all the bounds can be easily calculated in practice.

After we sent Tilouine an earlier version of this article, we learned from him that Stroh had been working on the liftability of cuspidal Siegel modular forms. In [63], Stroh obtained a liftability theorem based on a result of Shepherd-Barron [61] (for projective toroidal compactifications of Siegel modular schemes), and deduced from it certain results on classicality in Hida theory. We note that this liftability also follows as a special case from our results in Section 8.2.

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1. Decomposition theorems

We recall two decomposition theorems in this section.

1.1. **Illusie's theorem.** Let k be a perfect field of characteristic $p > 0$, (X, D) and (Y, E) two smooth varieties with simple normal crossings divisors over k, and $f : (X, D) \longrightarrow (Y, E)$ a proper semistable morphism [27, §1]; this in particular means $D = f^{-1}(E)$ as schemes. Then the relative log (= logarithmic) de Rham cohomology sheaves $H^m(f) = R^m f_*(\omega^\bullet_{X/Y})$ (here $\omega^\bullet_{X/Y} = \Omega^\bullet_{X/Y}(\log(D/E))$; see [27, §1.3]) are coherent sheaves equipped with the Gauss–Manin connection and the Hodge filtration.

Theorem 1.1 (Illusie)**.** *Assume that the morphism f lifts to $W_2(k)$ (in the obvious sense, see [27, §2]) and that Y is of dimension $\leq e$. Then*

(1) For $m < p$, the Hodge spectral sequence

$$E_1^{ij} = R^j f_* \omega^i_{X/Y} \Rightarrow H^{i+j}(f) \tag{1.2}$$

has $E_1^{ij} = E_\infty^{ij}$ whenever $i + j < p$, and the associated graded of the filtered coherent module $H^m(f)$ is locally free and of formation compatible with arbitrary base change $Z \longrightarrow Y$. When $m + e < p$, there is an isomorphism

$$\oplus_i \operatorname{gr}^i \omega^\bullet_{Y'}(H^m(f)') \simeq F_{Y/k*}\omega^\bullet_Y(H^m(f)),$$

in the derived category $D(\mathcal{O}_{Y'})$, where Y' and $H^m(f)'$ denote the base change of Y and $H^m(f)$, respectively, by the absolute Frobenius on k.

(2) If $\dim X < p$, then (1.2) degenerates and we have an isomorphism

$$\oplus_i \operatorname{gr}^i \omega^\bullet_{Y'}(H^*(f)') \simeq F_{Y/k*}\omega^\bullet_Y(H^*(f)),$$

where $H^*(f) = \oplus_m H^m(f)$ denotes the total cohomology.

For (2), see [27, Thm. 4.7] and the paragraph that follows it. For (1), the statements on the spectral sequence and the local freeness are in [27, Cor. 2.4]. This then verifies the condition $(*)$ of [27, Thm. 4.7] for $i + j < p$, which suffices for the constructions and calculations made in [27, §§3–4] for H^m. Moreover, the condition $m + e < p$ implies that the subcomplex G_{p-1} is the whole complex.

Remark 1.3. The decomposition extends to proper, smooth, and $W_2(k)$-liftable morphisms of Cartier type between fs log schemes in characteristic $p > 0$. (See [28, Thm. 4.12].)

1.2. **Ogus's theorem.** As mentioned in the Introduction, we will later need a decomposition theorem for morphisms that are proper, log smooth, and log integral, but not necessarily semistable. The goal of this subsection is to obtain such a decomposition, with the help of Ogus's machinery in his book [51]. We manage to do so only under a rather strong set of conditions that will be satisfied in our application to Shimura varieties. We note that Illusie already outlined, in [27, Ex. 4.21], a method for proving such a decomposition for crystals that satisfy certain divisibility conditions.

Notation. Let k be a perfect field of characteristic $p > 0$, $W(k)$ the ring of Witt vectors over k, and $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$ two smooth separated schemes of finite type over $W(k)$ with relative simple normal crossings divisors. We indicate by the subscript n the reduction of objects modulo p^n, and put $(X, D) = (\mathcal{X}_1, \mathcal{D}_1)$ and $(Y, E) = (\mathcal{Y}_1, \mathcal{E}_1)$. We regard the spectra of $W(k)$, $W_n(k)$, and k as log schemes with the trivial log structures, while we equip the usual log structures on the schemes with relative simple normal crossings divisors.

We consider a proper, log smooth morphism $\mathcal{F} : (\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{Y}, \mathcal{E})$ such that the reduced divisor $\mathcal{F}^{-1}(\mathcal{E})_{\mathrm{red}}$ is equal to $\mathcal{D}$. Throughout this subsection, we assume that $\mathcal{F}$ is *integral* in the logarithmic sense [28, Def. 4.3] ("log integral" in short); we note that the last condition implies that $\mathcal{F}$ is flat.

In order to apply Ogus's theory without having $\mathcal{F}$ "perfectly smooth" (see [51, p. 24], which includes the condition that the reduction f of $\mathcal{F}$ modulo p be of Cartier type [28, Def. 4.8]), we make the following

Assumption 1.4 (good crystalline cohomology). *(1) The Hodge spectral sequence*

$$E_1^{ij} = R^j \mathcal{F}_* \omega^i_{\mathcal{X}/\mathcal{Y}} \Rightarrow R^{i+j}\mathcal{F}_*(\omega^\bullet_{\mathcal{X}/\mathcal{Y}})$$

associated with the proper, log smooth, and log integral morphism $\mathcal{F}$ degenerates, and its $E_1 = E_\infty$ terms are locally free over $\mathcal{Y}$ (a fortiori p-torsion free).

(2) *For all $m \geq 0$, the crystalline Frobenius*

$$\Phi^m : F_{Y/W}^* \mathcal{H}'^m \longrightarrow \mathcal{H}^m,$$

where $\mathcal{H}^m$ denotes the m-th relative log crystalline cohomology of $(X, D)/(Y, E)/W$ and $\mathcal{H}'^m$ denotes the base change on $\mathcal{Y}'$ of $\mathcal{H}^m$ by the Frobenius on $W = W(k)$, is a p-isogeny.

Remark 1.5. Under Assumption 1.4(1), the relative log de Rham cohomology appearing in (1) and the log crystalline cohomology appearing in (2), which correspond (in the sense of [28, Thm. 6.2]) thanks to the log crystalline Poincaré Lemma (see [28, Thm. 6.4]), are locally free. We will abuse the notation and denote both by $\mathcal{H}^*$.

Remark 1.6. A projective smooth surface $\mathcal{X}/W(k)$ can already have nontrivial p-torsion in its cohomology and violate Assumption 1.4(1).

Here's an example where (1) of Assumption 1.4 is satisfied but (2) is not. Let $\mathcal{Y} = \mathbb{A}^1_{W(k)}$ be the affine line with coordinate y, and let $\mathcal{E}$ be the divisor defined by $y = 0$. Choose an integer $n > 0$ prime to p and consider $\mathcal{X} = \mathbb{A}^1_{W(k)}$ with coordinate x and the covering $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{Y}$ given by $y = x^n$. With $\mathcal{D}$ being the divisor defined by $x = 0$, $\mathcal{F}$ satisfies Assumption 1.4(1), since the cohomology of $\mathcal{F}$ is simply the direct image $\mathcal{F}_* \mathcal{O}_{\mathcal{X}}$ of the structure sheaf, which is locally free. However, unless $n = 1$, Assumption 1.4(2) is *not* satisfied.

In our application to Shimura varieties, we will verify Assumption 1.4 by using the perfect self-duality:

Proposition 1.7. *Let $\mathcal{F} : (\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{Y}, \mathcal{E})$ be a morphism of pure relative dimension n satisfying Assumption* 1.4(1)*. Suppose that:*

(1) *The pair $(\mathcal{H}^{2n}, \Phi^{2n})$ is isomorphic to the (Tate) F-span $\mathcal{O}(-n)$ defined in* [51, Def. 5.2.1].

(2) *The canonical cup product pairing*

$$\mathcal{H}^m \times \mathcal{H}^{2n-m} \longrightarrow \mathcal{H}^{2n}$$

is a perfect duality for each $m \geq 0$.

Then $\mathcal{F}$ also satisfies Assumption 1.4(2)*.*

Proof. The crystalline Frobenius is compatible with the pairing, and is a p-isogeny on $\mathcal{H}^{2n}$ by the assumption. These two facts imply that the image of Φ^m generates $\mathcal{H}^m$ once p is inverted, and hence that Φ^m is a p-isogeny. □

Now we come to a decomposition theorem. The terms in quotation marks in the proof below are those of Ogus, for which we refer to the book [51].

Theorem 1.8 (Ogus)**.** *Let $\mathcal{F}$ be a morphism satisfying Assumption* 1.4*, and suppose that $\mathcal{Y}/W(k)$ and $\mathcal{X}/\mathcal{Y}$ are of pure relative dimension e and n, respectively. Then for each integer m satisfying $m + e < p$, there is an isomorphism*

$$\oplus_i \operatorname{gr}^i \omega^\bullet_{Y'}(H^m(f)') \simeq F_{Y/k*} \omega^\bullet_Y(H^m(f)),$$

in the derived category $D(Y')$, where $'$ denotes the base change of objects by the absolute Frobenius on k.

If, in addition, the conditions of Proposition 1.7 *are satisfied, then there is a similar isomorphism for each integer m in the range* $(2n - m) + e < p$.

Proof. We first deal with the case $m + e < p$. Put $S = \mathrm{Spf}(W(k))$ and replace $\mathcal{X}$, $\mathcal{D}$, $\mathcal{Y}$ and $\mathcal{E}$ with the formal schemes over S obtained by p-adic completion. Then, first of all, $\mathcal{H} = \mathcal{H}^m(\mathcal{F})$ defines a "T-crystal" $(\mathcal{H}, A)$ on $\mathrm{Cris}((\mathcal{Y}, \mathcal{E})/S)$. To see this, apply [51, Thm. 6.3.2] to the trivial T-crystal $\mathcal{O}_{\mathcal{X}/S}$ (with the trivial filtration) on $(\mathcal{X}, \mathcal{D})$ and $\mathcal{F}$ (and this application requires the local freeness and the degeneration, as in Assumption 1.4(1), in certain degrees).

Then we restrict it to the T-crystal $(\mathcal{H}_1, A_1)$ on $\mathrm{Cris}((Y, E)/S)$ (cf. [51, Rem. 3.2.4]). It is "uniform" in the sense of *ibid.* Def. 3.2.1. Taking the base change by the absolute Frobenius on k and its canonical lifting on W (which are automorphisms as k is perfect), we get T-crystals $(\mathcal{H}', A')$ on $\mathrm{Cris}((\mathcal{Y}', \mathcal{E}')/S)$ and $(\mathcal{H}'_1, A'_1)$ on $\mathrm{Cris}((Y', E')/S)$. The latter is "admissible" in the sense of *ibid.* Def. 5.2.4, because $(\mathcal{H}'|_{Y'}, A')$ is uniform.

Thus the functor μ in *ibid.* §5.2 applies, and gives rise to an "F_γ-span" (see *ibid.* Prop. 5.2.5) and also an admissible "F-span" (see *ibid.* Def. 5.2.9 and the ensuing remark):

$$\Psi : F^*_{Y/S}\mathcal{H}'_1 \longrightarrow \mathcal{K}$$

for some crystal $\mathcal{K}$.

On the other hand, by Assumption 1.4(2), we have a p-isogeny

$$\Phi = \Phi^m : F^*_{Y/S}\mathcal{H}'_1 \longrightarrow \mathcal{H}_1$$

between crystals, which defines another — a priori different — F-span. By Assumption 1.4(1) and *ibid.* Thm. 5.2.9, it is also admissible; in other words, there exists a filtration B'_1 on $\mathcal{H}'_1$ making the pair $(\mathcal{H}'_1, B'_1)$ a T-crystal such that $\mu(\mathcal{H}'_1, B'_1)$ is equal to Φ.

Lemma 1.9. *The two F-spans Ψ and Φ are isomorphic.*

Proof of Lemma 1.9. By the paragraph following *ibid.* Def. 5.2.9, the association of the T-crystal with an admissible F-span is fully faithful, even an equivalence for those with small width. Thus, it suffices to prove that the two filtrations A'_1 and B'_1 are equal. Showing this equality is a local problem on the base.

The filtration B'_1 is defined, as in *ibid.* §5.1, in terms of the filtration

$$M^k := \Phi^{-1}(p^k\mathcal{H}_1)$$

on $F^*_{Y/S}\mathcal{H}'_1$, while A'_1 is the (p, γ)-saturation of A'.

The morphism $\mathcal{F}$ restricts, for a trivial reason, to a "perfectly smooth" morphism on the complement $\mathcal{U}$ of $\mathcal{E}$ in $\mathcal{Y}$. On this open dense $\mathcal{U}$, we know that the restrictions of A'_1 and of B'_1 coincide: It is the theorem of Mazur and Ogus, previously a conjecture of Katz. (One can apply *ibid.* Cor. 7.5.2 to the trivial crystal, using Assumption 1.4.)

One can deduce from this that B'_1 contains A'_1; for each integer k, the image of a local section of $(A'_1)^k$ under Φ is, at the same time, a local section of (the locally free) $\mathcal{H}_1$ and divisible by p^k over $\mathcal{U}$.

To prove the inclusion in the reverse direction, it suffices to show that for every integer k, $(A'_1)^k$, and $(B'_1)^k$ have the same image in $\mathcal{H}'_1 \bmod p^s$ for all $s > k$. Suppose, towards a contradiction, that the latter contains an element t that the former does not. Then there exists an element x in the monoid of the log structure of $(\mathcal{Y}', \mathcal{E}')$ mod p^s such that $x \cdot t$ belongs to $(A'_1)^k$. In other words, the module

$$\mathcal{H}'_1/(A'_1)^k$$

contains a nonzero element $[t]$ annihilated by x. While this module is not locally free over $\mathcal{Y}'$ mod p^s, we still have:

Lemma 1.10. *The module $\mathcal{H}'_1/(A'_1)^k$ is a successive extension of modules that are locally free over $\mathcal{O}_{Y'} = \mathcal{O}_{\mathcal{Y}'}/p\mathcal{O}_{\mathcal{Y}'}$.*

Proof. As remarked above, A'_1 is the (p, γ)-saturation of A'. Because the Hodge level is smaller than p, we only need to consider the case where k is smaller than p. Thus

$$A_1'^k = A'^k + pA'^{k-1} + \cdots + p^k A'^0,$$

Since $\mathcal{H}'_1/A_1'^k$ is a successive extension of $A_1'^j/A_1'^{j+1}$ for $j < k$, we are reduced to showing that $A_1'^{k-1}/A_1'^k$ is a locally free $\mathcal{O}_{Y'}$-module (note that the formula implies that $p \cdot A_1'^{k-1} \subseteq A_1'^k$). Then we use the chain of inclusions

$$A_1'^k = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_k = A_1'^{k-1}$$

in which the modules are of the form

$$C_i = \sum_{j=0}^{k} p^{c(i,j)} A'^j$$

where $c(i, j)$ is equal to $k - j - 1$ when $j < i$ and to $k - j$ when $j \geq i$. For each $i < k$, multiplication by p^{k-i-1} maps A'^i into C_{i+1}, and induces an isomorphism

$$\mathrm{gr}^i_{A'}(\mathcal{H}') \otimes \mathbb{Z}/p \xrightarrow{\sim} C_{i+1}/C_i.$$

Thus Assumption 1.4(1) implies that C_{i+1}/C_i is locally free over $\mathcal{O}_{Y'}$, and this completes the proof of Lemma 1.10. □

Returning to the proof of Lemma 1.9, on one hand Lemma 1.10 implies that multiplication by x is injective on this module, since $x \bmod p$ is nonzero. But on the other hand, x annihilates $[t] \neq 0$ in it. This is the desired contradiction completing the proof of Lemma 1.9. □

Now we are ready to apply *ibid.* Thm. 8.2.1. We put the assumption on m in our statement so as to meet Ogus's condition on the width, and the liftability condition is satisfied because the restriction $(\mathcal{H}'_2, A'_2)$ on $\mathrm{Cris}((\mathcal{Y}'_2, \mathcal{E}'_2)/S)$ of $(\mathcal{H}', A')$ further restricts to $(\mathcal{H}'_1, A'_1)$ by transitivity. Then the formula (2) in *loc. cit.* is exactly what we want.

Finally, when the conditions of Proposition 1.7 are satisfied, we can apply *ibid.* Thm. 8.2.1 and the duality. This finishes the proof of Theorem 1.8. □

Remark 1.11. From the proof, one sees that the crystalline Frobenius $F_Y^* \mathcal{H}^m \longrightarrow \mathcal{H}^m$ corresponds to an F-T-crystal in the sense of [51, Def. 5.3.1].

Example 1.12. Start with a noncompact PEL datum, a "good" (for the PEL datum) prime number p, and a neat and prime-to-p level. Denote by $\mathcal{U}$ the Shimura variety of this level over a suitable Witt vector ring W of residue characteristic p, choose a projective smooth toroidal compactification $\mathcal{Y}$ of $\mathcal{U}$ over W and let $\mathcal{E}$ be the complementary "boundary divisor". The universal abelian scheme, as well as its n-fold fiber product (for each integer $n \geq 0$), admits a *log smooth* toroidal compactification $(\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{Y}, \mathcal{E})$, but not necessarily integral in the log sense. However, one can *choose* suitable cone decompositions (at the expense of replacing both $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{Y}, \mathcal{E})$) such that the resulting $(\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{Y}, \mathcal{E})$ *are* integral in the log sense. (See Remark 7.17 below.) One can also prove that the conditions of Proposition 1.7 (hence also Assumption 1.4) are satisfied. (See [38, Thm. 2.15 and Prop. 6.9], Lemma 7.18, and the proof of Proposition 7.21 below.)

The following can sometimes be useful for verifying the condition (1) in Proposition 1.7:

Proposition 1.13. *The following conditions on $\mathcal{F}$:*

(a) $\mathcal{F}$ has pure relative dimension n and the canonical map

$$R^n\mathcal{F}_*\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E}) \longrightarrow R^{2n}\mathcal{F}_*\Omega^\bullet_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E})$$

is an isomorphism of locally free sheaves; and

(b) the structure map $\mathcal{O}_\mathcal{Y} \longrightarrow \mathcal{F}_*\mathcal{O}_\mathcal{X}$ is an isomorphism;

imply the condition (1) *in Proposition* 1.7.

Proof. We will denote the common sheaf in (a) by $\mathcal{H}^{2n}$; when Assumption 1.4(1) (which is stronger than (a)) is also satisfied, it agrees with our previous notation $\mathcal{H}^{2n}$ in Remark 1.5. Note also that (b) implies that $\mathcal{F}$ is surjective and has geometrically connected fibers [19, III, 4.3.2 and 4.3.4].

To prove the assertion, we may assume that $\mathcal{Y}$ has pure relative dimension e over $W = W(k)$. Putting $d = n + e$, the dualizing sheaf for $\mathcal{F}$ is given by:

$$\omega = \Omega^d_{\mathcal{X}/W} \otimes \big(\mathcal{F}^*\Omega^e_{\mathcal{Y}/W}\big)^{-1}.$$

From the Koszul exact sequence

$$0 \longrightarrow \mathcal{F}^*\Omega^1_{\mathcal{Y}/W}(\log \mathcal{E}) \longrightarrow \Omega^1_{\mathcal{X}/W}(\log \mathcal{D}) \longrightarrow \Omega^1_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E}) \longrightarrow 0$$

we get the relation

$$\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E}) = \Omega^d_{\mathcal{X}/W}(\mathcal{D}) \otimes \big(\mathcal{F}^*\Omega^e_{\mathcal{Y}/W}(\mathcal{E})\big)^{-1} = \omega(\mathcal{D} - \mathcal{F}^{-1}\mathcal{E})$$

by looking at the highest exterior powers. The divisor

$$\Delta := \mathcal{F}^{-1}\mathcal{E} - \mathcal{D},$$

is effective, and we have the natural inclusion $\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E}) \subseteq \omega$. Hence

$$R^n\mathcal{F}_*\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log \mathcal{D}/\mathcal{E}) \longrightarrow R^n\mathcal{F}_*\omega. \tag{1.14}$$

Note that the target is canonically isomorphic, via the trace map, to $\mathcal{O}_\mathcal{Y}$.

Lemma 1.15. *The map* (1.14) *is an isomorphism.*

Proof. From the inclusions

$$\mathcal{O}_\mathcal{X} \subseteq \mathcal{O}_\mathcal{X}(\Delta) \subseteq \mathcal{O}_\mathcal{X}(\mathcal{F}^{-1}\mathcal{E})$$

we get

$$\mathcal{F}_*\mathcal{O}_\mathcal{X} \subseteq \mathcal{F}_*\big(\mathcal{O}_\mathcal{X}(\Delta)\big) \subseteq \mathcal{F}_*\mathcal{O}_\mathcal{X}(\mathcal{F}^{-1}\mathcal{E}). \tag{1.16}$$

Thanks to (b), the first group identifies itself with $\mathcal{O}_\mathcal{Y}$ via the structure map, and the third with $\mathcal{O}_\mathcal{Y}(\mathcal{E})$ by the projection formula. Thus a local section of the sheaf in the middle of (1.16) is a rational function g on $\mathcal{Y}$ with at most simple poles along the branches of $\mathcal{E}$. However, if g did have a simple pole, then it would not belong to the middle sheaf, for its pullback to $\mathcal{X}$ would have a worse pole than is allowed by Δ. Therefore, the first inclusion is in fact an equality.

Grothendieck duality (see [25, VII 3.3 or Appendix]) provides us with a canonical isomorphism

$$R\mathcal{F}_*R\underline{\mathrm{Hom}}_{\mathcal{X}}(L,\omega[n]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathcal{Y}}(R\mathcal{F}_*L,\mathcal{O}_{\mathcal{Y}}),$$

functorial in L. Applying it to (1.14) and taking the cohomology sheaf in degree $-n$, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_*\mathcal{O}_{\mathcal{X}} \arrow[r, "\sim"] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{Y}}(R^n\mathcal{F}_*\omega,\mathcal{O}_{\mathcal{Y}}) \arrow[d] \\
\mathcal{F}_*\mathcal{O}_{\mathcal{X}}(\Delta) \arrow[r, "\sim"] & \underline{\mathrm{Hom}}_{\mathcal{Y}}(R^n\mathcal{F}_*\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log\mathcal{D}/\mathcal{E}),\mathcal{O}_{\mathcal{Y}}).
\end{tikzcd}$$

As $R^n\mathcal{F}_*\Omega^n_{\mathcal{X}/\mathcal{Y}}(\log\mathcal{D}/\mathcal{E})$ is assumed to be locally free in (a), the fact that the left vertical map is an isomorphism implies that (1.14) is also an isomorphism. This proves Lemma 1.15. $\square$

Now let $\gamma=\gamma_{\mathcal{Y}}$ (resp. $\gamma_{\mathcal{U}}$, resp. $\gamma'=\gamma_{\mathcal{Y}'}$, resp. $\gamma_{\mathcal{U}'}$) be the canonical generator of $\mathcal{H}^{2n}$ (resp. $\mathcal{H}^{2n}|_{\mathcal{U}}$, resp. $\mathcal{H}'^{2n}$, resp. $\mathcal{H}'^{2n}|_{\mathcal{U}'}$) mapping to the section 1 of $\mathcal{O}_{\mathcal{Y}}$ (resp. $\mathcal{O}_{\mathcal{U}}$, resp. $\mathcal{O}_{\mathcal{Y}'}$, resp. $\mathcal{O}_{\mathcal{U}'}$) under the composite of (1.14) and the trace map (resp. the analogous maps). Then γ (resp. γ') restricts to $\gamma_{\mathcal{U}}$ (resp. $\gamma_{\mathcal{U}'}$).

One knows that, as a coherent module with integrable connection with log poles on $(\mathcal{Y},\mathcal{E})$, the sheaf $\mathcal{H}^{2n}$ with the Gauss–Manin connection is isomorphic to the structure sheaf with the exterior differential. This can be seen from the fact that γ_U, and hence γ, are annihilated by the connection.

On the other hand, Φ maps $\gamma_{\mathcal{U}'}$ to $p^n\gamma_{\mathcal{U}}$ over $\mathcal{U}$, since the restriction of $\mathcal{F}$ over $\mathcal{U}$ is proper and smooth (the log structures being trivial) of pure relative dimension n. This implies that γ' is mapped to $p^n\gamma$, and proves Proposition 1.13. $\square$

2. A little log geometry

First, we record some facts from logarithmic geometry that will be used in proving the main theorems. Then we prove the nilpotence of the residue map in semistable families by adapting the method of Katz; this does not seem to be recorded in the literature.

Throughout this section, (X,D) and (Y,E) will denote smooth varieties over a field k with simple normal crossings divisors, and $f:X\longrightarrow Y$ a proper and log smooth (but not necessarily log integral) morphism. We will denote by $H^m(f)$ the relative log de Rham cohomology $R^mf_*\Omega^\bullet_{X/Y}(\log(D/E))$ equipped with the Gauss–Manin connection ∇ with log poles along E and the Hodge filtration F.

Let (M,∇) be a coherent module with an integrable connection on (Y,E) with log poles along E. For each branch E_i of E, we denote by $\mathrm{Res}_{E_i}(M,\nabla)$ the residue map.

2.1. **Log smoothness and Bertini's hyperplanes.** We will later employ inductions on dimension by choosing "good" hyperplane sections. For this, we need to know that any given log smooth morphism between smooth quasi-projective varieties with simple normal crossings divisors *restricts* to one of the same type over a *general* hyperplane.

Recall that an effective divisor D on a smooth variety X is called a *simple normal crossings divisor* if D is the union of its smooth, connected branches $(D_i)_{i\in I}$ that meet transversally. This means that for every closed point x of X, there is an open neighborhood U of x and, for each index i in the set

$$I_x := \{i \in I : D_i \text{ passes through } x\},$$

a section x_i of $\mathcal{O}_X$ over U which defines $D_i \cap U$, such that the sequence $(x_i)_{i\in I_x}$ is a part of a regular sequence in the maximal ideal $\mathbf{m}_x$ of the Zariski local ring $\mathcal{O}_{X,x}$. This is equivalent to saying that dx_i form a part of a basis of the free $\mathcal{O}_{X,x}$-module $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$, or still that the images $[dx_i]$ form a part of a basis of the free $k(x)$-module $\Omega^1_{X/k} \otimes_{\mathcal{O}_X} k(x)$, where $k(x) = \mathcal{O}_{X,x}/\mathbf{m}_x$ and $[\,\cdot\,]$ denotes reduction modulo $\mathbf{m}_x$. The last formulation follows easily from the fact that $\mathcal{O}_{X,x}$ is local and $\Omega^1_{X/k}$ is locally free.

A simple normal crossings divisor D defines the prototypical log structure on X (see [28]), and one forms the sheaf $\Omega^1_{X/k}(\log D)$ of Kähler differentials with log poles along D (see [11], [29], or [28]); locally, it is generated by $\Omega^1_{X/k}$ and $d\log x_i = (1/x_i)dx_i$ as an $\mathcal{O}_X$-module. One naturally associates with the divisor D a stratification of X into locally closed, smooth subvarieties, by taking intersections of the branches and their complements.

Definition 2.1. *Let D be a simple normal crossings divisor on a smooth variety X. We say that an effective divisor H on X* **meets** D **transversally***, or that H* **is transversal to** D*, if H is smooth, H does not coincide with any branch of D, and $D \cup H$ is again a simple normal crossings divisor on H. This is the same as requiring that H should not be a branch of D and that H should properly intersect each stratum defined by D in smooth subschemes.*

Proposition 2.2. *Let D be a simple normal crossings divisor on a smooth variety X/k, and let H be an effective divisor on X. Then H meets D transversally if and only if, for every closed point x in H and every local equation $h \in \mathcal{O}_{X,x}$ of H, the image of $[dh]$ in $\Omega^1_{X/k}(\log D) \otimes_{\mathcal{O}_X} k(x)$ is nonzero.*

Proof. Write $M = \Omega^1_{X/k}(\log D) \otimes \mathcal{O}_{X,x}$, $M' = \Omega^1_{X/k} \otimes \mathcal{O}_{X,x}$, $M(x) = M \otimes k(x)$, and $M'(x) = M' \otimes k(x)$. Renaming the branches of D if necessary, let $(D_i)_{i=1,\cdots,\ell}$ pass through x, with local equations $(x_i)_{i=1,\cdots,\ell}$, and let $(y_j)_{j=1,\cdots,r}$ complement the (x_i) into a coordinate system. We have a commutative diagram

$$\begin{tikzcd}
M' \arrow[r] \arrow[d] & M'(x) \arrow[r, equal] \arrow[d] & \left(\oplus_{i=1}^{\ell} k(x)[dx_i]\right) \bigoplus \left(\oplus_{j=1}^{r} k(x)[dy_j]\right) =: V_1' \bigoplus V_2 \arrow[d, "0\oplus 1"'] \\
M \arrow[r] & M(x) \arrow[r, equal] & \left(\oplus_{i=1}^{\ell} k(x)[d\log x_i]\right) \bigoplus \left(\oplus_{j=1}^{r} k(x)[dy_j]\right) =: V_1 \bigoplus V_2.
\end{tikzcd}$$

where the vertical map on the left is the natural inclusion. By the last of the equivalent formulations in the definition of a simple normal crossings divisor, if H meets D transversally, then $[dh]$ in $M'(x)$ does not belong to V_1', so $[dh]$ in $M(x)$ is nonzero. Conversely, if $[dh]$ in $M(x)$ is nonzero, then (i) $[dh]$ is nonzero in $M'(x)$, hence H is smooth at x; and (ii) $[dh]$ cannot belong to V_1', so $D \cup H$ defines a simple normal crossings divisor at x. With varying $x \in H$, the proof is complete. $\square$

Proposition 2.3. *Let $f : (X, D) \longrightarrow (Y, E)$ be a log smooth morphism between smooth k-varieties with simple normal crossings divisors. Let H be an effective divisor on Y that meets E transversally. Then the scheme-theoretic inverse image $f^{-1}(H)$ meets D transversally, and the restriction $f|_H : (X_H, D_H) \longrightarrow (H, E_H)$ is log smooth, where $X_H = f^{-1}(H)$, $D_H = D \cap f^{-1}(H)$, and $E_H = E \cap H$. Moreover, if f is integral in the log sense* [28, Def. 4.3] (*resp. if f is semistable* [27, Section 1]), *then so is $f|_H$.*

Proof. Let x be a closed point of X contained in $f^{-1}(H)$, and put $y = f(x)$. Choose a local equation $h \in \mathcal{O}_{Y,y}$ of H, and $f^{-1}(H)$ is locally defined by $f^*(h)$ in $\mathcal{O}_{X,x}$. By one half of [28, Prop. 3.12], $f^*\Omega^1_{Y/k}(\log E)$ is locally a direct summand of $\Omega^1_{X/k}(\log D)$, and as such the resulting map

$$f^*\Omega^1_{Y/k}(\log E) \otimes_{\mathcal{O}_X} k(x) \longrightarrow \Omega^1_{X/k}(\log D) \otimes_{\mathcal{O}_X} k(x)$$

is injective. So $[dh] \neq 0$ in the source implies $f^*[dh] = [df^*(h)] \neq 0$ in the target, and $f^{-1}(H)$ meets D transversally by Proposition 2.2.

To see that $f|_H$ is log smooth, use the commutative diagram

(2.4)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & (f|_H)^*\check{\mathcal{N}}_{H/Y} \arrow[r] \arrow[d, equal] & (f^*\omega^1_Y)|_{X_H} \arrow[r] \arrow[d] & (f|_H)^*\omega^1_H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{\mathcal{N}}_{X_H/H} \arrow[r] & (\omega^1_X)|_{X_H} \arrow[r] \arrow[d] & \omega^1_{X_H} \arrow[r] \arrow[d] & 0 \\
 & & (\omega^1_{X/Y})|_{X_H} \arrow[r] \arrow[d] & \omega^1_{X_H/H} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd},$$

where we suppressed $/k$ and used the notation ω for log differentials and $\check{\mathcal{N}}$ for conormal bundles, respectively. We know the two rows and the middle column are exact and (as all the terms are locally free) locally split. It then follows from the other half of [28, Prop. 3.12] that $f|_H$ is log smooth.

The statement about $f|_H$ being integral in the log sense (resp. semistable) follows easily from the local description of the sheaf of monoids of the log structures associated with the divisors. $\square$

Corollary 2.5. *Let $f : (X, D) \longrightarrow (Y, E)$ be as in Proposition* 2.3 *and L a very ample invertible sheaf on Y. Then a general hyperplane section H in the linear system $|L|$ satisfies the conclusions of Proposition* 2.3*; in particular, such H exists if k is infinite.*

Proof. In view of Proposition 2.3, the statement follows directly from the usual form of Bertini's theorem. $\square$

2.2. **Semistable morphisms: Katz's theorem.** In this subsection, we assume our $f : (X, D) \longrightarrow (Y, E)$ satisfies $f^{-1}(E) \subseteq D$ as *schemes*. This is satisfied for instance if f is semistable.

Given a coherent module with an integrable connection (M, ∇) on a smooth k-variety X with log poles along D and a monic polynomial $P(T) \in k[T]$, we will say that $\mathrm{Res}_{D_i}(M, \nabla)$ is a root of $P(T)$ if $P(\mathrm{Res}_{D_i}(M, \nabla))$, as an endomorphism of $M|_{D_i}$, is zero. The residue map on $(\mathcal{O}_X, d)$ along any branch is zero, and hence is a root of the polynomial T.

Theorem 2.6 (Katz). *Let f be as above, (M, ∇) a coherent module with an integrable connection on X with log poles along D, and E_1 a branch of E. Write $f^{-1}(E_1) = \sum_{i \in I_1} D_i \leq D$ and suppose that for each $i \in I_1$, $\mathrm{Res}_{D_i}(M, \nabla)$ is a root of a monic polynomial $P_i(T)$. Let $H = H^m(f, M, \nabla)$ be the m-th cohomology of the relative log de Rham complex attached to (M, ∇), equipped with the Gauss–Manin connection ∇_H. Then $\mathrm{Res}_{E_1}(H, \nabla_H)$ is a root of $Q(T)^{m+1}$, where*

$$Q(T) = \prod_{i \in I_1} P_i(T).$$

In particular, if (M, ∇) has nilpotent residues, so does the relative log de Rham cohomology of (M, ∇).

Proof. This is essentially [30, VII], minimally modified. The question being local on Y, we may assume E_1 is defined by a global section t of $\mathcal{O}_Y$ that can be completed to an affine coordinate system. We may also assume that X is of pure dimension n over k. Then Res_{E_1} is the reduction modulo t of the composite map

$$R : H \xrightarrow{\nabla_H} \Omega^1_Y(\log E) \otimes H \xrightarrow{t\frac{d}{dt} \otimes 1} H$$

and it suffices to show that $Q(R)^{m+1} H \subseteq tH$.

Choose an affine open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X such that: on each U_α, one has an affine coordinate system $x_1^{(\alpha)}, \cdots, x_n^{(\alpha)}$ and for each $i \in I_1$, $D_i \cap U_\alpha$ either is empty or is defined by $x^{(\alpha)}_{j(i)} = 0$ for some index $j(i)$. We have $f^*(t) = \prod_{i \in I_1} (x_i^{(\alpha)})^{e_i}$, where e_i is 0 or 1.

The cohomology of the total complex associated with the Čech bicomplex with terms $C^{a,b} = C^a(\mathcal{U}, \Omega^b_{X/Y}(\log(D/E)) \otimes_{\mathcal{O}_X} M)$ calculates H in degree m. One constructs (see [31] and [29]) an operator σ acting on the bicomplex and inducing the action of R upon passage to cohomology; one chooses a log derivation d on (X, D) extending $t\frac{d}{dt}$ and constructs two maps, of bidegree $(0, 0)$ and $(1, -1)$ respectively, and $\sigma = \sigma(d)$ is the sum of the two. Let F_Z denote the "Zariski" filtration F_Z^a defined by the first (i.e. Čech) degree on the bicomplex, and we use the same notation for the induced filtration on the total complex and the cohomology; we have $F_Z^a(H) = 0$ for $a > m$. Then the effect of σ on the graded pieces associated with F_Z is just the Lie derivative of $\nabla(d)$.

To prove $Q(R)^{m+1} H \subseteq tH$, it suffices to show that $Q(R)(tF_Z^a(H)) \subseteq t(F_Z^a(H))$ and $Q(R)(F_Z^a(H)) \subseteq tF_Z^a(H) + F_Z^{a+1}(H)$. For this, we choose suitable extensions of the derivation $t\frac{d}{dt}$ and use the resulting σ that realize R on the level of the Čech complex as above. The first follows easily from the product rule: $\sigma(hc) = (t\frac{d}{dt})h \cdot c + h \cdot \sigma(c)$ for $h \in \mathcal{O}_Y$ and $c \in C^{a,b}$, for any such σ. For the second, one proceeds just as in [30, VII]. Note that all the coefficients are 1. $\square$

Corollary 2.7. *With f as above, the relative log de Rham cohomology of f with trivial coefficients has nilpotent residues.*

3. A vanishing theorem

We first place our vanishing theorem for semistable morphisms in the natural context, by reviewing previously known theorems. After giving the proof in the second subsection, we give a variant — that will be applied later in the article — in the third. Then we record a corollary regarding the higher direct images of the canonical bundle.

3.1. **Review of vanishing theorems and the statement.** We start with the most classical:

Theorem 3.1 (Kodaira–Akizuki–Nakano)**.** *Let X be a projective smooth variety of pure dimension d over a field k of characteristic zero. For every ample line bundle L on X, we have*

$$H^i(X, \Omega^j_X \otimes L) = 0 \quad \text{for } i + j > d, \tag{3.2}$$

$$H^i(X, \Omega^j_X \otimes L^{\otimes -1}) = 0 \quad \text{for } i + j < d. \tag{3.3}$$

The two statements are equivalent by Serre duality.

When k has positive characteristic, (3.2) and (3.3) are known to be false; Raynaud constructed a counterexample [56]. However, under certain hypotheses, we still get the vanishing [12, Thm. 2.8]:

Theorem 3.4 (Deligne–Illusie–Raynaud)**.** *Let X be a projective smooth variety of pure dimension d over a perfect field k of characteristic $p > 0$, and assume that X lifts to $W_2(k)$ and that $\dim X \leq p$. Then for every ample line bundle L on X,* (3.2) *and* (3.3) *still hold.*

Without the assumption $\dim X \leq p$, one still has partial vanishing. When X is a surface, one may just assume that L is numerically effective with positive self-intersection. From this theorem and the technique of spreading out and reducing modulo p, the first purely algebraic proof of Theorem 3.1 was obtained.

There have been two generalizations of Theorem 3.4. In one direction, one wishes to relax the condition of L being ample to that of L being (only) nef and big. Using their method of integrable connections on line bundles associated with cyclic coverings, Esnault and Viehweg proved [14, Prop. 11.5]:

Theorem 3.5 (Esnault–Viehweg)**.** *Let X be a projective smooth variety over a perfect field k of characteristic $p > 0$, D a simple normal crossings divisor on X, and L a line bundle on X. Assume that X has pure dimension $d \leq p$, that (X, D) lifts to $(\tilde{X}, \tilde{D})$ over $W_2(k)$, that L lifts to $\tilde{L}$ on $\tilde{X}$, and that there exist an integer $\nu_0 > 0$ and an effective divisor D' supported on D such that*

$$L^{\otimes(\nu+\nu_0)}(-D') \text{ is ample for every integer } \nu \geq 0. \tag{3.6}$$

Then we have

$$H^i(X, \Omega^j_X(\log D) \otimes L^{\otimes -1}) = 0 \quad \text{for } i + j < d, \text{ and}$$
$$H^i(X, \Omega^j_X(\log D) \otimes L(-D)) = 0 \quad \text{for } i + j > d.$$

We note that the condition (3.6) implies that L is nef and big (see [14, Rem. 11.6(a)]) and that we need L to be liftable to $\tilde{X}$, unlike in Theorem 3.4. Theorem 3.5 can be regarded as a version of the Grauert–Riemenschneider vanishing theorem, modulo the embedded resolution of singularities over $W_2(k)$ (see [14, Rem. 11.6]). This overlaps with Theorem 3.4 when X is a surface.

In another direction, one wishes to allow more general coefficients in Theorem 3.4. Thus, Illusie [27] considered coefficients in the relative log de Rham cohomology of semistable morphisms $f : (X, D) \longrightarrow (Y, E)$. As a consequence of his decomposition theorem generalizing that of [12], he obtained [27, Cor. 4.16]:

Theorem 3.7 (Illusie)**.** *Let k be a perfect field of characteristic $p > 0$, (X, D) and (Y, E) proper smooth varieties over k with simple normal crossings divisors, and $f : (X, D) \longrightarrow (Y, E)$ a semistable morphism, with $D = f^{-1}(E)$. Assume that f lifts to $\tilde{f} : (\tilde{X}, \tilde{D}) \longrightarrow (\tilde{Y}, \tilde{E})$ over $W_2(k)$ and that Y has pure dimension $e \geq 0$.*

Suppose that $\dim X < p$. *Then for every ample line bundle L on Y,*

$$H^q(Y, L \otimes \mathrm{gr}^* \omega_Y^\bullet(H)) = 0 \ \text{ for } q > e, \tag{3.8}$$

$$H^q(Y, L^{\otimes -1}(-E) \otimes \mathrm{gr}^* \omega_Y^\bullet(H)) = 0 \ \text{ for } q < e, \tag{3.9}$$

where $H = H^*(f)$ denotes the relative log de Rham cohomology of f and gr *refers to the Hodge filtration.*

We prove a kind of common generalization of Theorems 3.5 and 3.7.

Theorem 3.10. *Keep the notation and the assumptions of the first paragraph of Theorem* 3.7*, and assume in addition that f has pure relative dimension $n \geq 0$.*

Let L be a line bundle on Y which lifts to $\tilde{L}$ on $\tilde{Y}$, and assume there exist an integer $\nu_0 > 0$ and an effective divisor E' supported on E satisfying the condition (3.6) (*with E' in place of D'*).

(1) For every integer $m < p - e$, we have

$$H^q\left(Y, L^{\otimes -1} \otimes \mathrm{gr}^* \omega_Y^\bullet(H^m(f))\right) = 0 \ \text{ for } q < e, \tag{3.11}$$

$$H^q\left(Y, L(-E) \otimes \mathrm{gr}^* \omega_Y^\bullet(H^{2n-m}(f))\right) = 0 \ \text{ for } q > e, \tag{3.12}$$

where $H^m(f)$ denotes the m-th relative log de Rham cohomology of f.

(2) Suppose $\dim X < p$*. Then we have*

$$H^q\left(Y, L^{\otimes -1} \otimes \mathrm{gr}^* \omega_Y^\bullet(H^*(f))\right) = 0 \ \text{ for } q < e, \tag{3.13}$$

$$H^q\left(Y, L(-E) \otimes \mathrm{gr}^* \omega_Y^\bullet(H^*(f))\right) = 0 \ \text{ for } q > e, \tag{3.14}$$

where $H^*(f) = \oplus_m H^m(f)$ denotes the total cohomology of f.

The proof will be given in the next section. It will be mostly an adaptation of the method of Esnault and Viehweg, with the theorems of Illusie and of Katz that we recalled in the previous two sections as additional ingredients.

Remark 3.15. Given Theorems 3.5, 3.7, and 3.10, it is natural to wonder whether (3.8) and (3.9) may be true under the conditions of Theorem 3.10(2). They are not, already when f is an identity morphism. In the rest of this subsection (Section 3.1), we give the simplest example that we can make.

Let k be an algebraically closed field of characteristic $p \geq 0$. Start with $X_0 = \mathbb{P}_k^2$, choose two distinct points P_1 and P_2 on X_0, and let C_0 be the line joining the two. Put $Z_0 = \{P_1, P_2\}$ and let X be the blow-up of X_0 along Z_0, with the structure morphism $\pi : X \longrightarrow X_0$ and the exceptional divisor $E = E_1 + E_2$ (E_i lying over P_i for $i = 1, 2$). We let C be the proper transform of C_0, so that π restricts to an isomorphism of C onto C_0. We denote by $H = \pi^*(\ell)$ the pullback of the hyperplane class on X_0, so that C belongs to the linear system $|H - E|$.

Let L be the line bundle $\mathcal{O}(2H - E) \simeq \mathcal{O}(H + C)$. The restriction $L|_C$ is trivial (being of degree zero on $C \simeq \mathbb{P}^1_k$) and $L(-C) \simeq \mathcal{O}(H)$ is base point free. Therefore L is numerically effective. The next lemma then shows that the pair (L, C) satisfies the condition (3.6).

Lemma 3.16. *For $r \geq 2$, $L^{\otimes r}(-C) = L^{\otimes(r-1)}(H) \simeq \mathcal{O}_X\big((2r-1)H - (r-1)E\big)$ is ample.*

Proof. We use the Nakai–Moishezon criterion. The divisor in the parentheses has self-intersection

$$(2r-1)^2(H \cdot H) + (r-1)^2(E \cdot E) = (2r-1)^2 - 2(r-1)^2 = 2r^2 - 1 > 0$$

whenever $r \geq 1$. Let Y be an irreducible divisor on X. If $Y \subseteq E$, then $H \cdot Y = 0$ and Y intersects the divisor positively whenever $r \geq 2$. Otherwise, $L^{\otimes(r-1)}$ is nef, and by the projection formula, we have $H \cdot Y > 0$. $\square$

Note that when $p > 0$, everything lifts to $W_2(k)$ and $p \geq 2 = \dim X$, therefore Theorem 3.5 of Esnault and Viehweg applies. When $p = 0$, we can apply the Kawamata–Viehweg vanishing theorem to the same effect.

Proposition 3.17. *With the notation as above, $H^1(X, \Omega^2_X(\log C) \otimes L) \neq 0$.*

Proof. By the adjunction formula, we have an exact sequence

$$0 \longrightarrow \Omega^2_X \otimes L \longrightarrow \Omega^2_X(C) \otimes L \longrightarrow \Omega^1_C \otimes L \longrightarrow 0.$$

Either by the Esnault–Viehweg theorem or by the Kawamata–Viehweg theorem, we have $H^i(X, \Omega^2_X \otimes L) = 0$ for $i = 1, 2$. From the long exact sequence of cohomology we get

$$H^1(X, \Omega^2_X(C) \otimes L) \simeq H^1(C, \Omega^1_C \otimes L).$$

But by construction, $L|_C \simeq \mathcal{O}_C$ and the last group is one dimensional over k. $\square$

3.2. **Proof.** Just as in the proof of [27, Cor. 4.16], (3.12) and (3.14) can be deduced from (3.11) and (3.13), respectively, by Serre duality. So we focus on (3.11) and (3.13). For the rest of this subsection, we write $H = H^m(f)$ in the case (1) and $H = H^*(f)$ in the case (2).

By extending scalars if necessary, we may and will assume that k is algebraically closed. In the case (2), there is nothing to prove when $m \notin [0, 2n]$ or when $X = \varnothing$, so we assume $m \in [0, 2n]$ and $X \neq \varnothing$. We also assume that Y is connected and that f is surjective; this implies, in either of the cases (1) or (2), that $e < p$.

First we settle the case when L is *ample*.

Lemma 3.18 (cf. Ogus, [51, Thm. 8.2.8])**.** *Keep the notation and the assumptions of the first paragraph of Theorem* 3.10*, and let L be an ample line bundle on X (without assuming the existence of $\tilde{L}$). Then* (3.11) *and* (3.13) *are true.*

Proof of Lemma 3.18*.* We proceed in the same way as Illusie proved (3.8) and (3.9), which are [27, (4.16.1) and (4.16.2)]. We use [27, (4.16.3)] with $M = L^{\otimes -1}$; the proof of the inequality only uses the decomposition of the de Rham complex (provided by Illusie's Theorem 1.1), and does not depend on whether M is ample or anti-ample. Then, as in *loc. cit.*, one concludes by applying Serre's vanishing theorem for ample line bundles. We note here that the statements are different from those of Illusie's when E is nonempty, and that they were considered by Ogus in the case of certain crystals. □

The proof of the general case closely follows that of Esnault and Viehweg [14, §§10–11]. Write the divisors

$$\tilde{E} = \sum_{\ell=1}^{r} \tilde{E}_\ell \quad \text{and} \quad E = \sum_{\ell=1}^{r} E_\ell$$

into sums of their smooth branches, and similarly

$$\tilde{E}' = \sum_{\ell=1}^{r} c_\ell \tilde{E}_\ell \quad \text{and} \quad E' = \sum_{\ell=1}^{r} c_\ell E_\ell, \tag{3.19}$$

for integers $c_\ell \geq 0$.

We will use induction on $e = \dim Y$, and for this we need to choose a suitable hyperplane section Z of Y. We recall how Esnault and Viehweg proceeded, on [14, p. 130]. As $L^{\otimes \nu_0}(-E')$ is ample, we can choose an integer $\mu_0 > 0$ such that $L^{\otimes \mu_0 \nu_0}(-\mu_0 E')$ is very ample. Enlarging μ_0 further if necessary, we will also assume that both

$$L^{\otimes \mu_0 \nu_0}(-\mu_0 E') \otimes K_Y^{\otimes -1} \quad \text{and} \quad L^{\otimes \mu_0 \nu_0}(-\mu_0 E' + E'_{\mathrm{red}}) \otimes K_Y^{\otimes -1}$$

are ample, where K_Y denotes the canonical bundle of Y/k. Apply [14, Cor. 11.4], and it follows that for all $\nu \geq 0$, both

$$L^{\otimes(\nu + \mu_0 \nu_0 (e+3))}(-\mu_0(e+3)E') \quad \text{and} \quad L^{\otimes(\nu + \mu_0 \nu_0 (e+3))}(-\mu_0(e+3)E' + E'_{\mathrm{red}})$$

are very ample. Putting $E'_{\mathrm{new}} = \mu_0(e+3)E'$ and $\nu_{0,\mathrm{new}} = \mu_0 \nu_0 (e+3)$, we have (the condition $(**)$ of Esnault and Viehweg):

$$\forall \nu \geq 0,\ L^{\otimes(\nu + \nu_{0,\mathrm{new}})}(-E'_{\mathrm{new}}) \quad \text{and} \quad L^{\otimes(\nu + \nu_{0,\mathrm{new}})}(-E'_{\mathrm{new}} + E'_{\mathrm{red}}) \quad \text{are very ample.}$$

Note that E' and E'_{new} have the same support (so $E'_{\mathrm{red}} = (E'_{\mathrm{new}})_{\mathrm{red}}$).

Lemma 3.20. *For all $\nu \geq 0$, the line bundle $L^{\otimes(\nu + \nu_{0,\mathrm{new}})}(-E'_{\mathrm{new}}) \otimes K_Y^{\otimes -1}$ is ample and for every $j > 0$, we have*

$$H^j\big(Y, L^{\otimes(\nu + \nu_{0,\mathrm{new}})}(-E'_{\mathrm{new}})\big) = 0.$$

Proof of Lemma 3.20. Write $L^{\otimes(\nu + \nu_{0,\mathrm{new}})}(-E'_{\mathrm{new}}) \otimes K_Y^{\otimes -1}$ as

$$\big[L^{\otimes \mu_0 \nu_0}(-\mu_0 E') \otimes K_Y^{\otimes -1}\big] \otimes \big[L^{\otimes(\nu+\nu_0)}(-E')\big] \otimes \big[\big(L^{\otimes \nu_0}(-E')\big)^{\otimes(\mu_0(e+2)-1)}\big];$$

the first of the three in the brackets is ample by the choice of μ_0 and the other two are ample by the condition (3.6). The cohomology vanishing then follows from Theorem 3.4 (note that $\dim Y = e < p$). □

Now replace E' with E'_{new} (and reset the coefficients c_ℓ in (3.19)) and ν_0 with $\nu_{0,\mathrm{new}}$. Then Lemma 3.20 implies that *any* global section of $L^{\otimes(\nu+\nu_0)}(-E')$ lifts to a global section of $\tilde{L}^{\otimes(\nu+\nu_0)}(-\tilde{E}')$; just look at the long exact sequence attached to the short exact sequence

$$0 \longrightarrow L^{\otimes(\nu+\nu_0)}(-E') \xrightarrow{\times p} \tilde{L}^{\otimes(\nu+\nu_0)}(-\tilde{E}') \xrightarrow{\bmod p} L^{\otimes(\nu+\nu_0)}(-E') \longrightarrow 0.$$

Still tracing the steps of Esnault and Viehweg, we choose an integer $\eta > 0$ such that $N := p^\eta + 1 > \nu_0$ and $N > c_\ell$ for all $\ell = 1, \cdots, r$, so that the integral part $[\frac{E'}{N}]$ is zero. By applying the facts we gathered in Section 2.1, we can arrange the following: Take a general section of $L^{\otimes N}(-E')$ corresponding to a smooth hypersurface Z, and choose a closed subscheme $\tilde{Z}$ of $\tilde{X}$ that lifts Z and is flat (equivalently, smooth) over $W_2(k)$. We may assume that $\tilde{E} + \tilde{Z}$ and $\tilde{D} + \tilde{f}^{-1}(\tilde{Z})$ are still relative simple normal crossings divisors, in $\tilde{X}$ and $\tilde{Y}$ respectively. Put $\tilde{X_Z} := \tilde{X} \times_{\tilde{Y}} \tilde{Z}$ and $\tilde{D_Z} := \tilde{D} \times_{\tilde{Y}} \tilde{Z}$, so that $\tilde{D_Z}$ is a relative simple normal crossings divisor in $\tilde{X_Z}$. The base change (restriction) of $\tilde{f}$ to $\tilde{Z}$

$$\tilde{f_Z} : (\tilde{X_Z}, \tilde{D_Z}) \longrightarrow (\tilde{Z}, \tilde{E} \cap \tilde{Z}),$$

is semistable. From the étale local description of $\tilde{f}$ in charts, one sees that there is a canonical isomorphism

$$\Omega^\bullet_{\tilde{X}/\tilde{Y}}(\log \tilde{D}/\tilde{E}) \otimes_{\mathcal{O}_{\tilde{Y}}} \mathcal{O}_{\tilde{Z}} \simeq \Omega^\bullet_{\tilde{X_Z}/\tilde{Z}}(\log \tilde{D_Z}/(\tilde{E} \cap \tilde{Z}))$$

of the relative log de Rham complexes. The same statements remain true in reduction modulo p, and the relative cohomology $H^m(f_Z)$ in the case (1) (resp. $H^*(f_Z)$ in the case (2)) is isomorphic to $H^m(f) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z$ (resp. $H^*(f) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z$), since the formation of H is compatible with any base change (see Theorem 1.1).

By the very construction, the relative de Rham complex of $f : (X, D) \longrightarrow (Y, E)$ is naturally isomorphic to that of $f : \big(X, D + f^{-1}(Z)\big) \longrightarrow (Y, E + Z)$. Therefore the relative cohomology sheaves are canonically isomorphic, degree by degree.

These three associated de Rham complexes fit in an exact sequence

$$0 \longrightarrow \Omega^\bullet_Y(\log E) \otimes H \longrightarrow \Omega^\bullet_Y(\log(E+Z)) \otimes H \longrightarrow \Omega^\bullet_Z(\log(E \cap Z))[-1] \otimes H \longrightarrow 0.$$

As the morphisms preserve the Hodge filtration, we get an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{gr}^i\,(\Omega^\bullet_Y(\log E) \otimes H) &\longrightarrow \mathrm{gr}^i\,(\Omega^\bullet_Y(\log(E+Z)) \otimes H) \\ &\longrightarrow \big(\mathrm{gr}^{i-1}\,(\Omega^\bullet_Z(\log(E \cap Z)) \otimes H)\big)\,[-1] \longrightarrow 0, \end{aligned}$$

hence the following fragment in the long exact sequence of cohomology:

$$\begin{aligned} &H^{q-2}\big(Z, (L|_Z)^{\otimes -1} \otimes \mathrm{gr}^{i-1}\,(\Omega^\bullet_Z(\log(E \cap Z)) \otimes H)\big) \\ \longrightarrow\ &H^q\big(Y, L^{\otimes -1} \otimes \mathrm{gr}^i\,(\Omega^\bullet_Y(\log E) \otimes H)\big) \\ \longrightarrow\ &H^q\big(Y, L^{\otimes -1} \otimes \mathrm{gr}^i\,(\Omega^\bullet_Y(\log(E+Z)) \otimes H)\big) \end{aligned}$$

By induction, we may apply the theorem to $(Z, E \cap Z, f_Z, L|_Z)$, which implies that the first term is zero whenever $q < e$. Thus, once we prove that the third term is zero, the proof of Theorem 3.10 will be complete.

We adapt the method of cyclic coverings of Esnault and Viehweg below, and for this let us recall their construction. For each integer a, put

$$L^{(a)} = L^{\otimes a}\Big(-\Big[\frac{a(E'+Z)}{N}\Big]\Big).$$

Then the dual line bundle $(L^{(a)})^{\otimes -1}$ is equipped with a canonical integrable connection $\nabla_{(a)}$ with log poles along $E + Z$ (see [14, §3] for the construction). Note that $L^{(1)} = L$ and that

$$L^{(N-1)} = L^{\otimes(N-1)}\Big(-\Big[\frac{(N-1)E'}{N}\Big]\Big) = L^{\otimes(N-1)}(-E' + E'_{\mathrm{red}})$$

is *ample*, by the choice of N.

Denote by Y_1 the base change of Y/k by the absolute Frobenius $F_k : k \longrightarrow k$, and let E_1, E'_1, Z_1 and L_1 be the base change of E, E', Z and L, respectively, by F_k. Let $F_{Y/k} : Y \longrightarrow Y_1$ be the relative Frobenius. Also denote by

$$L_1^{(a)} = F_k^* L^{(a)} = L_1^{\otimes a}\Big(-\Big[\frac{a(E'_1 + Z_1)}{N}\Big]\Big).$$

Pulling back by the absolute Frobenius F_Y of Y (which is equal to the composite $Y \xrightarrow{F_{Y/k}} Y_1 \xrightarrow{F_k} Y$) multiplies line bundles and Cartier divisors by p. Thus there is a natural identification

$$F_{Y/k}^*(L_1^{(a)})^{\otimes -1} = L^{\otimes -pa}\Big(p\Big[\frac{a(E' + Z)}{N}\Big]\Big), \tag{3.21}$$

and the right-hand side is naturally contained in $(L^{(pa)})^{\otimes -1}$:

$$L^{\otimes -pa}\Big(p\Big[\frac{a(E' + Z)}{N}\Big]\Big) \subseteq L^{\otimes -pa}\Big(\Big[\frac{pa(E' + Z)}{N}\Big]\Big) = (L^{(pa)})^{\otimes -1}; \tag{3.22}$$

we denote the difference between the two divisors in the parentheses by

$$B = B_a := \Big[\frac{pa(E' + Z)}{N}\Big] - p\Big[\frac{a(E' + Z)}{N}\Big],$$

so that the leftmost term of (3.22) is the rightmost term twisted by $-B$. When $0 < a < pa < N$, the coefficient of Z in B is zero, and the other coefficients in

$$B = \sum_{\ell=1} b_\ell E_\ell = \sum_{\ell=1}^{r} \Big(\Big[\frac{pac_\ell}{N}\Big] - p\Big[\frac{ac_\ell}{N}\Big]\Big) E_k \tag{3.23}$$

are all integers between 0 and $p - 1$.

There are two a priori different integrable connections on the common sheaf (3.21): On one hand, we can equip the left-hand side with the connection $1 \otimes d_{Y/k}$ (see, e.g., [29]). On the other hand, the connection $\nabla_{(pi)}$ on the rightmost term of (3.22) gives rise to the connection $\nabla_{(pi)}(-B)$ (see [14, Lem. 2.7]). These two turn out to be the same: It is the content of [14, Claim 10.25]. The first description will be useful in conjunction with the projection formula (see below), while the second allows the inclusion of sheaves with integrable connections.

Recall that given two sheaves with integrable connections with log poles $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$, their tensor product $\mathcal{E}_1 \otimes_{\mathcal{O}_Y} \mathcal{E}_2$ is endowed with the connection $\nabla_1 \otimes 1 + 1 \otimes \nabla_2$; thus $\mathrm{Res}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mathrm{Res}(\mathcal{E}_1) \otimes 1 + 1 \otimes \mathrm{Res}(\mathcal{E}_2)$. Also recall [14, Lem. 3.16(c)] that the residue of $\nabla_{(a)}$ on $(L^{(a)})^{\otimes -1}$ along E_ℓ is equal to the scalar $\{ac_\ell/N\} = ac_\ell/N - [ac_\ell/N]$.

In view of Theorem 2.6, the integrable connection on the sheaf $H \otimes F_{Y/k}^*(L_1^{(a)})^{\otimes -1}$ has nilpotent residues along E, while Res_{E_ℓ} of $H \otimes (L_1^{(pa)})^{\otimes -1}$ is the sum of a nilpotent endomorphism and the scalar

$$\{pac_\ell/N\} = pac_\ell/N - [pac_\ell/N] = -b_\ell \text{ in } \mathbb{F}_p,$$

with the notation of (3.23).

As in [14, Claim 10.24], we would like to see that the inclusion (3.22) results in a quasi-isomorphism of the log de Rham complexes

$$\Omega_Y^\bullet(\log(E + Z)) \otimes H \otimes F_{Y/k}^*(L_1^{(a)})^{\otimes -1} \xrightarrow{qis} \Omega_Y^\bullet(\log(E + Z)) \otimes H \otimes (L^{(pa)})^{\otimes -1}.$$

For this, we work with one ℓ at a time. If $b_\ell = 0$, there is nothing to do. Otherwise, we have $0 < b_\ell < p$, and we close the gap between the coefficients of E_ℓ in the two sides of (3.22) one by one, applying [14, Lem. 2.7 and 2.10] each step (for the isomorphism condition in [14, 2.10], note that the sum of a nonzero scalar and a nilpotent endomorphism is an automorphism).

Then by using the description (involving Frobenius) of the connection on (3.21) and the projection formula, we get the quasi-isomorphism

$$\begin{aligned} &F_{Y/k*}\Big(\Omega^\bullet_Y(\log(E+Z)) \otimes_{\mathcal{O}_Y} H\Big) \otimes_{\mathcal{O}_{Y_1}} (L_1^{(a)})^{\otimes -1} \\ &\longrightarrow F_{Y/k*}\Big(\Omega^\bullet_Y(\log(E+Z)) \otimes H \otimes (L^{(pa)})^{\otimes -1}\Big). \end{aligned}$$

Now Illusie's Theorem 1.1 applied to $f : (X, D + f^{-1}Z) \longrightarrow (Y, E + Z)$ gives a decomposition in the derived category $D(\mathcal{O}_{Y_1})$

$$F_{Y/k*}\left(\Omega^\bullet_Y(\log(E+Z)) \otimes_{\mathcal{O}_Y} H\right) \simeq \oplus_i \operatorname{gr}^i\left(\Omega^\bullet_{Y_1}(\log(E_1+Z_1)) \otimes H_1\right),$$

and we can twist it by any line bundle on Y_1. Therefore, for every integer a with $0 < a < pa < N$ and every integer q,

$$\begin{aligned} &\dim_k H^q\Big(Y, \operatorname{gr}^*\big[\Omega^\bullet_Y(\log(E+Z)) \otimes H\big] \otimes (L^{(a)})^{-1}\Big) \\ =\ &\dim_k H^q\Big(Y_1, \operatorname{gr}^*\big[\Omega^\bullet_{Y_1}(\log(E_1+Z_1)) \otimes H_1\big] \otimes (L_1^{(a)})^{-1}\Big) \\ =\ &\dim_k H^q\Big(Y_1, F_{Y/k*}\big[\Omega^\bullet_Y(\log(E+Z)) \otimes H\big] \otimes (L_1^{(a)})^{-1}\Big) \\ =\ &\dim_k H^q\Big(Y, \Omega^\bullet_Y(\log(E+Z)) \otimes H \otimes (L^{(pa)})^{-1}\Big) \\ \leq\ &\dim_k H^q\Big(Y, \operatorname{gr}^*\big[\Omega^\bullet_Y(\log(E+Z) \otimes H\big] \otimes (L^{(pa)})^{-1}\Big). \end{aligned}$$

The equalities follow from the base change by F_k, the decomposition theorem, and the quasi-isomorphism above, respectively. For the inequality, put the trivial filtration on $M := (L^{(pa)})^{\otimes -1}$ (namely $\mathrm{Fil}^i = M$ for $i \leq 0$ and $\mathrm{Fil}^i = 0$ for $i > 0$), and the tensor product filtration on $H \otimes M$; this still satisfies the Griffiths transversality. The associated graded pieces of the log de Rham complex are isomorphic to the ones in the last line (since $1 \otimes \nabla_M$ preserves the filtration and as such reduces to zero upon taking gr) and the inequality results from the spectral sequence associated with filtered complexes.

Start from $a = 1$ and apply the inequality η times. As noted above, $p^\eta = N - 1$ and $L^{(N-1)}$ is ample, so Lemma 3.18 applies for $(Y, E + Z)$ and $L^{(N-1)}$. Therefore, noting $L^{(1)} = L$, we get

$$H^q\big(Y, L^{\otimes -1} \otimes \operatorname{gr}^*(\Omega^\bullet_Y(\log(E+Z)) \otimes H)\big) = 0 \quad \text{for } q < e,$$

and we conclude the proof of Theorem 3.10. □

3.3. **A variant.**

Theorem 3.24. *Let $f : (X, D) \longrightarrow (Y, E)$ be a proper, log smooth, and log integral morphism of pure relative dimension n between proper smooth varieties with simple normal crossings divisors over a perfect field k of characteristic $p > 0$. Assume that f admits a lifting $\mathcal{F} : (\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{Y}, \mathcal{E})$ over $W(k)$ that satisfies Assumption 1.4. Suppose that Y is of pure dimension $e \geq 0$.*

Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ with restriction L on Y. Suppose that there exist an integer $\nu_0 > 0$ and an effective divisor E' supported on E such that

$$L^{\otimes \nu + \nu_0}(-E') \text{ is ample for every } \nu \geq 0. \tag{3.25}$$

For each integer m, consider the m-th relative log de Rham cohomology $H^m(f)$.

(1) If $m < p - e$ and if the residue map on $H^m(f)$ along every branch of E is nilpotent, then we have

$$H^q\left(Y, L^{\otimes -1} \otimes \operatorname{gr}^* \omega_Y^\bullet(H^m(f))\right) = 0 \quad \textit{for } q < e. \tag{3.26}$$

(2) Suppose that the conditions of Proposition 1.7 *are satisfied, so that the filtered module with connection $H^m(f)$ is isomorphic to the $\mathcal{O}_Y$-dual of $H^{2n-m}(f)$ with coefficients in $H^{2n}(f)$. Then, if either $m < p - e$ or $2n - m < p - e$, and if the residue map on $H^m(f)$ along every branch of E is nilpotent, then* (3.26) *still holds, and moreover, we also have*

$$H^q\left(Y, L(-E) \otimes \operatorname{gr}^* \omega_Y^\bullet(H^m(f))\right) = 0 \quad \textit{for } q > e. \tag{3.27}$$

Proof. We first deal with the case (1). The ample case (Lemma 3.18) and the cleaning of the notation and the assumptions in the beginning of Section 3.2 go without change. Then the main line of the proof in Section 3.2 carries over, with the necessary changes:

First, in the induction gambit, still choose a general hyperplane section Z, but this time lift it to $\mathcal{Z}$ over $W(k)$ and not merely to $\tilde{Z}$ over $W_2(k)$. This is still possible thanks to Lemma 3.20; *any* global section over Y lifts to $\mathcal{Y}$. The relative log crystalline and log de Rham cohomology groups are of formation compatible with the base change (restriction) to $\mathcal{Z}$ and Z, thanks to Assumption 1.4(1) (see [28, Thm. 6.10]). Furthermore, one deduces from this that the restriction of $\mathcal{F}$ to $\mathcal{Z}$ still satisfies Assumption 1.4.

In the semistable case, the nilpotence of the residue maps was provided by Katz's Theorem 2.6; this time we put it in the list of conditions. It still allows us to apply the method of Esnault and Viehweg using cyclic coverings.

Finally, the decomposition theorem necessary in the last step is provided by the first part of Theorem 1.8.

The proof of (3.26) in (2) is obtained by applying the second part of Theorem 1.8 to get the decomposition in the range $2n - m < p - e$. Then (3.27) follows from (3.26) by Serre duality. □

3.4. **Higher direct images of the canonical bundle.** We first note that by the usual method of reducing to characteristic $p > 0$ (cf. [12, Cor. 2.7] or [27, Cor. 4.17]) we get a purely algebraic proof of:

Corollary 3.28. *Let k be a field of characteristic zero, $f : (X, D) \longrightarrow (Y, E)$ a semistable morphism between proper smooth varieties over k, and L a line bundle over Y. Assume that there exist an effective divisor E' supported on E and an integer $\nu_0 > 0$ such that $L^{\otimes \nu}(-E')$ is ample whenever $\nu \geq \nu_0$. Then if Y has pure dimension $e \geq 0$,*

$$H^q\left(Y, L^{\otimes -1} \otimes \operatorname{gr}^* \omega_Y^\bullet(H^*(f))\right) = 0 \quad \textit{for } q < e,$$
$$H^q\left(Y, L(-E) \otimes \operatorname{gr}^* \omega_Y^\bullet(H^*(f))\right) = 0 \quad \textit{for } q > e.$$

This in turn allows us to recover (just as in [27, Rem. 4.18]) a vanishing statement regarding the higher direct images of the canonical bundle (see [52], [33, Thm. 2.1], [34, Thm. 2.14], and [13, Thm. 3.1]) in the semistable case:

Corollary 3.29. *Assume that $f : (X, D) \longrightarrow (Y, E)$ over k satisfy the assumptions of either Theorem* 3.10(2) *or Corollary* 3.28*, and that X has pure dimension $d = n + e$. Then we have*

$$H^j\big(Y, L \otimes R^k f_*(\Omega^d_X)\big) = 0 \ \textit{ for } j > 0 \textit{ and every } k. \tag{3.30}$$

Proof. Take the direct summand $H^{k+n} = R^{k+n} f_*(\Omega^\bullet_{X/Y}(\log D/E))$ of H^*. By unwinding the definitions, we have

$$\begin{aligned} L(-E) \otimes \big(\mathrm{gr}^d\, \omega^\bullet_Y(H^{k+n})\big) &= L(-E) \otimes \big(\Omega^e_Y(E) \otimes R^k f_*(\Omega^n_{X/Y}(\log D/E))[-e]\big) \\ &= L \otimes R^k f_*\big(f^*(\Omega^e_Y) \otimes \Omega^n_{X/Y}(\log D/E)\big)[-e] \\ &= L \otimes R^k f_* \Omega^d_X[-e]. \end{aligned}$$

Thus the second vanishing statements imply (3.30). □

4. Canonical extensions of automorphic bundles

The remainder of this article will be built heavily on [42]. We shall follow the conventions and notations there (which are different from those used in the preceding three sections) unless otherwise specified. For references to results proved in [35], we will cite the published revision [39] instead (with updated citation numbers).

4.1. **Toroidal compactifications.** As in [42, §1.1], let $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ be an integral PEL datum, let F denote the center of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$; let the group functor G over $\mathrm{Spec}(\mathbb{Z})$ be as in [39, Def. 1.2.1.6] (and [42, Def. 1.1]), and let the reflex field F_0 be as in [39, Def. 1.2.5.4]; let $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$) denote the ring of integers in F (resp. F_0); choose a *good prime* p, choose R_1, and define $\mathcal{O}_{F,1}$, $\mathcal{O}_1$, L_1, $L_{0,1}$, G_1, P_1, and M_1 accordingly. We shall denote the residue field of R_1 by k_1 (rather than by κ_1; we will use κ for a different purpose).

Let $\mathcal{H}$ be a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ (see [53, 0.6] or [39, Def. 1.4.1.8]). By [39, Def. 1.4.1.4] (with $\Box = \{p\}$ there), the data of $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$. By [39, Thm. 1.4.1.11 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over S_0 (because $\mathcal{H}$ is neat). Let d be the relative dimension of $\mathsf{M}_{\mathcal{H}}$ over S_0. Let $\mathsf{M}_{\mathcal{H},0}$ be defined as in [42, §1.2], which is smooth over S_0.

Remark 4.1. In this article, we shall *not* retain [42, Ass. 1.9]; i.e., we do *not* assume that $\mathsf{M}_{\mathcal{H},0}$ is proper (or equivalently projective) over S_0 (cf. [36, §4]). (Nevertheless, the relative dimension d can still be effectively calculated using the PEL datum, as explained in [42, Rem. 1.10].)

Let $\mathsf{S}_1 := \mathrm{Spec}(R_1)$, and let $\mathsf{M}_{\mathcal{H},1} := \mathsf{M}_{\mathcal{H},0} \underset{\mathsf{S}_0}{\times} \mathsf{S}_1$. Let us denote the universal object over $\mathsf{M}_{\mathcal{H}}$ by $(A, \lambda, i, \alpha_{\mathcal{H}}) \longrightarrow \mathsf{M}_{\mathcal{H}}$, and denote (abusively) its pullback to $\mathsf{M}_{\mathcal{H},0}$ (resp. $\mathsf{M}_{\mathcal{H},1}$) by $(A, \lambda, i, \alpha_{\mathcal{H}}) \longrightarrow \mathsf{M}_{\mathcal{H},0}$ (resp. $(A, \lambda, i, \alpha_{\mathcal{H}}) \longrightarrow \mathsf{M}_{\mathcal{H},1}$).

Proposition 4.2. *By* [39, Thm. 6.4.1.1 and 7.3.3.4]*, when $\mathcal{H}$ is neat, $\mathsf{M}_{\mathcal{H}}$ admits a* **toroidal compactification** $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$*, a scheme projective and smooth over S_0, depending on a compatible collection Σ (of the so-called cone decompositions)* *that is* **projective** *and* **smooth** *in the sense of* [39, Def. 6.3.3.4 and 7.3.1.3]. (*By abuse of language we sometimes simply call Σ a cone decomposition, even though it is technically a compatible collection of cone decompositions along various boundary components.*) *It satisfies the following properties:*

(1) *The universal abelian scheme $A \longrightarrow \mathsf{M}_{\mathcal{H}}$ extends to a semi-abelian scheme $A^{\mathrm{ext}} \longrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, the polarization $\lambda : A \longrightarrow A^\vee$ extends to a prime-to-p isogeny $\lambda^{\mathrm{ext}} : A^{\mathrm{ext}} \longrightarrow (A^{\mathrm{ext}})^\vee$ between semi-abelian schemes, and the endomorphism structure $i : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H}}}(A)$ extends to an endomorphism structure $i^{\mathrm{ext}} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{ext}}}(A^{\mathrm{ext}})$.* (*These extensions are unique because the base is normal. See* [18, Ch. I, Prop. 2.7].)

(2) *The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$* (*with its reduced structure*) *is a relative Cartier divisor $\mathsf{D} = \mathsf{D}_{\infty,\mathcal{H}}$ with* **simple** *normal crossings. Here simpleness of the normal crossings uses* [39, Cond. 6.2.5.25 and Lem. 6.2.5.27] (*cf.* [18, Ch. IV, Rem. 5.8(a)]) *and the neatness of $\mathcal{H}$.*

(3) *Let $\underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} := \underline{\mathrm{KS}}_{(A^{\mathrm{ext}},\lambda^{\mathrm{ext}},i^{\mathrm{ext}})/\mathsf{M}_{\mathcal{H}}}$ be the quotient of $\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ by the $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$-submodule generated by $(\lambda^{\mathrm{ext}})^*(y) \otimes z - (\lambda^{\mathrm{ext}})^*(z) \otimes y$ and $i^{\mathrm{ext}}(b)^*(x) \otimes y - x \otimes (i^{\mathrm{tor}}(b)^\vee)^*(y)$ for $x \in \underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$, $y, z \in \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$, and $b \in \mathcal{O}$. Let $\overline{\Omega}^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0} := \Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}(\log \mathsf{D}) = \Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}[\mathrm{d}\log \mathsf{D}]$ be the sheaf of modules of log 1-differentials on $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over S_0, with respect to D. Then the usual Kodaira–Spencer morphism $\mathrm{KS}_{A/\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H}}} \longrightarrow \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$* (*cf.* [39, Def. 2.1.7.9]) *extends to the* **extended Kodaira–Spencer morphism**

$$\mathrm{KS}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \longrightarrow \overline{\Omega}^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}, \tag{4.3}$$

which factors through $\underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ and induces the **extended Kodaira–Spencer isomorphism**

$$\mathrm{KS}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \xrightarrow{\sim} \overline{\Omega}^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}. \tag{4.4}$$

(*These morphisms and isomorphisms are up to a Tate twist often suppressed in the notation.*)

(4) *Let $\omega := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$. Then by* [39, Thm. 7.2.4.1], *the scheme $\mathrm{Proj}\Big(\underset{r\geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, \omega^{\otimes r})\Big)$ is normal and projective over S_0, contains $\mathsf{M}_{\mathcal{H}}$ as an open dense subscheme, and defines the* **minimal compactification** *$\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ of $\mathsf{M}_{\mathcal{H}}$* (*independent of the choice of Σ*). *Moreover, the line bundle ω descends to an* **ample line bundle** *over $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$.*

(5) *Under the assumption that Σ is projective,* [39, Thm. 7.3.3.4] *asserts more precisely that $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is the normalization of the blow-up of $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ along a coherent sheaf of ideals $\mathcal{J}$ of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}}$ whose pullback to a coherent sheaf of ideals $\jmath$ of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is of the form $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(-\mathsf{D}')$, for some relative Cartier divisor D' with normal crossings on $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ such that $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$. In particular:*

$$\exists\, r_0 > 0 \text{ such that } \omega^{\otimes r}(-\mathsf{D}') \text{ is ample for every } r \geq r_0. \tag{4.5}$$

In what follows, we shall sometimes omit Σ when the choice is clear.

Remark 4.6. In previous works of the first author, there has been different notations for the extension of $A \longrightarrow \mathsf{M}_{\mathcal{H}}$ to the semi-abelian scheme $A^{\mathrm{ext}} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, sometimes even denoted $A^{\mathrm{tor}} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. In this article, we shall maintain the convention that the superscript "tor" always means toroidal compactifications, while the superscript "ext" means semi-abelian (group scheme) extensions.

Remark 4.7. In what follows, when we consider refinements Σ' of Σ (see [39, Def. 6.4.2.2]), we shall only consider those Σ' that are still *projective* and *smooth* (together with all other running assumptions on Σ), so that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ will remain a scheme projective and smooth over S_0, and so that (5) of Proposition 4.2 will continue to hold (with new D and D' defined by Σ').

Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ denote the schematic closure of Sh_H in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, and let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ denote the pullback of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ under $\mathsf{S}_1 \longrightarrow \mathsf{S}_0$. Then $\mathsf{M}_{\mathcal{H},1}$ is smooth over S_1, and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1} \longrightarrow \mathsf{S}_1$ is proper and smooth and shares the properties of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \longrightarrow \mathsf{S}_0$ listed above. By abuse of notation, we denote the pullback of D to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ still by D. Similarly, let $\mathsf{M}^{\min}_{\mathcal{H},0}$ denote the schematic closure of Sh_H in $\mathsf{M}^{\min}_{\mathcal{H}}$, and let $\mathsf{M}^{\min}_{\mathcal{H},1}$ denote the pullback of $\mathsf{M}^{\min}_{\mathcal{H},0}$ under $\mathsf{S}_1 \longrightarrow \mathsf{S}_0$. Then $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1} \longrightarrow \mathsf{M}^{\min}_{\mathcal{H},1}$ enjoys the same properties of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \longrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ described in Proposition 4.2.

Remark 4.8. The smooth toroidal compactifications (stratified into smooth locally closed subschemes determined by the boundary divisor) are nice geometric objects suitable for the study of (log) Hodge, de Rham, and crystalline cohomology. However, as we shall see in Section 7.1, while ample automorphic line bundles over $\mathsf{M}_{\mathcal{H}}$ extend to ample line bundles on the (generally singular) minimal compactification $\mathsf{M}^{\min}_{\mathcal{H}}$, their canonical or subcanonical extensions over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ are almost never ample, except in very special cases. Therefore, it is crucial that we have developed a more refined vanishing theorem in the first three sections of this article.

Let $m \geq 0$ be any integer, and let $\mathsf{N}_m := A^m$ be the m-fold fiber product of $A \longrightarrow \mathsf{M}_{\mathcal{H}}$. By [38, Thm. 2.15], by taking $Q := \mathcal{O}^{\oplus m}$ there (cf. [38, Ex. 2.2]), the abelian scheme $\mathsf{N}_m \longrightarrow \mathsf{M}_{\mathcal{H}}$ admits a collection of (non-canonical) toroidal compactifications $\mathsf{N}^{\mathrm{tor}}_{m,\kappa}$, with indices κ in a partially ordered set $\mathbf{K}_{m,\mathcal{H},\Sigma}$, such that the (smooth) structural morphism $\mathsf{N}_m \longrightarrow \mathsf{M}_{\mathcal{H}}$ extends to a proper *log smooth* morphism $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ for each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$. This collection $\{\mathsf{N}^{\mathrm{tor}}_{m,\kappa}\}_{\kappa\in\mathbf{K}_{m,\mathcal{H},\Sigma}}$ enjoys a long list of nice properties (see the statements of [38, Thm. 2.15]); we will give precise references to them when needed.

Remark 4.9. In our relative setup, the fiber products of "good" toroidal compactifications are in general not "good" toroidal compactifications. The expected properties of the structural morphisms such as $f^{\mathrm{tor}}_{m,\kappa}$ can be destroyed by taking fiber products. This is why we maintain m, the number of copies, in the notation.

By abuse of notation, we shall denote the pullbacks of $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ by $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ and $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$, respectively (and similarly over other base schemes).

4.2. **Canonical extensions.** As explained in [38, §6B], the locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})$ over $\mathsf{M}_{\mathcal{H},1}$ extends to a *unique* locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ satisfying the properties stated in [38, Prop. 6.9] (with $\mathsf{M}_{\mathcal{H}}$, etc, there replaced with their base changes $\mathsf{M}_{\mathcal{H},1}$, etc, here); in particular, if we set

$\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}})$, the Gauss–Manin connection of $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ over $\mathsf{M}_{\mathcal{H},1}$ (see [42, Def. 1.11]) extends to the *extended Gauss–Manin connection*

$$\nabla : \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \longrightarrow \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1} \tag{4.10}$$

over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$, an integrable connection with log poles along D, which induces the extended Kodaira–Spencer morphism (4.3) (and the extended Kodaira–Spencer isomorphism (4.4)) as in [38, (5) of Prop. 6.9].

Let the principal bundles $\mathcal{E}_{\mathrm{G}_1}$, $\mathcal{E}_{\mathrm{P}_1}$, and $\mathcal{E}_{\mathrm{M}_1}$ be defined as in [42, Def. 1.12, 1.13, and 1.14]. Then, also as explained in [38, §6B], the principal bundle $\mathcal{E}_{\mathrm{P}_1}$ (resp. $\mathcal{E}_{\mathrm{M}_1}$) extends to a princple bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1}$ (resp. $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1}$) over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (with their definitions in [38, §6B] replaced with their base changes to S_1 here), and the principal bundle $\mathcal{E}_{\mathrm{G}_1}$ extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ by setting

$$\begin{aligned} \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}} \Big(& (\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(1)), \\ & ((L_{0,1} \oplus L^{\vee}_{0,1}(1)) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(1))\Big). \end{aligned} \tag{4.11}$$

As in the cases of the principal bundles $\mathcal{E}_{\mathrm{G}_1}$, $\mathcal{E}_{\mathrm{P}_1}$, and $\mathcal{E}_{\mathrm{M}_1}$, these define étale torsors by Artin's theory of approximations (cf. [1, Thm. 1.10 and Cor. 2.5]), because they have sections over formal completions (because $\underline{\mathrm{Lie}}^{\vee}_{(A^{\mathrm{ext}})^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ and $L^{\vee}_{0,1}(1) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ can be compared using the Lie algebra condition [39, Def. 1.3.4.1 and Lem. 1.2.5.11], and because the pairings $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}$ can be compared using [39, Cor. 1.2.3.10]).

For each R_1-algebra R, we define $\mathrm{Rep}_R(\mathrm{G}_1)$, $\mathrm{Rep}_R(\mathrm{P}_1)$, and $\mathrm{Rep}_R(\mathrm{M}_1)$ as in [42, Def. 1.15], and define the functors $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$, $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$, and $\mathcal{E}_{\mathrm{M}_1,R}(\,\cdot\,)$ (of automorphic sheaves or bundles) as in [42, Def. 1.16].

Definition 4.12. *Let R be any R_1-algebra. For each $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, we define*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1} \underset{R_1}{\otimes} R) \overset{\mathrm{G}_1 \underset{R_1}{\otimes} R}{\times} W, \tag{4.13}$$

called the **canonical extension** *of $\mathcal{E}_{\mathrm{G}_1,R}(W)$, and define*

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \mathscr{I}_{\mathsf{D}},$$

called the **subcanonical extension** *of $\mathcal{E}_{\mathrm{G}_1,R}(W)$, where $\mathscr{I}_{\mathsf{D}}$ is the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$-ideal defining the relative Cartier divisor D (in (2) of Proposition 4.2). We define similarly $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(W)$ with G_1 (and its principal bundle) replaced with P_1 and M_1 (and their respective principal bundles). (See [38, Def. 6.13].)*

Lemma 4.14. *Lemmas 1.18, 1.19, and 1.20, and Corollary 1.21 in [42] remain true if we replace the automorphic sheaves with their canonical or subcanonical extensions.*

Proof. The same proofs in *loc. cit.* for these results also work here. ∎

4.3. **De Rham complexes.** Let R be any R_1-algebra. For simplicity, we shall denote pullbacks of objects from R_1 to R by replacing the subscript 1 with R, although we shall use the same notation D for its pullback.

Following the definition of Gauss–Manin connections on automorphic bundles in [42, Def. 1.24], the extended Gauss–Manin connection (4.10) induces integrable connections on canonical and subcanonical extensions (extending the ones on the automorphic bundles) as follows:

Let $\overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1} := \Omega^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}(\log \mathsf{D}) \cong \wedge^{\bullet}(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}[\mathrm{d}\log \mathsf{D}])$. Let $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$ be the subsheaf of $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_*\mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$ corresponding to the subsheaf $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \oplus \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$ of $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})_*(\mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0})$ under the canonical splitting $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \cong \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \oplus \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, with the summand $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$ given by the image of $\mathrm{pr}_2^* : \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \longrightarrow \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, and with the summand $\Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$ spanned by the image of $(\mathrm{pr}_1^* - \mathrm{pr}_2^*) = (s^* - \mathrm{Id}^*) \circ \mathrm{pr}_2^* : \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \longrightarrow \mathscr{P}^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$. Then the morphisms $\mathrm{pr}_1^*, \mathrm{pr}_2^*, \mathrm{Id}^*, s^*$ induce respectively morphisms $\overline{\mathrm{pr}}_1^*, \overline{\mathrm{pr}}_2^* : \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \longrightarrow \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$ and $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0} \xrightarrow{\sim} \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$ such that $\overline{s}^* - \overline{\mathrm{Id}}^*$ induces the universal log differential $\mathrm{d} : \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \longrightarrow \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$. Note that these objects are uniquely determined by their restrictions to $\mathsf{M}_{\mathcal{H},1}$. (Therefore, although we defined them as induced objects, they agree with the corresponding objects defined in log geometry.)

Since the Gauss–Manin connection (see [42, Def. 1.11]) induces the extended Gauss–Manin connection (4.10), the extended Gauss–Manin connection (4.10) is the difference between the two isomorphisms $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathrm{pr}}_2^*(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}) \xrightarrow{\sim} \overline{\mathrm{pr}}_1^*(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$. (We can interpret $\overline{\mathrm{Id}}^*$ as induced by the extended Gauss–Manin connection (4.10) and $\overline{s}^*$.) Note that here $\overline{\mathrm{pr}}_1^*$ and $\overline{\mathrm{pr}}_2^*$ are morphisms with target tensored with $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$, but not $\mathscr{P}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}/\mathsf{S}_0}$ (which can be identified with the structural sheaf of the first infinitesimal neighborhood of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathsf{S}_0}{\times} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$). By construction of $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(\,\cdot\,)$ (cf. (4.13)), for each $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, the two isomorphisms above induce two isomorphisms $\overline{\mathrm{Id}}^*, \overline{s}^* : \overline{\mathrm{pr}}_2^*(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)) \xrightarrow{\sim} \overline{\mathrm{pr}}_1^*(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W))$ lifting the identity morphism on $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$. Hence the difference $\overline{s}^* - \overline{\mathrm{Id}}^*$ induces a morphism

$$\nabla : \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \longrightarrow \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R} \tag{4.15}$$

of abelian sheaves. Since the connection ∇ in (4.15) is induced by the connection [42, (1.23)], the integrability condition is trivially verified. By applying $\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \mathscr{I}_{\mathsf{D}}$, we obtain an integrable connection with log poles

$$\nabla : \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \longrightarrow \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}. \tag{4.16}$$

Definition 4.17. *The integrable connection* ∇ (*with log poles*) *in* (4.15) (*resp.* (4.16)) *is called the* **extended Gauss–Manin connection** *for* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$ (*resp.* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W)$).

Hence we obtain the following extension of the de Rham complex $(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ (see [42, Def. 1.24]):

Definition 4.18. *The integrable connections* (4.15) *and* (4.16) (*with log poles*) *define the* (*log*) **de Rham complexes** $(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$ *and* $(\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$.

4.4. **Hodge filtrations.** For each $\tau \in \Upsilon$, let $\mathcal{O}_\tau$, V_τ, p_τ, q_τ, m_τ, L_τ, G_τ, P_τ, and M_τ be defined as in [42, §§2.1–2.2]. Fix any choice of a cocharacter $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \longrightarrow \mathrm{G}_1$ (splitting the similitude character $\upsilon : \mathrm{G}_1 \longrightarrow \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$) as in [42, Def. 2.4], and consider its reciprocal $H : \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \longrightarrow \mathrm{G}_1$ as in [42, §2.3]. Then, for each R_1-algebra R and each $W \in \mathrm{Rep}_R(\mathrm{P}_1)$, we can define the Hodge filtrations on W, $\mathcal{E}_{\mathrm{P}_1,R}(W)$, and $\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ as in [42, §2.3]. By Lemma 4.14, the Hodge filtration on W defines filtrations on $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$ (extending the Hodge filtration $\mathtt{F}^a(\mathcal{E}_{\mathrm{P}_1,R}(W))$ on $\mathcal{E}_{\mathrm{P}_1,R}(W)$), which we denote by $\mathtt{F}^a(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W))$ and $\mathtt{F}^a(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W))$, for $a \in \mathbb{Z}$.

Definition 4.19. *The filtration* $\mathtt{F}(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$ (*resp.* $\mathtt{F}(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$) *is called the Hodge filtration on* $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$ (*resp.* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$).

By Lemma 4.14, $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$ and $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$.

Definition 4.20. *Let* $W \in \mathrm{Rep}_R(\mathrm{G}_1)$. *By considering* W *as an object of* $\mathrm{Rep}_R(\mathrm{P}_1)$ *by restriction from* G_1 *to* P_1, *we can define the Hodge filtrations on* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$ (*resp.* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$) (*see Lemma* 4.14) *as in Definition* 4.19. *The Hodge filtration on the de Rham complex* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ *is defined by*

$$\mathtt{F}^a(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}) := \mathtt{F}^{a-\bullet}(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R},$$

extending the Hodge filtration on $\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ (*see* [42, Def. 2.13]). *The Hodge filtration on* $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ *is defined similarly.*

These form subcomplexes of the de Rham complexes under the Gauss–Manin connections, thanks to the Griffiths transversality. (The only de Rham complexes we will need for our main results are those realized by weak geometric plethysm as in Proposition 5.8, for which the Griffiths transversality is clear. For de Rham complexes attached to an arbitrary $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, see [40].)

Lemma 4.21 (cf. [42, Lem. 2.14])**.** *Suppose* W_1 *and* W_2 *are two objects in* $\mathrm{Rep}_R(\mathrm{G}_1)$ *such that the induced actions of* P_1 *and* $\mathrm{Lie}(\mathrm{G}_1)$ *on them satisfy*

$W_1|_{\mathrm{P}_1} \cong W_2|_{\mathrm{P}_1}$ and $W_1|_{\mathrm{Lie}(\mathrm{G}_1)} \cong W_2|_{\mathrm{Lie}(\mathrm{G}_1)}$. Then we have a canonical isomorphism

$$(\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W_1) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla) \cong (\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W_2) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla) \tag{4.22}$$

respecting the Hodge filtrations on both sides. (*Consequently, the same is true with $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(\,\cdot\,)$ replaced with $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(\,\cdot\,)$.*)

Proof. The same proof for [42, Lem. 2.14] works here. □

5. Geometric realizations of canonical extensions

The aim of this section is to explain how the constructions in [42, §3] for automorphic bundles can be extended to their canonical extensions.

5.1. **Setup.** We retain the notation of [42, §§2.4–2.6]. (See especially [42, Rem. 2.17], which is justified in our context by Lemma 4.21, generalizing [42, Lem. 2.14].)

Let R be any R_1-algebra. Let $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ (as in [42, Def. 2.29]). Since the underlying R-module of $V_{[\mu],R}$ is locally free, we can consider the contragradient representation $V^{\vee}_{[\mu],R} \in \mathrm{Rep}_R(\mathrm{G}_1)$. Then we have the associated automorphic bundles $\underline{V}_{[\mu],R} := \mathcal{E}_{\mathrm{G}_1,R}(V_{[\mu],R})$ and $\underline{V}^{\vee}_{[\mu],R} := \mathcal{E}_{\mathrm{G}_1,R}(V^{\vee}_{[\mu],R})$ over $\mathsf{M}_{\mathcal{H},R}$ (cf. [42, Prop. 3.7]), and their canonical (resp. subcanonical) extensions $\underline{V}^{\mathrm{can}}_{[\mu],R} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(V_{[\mu],R})$ and $(\underline{V}^{\vee}_{[\mu],R})^{\mathrm{can}} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(V^{\vee}_{[\mu],R})$ (resp. $\underline{V}^{\mathrm{sub}}_{[\mu],R} := \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(V_{[\mu],R})$ and $(\underline{V}^{\vee}_{[\mu],R})^{\mathrm{sub}} := \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(V^{\vee}_{[\mu],R})$) over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}$.

The connections (4.15) and (4.16) define respectively the de Rham complexes $((\underline{V}^{\vee}_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$ and $((\underline{V}^{\vee}_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$, both extending the de Rham complex $(\underline{V}^{\vee}_{[\mu],R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$, with their Hodge filtrations as in Definition 4.20 extending that on $\underline{V}^{\vee}_{[\mu],R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^{\bullet}_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$.

Our goal in this section is to explain how the canonical extensions of automorphic bundles can be realized as summands of the relative log de Rham cohomology of compactifications of Kuga families (cf. [42, §3]). There will be two versions, one weaker version (Proposition 5.8) to be applied to the study of (log) Hodge and de Rham cohomology in Section 7, and a stronger version (Proposition 5.14) to be applied to the comparison with étale and Betti cohomology in Section 9.

For simplicity, we write, for each integer $m \geq 0$, each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$, and each degree $i \geq 0$, the relative cohomology

$$\underline{H}^i_{m,\kappa} := \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}) := R^i(f^{\mathrm{tor}}_{m,\kappa})_*(\overline{\Omega}^{\bullet}_{\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}) \tag{5.1}$$

and the relative homology

$$\begin{aligned} \underline{H}_{m,\kappa,i} &:= \underline{H}^{\text{log-dR}}_i(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}) \\ &:= \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}), \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}). \end{aligned} \tag{5.2}$$

By [38, Thm. 2.15], these are locally free sheaves over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$, with a long list of nice properties. In particular, they admit Gauss–Manin connections and de Rham complexes (with log poles along D, by the same arguments in [42, §4.1] and [38, §4D]), which are compatible with Definitions 4.17 and 4.20 when $R = R_1$ and $W = L_1$ (because their restrictions to the open dense subscheme $\mathsf{M}_{\mathcal{H},1}$ are the canonical ones). We shall denote these de Rham complexes by $(\underline{H}^i_{m,\kappa} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}, \nabla)$ and $(\underline{H}_{m,\kappa,i} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}, \nabla)$, the latter being dual to the former. The former agrees (up to canonical isomorphism) with the i-th exterior power of the m-fold direct sum of the de Rham complex defined by (4.10) (by uniqueness of such extensions, again by restriction to $\mathsf{M}_{\mathcal{H},1}$), with its Hodge filtration (denoted by $\mathtt{F}$ as always) the same as the m-th tensor product of the two-step filtration as in [38, Prop. 6.9]. Hence they are independent of the choice of κ (up to canonical isomorphism), and we can define unambiguously

$$\underline{H}^i_{m,\mathcal{H},\Sigma} := \underline{H}^i_{m,\kappa} \tag{5.3}$$

and

$$\underline{H}_{m,\mathcal{H},\Sigma,i} := \underline{H}_{m,\kappa,i} \tag{5.4}$$

using any κ in the partially ordered set $\mathbf{K}_{m,\mathcal{H},\Sigma}$, together with well-defined Gauss–Manin connections and de Rham complexes (with obvious notations).

Remark 5.5. Then we have $\underline{H}_{m,\mathcal{H},\Sigma,\bullet} \cong \underline{H}_{m,\kappa,\bullet} \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\wedge^\bullet(L_1^{\oplus m}))$ (horizontally over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$; i.e., respecting not only the Gauss–Manin connections and the de Rham complexes, but also the Hodge filtrations).

For any R_1-algebra R, we denote the base changes of these sheaves or complexes by adding R to the end of the subscripts. For simplicity, we shall only explain the case $R = R_1$ in the remainder of this section. Nevertheless, it will be clear from the constructions that they also work over general R.

5.2. **Refinements of cone decompositions.**

Proposition 5.6. *Let $m \geq 0$ be any integer. Suppose we have two choices Σ' and Σ for $\mathsf{M}_{\mathcal{H}}$, such that Σ' is a refinement of Σ (see* [39, Def. 6.4.2.2]*). Then there is a canonical proper log étale morphism $[1]_{\Sigma',\Sigma} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma',1} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (cf.* [39, Prop. 6.4.3.4]*), and, for each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$, there exists some $\kappa' \in \mathbf{K}_{m,\mathcal{H},\Sigma'}$, together with a proper log étale morphism $[1]_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa'} \longrightarrow \mathsf{N}^{\mathrm{tor}}_{m,\kappa}$ covering $[1]_{\Sigma',\Sigma} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma',1} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (compatible with $f^{\mathrm{tor}}_{m,\kappa'}$ on the source and with $f^{\mathrm{tor}}_{m,\kappa}$ on the target), inducing canonical horizontal isomorphisms $\underline{H}^\bullet_{m,\mathcal{H},\Sigma'} \xrightarrow{\sim} [1]^*_{\Sigma',\Sigma}\underline{H}^\bullet_{m,\mathcal{H},\Sigma}$ and $([1]_{\Sigma',\Sigma})_*\underline{H}_{m,\mathcal{H},\Sigma',\bullet} \xrightarrow{\sim} \underline{H}_{m,\mathcal{H},\Sigma,\bullet}$.*

Proof. This follows from (a special case of) [38, (4) of Thm. 2.15], with $g_h = 1 \in \mathrm{G}(\mathbb{A}^{\infty,p})$ there. (It is essentially by definition (see [28, Thm. 3.5]) that morphisms between toroidal compactifications of $\mathsf{M}_{\mathcal{H}}$ are log étale.) $\square$

Lemma 5.7. *Consider the canonical morphism $[1]_{\Sigma',\Sigma} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma',1} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (cf.* [39, Prop. 6.4.3.4]*). By abuse of notation, let us denote the boundary of both $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma',1}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (namely the complement of $\mathsf{M}_{\mathcal{H},1}$ in them with their reduced structures) by D, defined by the coherent ideals $\mathscr{I}_{\mathsf{D}}$ in the structural sheaves of their ambient spaces. Then we have a canonical isomorphism $([1]_{\Sigma',\Sigma})_*\mathscr{I}_{\mathsf{D}} \xrightarrow{\sim} \mathscr{I}_{\mathsf{D}}$; moreover, $R^i([1]_{\Sigma',\Sigma})_*\mathscr{I}_{\mathsf{D}} = 0$ for each $i > 0$. The same are true with these sheaves of ideals replaced with the corresponding structural sheaves (which have already been in* [38, Thm. 2.15]*, or rather* [39, proof of Lem. 7.1.1.5]*).*

Proof. All of these statements follow from the argument in [32, Ch. I, §3], which has been used (in some different languages) in [23, Lem. 1.3.6] and [38, Thm. 2.15]. The proof of [23, Lem. 1.3.6] (using closed coverings defined by cone decompositions) explains both the cases of the structural sheaves and the coherent ideals defining boundary divisors, from which our case follows easily, as explained in proof of [23, Lem. 1.6.8(iii)]. (Alternatively, one can pass to formal completions along boundary strata and use open coverings defined by cone decompositions as in [38].) $\square$

5.3. **Weak geometric plethysm.**

Proposition 5.8 (cf. [42, Prop. 3.7]). *Suppose* $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<\mathrm{w}p}$, *with* $0 \leq n := |\mu|_{\mathrm{L}} < p$, *as in* [42, Def. 3.2]. *Suppose moreover that* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$. *Then* $\underline{H}_{n,\mathcal{H},\Sigma,\bullet} \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\wedge^\bullet(L_1^{\oplus n}))$ (*horizontally over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$) (*cf. Remark* 5.5), *and the functorial images under* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ *of idempotent endomorphisms of* $L_1^{\oplus n}$ *as an object in* $\mathrm{Rep}_{R_1}(\mathrm{G}_1)$ *define* **horizontal** *idempotent endomorphisms of* $\underline{H}_{n,\mathcal{H},\Sigma,\bullet}$. *In particular, the commuting elements* $\varepsilon^{\mathrm{L}}_n$, $\varepsilon^{\mathrm{S}}_\mu$, $\varepsilon^{\mathrm{Y}}_\mu$, *and* ε^λ_μ *defined in* [42, §§3.1–3.4] *define* (*commuting*) *idempotents* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\varepsilon^{\mathrm{L}}_n)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\varepsilon^{\mathrm{S}}_n)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\varepsilon^{\mathrm{Y}}_n)$, *and* $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\varepsilon^\lambda_n)$, *respectively, all acting as horizontal idempotent endomorphisms on* $\underline{H}_{n,\mathcal{H},\Sigma,\bullet} \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\wedge^\bullet(L_1^{\oplus n}))$. *By abuse of notation, let us denote these idempotents by* $(\varepsilon^{\mathrm{L}}_n)_*$, $(\varepsilon^{\mathrm{S}}_\mu)_*$, $(\varepsilon^{\mathrm{Y}}_\mu)_*$, *and* $(\varepsilon^\lambda_\mu)_*$, *and denote their dual idempotents on* $\underline{H}^\bullet_{n,\mathcal{H},\Sigma}$ *by* $(\varepsilon^{\mathrm{L}}_n)^*$, $(\varepsilon^{\mathrm{S}}_\mu)^*$, $(\varepsilon^{\mathrm{Y}}_\mu)^*$, *and* $(\varepsilon^\lambda_\mu)^*$, *respectively. Hence, by defining* $\varepsilon_\mu = \varepsilon^\lambda_\mu\, \varepsilon^{\mathrm{Y}}_\mu\, \varepsilon^{\mathrm{S}}_\mu\, \varepsilon^{\mathrm{L}}_n$ *and* t_μ *as in* [42, Prop. 3.7], *and by defining* $(\varepsilon_\mu)_*$ *and* $(\varepsilon_\mu)^*$ *in the obvious way, we obtain canonical* **horizontal** *isomorphisms*

$$\underline{V}^{\mathrm{can}}_{[\mu]} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}(V_{[\mu]}) \cong (\varepsilon_\mu)_*\, \underline{H}_{n,\mathcal{H},\Sigma,n}$$

and (*by duality*)

$$(\underline{V}^\vee_{[\mu]})^{\mathrm{can}} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}(V^\vee_{[\mu]}) \cong (\varepsilon_\mu)^*\, \underline{H}^n_{n,\mathcal{H},\Sigma}, \tag{5.9}$$

realizing in particular the de Rham complex of $(\underline{V}^\vee_{[\mu]})^{\mathrm{can}}$ *as a horizontal summand of the de Rham complex of* $\underline{H}^n_{n,\mathcal{H},\Sigma}$. (*The same is true if we base change everything to an* R_1*-algebra* R.)

Proof. Since the idempotent actions of $\varepsilon^{\mathrm{L}}_n$, $\varepsilon^{\mathrm{S}}_\mu$, $\varepsilon^{\mathrm{Y}}_\mu$, and ε^λ_μ on $\wedge^\bullet(L_1^{\oplus n})$ commute with the actions of G_1, their kernels and images split in $\mathrm{Rep}_{R_1}(\mathrm{G}_1)$ and allow us to deduce that their functorial images under $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ are horizontal (because the corresponding de Rham complexes split as well). $\square$

As a result, for realizing de Rham complexes of $(\underline{V}^\vee_{[\mu]})^{\mathrm{can}}$ as a horizontal summand of the de Rham complexes of $\underline{H}^n_{n,\mathcal{H},\Sigma}$, Proposition 5.8 explains the simple fact that we do not need true endomorphisms of $f^{\mathrm{tor}}_{n,\kappa} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ for any $\kappa \in \mathbf{K}_{n,\mathcal{H},\Sigma}$. This will be all that we need until the end of Section 8.

However, for applying the crystalline comparison theorem in Section 9.1, we will need a more precise construction, involving morphisms between $f^{\mathrm{tor}}_{n,\kappa} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ and $f^{\mathrm{tor}}_{n,\kappa'} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa'} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ for different $\kappa, \kappa' \in \mathbf{K}_{n,\mathcal{H},\Sigma}$. We will explain this in the following subsections. (These results will only be used in Section 9.2 below.)

5.4. **Extensions of endomorphisms.** The aim of this subsection is to explain how the idempotents $(\varepsilon_n^{\mathrm{L}})_*$, $(\varepsilon_\mu^{\mathrm{S}})_*$, and $(\varepsilon_\mu^{\mathrm{Y}})_*$ can be extended.

Proposition 5.10. *Let $m \geq 0$ be an integer and let h be an element of $\mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus m})$ such that the cokernel $\mathcal{O}^{\oplus m}/(h(\mathcal{O}^{\oplus m}))$ has finite cardinality prime to p. In this case, by identifying N_m with $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O}^{\oplus m}, A)$, the endomorphism h of the lattice $\mathcal{O}^{\oplus m}$ induces an endomorphism $h^* : \mathsf{N}_m \longrightarrow \mathsf{N}_m$ which is an isogeny of degree prime to p, and, for each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$, there exists some $\kappa' \in \mathbf{K}_{m,\mathcal{H},\Sigma}$ (with the same Σ), together with a proper log étale morphism $h^*_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa'} \longrightarrow \mathsf{N}^{\mathrm{tor}}_{m,\kappa}$ (compatible with $f^{\mathrm{tor}}_{m,\kappa'}$ on the source and with $f^{\mathrm{tor}}_{m,\kappa}$ on the target), inducing canonical horizontal isomorphisms $(h^*)^* : \underline{H}^\bullet_{m,\mathcal{H},\Sigma} \xrightarrow{\sim} \underline{H}^\bullet_{m,\mathcal{H},\Sigma}$ and $(h^*)_* : \underline{H}_{m,\mathcal{H},\Sigma,\bullet} \xrightarrow{\sim} \underline{H}_{m,\mathcal{H},\Sigma,\bullet}$. (See* (5.3) *and* (5.4).) *The canonical isomorphism $(h^*)_* : \underline{H}_{m,\mathcal{H},\Sigma,\bullet} \xrightarrow{\sim} \underline{H}_{m,\mathcal{H},\Sigma,\bullet}$ agree with the functorial image under $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ of the canonical isomorphism $h^* : \wedge^\bullet(L_1^{\oplus m}) \cong \wedge^\bullet(\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m}, L_1)) \xrightarrow{\sim} \wedge^\bullet(\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m}, L_1)) \cong \wedge^\bullet(L_1^{\oplus m})$ induced by $h \underset{\mathbb{Z}}{\otimes} R_1$.*

Proof. Let $Q := \mathcal{O}^{\oplus m}$, and let $Q' \subset Q \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ be the $\mathcal{O}$-lattice such that $Q' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p = h(Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p)$ as submodules of $Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}$. Let $g_l \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}}(Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p})$ and $g'_l \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}}(Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p})$ be elements such that $Q' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p = g_l(Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p)$ and $Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p = g'_l(Q' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p)$ corresponds respectively the two maps $h : Q \longrightarrow Q'$ and $\mathrm{incl} : Q' \longrightarrow Q$ (which exist by approximation, or rather by the theory of lattices). Then the proposition follows from (a special case of) [38, (5) of Thm. 2.15], by composing the two proper log étale morphisms obtained with $g_l \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}}(Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p})$ and $g'_l \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}}(Q' \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p})$, respectively. The last statement is true because (by definition) it is so after pullback to the open dense subscheme $\mathsf{M}_{\mathcal{H}}$, and because of the local freeness of the canonical extensions $\underline{H}_{m,\mathcal{H},\Sigma,\bullet} \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\wedge^\bullet(L_1^{\oplus m}))$. □

Corollary 5.11. *There is a canonical* **right action** *of $\mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})$ on $\underline{H}_{m,\mathcal{H},\Sigma,\bullet}$, or rather a* **left action** *of $\mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})^{\mathrm{op}}$, given by horizontal morphisms, which can be realized as an R_1-linear combination of automorphisms $(h^*)_*$ as in Corollary* 5.11. *This left action agrees with the functorial image under $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ of the left action of the natural left action of $\mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})^{\mathrm{op}}$ on $\wedge^\bullet(L_1^{\oplus m}) \cong \wedge^\bullet(\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m}, L_1))$.*

Proof. Let us take the action to be the functorial image under $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ of the right action of $\mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})$ on $\wedge^\bullet(L_1^{\oplus m}) \cong \wedge^\bullet(\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m}, L_1))$. As in the proof of [39, Lem. 5.2.2.3], by adding suitable integers to generators, $\mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus m})$ is spanned (over $\mathbb{Z}$) by elements invertible in $\mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus m}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$. Hence the corollary follows by taking R_1-linear combinations. □

The natural right action of $\mathcal{O}_1^m \rtimes \mathfrak{S}_m$ on $\mathcal{O}_1^{\oplus m}$ (by multiplication and permutation) defines a ring homomorphism $R_1[\mathcal{O}_1^m \rtimes \mathfrak{S}_m]^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})$, or equivalently a ring homomorphism $R_1[\mathcal{O}_1^m \rtimes \mathfrak{S}_m] \longrightarrow \mathrm{End}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus m})^{\mathrm{op}}$. Therefore, by Corollary 5.11, we obtain a left action of $R_1[\mathcal{O}_1^m \rtimes \mathfrak{S}_m]$ on $\underline{H}_{m,\mathcal{H},\Sigma,\bullet}$, given by horizontal morphisms, which can be realized as an R_1-linear combination of isomorphisms $(h^*)_*$ as in Corollary 5.11. In fact, the same proof of Corollary 5.11 (by the argument of adding suitable integers) shows that we can realize the morphisms as an R_1-linear combination of morphisms $(h^*)_*$ as in Corollary 5.11 with h in the homomorphic image of $\mathbb{Z}[\mathcal{O}^m \rtimes \mathfrak{S}_m]^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus n})$. As a result:

Corollary 5.12. *Let $\varepsilon_n^{\mathrm{L}}$, $\varepsilon_\mu^{\mathrm{S}}$, and $\varepsilon_\mu^{\mathrm{Y}}$ be the elements in $R_1[\mathcal{O}_1^n \rtimes \mathfrak{S}_n]$ defined in* [42, §§3.2–3.3] *which act as idempotents on $L_1^{\oplus n} \cong \mathrm{Hom}_{\mathcal{O}_1}(\mathcal{O}_1^{\oplus n}, L_1)$. Then the horizontal idempotent endomorphisms $(\varepsilon_n^{\mathrm{L}})_*$, $(\varepsilon_\mu^{\mathrm{S}})_*$, and $(\varepsilon_\mu^{\mathrm{Y}})_*$ of $\underline{H}_{n,\mathcal{H},\Sigma,\bullet}$ in Proposition* 5.8 *can be realized as an R_1-linear combination of automorphisms $(h^*)_*$ as in Corollary* 5.11 *(with h in the image of $\mathbb{Z}[\mathcal{O}^n \rtimes \mathfrak{S}_n]^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus n})$). By duality, the horizontal idempotent endomorphisms $(\varepsilon_n^{\mathrm{L}})^*$, $(\varepsilon_\mu^{\mathrm{S}})^*$, and $(\varepsilon_\mu^{\mathrm{Y}})^*$ of $\underline{H}^\bullet_{n,\mathcal{H},\Sigma}$ in Proposition* 5.8 *can be realized as R_1-linear combinations of automorphisms $(h^*)^*$ as in Corollary* 5.11 *(with similar h).*

5.5. **Extensions of polarizations.** To extend the idempotents $(\varepsilon_\mu^\lambda)_*$ or $(\varepsilon_\mu^\lambda)^*$, we have to extend the Chern classes we are using. According to [42, §3.4 and §3.6], we need the Chern classes of the pullbacks of Poincaré line bundles, or rather the Chern classes of the pullbacks of $(\mathrm{Id} \times \lambda)^*\mathcal{P}_A$. (We prefer to work with this line bundle, so that technically we no longer have to consider the polarization λ, for which we do not have a ready extension over the toroidal compactificaitons of Kuga families.)

Let $m \geq 0$ be an integer. For $1 \leq i < j \leq m$, let $\mathcal{L}_{m,i,j} := \mathrm{pr}_{ij}^*(\mathrm{Id} \times \lambda)^*\mathcal{P}_A$, where $\mathrm{pr}_{ij} : \mathsf{N}_m = A^m \longrightarrow A \underset{\mathsf{M}_\mathcal{H}}{\times} A$ is the projection to the i-th and j-th factors. For each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$, since $\mathsf{N}^{\mathrm{tor}}_{m,\kappa}$ is projective and smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, it is noetherian and locally factorial. Then, by [19, IV-4, 21.6.11], the canonical restriction morphism $\mathrm{Pic}(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}) \longrightarrow \mathrm{Pic}(\mathsf{N})$ is *surjective*, and there exists some line bundle $\mathcal{L}_{m,i,j,\kappa}$ over $\mathsf{N}^{\mathrm{tor}}_{m,\kappa}$ extending $\mathcal{L}_{m,i,j}$ over N_m.

Proposition 5.13. *Let $\mathcal{L}_{m,i,j,\kappa}$ be any extension of $\mathcal{L}_{m,i,j}$ as above. The (log) first Chern class c_{ij} of $\mathcal{L}_{m,i,j,\kappa}$ in $\underline{H}^2_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1})(1) = \underline{H}^2_{m,\kappa}(1) \cong \underline{H}^2_{m,\mathcal{H},\Sigma}(1)$ is independent of the choice of $\mathcal{L}_{m,i,j,\kappa}$. Consequently, since $\underline{H}^1_{m,\mathcal{H},\Sigma} \cong (\underline{H}^1_{1,\mathcal{H},\Sigma})^{\oplus m} \cong (\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}})^{\oplus m}$ (by the characterization as in* [38, Prop. 6.9]*) and $\underline{H}^\bullet_{m,\mathcal{H},\Sigma} \cong \wedge^\bullet \underline{H}^1_{m,\mathcal{H},\Sigma}$ (by* [38, Thm. 2.15]*), cup product with c_{ij} induces a well-defined morphism $(\underline{H}^1_{1,\mathcal{H},\Sigma})^{\otimes(m-2)} \longrightarrow (\underline{H}^1_{1,\mathcal{H},\Sigma})^{\otimes m}(1)$, canonically dual to the morphism $\underline{H}^{\otimes m}_{1,\mathcal{H},\Sigma,1} \longrightarrow \underline{H}^{\otimes(m-2)}_{1,\mathcal{H},\Sigma,1}(1)$ induced by evaluating the canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_\lambda : \underline{H}_{1,\mathcal{H},\Sigma,1} \times \underline{H}_{1,\mathcal{H},\Sigma,1} \longrightarrow \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(1)$ as in* [38, Prop. 6.9] *on the i-th and j-th factors.*

Proof. The (log) first Chern class c_{ij} is independent of the choice of the extension $\mathcal{L}_{m,i,j,\kappa}$ of $\mathcal{L}_{m,i,j}$ because the extension is unique up to a divisor supported on the boundary. The second statement then follows because the pairings (once defined) are canonically determined by their restrictions to $\mathsf{M}_{\mathcal{H},1}$. $\square$

5.6. **Strong geometric plethysm.**

Proposition 5.14. *Suppose $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$, with $0 \leq n := |\mu|_{\mathrm{L}} < p$, as in* [42, Def. 3.2]*. Suppose moreover that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Let $(\varepsilon_\mu)^*$ be the horizontal idempotent endomorphism of $\underline{H}^\bullet_{n,\mathcal{H},\Sigma}$ as in Proposition* 5.8*, and let t_μ be as in* [42, Prop. 3.7]*, so that we have the canonical* **horizontal** *isomorphism $(\underline{V}^\vee_{[\mu]})^{\mathrm{can}} \cong (\varepsilon_\mu)^*\,\underline{H}^n_{n,\mathcal{H},\Sigma}$ in* (5.9) *realizing in particular the de Rham complex of $(\underline{V}^\vee_{[\mu]})^{\mathrm{can}}$ as a horizontal summand of the de Rham complex of $\underline{H}^n_{n,\mathcal{H},\Sigma}$. Then there* *exists a sequence of cone decompositions* $\{\Sigma_i\}_{0\leq i\leq t}$*, a sequence* $\{h_i\}_{1\leq i<t}$ *of elements in* $\mathrm{End}_{\mathcal{O}}(\mathcal{O}^{\oplus n})$*, two sequences* $\{\kappa_i\}_{0\leq i\leq t}$ *and* $\{\kappa'_i\}_{0\leq i<t}$ *(whose meanings will be explained below), and a sequence of global sections* c_i *of either* $\underline{H}^0_{n,\mathcal{H},\Sigma}$ *or* $\underline{H}^2_{n,\mathcal{H},\Sigma}(1)$*, such that:*

(1) $\Sigma_0 = \Sigma$*, and, for each* $0 \leq i < t$*,* Σ_{i+1} *is a refinement of* Σ_i*.*
(2) For each $0 \leq i \leq t$ *(resp.* $0 \leq i < t$*),* κ_i *(resp.* κ'_i*) is an element in* $\mathbf{K}_{n,\mathcal{H},\Sigma_i}$*.*
(3) For each $0 \leq i < t$*, there exists a proper log étale morphism* $(h_i)^*_{\kappa'_i,\kappa_i} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa'_i} \longrightarrow \mathsf{N}^{\mathrm{tor}}_{n,\kappa_i}$ *as in Proposition* 5.10*.*
(4) For each $0 \leq i < t$*, we have* $\kappa'_i \succ \kappa_i$ *in* $\mathbf{K}_{n,\mathcal{H},\Sigma_i}$*, so that there is a proper log étale morphism* $f_{\kappa'_i,\kappa_i} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa'_i} \longrightarrow \mathsf{N}^{\mathrm{tor}}_{n,\kappa_i}$ *as in* [38, Thm. 2.15]*.*
(5) For each $0 \leq i < t$*, there exists a proper log étale morphism* $[1]_{\kappa_{i+1},\kappa'_i} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa_{i+1}} \longrightarrow \mathsf{N}^{\mathrm{tor}}_{n,\kappa'_i}$ *as in Proposition* 5.6 *(with* (Σ_i,κ'_i) *and* $(\Sigma_{i+1},\kappa_{i+1})$ *here standing respectively for* (Σ,κ) *and* (Σ',κ') *there).*
(6) For each $0 < i \leq t$*,* c_i *is either a scalar in* $\underline{H}^0_{n,\mathcal{H},\Sigma_i}$*, or the (log) Chern class of some line bundle in* $\underline{H}^2_{n,\mathcal{H},\Sigma_i}(1)$*.*
(7) By abuse of notation, for each $0 \leq i < t$*, let* h_i^* *be the composition of the morphisms* $[1]_{\kappa_t,\kappa'_{t-1}} \circ f_{\kappa'_{t-1},\kappa_{t-1}}$*,* $[1]_{\kappa_{t-1},\kappa'_{t-2}} \circ f_{\kappa'_{t-2},\kappa_{t-2}}$*, ...,* $[1]_{\kappa_3,\kappa'_2} \circ f_{\kappa'_2,\kappa_2}$*, and* $[1]_{\kappa_2,\kappa'_1} \circ f_{\kappa'_1,\kappa_1}$*, but with (only one term)* $[1]_{\kappa_{i+1},\kappa'_i} \circ f_{\kappa'_i,\kappa_i}$ *replaced with* $[1]_{\kappa_{i+1},\kappa'_i} \circ (h_i)^*_{\kappa'_i,\kappa_i}$*. (We group the morphisms two-by-two so that the choice of* κ'_i *is irrelevant in practice.)*
(8) The summand $(\varepsilon_\mu)^* \underline{H}^n_{n,\mathcal{H},\Sigma}$ *can be constructed as in* [42, §§3.1–3.4 and §3.6] *using a finite number of* R_1*-linear combinations of the elements* $(h_i^*)^*(c_i)$*, for* $0 < i \leq t$*.*

(The same is true if we base change everything to an R_1*-algebra* R*.)*

Proof. This follows by the very construction in [42, §§3.1–3.4 and §3.6], and by Corollary 5.12 and Proposition 5.13. □

Remark 5.15. By Lemma 5.7, all the refinements of cone decompositions in Proposition 5.14 are harmless for our purpose.

Remark 5.16. Proposition 5.14 will not be used until Section 9.2 below.

6. Nilpotent residues: the case of automorphic bundles

In this section, we give two proofs of the nilpotence of the residue maps for automorphic bundles over $\mathsf{M}_{\mathcal{H},1}$. The first proof, in Section 6.1, will be a simple argument using the comparison in [37] and the explicit analytic local charts of toroidal compactifications in Ash–Mumford–Rapoport–Tai [2]. The second proof, in Section 6.2, will be a purely algebraic argument (in that it does not use any transcendental techniques), using only the definition of canonical extensions of automorphic bundles. With this, the proof of our main vanishing results becomes purely algebraic. (Nevertheless, this is inspired by the analytic argument, because it uses the étale local charts along the boundary.)

6.1. **Analytic local charts.** Since a morphism between locally free sheaves of finite rank is zero if and only if its restriction to an open dense subset is zero, and since it is zero if and and only if it is so after a faithfully flat base change, it suffices to show that the residue maps over $\mathbb{C}$ are nilpotent. By [30, VI and VII], this is equivalent to showing that the monodromy transformations about irreducible components of the boundary divisor D on the analytifications of automorphic bundles (base changed to $\mathbb{C}$) are all *unipotent*.

By [37, Thm. 4.1.1], the analytification of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0,\mathbb{C}} := \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$ is an analytic toroidal compactification of the analytification of $\mathsf{M}_{\mathcal{H},0,\mathbb{C}} := \mathsf{M}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$. Therefore, according to [2, Ch. III, §5, Main Thm. I and its proof], the connected local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0,\mathbb{C}}$ about a boundary divisor are partial toroidal embeddings of punctured polydisk bundles with fundamental group canonically identified with an arithmetic subgroup of the unipotent radical of some maximal parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. (As usual, the rational structure of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ for this arithmetic subgroup depends on the connected component over which the chart lies.) As explained in [42, Lem. 8.14], the analytification of automorphic bundles are constructed tautologically using algebraic representations of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. Hence the monodromy transformations along irreducible components of the boundary divisor D are all unipotent, as desired. As a result:

Proposition 6.1. *For every $W \in \mathrm{Rep}_{R_1}(\mathrm{G}_1)$, the extended Gauss–Manin connection ∇ on $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R_1}(W)$ (see Definition 4.17) has nilpotent residues.*

Remark 6.2. The unipotence of monodromy (specifically in the center of the unipotent radical of some maximal parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$) is by no means an accident. Firstly, this is a consequence of the very construction — the so-called *toroidal compactifications* make use of algebraic tori built *exactly* from these centers of unipotent radicals of maximal parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$. Secondly, these unipotent monodromy transformations along the boundary also characterize the cone decompositions chosen in the construction of the toroidal compactification — see in particular [2, Ch. III, §7, Main Thm. II].

6.2. **A purely algebraic argument.** Inspired by the (still transcendental) argument in Section 6.1, we now present a purely algebraic argument, using only the construction of automorphic bundles and the (algebraic) construction of toroidal compactifications. The idea is not complicated, but unlike the transcendental local charts in [2, Ch. III], their algebraic analogues are noncanonical and can only be described étale locally.

Let U_1 be the unipotent radical of the parabolic subgroup P_1 of G_1. Then $\mathfrak{u}_1 := \mathrm{Lie}(\mathrm{U}_1)$ is the unipotent radical of the parabolic subalgebra $\mathfrak{p}_1 := \mathrm{Lie}(\mathrm{P}_1)$ of $\mathfrak{g}_1 := \mathrm{Lie}(\mathrm{G}_1)$. Let $\mathfrak{p}_1^-$ be the parabolic subalgebra of $\mathfrak{g}_1$ *opposite* to $\mathfrak{p}_1$, and let $\mathfrak{u}_1^-$ be the unipotent radical of $\mathfrak{p}_1^-$. Let us abusively denote $(\mathfrak{u}_1^-)^\vee$ as $\mathfrak{u}_1^\#$, which can be identified with $\mathfrak{u}_1$ when $p > 2$, or when $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau,R_1}$ for some $\tau \in \Upsilon$.

For each R_1-algebra R, let $\mathrm{Sym}_{\leq 1}(\mathfrak{u}_R^\#) := \mathrm{Sym}(\mathfrak{u}_R^\#)/\mathrm{Sym}^{\geq 2}(\mathfrak{u}_R^\#) \cong (\mathrm{Sym}^{\leq 1}(\mathfrak{u}_R^-))^\vee$ (viewed as filtered, rather than graded, objects in $\mathrm{Rep}_R(\mathrm{P}_1)$).

Lemma 6.3. *For any isomorphism $\phi : R_1(1) \xrightarrow{\sim} R_1$ inducing an isomorphism $L_{0,1}(1) \longrightarrow L_{0,1}^\vee$ which we also denote by ϕ, the R_1-module $\mathfrak{u}_1^\#$ is isomorphic to*

$$(L_{0,1} \underset{R_1}{\otimes} L_{0,1}^\vee(1))/\begin{pmatrix} \phi(y) \otimes z - \phi(z) \otimes y \\ (b^\star x) \otimes y - x \otimes (by) \end{pmatrix}_{x \in L_{0,1}^\vee, y,z \in L_{0,1}^\vee(1), b \in \mathcal{O}_1} \tag{6.4}$$

in $\mathrm{Rep}_{R_1}(\mathrm{P}_1)$*. The same is true with* R_1 *replaced with any* R_1*-algebra* R*.*

Proof. The first statement is clear from the definition of P_1. The second statement follows from the freeness of the module in (6.4) over R_1, which follows from [39, Prop. 1.2.2.3], because we assume that p is a good prime. $\square$

Lemma 6.5. *Consider the morphism*

$$(6.6)\quad (\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1} \underset{R_1}{\otimes} R) \times \mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R) \longrightarrow \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}} \oplus \overline{\Omega}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} : (\xi, (r, u)) \mapsto (r, [\xi^{-1} u \xi]),$$

where ξ *is any section of* $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$*, where* $(r, u) \in \mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R)$*, with* r *in degree zero and* u *in degree one, and where* $[\xi^{-1} u \xi]$ *is defined as follows: (For simplicity of notation, let us treat sections as global sections, although they might only exist locally.) Each section* ξ *of* $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$ *induces by definition (see Section 4.2) an isomorphism* $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R})^{\mathrm{can}} \overset{\sim}{\longrightarrow} (L_{0,1} \oplus L^{\vee}_{0,1}(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$ *(which we again denote by* ξ*) matching the natural filtrations, and hence also induces a splitting* $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R})^{\mathrm{can}} \cong \underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}} \oplus \underline{\mathrm{Lie}}^{\vee}_{(A^{\mathrm{ext}})^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$ *(corresponding to the canonical splitting of* $L_{0,1} \oplus L^{\vee}_{0,1}(1)$*). Then* $\xi^{-1} u \xi$ *induces a morphism* $\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}} \longrightarrow \underline{\mathrm{Lie}}^{\vee}_{(A^{\mathrm{ext}})^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}$*, and hence induces a section of* $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ *under the extended Kodaira–Spencer morphism (4.3), which we denote by* $[\xi^{-1} u \xi]$*.*

The morphism (6.6) induces a morphism (see Definition 4.12)

$$(6.7)\qquad \mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R)) \longrightarrow \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}} \oplus \overline{\Omega}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R} \cong \overline{\mathscr{P}}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R},$$

because $p(\xi, (r, u)) = (p\xi, (r, pup^{-1}))$ *and* $(\xi, (r, u))$ *have the same image* $(r, [(p\xi)^{-1}(pup^{-1})(p\xi)]) = (r, [\xi^{-1} u \xi])$ *for each section* p *of* $\mathrm{P}_1 \underset{R_1}{\otimes} R$*. This morphism is an isomorphism of* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$*-modules.*

Proof. By trivializing $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}$ étale locally, we see that the morphism (6.7) is a morphism of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules. By definition, it sends the canonical submodule $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathfrak{u}^{\#}_R)$ of $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R))$ to the canonical submodule $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}} \oplus \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$, and the induced morphism $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathfrak{u}^{\#}_R) \longrightarrow \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}$ is an isomorphism by Lemma 6.3 and by the extended Kodaira–Spencer isomorphism (4.4). Also by definition, the induced morphism $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(R) \longrightarrow \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$ between quotient modules is an isomorphism. Hence (6.7) is an isomorphism. $\square$

Proposition 6.8. *Let* R *be any* R_1*-algebra, and let* $W \in \mathrm{Rep}_R(\mathrm{G}_1)$*. Under the functor* $\mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\,\cdot\,)$*, the canonical morphism*

$$(6.9)\quad W^{\vee} \underset{R}{\otimes} \mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R) \longrightarrow W^{\vee} \underset{R}{\otimes} \mathfrak{u}^{\#}_R : w \otimes (c + e) \mapsto w \otimes e + \sum_{1 \leq j \leq d} (y_j w) \otimes (c f_j)$$

for $c \in R_1$*,* $e \in \mathfrak{u}^{-}_R$*,* $w \in W^{\vee}$*, and any free* R*-basis* $y_1, \ldots, y_d$ *of* $\mathfrak{u}^{-}_R$ *dual to a free* R*-basis* $f_1, \ldots, f_d$ *of* $\mathfrak{u}^{\#}_R$*, is associated with the canonical morphism*

$$(6.10)\qquad \mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^{\vee}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R} \longrightarrow \mathscr{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^{\vee}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$$

inducing (by composition with the canonical morphism $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^\vee) \longrightarrow \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ *induced by the canonical morphism* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}} \longrightarrow \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$*) the extended Gauss–Manin connection* (4.15) *(see Definition* 4.17*).*

Proof. The canonical morphism $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R} \longrightarrow \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ corresponds to $\mathrm{Sym}_{\leq 1}(\mathfrak{u}^{\#}_R) \longrightarrow \mathfrak{u}^{\#}_R : c + e \mapsto e$ for $c \in R$ and $e \in \mathfrak{u}^-_R$. The canonical morphism (6.10) (inducing the extended Gauss–Manin connection (4.15)) is defined by (the restriction to $\overline{\mathrm{pr}}_2^*(\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^\vee))$ of) $\overline{s}^* - \overline{\mathrm{Id}}^*$ on $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W^\vee) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$, satisfying $(\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes x) = ((\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes 1))x + z \otimes ((\overline{s}^* - \overline{\mathrm{Id}}^*)x)$ for all section z of $\mathcal{E}_{\mathrm{P}_1,R}(W^\vee)$ and all section x of $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ (because $((\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes 1)) \otimes ((\overline{s}^* - \overline{\mathrm{Id}}^*)x) = 0$). Since $(\overline{s}^* - \overline{\mathrm{Id}}^*)x$ is known to agree with the image of the canonical morphism $\overline{\mathscr{P}}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R} \longrightarrow \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ when restricted to sections x in $\overline{\mathrm{pr}}_2^*(\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}})$, it remains to study $(\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes 1)$.

Consider the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},R} \longrightarrow \mathsf{M}_{\mathcal{H},R}$. Then $\mathrm{pr}_i^* \, \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R}) \cong \underline{H}^{\mathrm{dR}}_1(\mathrm{pr}_i^* \, A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},R})$, and we obtain a morphism $(s^* - \mathrm{Id}^*) : \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R}) \longrightarrow \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$. For each section v of $\underline{\mathrm{Der}}^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$, we obtain a morphism $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R}) \longrightarrow \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R})$ respecting $\langle \cdot, \cdot \rangle_\lambda$, and inducing a trivial action on the top Hodge graded piece. If we identify $(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},R}), \langle \cdot, \cdot \rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},R}})$ with $((L_R \oplus L_R^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}, \langle \cdot, \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}(1), L_R^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}))$ by any section of $\mathcal{E}_{\mathrm{P}_1} \underset{R_1}{\otimes} R$, this morphism induced by v defines a section u_v of the pullback of $\mathfrak{u}^-_R$ to $\mathsf{M}_{\mathcal{H},R}$. Since everything is canonically extended (by open density of $\mathsf{M}_{\mathcal{H},R}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}$), the same is true over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}$. (This is compatible with the identification $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(\mathfrak{u}^{\#}_R) \cong \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ based on Lemma 6.3 and the (extended) Kodaira–Spencer isomorphism (4.4).) Hence $v(\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes 1) = u_v(z)$, and so $(\overline{s}^* - \overline{\mathrm{Id}}^*)(z \otimes 1) = \sum_{1 \leq j \leq d} (y_j z) \otimes f_j$ by duality, as desired. □

Proposition 6.11. *For each R_1-algebra R and each $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, the extended Gauss–Manin connection ∇ on $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$ (see Definition* 4.17*) has nilpotent residues.*

Proof. By passing to the complement of the branches of D other than any given one, it suffices to verify the nilpotence of the residue maps over open dense subschemes of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}$ over which the pullback of D is an irreducible smooth divisor. By étale localization, we may assume that D is a coordinate hyperplane in an affine space, defined by the vanishing of some variable t. Consider the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}$-dual $\overline{\underline{\mathrm{Der}}}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ of $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$. Then the section $v := t(\partial/\partial t)$ defines a section of $\overline{\underline{\mathrm{Der}}}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$ (dual to the section $\mathrm{d}(\log(t)) = t^{-1}(\mathrm{d}\,t)$ of $\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}$), which corresponds to a section u_v of the pullback of $\mathfrak{u}_R^-$ to $\mathsf{M}_{\mathcal{H},\Sigma,R}^{\mathrm{tor}}$ under the (extended) Kodaira–Spencer isomorphism (4.4).

By Proposition 6.8, the evaluation of the extended Gauss–Manin connection ∇ at u_v is defined étale locally by $w \otimes x \mapsto w \otimes (t(\partial x/\partial t)) + (u_v(w)) \otimes x$, for each $w \in W$ and each local section x of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma,R}^{\mathrm{tor}}}$. By reduction modulo t, this induces the residue map $w \otimes \bar{x} \mapsto u_v(w) \otimes \bar{x}$, which is nilpotent because u_v is a section of the pullback of the nilpotent Lie algebra $\mathfrak{u}_R^-$. (In particular, if $(\mathfrak{u}_R^-)^{i_0}(W) = 0$ for some $i_0 \geq 0$, then the i_0-th power of the residue map of $\mathcal{E}_{\mathrm{G}_1,R}^{\mathrm{can}}(W)$ along each irreducible component of the boundary divisor is also zero.) □

Remark 6.12. For readers familiar with the constructions of both [2] and [39], we point out that we do not need the full Lie algebra $\mathfrak{u}_R^-$ in the proof of Proposition 6.11: (We shall use the notations of [39] freely in the remainder of this remark.) According to [39, Prop. 6.2.5.18], along a codimension-one boundary stratum labeled by the class of some $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (with $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}^+$ a cone of dimension one in $\Sigma_{\Phi_{\mathcal{H}}}$), we only need a Lie subalgebra of $\mathfrak{u}_R^-$ isomorphic to $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee \underset{\mathbb{Z}}{\otimes} R_1$. (This is consistent with the proof of Proposition 6.1: By [37, Cor. 3.6.10], when $R_1 = \mathbb{C}$, the comparison in the second paragraph of Section 6.1 using [37, Thm. 4.6.1] matches $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ with the Lie algebra of the center of the unipotent radical of the parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ mentioned in the proof of Proposition 6.1. However, this "consistency" is by no means tautological, because it involves objects of completely disparate natures.)

7. Vanishing with automorphic coefficients

Let R be an R_1-algebra. We have defined the subset $\mathrm{W}^{\mathrm{M}_1}$ of the Weyl group in [42, Def. 2.29], and the sets of p-small weights $\mathrm{X}_{\mathrm{G}_1}^{+,<p}$ and $\mathrm{X}_{\mathrm{M}_1}^{+,<p}$ in [42, Def. 2.29]. Let w, μ, and ν be elements of these three sets, respectively. Then we define as in [42, §2.6 and §6.3] the objects $W_{\nu,R}$, $W_{\nu,R}^\vee$, $W_{w\cdot[\mu],R}$, and $W_{w\cdot[\mu],R}^\vee$ in $\mathrm{Rep}_R(\mathrm{G}_1)$, which give rise to the automorphic bundles $\underline{W}_{\nu,R}$, $\underline{W}_{\nu,R}^\vee$, $\underline{W}_{w\cdot[\mu],R}$, and $\underline{W}_{w\cdot[\mu],R}^\vee$ over $\mathsf{M}_{\mathcal{H},R}$. Their canonical and subcanonical extensions will be denoted with superscripts "can" and "sub", respectively. By the functoriality of the constructions, these sheaves satisfy obvious compatibilities with each other.

7.1. Automorphic line bundles and positivity.

Recall (see [42, Def. 7.1]) that we say an element ν in $\mathrm{X}_{\mathrm{M}_1}^{+,<p}$ is a generalized parallel weight if W_ν is a rank one free R_1-module, and we say ν is *positive* if $\underline{W}_\nu$ is ample over $\mathsf{M}_{\mathcal{H},\Sigma,1}$. (Note that this does *not* imply the ampleness of either $\underline{W}_\nu^{\mathrm{can}}$ or $\underline{W}_\nu^{\mathrm{sub}}$, except in very special cases.)

Lemma 7.1 (cf. [42, Lem. 7.9])**.** *The line bundle ω over $\mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}$ is isomorphic to $\underline{W}_{\nu_\omega}^{\mathrm{can}}$ for the generalized parallel weight ν_ω with coefficients $(k_{\omega,\tau})_{\tau\in\Upsilon}$ satisfying $k_{\omega,\tau} = \mathrm{rk}_{R_1}(V_\tau)$.*

Proof. This is because $\omega = \wedge^{\mathrm{top}} \ \underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}^\vee$ by definition, and because $\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}}^\vee \cong \underline{\mathrm{Lie}}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}}^\vee \cong \mathcal{E}_{\mathrm{M}_1,R}^{\mathrm{can}}(L_{0,1}^\vee)$ as vector bundles over $\mathsf{M}_{\Sigma,\mathcal{H}}^{\mathrm{tor}}$ (ignoring Tate twists). (See Section 4.2, [38, §6B], and [42, Ex. 1.22].) □

Proposition 7.2 (Correction of the original published version; cf. [42, Prop. 7.10; see also the errata])**.** *The canonical extension $\underline{W}_\nu^{\mathrm{can}}$ defines a torsion element in* *the Picard group of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ *if its coefficients satisfy the condition that* $(k_\tau)_{\tau\in\Upsilon}$ *of* ν *satisfy* $k_\tau + k_{\tau\circ c} = 0$.

Proof. The proof of [42, Prop. 7.10 in the errata] works here as well, with the automorphic bundles replaced with their canonical extensions. □

Corollary 7.3. *If* $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ *is* **$\mathbb{Q}$-simple** (*as in* [42, Def. 7.11]*; i.e.,* $\Upsilon = \mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$ *has a single equivalence class under* $\sim_{\mathbb{Q}}$ *as in* [42, Def. 7.12]), *and if* $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *is* **parallel** (*see* [42, Def. 7.14]) *with coefficients* $(k_\tau)_{\tau\in\Upsilon}$ *satisfying* $[k]_\tau = k_\tau + k_{\tau\circ c} > 0$ *for all* $\tau \in \Upsilon$, *then* $\underline{W}^{\mathrm{can}}_\nu$ *satisfies the following condition:*

$$\exists\, r_0 > 0 \text{ such that } (\underline{W}^{\mathrm{can}}_\nu)^{\otimes r}(-\mathsf{D}') \text{ is ample for every } r \geq r_0, \tag{7.4}$$

where D' *is the divisor* D' *on* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *in* (5) *of Proposition* 4.2.

Proof. Under the assumptions, a positive tensor power of $\underline{W}^{\mathrm{can}}_\nu$ is isomorphic to a positive tensor power of ω by Lemma 7.1 and Proposition 7.2. Then the desired condition (7.4) follows from (4.5). □

The rest of this subsection will be devoted to proving an analogue of Corollary 7.3 when $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ is *not* necessarily $\mathbb{Q}$-simple. Readers interested only in the $\mathbb{Q}$-simple case can skip the rest of this subsection.

Over $\mathsf{M}_{\mathcal{H}}$, as in [42, Prop. 7.15], we argued by decomposing F into $\mathbb{Q}$-simple factors, by decomposing $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ accordingly, and by replacing $\mathcal{H}$ with a finite index subgroup (which results in passing to a finite cover of $\mathsf{M}_{\mathcal{H}}$, which does not affect ampleness of line bundles), such that there exists a finite morphism from $\mathsf{M}_{\mathcal{H},0}$ to a product of (base changes from possibly smaller rings of) analogous moduli problems defined by $\mathbb{Q}$-simple data. A natural question is whether similar statements hold for the toroidal or minimal compactifications.

Let us denote by $\Upsilon/{\sim_{\mathbb{Q}}}$ the set of equivalence classes $[\tau]_{\mathbb{Q}}$ modulo the relation $\sim_{\mathbb{Q}}$. Then the following is self-explanatory (which spells out what we have already used in the proof of [42, Prop. 7.15]):

Lemma 7.5. *Suppose a finite index suborder of* $\mathcal{O}$ *decomposes into a product* $\prod\limits_{[\tau]_{\mathbb{Q}}\in\Upsilon/\sim_{\mathbb{Q}}} \mathcal{O}_{[\tau]_{\mathbb{Q}}}$ *of orders in simple algebras over* $\mathbb{Q}$ (*stable under* $\star$). *Let* $1_{[\tau]_{\mathbb{Q}}}$ *be the identity element of* $\mathcal{O}_{[\tau]_{\mathbb{Q}}}$, *defining canonically an idempotent element of* $\mathcal{O}$. *By replacing* $\mathcal{H}$ *with a finite index subgroup, we may assume that the assignment*

$$(A, \lambda, i, \alpha_{\mathcal{H}}) \mapsto \big((A_{[\tau]_{\mathbb{Q}}}, \lambda_{[\tau]_{\mathbb{Q}}}, i_{[\tau]_{\mathbb{Q}}}, \alpha_{\mathcal{H},[\tau]_{\mathbb{Q}}})\big)_{[\tau]_{\mathbb{Q}}\in\Upsilon/\sim_{\mathbb{Q}}}, \tag{7.6}$$

where $A_{[\tau]_{\mathbb{Q}}} := 1_{[\tau]_{\mathbb{Q}}}(A)$ (*the image of the endomorphism* $i(1_{[\tau]_{\mathbb{Q}}})$ *of* A), *and* $\lambda_{[\tau]_{\mathbb{Q}}}$, $i_{[\tau]_{\mathbb{Q}}}$, $\alpha_{\mathcal{H},[\tau]_{\mathbb{Q}}}$ *are defined by restrictions* (*which are well defined thanks to the Rosati condition of* i), *defines a quasi-finite morphism*

$$\mathsf{M}_{\mathcal{H}} \longrightarrow \prod_{[\tau]_{\mathbb{Q}}\in\Upsilon/\sim_{\mathbb{Q}}} \mathsf{M}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}}}, \tag{7.7}$$

where each $\mathsf{M}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}}}$ *is the pullback* (*from a ring possibly smaller than* $\mathcal{O}_{F_0}$) *of a moduli problem defined by some* **simple** *linear algebraic datum.*

Proposition 7.8. *Keep the setting of Lemma* 7.5. *For any given cone decompositions* $\{\Sigma_{[\tau]_{\mathbb{Q}}}\}_{[\tau]_{\mathbb{Q}}\in\Upsilon/\sim_{\mathbb{Q}}}$, *up to replacing* Σ *with a refinement if necessary, we may* *assume that there exists a* **proper** *morphism*

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \longrightarrow (\mathsf{M}^{\mathrm{tor}})' := \prod_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}} \tag{7.9}$$

compatible with (7.7) (*and compatible among each other for different choices of cone decompositions*).

Proof. For any given $\{\Sigma_{[\tau]_{\mathbb{Q}}}\}_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}}$, the decomposition of our linear algebraic data into simple factors allows us to make sense of the *fiber product* $\prod\limits_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}} \Sigma_{[\tau]_{\mathbb{Q}}}$ in the obvious way. Then we can take Σ to be any refinement of this fiber product. Since the assignment (7.6) makes sense if we apply it to the data $(A^{\mathrm{ext}}, \lambda^{\mathrm{ext}}, i^{\mathrm{ext}})$ in (1) of Proposition 4.2, the *extension property* in the last statement of [39, Thm. 6.4.1.1], or rather the *universal property* of toroidal compactifications among degenerations over noetherian normal base schemes of a particular pattern controlled by the cone decompositions, asserts the (unique) existence of the morphism (7.9), which is proper because $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is proper over S_0. Since this morphism is defined by the universal property, it is compatible with similar morphisms defined by different choices of cone decompositions. (Since we allow replacements of $\mathcal{H}$ with finite index subgroups, we need to allow replacement of Σ with refinements due to the necessity in checking running technical assumptions on Σ, such as smoothness.) $\square$

Proposition 7.10. *With the setting of Lemma* 7.5 *and Proposition* 7.8*, there is a* **finite** *morphism*

$$\mathsf{M}^{\min}_{\mathcal{H}} \longrightarrow (\mathsf{M}^{\min})' := \prod_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}} \mathsf{M}^{\min}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}}} \tag{7.11}$$

extending (7.7) *and compatible with* (7.9) (*with any cone decompositions*).

Proof. By (4) of Proposition 4.2, $\mathsf{M}^{\min}_{\mathcal{H}} \cong \mathrm{Proj}\Big(\underset{r\geq 0}{\oplus}\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes r})\Big)$. Similarly, if we denote by $\omega_{[\tau]_{\mathbb{Q}}}$ the corresponding line bundle over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}}$, then $\mathsf{M}^{\min}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}}} \cong \mathrm{Proj}\Big(\underset{r\geq 0}{\oplus}\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}}, \omega_{[\tau]_{\mathbb{Q}}}^{\otimes r_{[\tau]_{\mathbb{Q}}}})\Big)$. Since ω is naturally isomorphic to the pullback of

$$\omega' := \underset{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}}{\otimes} \big((\mathsf{M}^{\mathrm{tor}})' \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}}\big)^* \omega_{[\tau]_{\mathbb{Q}}}$$

under (7.9), we have a canonical morphism

$$\begin{aligned} &\mathrm{Proj}\Big(\underset{r\geq 0}{\oplus}\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes r})\Big) \longrightarrow \mathrm{Proj}\Big(\underset{r\geq 0}{\oplus}\Gamma\big((\mathsf{M}^{\mathrm{tor}})', (\omega')^{\otimes r}\big)\Big) \\ &\cong \prod_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}} \mathrm{Proj}\Big(\underset{r\geq 0}{\oplus}\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}}, \omega_{[\tau]_{\mathbb{Q}}}^{\otimes r_{[\tau]_{\mathbb{Q}}}})\Big) \end{aligned} \tag{7.12}$$

realizing the morphism (7.11). By [39, Prop. 7.2.2.3], with $Z = (\mathsf{M}^{\mathrm{tor}})'$, $\mathcal{M} = \omega'$, and $\mathcal{E} = \mathscr{O}_{(\mathsf{M}^{\mathrm{tor}})'}$, and with $\mathcal{F}$ being the (coherent) pushforward of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ under the proper morphism (7.9), we see that (7.12) and hence (7.11) are *finite*. $\square$

Proposition 7.13. *Suppose* $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *is a generalized parallel weight with coefficients* $(k_\tau)_{\tau\in\Upsilon}$*. If* ν *is* **parallel** (*see* [42, Def. 7.14]) *in the sense that* $[k]_\tau = k_\tau + k_{\tau\circ c}$ *depends only on the equivalence class of* τ *in* $\Upsilon/\sim_{\mathbb{Q}}$*, then a nonzero tensor power of* $\underline{W}^{\mathrm{can}}_\nu$ *descends to* $\mathsf{M}^{\min}_{\mathcal{H},1}$*. If moreover* ν *is* **positive** *in the sense that* $[k]_\tau > 0$

for every $\tau \in \Upsilon$ (*cf.* [42, Def. 7.1 and Prop. 7.15]), *then a positive tensor power of* $\underline{W}_{\nu}^{\mathrm{can}}$ *descends to an* **ample line bundle** *over* $\mathsf{M}_{\mathcal{H},1}^{\min}$.

Proof. Let us retain the notation in the proof of Proposition 7.10. By Lemma 7.1 and Proposition 7.2, there exists some *even* integer $r > 0$ such that $(\underline{W}_{\nu}^{\mathrm{can}})^{\otimes r}$ is isomorphic to the pullback (under the composition of the canonical morphism $\mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}} \longrightarrow \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ and (7.9)) of the line bundle

$$\bigotimes_{[\tau]_{\mathbb{Q}} \in \Upsilon/\sim_{\mathbb{Q}}} \big((\mathsf{M}^{\mathrm{tor}})' \longrightarrow \mathsf{M}_{\mathcal{H}_{[\tau]_{\mathbb{Q}}},\Sigma_{[\tau]_{\mathbb{Q}}}}^{\mathrm{tor}}\big)^{*} \omega_{[\tau]_{\mathbb{Q}}}^{\otimes r_{[\tau]_{\mathbb{Q}}}}$$

for some integers $r_{[\tau]_{\mathbb{Q}}}$. By (4) of Proposition 4.2, this line bundle descends to a line bundle ω'' over $(\mathsf{M}^{\min})'$ because each $\omega_{[\tau]_{\mathbb{Q}}}$ does, and the pullback ω''' of this descended line bundle to $\mathsf{M}_{\mathcal{H},1}^{\min}$ (under the composition of the canonical morphism $\mathsf{M}_{\mathcal{H},1}^{\min} \longrightarrow \mathsf{M}_{\mathcal{H}}^{\min}$ and (7.11)) gives a descended form of $(\underline{W}_{\nu}^{\mathrm{can}})^{\otimes r}$, because (7.9) and (7.11) are compatible by Proposition 7.10. If ν is positive, then we may assume that $r_{[\tau]_{\mathbb{Q}}} > 0$ for every $[\tau]_{\mathbb{Q}}$. Then ω'' is *ample* over $(\mathsf{M}^{\min})'$ because the descended form of each $\omega_{[\tau]_{\mathbb{Q}}}$ is ample, and ω''' is ample over $\mathsf{M}_{\mathcal{H},1}^{\min}$ because (7.11) is finite, by Proposition 7.10 again. □

Now we can generalize Corollary 7.3 into:

Corollary 7.14. *If* $\nu \in \mathrm{X}_{\mathrm{M}_1}^{+,<p}$ *is a* **positive** *parallel weight, then the line bundle* $\underline{W}_{\nu}^{\mathrm{can}}$ *over* $\mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}$ *satisfies the following analogue of* (4.5):

(7.15) $\exists\, r_0 > 0$ *such that* $(\underline{W}_{\nu}^{\mathrm{can}})^{\otimes r}(-\mathsf{D}')$ *is ample for every* $r \geq r_0$,

where D' *is* (*by abuse of notation*) *the pullback of the divisor* D' *on* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *in* (5) *of Proposition* 4.2. *In particular,* $\underline{W}_{\nu}^{\mathrm{can}}$ *satisfies the condition* (3.25) *in Theorem* 3.24.

Proof. This is a combination of the second half of Proposition 7.13 with (5) of Proposition 4.2. □

7.2. Log integrality.

Proposition 7.16. *Let* $m \geq 0$ *be an integer. Up to replacing* Σ *with a refinement* (*see* [39, Def. 6.4.2.2]; *cf. Proposition* 5.6), *there exists an element* $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$ *such that the structural morphism* $f_{m,\kappa}^{\mathrm{tor}} : \mathsf{N}_{m,\kappa}^{\mathrm{tor}} \longrightarrow \mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}$ *is not only proper and log smooth, but also log integral* (*see* [28, Def. 4.3]). (*Then* $f_{m,\kappa}^{\mathrm{tor}}$ *is flat by* [28, Cor. 4.5].)

More precisely, $f_{m,\kappa}^{\mathrm{tor}}$ *is étale locally a morphism between toric schemes* (*equivariant under a morphism between tori*) *such that under the induced map between the* $\mathbb{R}$-*spans of cocharacter groups, the image of each cone in the cone decomposition used in the source is equal to* (*rather than just contained in*) *some cone in the cone decomposition used in the target.*

Proof. These follow from [38, Prop. 3.18 and 3.19], where the second pagraph follows from the explicit construction there (cf. also [18, Ch. VI, §§1–2]). □

Remark 7.17. For a fixed Σ, it is not known if there always exists a $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$ such that the structural morphism $f_{m,\kappa}^{\mathrm{tor}} : \mathsf{N}_{m,\kappa}^{\mathrm{tor}} \longrightarrow \mathsf{M}_{\mathcal{H},\Sigma,1}^{\mathrm{tor}}$ is log integral. Even if we only require *equidimensionality* (allowing other constructions, not necessarily given by [18, Ch. VI, §§1–2] or [38, §3]), it is still not known in general (even in the Siegel case) if there exist such equidimensional compactifying families *without* refining Σ (cf. [18, Ch. VI, Rem. 1.4] and [61, pp. 26–27]).

7.3. **Restatement of vanishing.** For simplicity, let us replace R_1 with its p-adic completion in this section, so that $R_1 = W(k_1)$.

Lemma 7.18. *Let $m \geq 0$ be any integer. For each Σ and each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$ such that the structural morphism $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ is log integral as in Proposition* 7.16*, the relative cohomology $\underline{H}^{\bullet}_{m,\kappa}$ satisfies Assumption* 1.4*.*

Proof. Let $\tilde{d}$ be the relative dimension of $f^{\mathrm{tor}}_{m,\kappa}$. By [38, Thm. 2.15], $\underline{H}^{\bullet}_{m,\kappa}$ satisfies Assumption 1.4(1). Hence it remains to show that $\underline{H}^{\bullet}_{m,\kappa}$ satisfies Assumption 1.4(2). Let us verify the conditions in Proposition 1.7: The condition (1) there is true by Proposition 1.13. On the other hand, by [38, Thm. 2.15] again, the canonical cup product pairing $\underline{H}^{\bullet}_{m,\kappa} \times \underline{H}^{2\tilde{d}-\bullet}_{m,\kappa} \longrightarrow \underline{H}^{2\tilde{d}}_{m,\kappa}$ can be identified with the canonical pairing $\wedge^{\bullet}\, \underline{H}^{1}_{m,\kappa} \times \wedge^{2\tilde{d}-\bullet}\, \underline{H}^{1}_{m,\kappa} \longrightarrow \wedge^{2\tilde{d}}\, \underline{H}^{1}_{m,\kappa}$, which is a perfect duality because $\underline{H}^{1}_{m,\kappa}$ is locally free of rank $2\tilde{d}$. This verifies the condition (2) in Proposition 1.7, as desired. □

Remark 7.19. According to [38, §5], there is a "log polarization"

$$\begin{aligned}(7.20)\qquad &\underline{\mathrm{Lie}}^{\oplus m}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}} \cong (f^{\mathrm{tor}}_{m,\kappa})_*(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}) \\ &\xrightarrow{\sim} R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong \underline{\mathrm{Lie}}^{\oplus m}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}\end{aligned}$$

whose restriction to $\mathsf{M}_{\mathcal{H},1}$ is the differential of a separable polarization of the abelian scheme $\mathsf{N}_m \longrightarrow \mathsf{M}_{\mathcal{H},1}$. Then (by [38, Thm. 2.15]) the determinant of (7.20) defines an $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$-module generator δ of $\underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}(\wedge^{\tilde{d}}\, \underline{\mathrm{Lie}}^{\oplus m}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}, \wedge^{\tilde{d}}\, \underline{\mathrm{Lie}}^{\oplus m}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}) \cong R^{\tilde{d}}(f^{\mathrm{tor}}_{m,\kappa})_*(\overline{\Omega}^{\tilde{d}}_{\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}) \cong \underline{H}^{2\tilde{d}}_{\log\text{-}\mathrm{dR}}(\mathsf{N}^{\mathrm{tor}}_{m,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}) = \underline{H}^{2\tilde{d}}_{m,\kappa}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1} \longrightarrow \mathsf{S}_1$ is proper, δ is up to multiplication by a unit in $R_1 = W(k_1)$ the canonical generator γ defined at the end of the proof of Proposition 1.13. Since $\delta|_{\mathsf{M}_{\mathcal{H},1}}$ is nothing but the determinant of a differential of a separable polarization of an abelian scheme, this gives an alternative proof of (1) in Proposition 1.7, without resorting to Grothendieck duality and the trace map.

Proposition 7.21 (cf. [42, Cor. 6.2])**.** *Suppose that $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$ with $n := |\mu|_{\mathrm{L}}$, and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Recall that $d = \dim_{\mathsf{S}_1}(\mathsf{M}_{\mathcal{H},1})$.* (*See* [42, Def. 3.9].) *Suppose moreover that $|\mu|_{\mathrm{re}} = d + n < p$. Let $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ be a* **positive** *parallel weight. Then we have:*

(1) $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, \underline{W}^{\mathrm{can}}_{-\nu,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \mathrm{Gr}_{\mathsf{F}}((\underline{V}^\vee_{[\mu],k_1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})) = 0$ *for every $i < d$.*

(2) $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, \underline{W}^{\mathrm{sub}}_{\nu,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \mathrm{Gr}_{\mathsf{F}}((\underline{V}^\vee_{[\mu],k_1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^{\bullet}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})) = 0$ *for every $i > d$.*

Proof. We apply Theorem 3.24. Let us verify the conditions:

(a) By Proposition 5.6 and Lemma 5.7, we are free to refine Σ, and to take any $\kappa \in \mathbf{K}_{n,\mathcal{H},\Sigma}$.

(b) By Proposition 7.16, up to replacing Σ with a refinement, there exists $\kappa \in \mathbf{K}_{n,\mathcal{H},\Sigma}$ such that the structural morphism $f^{\mathrm{tor}}_{n,\kappa} : \mathsf{N}^{\mathrm{tor}}_{n,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ is proper, log smooth, and log integral.

(c) By Lemma 7.18, the relative cohomology $\underline{H}^\bullet_{n,\kappa}$ satisfies Assumption 1.4.

(d) Corollary 7.14 verifies the condition (3.25) for the line bundle $\underline{W}^{\mathrm{can}}_{\nu,k_1}$, and the liftability properties are satisfied by our compactifications over S_1.

(e) Finally, the nilpotence of residue maps has been established in Section 6.

Thus Theorem 3.24 applies and yields:

(1) $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, \underline{W}^{\mathrm{can}}_{-\nu,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \mathrm{Gr}_{\mathtt{F}}(\underline{H}^n_{n,\kappa,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})) = 0$ for every $i < d$. (By Lemma 4.14, the line bundle $\underline{W}^{\mathrm{can}}_{-\nu,k_1}$ is dual to $\underline{W}^{\mathrm{can}}_{\nu,k_1}$.)

(2) $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, \underline{W}^{\mathrm{sub}}_{\nu,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \mathrm{Gr}_{\mathtt{F}}(\underline{H}^n_{n,\kappa,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})) = 0$ for every $i > d$. (By definition, $\underline{W}^{\mathrm{sub}}_{\nu,k_1} = \underline{W}^{\mathrm{can}}_{\nu,k_1}(-\mathsf{D})$; see Definition 4.12.)

Then the proposition follows because, by realizing $\underline{H}^\bullet_{n,\mathcal{H},\Sigma,k_1}$ as $\underline{H}^\bullet_{n,\kappa,k_1}$, Proposition 5.8 implies that $\mathrm{Gr}_{\mathtt{F}}((\underline{V}^\vee_{[\mu],k_1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})$ is a horizontal summand of

$\mathrm{Gr}_{\mathtt{F}}(\underline{H}^n_{n,\kappa,k_1} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}/\mathsf{S}_{k_1}})$. □

7.4. **Reformulations using dual BGG complexes.** For each integer $a \geq 0$, we denote by $\mathrm{W}^{\mathrm{M}_1}(a)$ the set of elements w in $\mathrm{W}^{\mathrm{M}_1}$ of length $l(w) = a$.

Theorem 7.22 (Faltings; cf. [15, §3 and §7], [18, Ch. VI, §5], and [48, §5])**.** *Let R be any R_1-algebra. For each $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$, there is an* F*-filtered complex* $\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})$*, with trivial differentials on the* F*-graded pieces, such that*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})) \cong \underset{w\in\mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}}$$

as $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules, together with a canonical quasi-isomorphic embedding

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^\bullet((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}})) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R})$$

in the category of complexes of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}$-modules, realizing the left-hand side as a summand of the right-hand side. The same is true if we replace $(\underline{V}^\vee_{[\mu],R})^{\mathrm{can}}$ (resp. $(\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}}$) with $(\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}}$ (resp. $(\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}}$).

As remarked after [42, Thm. 6.4], if G_1 has no type D factors, then this is well known. The same method as the one in [18, Ch. VI, §5] and [48, §5], using [54, Thm. D] as the main representation-theoretic input, carries over with little modification. The cases involving type D factors are no more difficult, since the method involves only the (compatible) actions of P_1 and $\mathrm{Lie}(\mathrm{G}_1)$ (cf. Lemma 4.21), and if one uses a simple variant of [54, Thm. A] instead of [54, Thm. D], the method also works when G_1 has type D factors. (For more detail, see [40].)

Corollary 7.23 (cf. [42, Cor. 6.5])**.** *For each $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ and each R_1-algebra R, and for $? = \mathrm{can}$ or sub, we have a canonical isomorphism*

$$H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \mathrm{Gr}_{\mathtt{F}}((\underline{V}^\vee_{[\mu],R})^? \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}))$$
$$\cong \underset{w\in\mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^?).$$

Combining Proposition 7.21 and Theorem 7.22, we obtain:

Corollary 7.24 (cf. [42, Cor. 7.4])**.** *Suppose that* $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$*, and that* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$*. If* $w \in \mathrm{W}^{\mathrm{M}_1}$ *and* $\nu \in \mathrm{X}_{\mathrm{M}_1}^{+,<p}$ *is a positive parallel weight, then:*

(1) $H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, (\underline{W}^\vee_{w\cdot[\mu]+\nu,k_1})^{\mathrm{can}}) = 0$ *for every* $i < d$*.*

(2) $H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, (\underline{W}^\vee_{w\cdot[\mu]-\nu,k_1})^{\mathrm{sub}}) = 0$ *for every* $i > d$*.*

Changing our perspective a little bit:

Corollary 7.25 (cf. [42, Cor. 7.5])**.** *Suppose that* $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{w}}p}$*,* $w \in \mathrm{W}^{\mathrm{M}_1}$*, and that* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$*. Suppose that, for each* $\mu' \in [\mu]$*, there exist positive parallel weights* $\nu_+, \nu_- \in \mathrm{X}_{\mathrm{M}_1}^{+,<p}$ *such that the condition* $\mu' \pm w^{-1}(\nu_\pm) \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$ *is satisfied.* (*The choices of* $\nu_\pm$ *may depend on* μ'*.*) *Then:*

(1) $H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, (\underline{W}^\vee_{w\cdot[\mu],k_1})^{\mathrm{can}}) = 0$ *for every* $i < d$*.*

(2) $H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}, (\underline{W}^\vee_{w\cdot[\mu],k_1})^{\mathrm{sub}}) = 0$ *for every* $i > d$*.*

Definition 7.26. *Let* $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{w}}p}$ *and let* R *be an* R_1*-algebra. We define the* **de Rham cohomology** $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ (*resp. the* **de Rham cohomology with compact support** $H^i_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$) *as the* i*-th hypercohomology of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}$ *with coefficients in the log de Rham complex* $((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$ (*resp. with coefficients in the complex* $((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R}, \nabla)$)*.*

Then we define the **interior de Rham cohomology** *as*

$$\begin{aligned} &H^i_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \\ &:= \mathrm{image}(H^i_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \xrightarrow{\mathrm{can.}} H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})). \end{aligned}$$

We define the various **Hodge cohomology groups** *similarly:*

$$\begin{aligned} &H^i_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \\ &:= H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \mathrm{Gr}_{\mathsf{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R})) \\ &\cong \underset{w \in \mathrm{W}^{\mathrm{M}_R}}{\oplus} H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}}), \end{aligned} \tag{7.27}$$

$$\begin{aligned} &H^i_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \\ &:= H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \mathrm{Gr}_{\mathsf{F}}((\underline{V}^\vee_{[\mu],R})^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}/\mathsf{S}_R})) \\ &\cong \underset{w \in \mathrm{W}^{\mathrm{M}_R}}{\oplus} H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}}) \end{aligned} \tag{7.28}$$

and

$$\begin{aligned} &H^i_{\mathrm{Hodge,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \\ &:= \mathrm{image}(H^i_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \xrightarrow{\mathrm{can.}} H^i_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})) \\ &\cong \underset{w \in \mathrm{W}^{\mathrm{M}_R}}{\oplus} H^{i-l(w)}_{\mathrm{int}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}}), \end{aligned} \tag{7.29}$$

where $H^{i-l(w)}_{\mathrm{int}}(\mathsf{M}_{\mathcal{H},R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})$ *is defined as*

$$\mathrm{image}(H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}}) \xrightarrow{\mathrm{can.}} H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})).$$

By Lemma 5.7, these groups are canonically independent of the choice of Σ.

Remark 7.30. These definitions involve a somewhat serious abuse of notation: For one, by H^i_{dR} we do *not* mean the hypercohomology of $\mathsf{M}_{\mathcal{H},R}$ with coefficients in the de Rham complex attached to $\underline{V}^\vee_{[\mu],R}$. The last group coincides with the one in our definition when R is a $\mathbb{Q}$-algebra, but when $R = k_1$, it is in general infinite dimensional over k_1. Also, for want of the embedded resolution of singularities in characteristic $p > 0$, we do not know whether every non-toroidal smooth compactification of $\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}$ with a simple normal crossings divisor as boundary would yield cohomology groups isomorphic to the ones in our definition, *even* for the trivial coefficient $(\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_1}}, \mathrm{d})$.

Remark 7.31. The notion of *interior cohomology* ("*innere Kohomologie*" in German) was first defined by Harder in [20, p. 41] as the image of the cohomology with compact support in the ordinary cohomology (which makes sense for all reasonable cohomology theories). We learned this notion (in English) from [23].

By construction, we have (for each R_1-algebra R) the Hodge spectral sequences

$$E_1^{a,b} := \mathrm{Gr}^a_{\mathsf{F}}(H^i_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})) \Rightarrow H^{a+b}_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \tag{7.32}$$

and

$$E_1^{a,b} := \mathrm{Gr}^a_{\mathsf{F}}(H^i_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})) \Rightarrow H^{a+b}_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \tag{7.33}$$

associated with the hypercohomology of filtered complexes.

Combining Corollaries 7.23 and 7.25, we obtain:

Theorem 7.34 (cf. [42, Thm. 7.6])**.** *Suppose that $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$, and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Also suppose that, for each $w \in \mathrm{W}^{\mathrm{M}_1}$ and each $\mu' \in [\mu]$, there exist positive parallel weights $\nu_+, \nu_- \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ such that the condition $\mu' \pm w^{-1}(\nu_\pm) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ is satisfied. Then, for $? = \mathrm{Hodge}$ or dR, we have:*

(1) $H^i_?(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ for every $i < d$.
(2) $H^i_{?,\mathrm{c}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ for every $i > d$.
(3) $H^i_{?,\mathrm{int}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ for every $i \neq d$.

Proof. When $? = \mathrm{Hodge}$, (1) and (2) follow from Corollaries 7.23 and 7.25. Then the case when $? = \mathrm{dR}$ follows from the spectral sequences (7.32) and (7.33) with $R = k_1$. Finally, (3) is true because, when $i \neq d$, either the source or the target of $H^i_{?,\mathrm{c}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) \xrightarrow{\mathrm{can.}} H^i_?(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1})$ is zero. □

Note that all the conditions on μ and p in Theorem 7.34 are the same as in the compact case. We shall make use of the definitions and arguments in [42, §7.3] without further remarks.

8. Main results: part I

8.1. **Main results for (log) de Rham and Hodge cohomology.** For the sake of clarity, we emphasize that the cohomology groups used in this section are those defined in Definition 7.26, and as such may not coincide with the *usual* definition (see Remark 7.30).

Theorem 8.1 (cf. [42, Thm. 8.1])**.** *Suppose that $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ satisfies $|\mu|_{\mathrm{re},+} < p$. (See [42, Def. 7.18 and (7.22)].) Then, for $? = \mathrm{Hodge}$ or dR, we have:*

(1) $H^i_?(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ *for every* $i < d$.
(2) $H^i_{?,\mathrm{c}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ *for every* $i > d$.
(3) $H^i_{?,\mathrm{int}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ *for every* $i \neq d$.

Proof. This follows from Theorem 7.34, and [42, Prop. 7.21 and Lem. 7.24]. □

Theorem 8.2 (cf. [42, Thm. 8.2]). *Keep the assumptions of Theorem* 8.1, *and let* R *be any* R_1*-algebra. Then the Hodge cohomology groups in Definition* 7.26 *satisfy:*

(1) $H^i_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every* $i < d$. *Moreover,* (*when* $i = d$) *if* R *is flat over* R_1, *then each summand* $H^{d-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})$ *in* (7.27) *is a free* R*-module of finite rank.*
(2) $H^i_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every* $i > d$. *Moreover,* (*when* $i = d$) *each summand* $H^{d-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{sub}})$ *in* (7.28) *surjects onto* $H^{d-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_R}, (\underline{W}^\vee_{w\cdot[\mu],k_R})^{\mathrm{sub}})$, *where* $k_R := R \underset{R_1}{\otimes} k_1$, *under the canonical morphism induced by the reduction-modulo-*p *homomorphism* $R_1 \longrightarrow k_1$.
(3) $H^i_{\mathrm{Hodge,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for* $i \neq d$. *Moreover,* (*when* $i = d$) *each summand* $H^{d-l(w)}_{\mathrm{int}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})$ *of* $H^d_{\mathrm{Hodge,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ *in* (7.29) *surjects onto* $H^{d-l(w)}_{\mathrm{int}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,k_R}, (\underline{W}^\vee_{w\cdot[\mu],k_R})^{\mathrm{can}})$. *If* R *is flat over* R_1, *then* $H^{d-l(w)}_{\mathrm{int}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, (\underline{W}^\vee_{w\cdot[\mu],R})^{\mathrm{can}})$ *is a free* R*-module of finite rank.*

Consequently, the spectral sequences (7.32) *and* (7.33) *induce, respectively, an injection*

$$\mathrm{Gr}_{\mathsf{F}}(H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})) \hookrightarrow H^d_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \tag{8.3}$$

and a surjection

$$H^d_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \twoheadrightarrow \mathrm{Gr}_{\mathsf{F}}(H^d_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})). \tag{8.4}$$

Proof. As a preparation, let us explain the base change for both the Hodge and de Rham cohomology. By [19, III, 6.2.2], the hypercohomology groups in Definition 7.26 (with $R = R_1$ there) can be calculated by the Čech complex defined by a finite affine open covering of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ (for any Σ satisfying the running assumptions). (As remarked in [31, p. 206], this is valid even when the differentials are only $\mathscr{O}_{\mathsf{S}_1}$-linear.) Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ is proper and flat over $\mathsf{S}_1 = \mathrm{Spec}(R_1)$, essentially the same proof as that of [49, §5, Thm.] (see also [5, III, 3.7 and 3.7.1]) shows that there exist bounded complexes whose components are free R_1-modules of finite type (i.e., strictly perfect complexes) $\mathscr{L}_{\mathrm{Hodge}}$, $\mathscr{L}_{\mathrm{dR}}$, $\mathscr{L}_{\mathrm{Hodge,c}}$, $\mathscr{L}_{\mathrm{dR,c}}$ that *universally calculate* the corresponding cohomology groups in the following sense: For each R_1-module E, each integer i, $?_1 =$ Hodge or dR, and $?_2 = \varnothing$ or c, we have

$$\mathscr{H}^i(\mathscr{L}_{?_1,?_2} \underset{R_1}{\otimes} E) \cong H^i_{?_1,?_2}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]} \underset{R_1}{\otimes} E), \tag{8.5}$$

where $\mathscr{H}^i$ denotes the i-th cohomology module of the complex of R_1-modules (note that the tensor product on the left-hand side is also the derived tensor product), and where (by abuse of notation) $H^i_{?_1,?_2}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]} \underset{R_1}{\otimes} E)$ is defined using canonical or subcanonical extensions of $\underline{V}^\vee_{[\mu]}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ as in Definition 7.26. In particular, $\mathscr{H}^i(\mathscr{L}_{?_1,?_2}) \cong H^i_{?_1,?_2}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]})$, and $\mathscr{H}^i(\mathscr{L}_{?_1,?_2} \underset{R_1}{\otimes} R) \cong$

$H^i_{?_1,?_2}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ for every R_1-algebra R. Consequently, since R_1 is a discrete valuation ring, we obtain (the "universal coefficient theorem"):

(8.6)
$$0 \longrightarrow \mathscr{H}^i(\mathscr{L}_{?_1,?_2}) \underset{R_1}{\otimes} E \longrightarrow \mathscr{H}^i(\mathscr{L}_{?_1,?_2} \underset{R_1}{\otimes} E) \longrightarrow \mathrm{Tor}_1^{R_1}(\mathscr{H}^{i+1}(\mathscr{L}_{?_1,?_2}), E) \longrightarrow 0.$$

Now let us turn to the proof of the theorem. Until the end of this proof, we will always take $?_1 = \text{Hodge}$. We begin with the case $R = R_1$. Applying (8.6) with $E = k_1$, we obtain, as a consequence, the usual *upper semicontinuity* of dimensions of cohomology (cf. [49, §5, Cor. (a)]). Combining this with Theorem 8.1, we obtain $H^i_{\text{Hodge}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) = 0$ for every $i < d$ and $H^i_{\text{Hodge,c}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) = 0$ for every $i > d$. The same holds for each summand of these modules in (7.27) and (7.28). Hence the cohomology long exact sequence attached to the canonical short exact sequence $0 \longrightarrow (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}} \xrightarrow{[p]} (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}} \longrightarrow (\underline{W}^\vee_{w\cdot[\mu],k_1})^{\text{can}} \longrightarrow 0$ shows that each summand $H^{d-l(w)}(\mathsf{M}^{\text{tor}}_{\mathcal{H},\Sigma,1}, (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}})$ in (7.27) is p-torsion-free, and hence free as an R_1-module, because R_1 is a discrete valuation ring whose maximal ideal is generated by p. This shows (1). A similar long exact sequence, using the subcanonical extension this time, shows (2). Since $H^{i-l(w)}_{\text{int}}(\mathsf{M}^{\text{tor}}_{\mathcal{H},\Sigma,1}, (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}})$ is at the same time a submodule of $H^{i-l(w)}(\mathsf{M}^{\text{tor}}_{\mathcal{H},\Sigma,1}, (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}})$ and a quotient module of $H^{i-l(w)}(\mathsf{M}^{\text{tor}}_{\mathcal{H},\Sigma,1}, (\underline{W}^\vee_{w\cdot[\mu]})^{\text{sub}})$, (3) follows from (1) and (2).

If R is flat over R_1, then (8.6) (with $E = R$) shows that all the cohomology groups over R in the statements are obtained by taking the tensor product of the corresponding cohomology groups over R_1 with R (over R_1). Therefore, each statement we have proved over R_1 remains valid over R, since the extension of scalars by a flat algebra preserves both the freeness (and the rank) of a module and the exactness of sequences (in particular, injectivity and surjectivity of a morphism). Note also that the Hodge filtration and the associated graded functor in (8.3) and (8.4) are compatible with the same extension of scalars.

Finally, consider a general R, not necessarily flat over R_1. To prove (1), we use (8.6) with $E = R$, the vanishing $H^i_{\text{Hodge}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]})$ for every $i < d$, and the freeness of $H^d_{\text{Hodge}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) = 0$ over R_1. To prove the vanishing statement in (2), we use (8.6) with $E = R$, and the vanishing $H^i_{\text{Hodge,c}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) = 0$ for every $i > d$. To prove the surjectivity statement in (2), we take the cohomology long exact sequence attached to the canonical short exact sequence $0 \longrightarrow p((\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}}) \longrightarrow (\underline{W}^\vee_{w\cdot[\mu]})^{\text{can}} \longrightarrow (\underline{W}^\vee_{w\cdot[\mu],k_1})^{\text{can}} \longrightarrow 0$, and use (8.6) with $E = pR$ and the vanishing $H^i_{\text{Hodge,c}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) = 0$ for every $i > d$. Then (3) follows from (1) and (2), as usual. □

Corollary 8.7 (cf. [42, Cor. 8.3])**.** *With the assumptions of Theorem* 8.1*, the following are true for every R_1-algebra R:*

(1) $H^i_{\text{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every $i < d$.*
(2) $H^i_{\text{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every $i > d$.*
(3) If R is flat over R_1, then $H^d_{\text{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ *is a free R-module of finite rank.*
(4) The morphism $H^d_{\text{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \longrightarrow H^d_{\text{dR,c}}(\mathsf{M}_{\mathcal{H},k_R}/\mathsf{S}_{k_R}, \underline{V}^\vee_{[\mu],k_R})$ *induced by the canonical morphism $R \longrightarrow k_R = R \underset{R_1}{\otimes} k_1$ is a surjection. In*

other words, every element of $H^d_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},k_R}/\mathsf{S}_{k_R}, \underline{V}^\vee_{[\mu],k_R})$ *is the reduction modulo* p *of some element of* $H^d_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$.

Proof. The spectral sequences (7.32) and (7.33) have the Hodge cohomology groups in Theorem 8.2 as their E_1 terms and abut to the de Rham cohomology groups in (1) and (2), respectively. Thus, the vanishing of the latter groups follows from Theorem 8.2.

The following argument also proves these, as well as the rest of the corollary. Let us begin with the case $R = R_1$ (and $k_R = k_1$). Since all the terms in the long exact sequence associated with the tensor product of the de Rham complex of $(\underline{V}^\vee_{[\mu]})^{\mathrm{can}}$ with the canonical short exact sequence $0 \longrightarrow R_1 \xrightarrow{[p]} R_1 \longrightarrow k_1 \longrightarrow 0$ are finitely generated R_1-modules, and since $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ for all $i < d$ by Theorem 8.1, we obtain (1) and (3) by Nakayama's lemma. A similar argument for the de Rham complex of $(\underline{V}^\vee_{[\mu]})^{\mathrm{sub}}$, using $H^i_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},k_1}/\mathsf{S}_{k_1}, \underline{V}^\vee_{[\mu],k_1}) = 0$ for all $i > d$ by Theorem 8.1, shows (2) and (4). For a general R, we proceed as in the proof of Theorem 8.2, using (8.6) with $?_1 = \mathrm{dR}$. □

Corollary 8.8 (cf. [42, Cor. 8.3])**.** *With the assumptions of Theorem* 8.1*, the following are true for every* R_1*-algebra* R *(with* $k_R = R \underset{R_1}{\otimes} k_1$*):*

(1) $H^i_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every* $i \neq d$.
(2) *If* R *is flat over* R_1*, then* $H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ *is a free* R*-module of finite rank.*
(3) *The morphism* $H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \longrightarrow H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},k_R}/\mathsf{S}_{k_R}, \underline{V}^\vee_{[\mu],k_R})$ *induced by the canonical morphism* $R \longrightarrow k_R = R \underset{R_1}{\otimes} k_1$ *is surjective. In other words, every element of* $H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},k_R}/\mathsf{S}_{k_R}, \underline{V}^\vee_{[\mu],k_R})$ *is the reduction modulo* p *of some element in* $H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$.

Proof. Since $H^i_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ is at the same time a submodule of $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ and a quotient module of $H^i_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$, the statement (1) follows from (1) and (2) of Corollary 8.7, and the statement (3) follows from (4) of Corollary 8.7. As for (2), since R is flat over R_1 in this case, we may assume that $R = R_1$, because taking images of morphisms of R_1-modules is also compatible with the flat base change from R_1 to R. Since $H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]})$ is by definition a submodule of $H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]})$, (2) follows from (3) of Corollary 8.7, because all submodules of a p-torsion-free module are p-torsion-free. □

Remark 8.9. We *do not* claim (even when R is flat over R_1) that the natural sequence

$$0 \longrightarrow H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \xrightarrow{[p]} H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \longrightarrow H^d_{\mathrm{dR,int}}(\mathsf{M}_{\mathcal{H},k_R}/\mathsf{S}_{k_R}, \underline{V}^\vee_{[\mu],k_R}) \longrightarrow 0$$

is exact (in the middle).

8.2. **Main results for cohomological automorphic forms.** Let w_0 be the unique Weyl element in $\mathrm{W}_{\mathrm{M}_1}$ such that $w_0\Phi^+_{\mathrm{M}_1} = \Phi^-_{\mathrm{M}_1}$ and $W_\nu \cong W^\vee_{-w_0(\nu)}$ for all $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. As in [42, Def. 8.4], we say that a weight $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ is *cohomological*

if there exist $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ and $\mu' \in [\mu]$ such that $-w_0(\nu) = w \cdot \mu'$ for some $w \in \mathrm{W}^{\mathrm{M}_1}$. We write in this case that $\mu' = \mu(\nu)$, $[\mu] = [\mu(\nu)]$, and $w = w(\nu)$.

Definition 8.10 (cf. [42, Def. 8.5])**.** *Let $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. Let R be any R_1-algebra. Consider the following graded modules of R-valued* **algebraic automorphic forms of weight** *ν:*

(1) $A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; R) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{can}}_{\nu,R})$. We call these forms **canonical**.

(2) $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; R) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{sub}}_{\nu,R})$. We call these forms **subcanonical**.

(3) $A^\bullet_{\nu,\mathrm{int}}(\mathcal{H}; R) := H^\bullet_{\mathrm{int}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R}, \underline{W}^{\mathrm{can}}_{\nu,R})$. We call these forms **interior**.

In all three cases, the choice of Σ is immaterial (cf. Proposition 5.6 *with $m = 0$, or rather* [39, proof of Lem. 7.1.1.5]*; and cf. Lemma* 5.7*).*

It is convenient to also introduce, for each R_1-module E, the E-valued forms $A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; E) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}, \underline{W}^{\mathrm{can}}_{\nu} \underset{R_1}{\otimes} E)$, $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; E) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}, \underline{W}^{\mathrm{sub}}_{\nu} \underset{R_1}{\otimes} E)$, and $A^\bullet_{\nu,\mathrm{int}}(\mathcal{H}; E) := \mathrm{image}(A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; E) \longrightarrow A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; E))$. (These are compatible with Definition 8.10 when $E = R$.)

Remark 8.11*.* We hasten to add that our terminology (canonical, subcanonical, and interior) are *not* standard; there do not seem to be standard names for these spaces, except when the cohomology degree is 0 or d (thanks to the prototypical case $R = \mathbb{C}$). In degree 0, forms in $A^0_{\nu,\mathrm{can}}(\mathcal{H}; R)$ can be called *holomorphic*, while forms in $A^0_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ can be called *cuspidal holomorphic*. Since $A^0_{\nu,\mathrm{int}}(\mathcal{H}; R)$ is canonically isomorphic to $A^0_{\nu,\mathrm{sub}}(\mathcal{H}; R)$, there is no special terminology for its forms. In degree d, forms in $A^d_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ can be called *anti-holomorphic* (thanks to Hodge theory over $\mathbb{C}$), while forms in $A^d_{\nu,\mathrm{can}}(\mathcal{H}; R)$ can be called *cuspidal anti-holomorphic* (thanks to Serre duality with the case of degree 0). This time $A^d_{\nu,\mathrm{int}}(\mathcal{H}; R)$ is equal to $A^d_{\nu,\mathrm{can}}(\mathcal{H}; R)$ as a submodule. We refrain from calling $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ cuspidal because this is not justified in degrees higher than 0. In general, $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ is not a submodule of $A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; R)$.

Proposition 8.12 (cf. [42, Prop. 8.6])**.** *Let R be any R_1-algebra. If $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ is cohomological and satisfies $\mu(\nu) \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ and $|\mu(\nu)|_{\mathrm{re},+} < p$, then:*

(1) $A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; R)$ is (by definition) a summand of $H^{\bullet+l(w(\nu))}_{\mathrm{Hodge}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ and vanishes below degree $d - l(w(\nu))$.

(2) $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ is (by definition) a summand of $H^{\bullet+l(w(\nu))}_{\mathrm{Hodge,c}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ and vanishes above degree $d - l(w(\nu))$.

(3) $A^\bullet_{\nu,\mathrm{int}}(\mathcal{H}; R)$ is concentrated in degree $d - l(w(\nu))$, and a summand of $H^d_{\mathrm{Hodge,int}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$.

Proof. This is a special case of Theorem 8.2. □

Theorem 8.13 (cf. [42, Thm. 8.7])**.** *Let R be any R_1-module, and let $k_R := R \underset{R_1}{\otimes} k_1$. Suppose that $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$, and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Then $A^\bullet_{\nu,\mathrm{can}}(\mathcal{H}; R)$ and $A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H}; R)$ have the following properties:*

(1) If there exists a positive parallel weight ν_- such that $\nu + \nu_-$ is cohomological and $\mu(\nu + \nu_-) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$, then $A^i_{\nu,\mathrm{can}}(\mathcal{H}; R) = 0$ for every $i < d - l(w(\nu + \nu_-))$.

(2) *If there exists a positive parallel weight* ν_+ *such that* $\nu-\nu_+$ *is cohomological and* $\mu(\nu-\nu_+) \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$*, then* $A^i_{\nu,\mathrm{sub}}(\mathcal{H};R) = 0$ *for every* $i > d - l(w(\nu-\nu_+))$.

(3) *For* $? = \mathrm{can}$ *or* sub*, if* R *is flat over* R_1*, and if* $A^{i-1}_{\nu,?}(\mathcal{H};k_1) = 0$ *for some degree* i*, then* $A^{i-1}_{\nu,?}(\mathcal{H};R) = 0$ *and* $A^i_{\nu,?}(\mathcal{H};R)$ *is a free* R*-module of finite rank.*

(4) *For* $? = \mathrm{can}$ *or* sub*, if* $A^{i+1}_{\nu,?}(\mathcal{H};R_1) = 0 = \mathrm{Tor}_1^{R_1}(A^{i+2}_{\nu,?}(\mathcal{H};R_1), pR)$ *for some degree* i*, then* $A^{i+1}_{\nu,?}(\mathcal{H};pR) = 0$ *and the natural morphism* $A^i_{\nu,?}(\mathcal{H};R) \longrightarrow A^i_{\nu}(\mathcal{H};k_R)$ *induced by* $R_1 \twoheadrightarrow k_1$ *is surjective; in other words, every section of* $A^i_{\nu,?}(\mathcal{H};k_R)$ *is* **liftable***, in the sense that it is the reduction modulo* p *of some section in* $A^i_{\nu,?}(\mathcal{H};R)$*. (The condition* $\mathrm{Tor}_1^{R_1}(A^{i+2}_{\nu,?}(\mathcal{H};R_1), pR) = 0$ *holds, for example, when either* $A^{i+2}_{\nu,?}(\mathcal{H};R_1)$ *or* pR *is flat over* R_1*. In particular, by* (3)*, the full condition* $A^{i+1}_{\nu,?}(\mathcal{H};R_1) = 0 = \mathrm{Tor}_1^{R_1}(A^{i+2}_{\nu,?}(\mathcal{H};R_1), pR)$ *holds when* $A^{i+1}_{\nu,?}(\mathcal{H};k_1) = 0$*.)*

(5) *For* $? = \mathrm{can}$ *or* sub*, if* $A^{i-1}_{\nu,?}(\mathcal{H};k_1) = 0$ *and* $A^{i+1}_{\nu,?}(\mathcal{H};R_1) = 0 = \mathrm{Tor}_1^{R_1}(A^{i+2}_{\nu,?}(\mathcal{H};R_1), pR)$ *for some degree* i*, then* $A^i_{\nu,?}(\mathcal{H};R)$ *is a free* R*-module of finite rank, and we have an exact sequence*

$$0 \longrightarrow A^i_{\nu,?}(\mathcal{H};pR) \longrightarrow A^i_{\nu,?}(\mathcal{H};R) \longrightarrow A^i_{\nu,?}(\mathcal{H};k_R) \longrightarrow 0.$$

Proof. Let us begin with the case $R = R_1$ (and $k_R = k_1$). By upper semicontinuity of dimensions of cohomology as in the proof of Theorem 8.2, (1) and (2) follow from reformulating Corollary 7.24. Then (3), (4), and (5) all follow from taking the long exact sequence induced by the canonical short exact sequence $0 \longrightarrow (\underline{W}_{\nu,R})^? \xrightarrow{[p]} (\underline{W}_{\nu,R})^? \longrightarrow (\underline{W}_{\nu,k_R})^? \longrightarrow 0$, as in the proof of Corollary 8.7.

In general, if R is flat over R_1, then the same statements hold by flat base change. Otherwise, the statements (without flatness assumption) follow from an analogue of the "universal coefficient theorem" in the proof of Theorem 8.2. ☐

Corollary 8.14 (cf. [42, Thm. 8.7])**.** *Let* R *be any* R_1*-algebra, and let* $k_R := R \underset{R_1}{\otimes} k_1$*. Suppose that* $\nu \in \mathrm{X}_{\mathrm{M}_1}^{+,<p}$*, and that* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$*. Then* $A^\bullet_{\nu,\mathrm{int}}(\mathcal{H};R)$ *has the following properties:*

(1) *If there exist positive parallel weights* ν_+ *and* ν_- *such that* $\nu-\nu_+$ *and* $\nu+\nu_-$ *are cohomological and such that* $\mu(\nu-\nu_+) \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$ *and* $\mu(\nu+\nu_-) \in \mathrm{X}_{\mathrm{G}_1}^{+,<_{\mathrm{re}}p}$*, then* $A^i_{\nu,\mathrm{int}}(\mathcal{H};R) = 0$ *for every* $i \notin [d - l(w(\nu+\nu_-)), d - l(w(\nu-\nu_+))]$.

(2) *If* R *is flat over* R_1*, and if* $A^{i-1}_{\nu,\mathrm{can}}(\mathcal{H};k_1) = 0$ *for some degree* i*, then* $A^{i-1}_{\nu,\mathrm{int}}(\mathcal{H};R) = 0$ *and* $A^i_{\nu,\mathrm{int}}(\mathcal{H};R)$ *is a free* R*-module of finite rank.*

(3) *If* $A^{i+1}_{\nu,\mathrm{sub}}(\mathcal{H};R_1) = 0 = \mathrm{Tor}_1^{R_1}(A^{i+2}_{\nu,\mathrm{sub}}(\mathcal{H};R_1), pR)$ *for some degree* i*, then* $A^{i+1}_{\nu,\mathrm{int}}(\mathcal{H};pR) = 0$ *and the natural morphism* $A^i_{\nu,\mathrm{int}}(\mathcal{H};R) \longrightarrow A^i_{\nu,\mathrm{int}}(\mathcal{H};k_R)$ *induced by* $R_1 \twoheadrightarrow k_1$ *is surjective; in other words, every section of* $A^i_{\nu,\mathrm{int}}(\mathcal{H};k_R)$ *is* **liftable***, in the sense that it is the reduction modulo* p *of some section in* $A^i_{\nu,\mathrm{int}}(\mathcal{H};R)$*. (See the parenthetical remark in* (4) *of Theorem* 8.13*.)*

Proof. These statements follow from the corresponding statements in Theorem 8.13, for reasons similar to those in the proof of Corollary 8.8. ☐

Remark 8.15. We *do not* claim that the natural sequence

$$0 \longrightarrow A^i_{\nu,\mathrm{int}}(\mathcal{H}; R) \xrightarrow{[p]} A^i_{\nu,\mathrm{int}}(\mathcal{H}; R) \longrightarrow A^i_{\nu,\mathrm{int}}(\mathcal{H}; k_R) \longrightarrow 0$$

is exact.

Remark 8.16. An analogue of Theorem 8.13 for certain automorphic *line bundles* over general (not necessarily compact) Shimura varieties can be found in [41]. However, our treatment of automorphic line bundles here (and in [42, §7]) is more complete.

Remark 8.17. Theorem 8.13 is sharp for compact Picard modular surfaces. (See [64], [41, Rem. 4.5], and [42, Rem. 8.10].)

8.3. **Comparison with transcendental results.** As remarked in the beginning of [42, §8.4], just as Deligne and Illusie deduced vanishing theorems of Kodaira type in characteristic zero from the vanishing statements in positive characteristics (see [12] and [27]), we now obtain *purely algebraic* proofs of (cruder forms of) certain vanishing theorems that have, so far, been obtained only via transcendental methods. (For comparison of our results with transcendental ones in the compact case, see [42, §8.4].)

As in [42, §8.4], let us write $\mathrm{X}^{++,<p}_{\mathrm{G}_\mathbb{C}} = \mathrm{X}^{++,<p}_{\mathrm{G}_1}$, and write $\mathrm{G}_\mathbb{C}$ in place of G_1. By [42, Cor. 8.15], it makes sense to write objects ${}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}$, $\underline{V}^\vee_{[\mu],\mathbb{C}}$, and $\underline{W}^\vee_{\nu,\mathbb{C}}$ for all dominant weights of $\mathrm{G}_\mathbb{C}$, and we have a canonical isomorphism $H^i_{\mathrm{B},?}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) \cong H^i_{\mathrm{dR},?}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, \underline{V}^\vee_{[\mu],\mathbb{C}})$ for each i and for $? = \varnothing$, c, or int.

Theorem 8.18 (cf. [42, Thm. 8.16])**.** *Suppose $\mu \in \mathrm{X}^{++}_{\mathrm{G}_\mathbb{C}}$. Then the following are true:*

(1) $H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) = 0$ *for every $i < d$.*
(2) $H^i_{\mathrm{B,c}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) = 0$ *for every $i > d$.*
(3) $H^i_{\mathrm{B,int}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) = 0$ *for every $i \neq d$.*

Proof. By [42, Cor. 8.15], we can choose a good prime p so large that $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_\mathbb{C}}$ and $|\mu|_{\mathrm{re},+} < p$. Then the results follow from Theorem 8.2. □

Remark 8.19 (cf. [42, Rem. 8.17]). To the best of our knowledge, the first analytic proofs of Theorem 8.18 (in the general noncompact case) were given by Li and Schwermer's work on the Eisenstein cohomology of arithmetic groups (see [44, Cor. 5.6]), and (independently) by Saper's work on $\mathcal{L}$-modules (see [59, §11, Thm. 5]). Before them, the important special case of symplectic groups with factors of rank two was treated using Franke's method in [66, Appendix A].

Remark 8.20. Both Li and Schwermer's and Saper's works apply to arithmetic quotients of Riemannian symmetric spaces *without* Hermitian structures. In this case, the first possible degree of non-vanishing is lower than half of the real dimension of the symmetric space by half of the difference of the real ranks of the semisimple part and its maximal compact subgroup. (In the Hermitian case, this difference is zero.) Before [44], Li and Schwermer also wrote a helpful summary [43, §2] on the vanishing theorems (e.g., that in [68]) known by then.

Remark 8.21. It is worth pointing out that (3) of Theorem 8.18 has a simple transcendental proof in [15, §6, Cor. to Thm. 9], the same work of Faltings we already cited in [42, Rem. 8.17]. This is because, in the Hermitian case, the canonical morphism $H^i_{\mathrm{B,c}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) \longrightarrow H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}})$ factors through the much better understood L_2 *cohomology* (see [15, §6, Thm. 9 a)]), and hence the vanishing of the interior cohomology can be deduced from the vanishing of the L_2 cohomology (see, e.g., [15, §6, Cor. to Thm. 8]), without referring to the much later [44, Cor. 5.6] or [59, §11, Thm. 5]. (In fact, by [60, Cor. 2.3], for regular weights, the interior cohomology and L_2 cohomology are the same because they both coincide with the cuspidal cohomology.)

Remark 8.22 (cf. [42, Rem. 8.19]). In works mentioned in Remarks 8.19 and 8.21 (and in [42, Rem. 8.17 and 8.18]), it suffices to assume that μ is *regular*, a weaker (and hence better) condition than ours when $\mathrm{G}_\mathbb{C}$ has factors of types C or D. As we mentioned in the compact case (see [42]), this is a limitation of our technique using positive parallel weights of minimal size.

Similarly (to the case of G_1), let us also write $\mathrm{X}^+_{\mathrm{M}_\mathbb{C}} = \mathrm{X}^+_{\mathrm{M}_1}$, and write $\mathrm{M}_\mathbb{C}$ in place of M_1 in the remainder of this subsection.

For ? = can, sub, or int, we can extend the definition of $A^\bullet_{\nu,?}(\mathcal{H},\mathbb{C})$ to all $\nu \in \mathrm{X}^+_{\mathrm{M}_\mathbb{C}}$, and deduce from Theorem 8.13 the following:

Theorem 8.23 (cf. [42, Thm. 8.20]). *(1) If there exists a positive parallel weight ν_- such that $\nu + \nu_-$ is cohomological and $\mu(\nu + \nu_-) \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$, then $A^i_{\nu,\mathrm{can}}(\mathcal{H};\mathbb{C}) = 0$ for every $i < d - l(w(\nu + \nu_-))$.*

(2) If there exists a positive parallel weight ν_+ such that $\nu - \nu_+$ is cohomological and $\mu(\nu - \nu_+) \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$, then $A^i_{\nu,\mathrm{sub}}(\mathcal{H};\mathbb{C}) = 0$ for every $i > d - l(w(\nu - \nu_+))$.

(3) If there exist positive parallel weights ν_+ and ν_- such that $\nu - \nu_+$ and $\nu + \nu_-$ are cohomological and such that $\mu(\nu - \nu_+) \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$ and $\mu(\nu + \nu_-) \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$, then $A^i_{\nu,\mathrm{int}}(\mathcal{H};\mathbb{C}) = 0$ for every $i \notin [d - l(w(\nu + \nu_-)), d - l(w(\nu - \nu_+))]$.

Remark 8.24 (cf. [42, Rem. 8.21]). When ν is cohomological and $\mu(\nu)$ is regular, one can use the mixed Hodge theory as in [18, Ch. VI, §5] and [24, Cor. 4.2.3] to show that Faltings's dual BGG spectral sequences (see [15, §3 and §7]; cf. Theorem 7.22) degenerate, and from this one can obtain an analytic analogue of Proposition 8.12 by using [44, Cor. 5.6] or [59, §11, Thm. 5], analogous to Theorem 8.23.

9. Crystalline comparison

To translate the vanishing results on the (log) de Rham (or crystalline) and Hodge cohomology into those on the Betti (or étale) cohomology, we will use the crystalline comparison theorem. This is similar to what we did in [42], but this time we will explain how to use the relative comparison due to Faltings, which yields a better lower bound for p. (See Remarks 9.8 and 9.9 below.)

We inherit the notations of [42, §5]. So we will denote by W, $K = \mathrm{Frac}(W)$, K^{ac}, $\iota : K^{\mathrm{ac}} \xrightarrow{\sim} \mathbb{C}$, and F_0^{ac} those we defined in *loc. cit.* We still write $\mathsf{M}_{\mathcal{H},W} := \mathsf{M}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} W$, and denote by A_W the pullback to $\mathsf{M}_{\mathcal{H},W}$ of the universal family from $\mathsf{M}_{\mathcal{H},0}$ (rather than from $\mathsf{M}_{\mathcal{H},1}$). Similar notations will be used for Kuga families and their toroidal compactifications, and for the base changes of objects from S_0 to K or K^{ac}.

We shall keep on using the correspondence between crystals and quasi-coherent sheaves with integrable and quasi-nilpotent (log) connections in [28, Thm. 6.2] and denote, by abuse of notation, the corresponding objects with the same symbols.

9.1. **Relative crystalline comparison.** For an integer $s \geq 1$, we write $W_s = W/p^s W$.

Proposition 9.1 (Faltings). *Let $m \geq 0$ be an integer, and let Σ and $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$ be as in Proposition* 7.16 *such that the structural morphism $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \longrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ is proper, log smooth, and log integral* (*see* [28, Def. 4.3])*, and satisfies the refined description in the second paragraph there.*

For $0 \leq j < p-2$, the crystal $\underline{H}^j_{m,\kappa,W_s} := R^j(f^{\mathrm{tor}}_{m,\kappa,W_s})_*(\overline{\Omega}^{\bullet}_{\mathsf{N}^{\mathrm{tor}}_{m,\kappa,W_s}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,W_s}})$ (by the abuse of notation explained above and in Remark 1.5) *is* **associated** *with the étale sheaf $R^i(f_{m,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\mathbb{Z}/p^s\mathbb{Z})$ via Faltings's contravariant functor $\mathbf{D}$* (*see* [16, Thm. 2.6* and §II i)])*. That is,*

$$\mathbf{D}^*(\underline{H}^j_{m,\kappa,W_s}) \cong R^j(f_{m,K^{\mathrm{ac}}})_{*,\text{ét}}(\mathbb{Z}/p^s\mathbb{Z}),$$

where $\mathbf{D}^*$ is the composite of $\mathbf{D}$ followed by the Pontryagin dual.

This association is functorial with respect to the proper log étale morphisms refining cone decompositions in the source and the target of $f^{\mathrm{tor}}_{m,\kappa}$ (*such as under replacements of κ and Σ with κ' and Σ' in Proposition* 7.16)*, and is compatible with the cup product and the formation of the Chern classes of line bundles.*

We note that such a comparison (for relative morphisms having similar properties as $f^{\mathrm{tor}}_{m,\kappa}$ does) was stated without details in the case of Siegel modular varieties in [18, p. 241, paragraph 1] (with a slightly worse bound for p, which can, however, be ameliorated by the Lefschetz-type argument as in [16, §V e)]). The essential ingredients are explained in Faltings's works on p-adic Hodge theory, and all subsequent works on cohomology of Shimura varieties (including ours) follow the same idea. Here we simply indicate a few points.

Proof of Proposition 9.1. First, by Lemma 7.18, $\underline{H}^{\bullet}_{m,\kappa,W}$ satisfies Assumption 1.4. Hence, by Remark 1.11, the proof of Theorem 1.8 shows that $\underline{H}^j_{m,\kappa,W}$ defines an *F-T-crystal* (see [51, Def. 5.3.1]), which verifies the implicit claim that (after reduction modulo p^s) $\underline{H}^j_{m,\kappa,W_s}$ defines a crystal (or rather "Fontaine module") for Faltings's theory (see [51, Prop. 5.3.9]). Thus it makes sense to apply the functor $\mathbf{D}^*$.

Then we use the theory of [17], which improves that of [16]. The étale local description of $f^{\mathrm{tor}}_{m,\kappa}$ in the second paragraph of Proposition 7.16 verifies the requirements in [17, §6]. This allows us to apply [17, 2. Thm. in §5], [17, 6. Thm. in §6], and the remark following the last (on adapting the arguments in [16]; although $f^{\mathrm{tor}}_{m,\kappa}$ is not of Cartier type, we verified above that our relative crystals satisfy the conditions required by Faltings). Given these, one can show that the étale and crystalline cohomology correspond under $\mathbf{D}^*$ by adapting the argument in the proof of [16, Thm. 6.2, with the remark following it]. $\square$

9.2. **Crystalline comparison with automorphic coefficients.** As in [42, §4.3], let Λ be an integral domain that is finite and flat over the p-adic completion of R_1 (and hence finite flat over $\mathbb{Z}_p$). For each integer $s \geq 1$, set $\Lambda_s = \Lambda/p^s\Lambda$. Moreover, as in [42, §5.2], assume that the set $\Omega := \mathrm{Hom}_{\mathbb{Z}_p\text{-alg.}}(W, \Lambda)$ has cardinality $[F_0 : \mathbb{Q}]$, so that we have a decomposition $W \underset{\mathbb{Z}_p}{\otimes} \Lambda \cong \prod_{\sigma \in \Omega} W_\sigma$ as in [42, (5.3)], and hence a similar decomposition $W_s \underset{\mathbb{Z}/p^s\mathbb{Z}}{\otimes} \Lambda_s \cong \prod_{\sigma \in \Omega} W_{\sigma,s}$ for each integer $s \geq 1$. By Proposition 9.1, with Σ and κ as in the statements there, we have

$$\mathbf{D}^*\Big(\underset{\sigma\in\Omega}{\oplus}\underline{H}^j_{m,\kappa,W_{\sigma,s}}\Big)\cong R^j(f_{m,K^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda_s) \tag{9.2}$$

(cf. [42, (5.6)]).

Let $\mu\in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$ with $n:=|\mu|_{\mathrm{L}}$. Suppose $\max(2,r_\tau)<p$ whenever $\tau=\tau\circ c$. In [42, §4.3], we defined

$${}_{\text{ét}}\underline{V}^\vee_{[\mu]}:=(\varepsilon_\mu)^*\,R^n(f_{n,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda)(-t_\mu)$$

and

$${}_{\mathrm{B}}\underline{V}^\vee_{[\mu]}:=(\varepsilon_\mu)^*\,R^n(f_{n,\mathbb{C}})_{*,\mathrm{B}}(\Lambda)(-t_\mu).$$

Therefore, ${}_{\text{ét}}\underline{V}^\vee_{[\mu]}$ is a summand of $R^n(f_{n,K^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda_s)$, cut out by (the Tate twist by $-t_\mu$ of) the idempotent $(\varepsilon_\mu)^*$. On the other hand, by (5.9), $(\underline{V}^\vee_{[\mu],W_{\sigma,s}})^{\mathrm{can}}$ is a summand of $\underline{H}^n_{n,\mathcal{H},\Sigma,W_{\sigma,s}}\cong\underline{H}^n_{n,\kappa,W_{\sigma,s}}$ cut out by (the Tate twist by $-t_\mu$ of) the idempotent $(\varepsilon_\mu)^*$ (defined in Proposition 5.8). It is natural to ask whether the functor $\mathbf{D}^*$ in (9.2) is compatible with the applications of $(\varepsilon_\mu)^*$; or, if not obviously so, whether we can at least justify this indirectly.

Proposition 9.3. *With the assumptions on μ above, if $n<p-2$, then the isomorphism* (9.2) *induces an isomorphism*

$$\mathbf{D}^*\Big(\underset{\sigma\in\Omega}{\oplus}(\underline{V}^\vee_{[\mu],W_{\sigma,s}})^{\mathrm{can}}\Big)\cong{}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_s} \tag{9.4}$$

(*after possibly replacing Σ with some refinement in the construction of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$).

Proof. As in the proof of Proposition 7.21, by applying Proposition 5.6, Lemma 5.7, and Proposition 7.16, up to replacing Σ with a refinement (and replacing κ accordingly), we may assume that $f^{\mathrm{tor}}_{n,\kappa}:\mathsf{N}^{\mathrm{tor}}_{n,\kappa}\longrightarrow\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ is log integral. Moreover, in step (5) of Proposition 5.14, for each $0\le i<t$, we can always choose Σ_{i+1} and $\kappa_{i+1}\in\mathbf{K}_{m,\mathcal{H},\Sigma_{i+1}}$ such that $f^{\mathrm{tor}}_{n,\kappa_{i+1}}:\mathsf{N}^{\mathrm{tor}}_{n,\kappa_{i+1}}\longrightarrow\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma_{i+1},1}$ is also log integral, regardless of the choice of $\kappa'_i\in\mathbf{K}_{m,\mathcal{H},\Sigma_i}$. As a result, the corresponding classes $(h_i^*)^*(c_i)$ in Proposition 5.14, defined by composing the morphisms $[1]_{\kappa_{i+1},\kappa'_i}\circ f_{\kappa'_i,\kappa_i}$ or $[1]_{\kappa_{i+1},\kappa'_i}\circ(h_i)^*_{\kappa'_i,\kappa_i}$ in step (7) there, are respected by the functor $\mathbf{D}^*$ by the functoriality and compatibility stated in Proposition 9.1. Hence (9.4) follows from Proposition 5.14 and (9.2), as desired. □

Proposition 9.5 (cf. [42, (5.7)])**.** *With the assumptions on μ above, if $d+n<p-2$, then the isomorphism* (9.4) *induces isomorphisms*

$$\mathbf{D}^*\Big(\underset{\sigma\in\Omega}{\oplus}H^\bullet_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},W_{\sigma,s}}/\mathsf{S}_{W_{\sigma,s}},\underline{V}^\vee_{[\mu],W_{\sigma,s}})\Big)\cong H^\bullet_{\text{ét}}(\mathsf{M}_{\mathcal{H},K^{\mathrm{ac}}},{}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_s})$$

and

$$\mathbf{D}^*\Big(\underset{\sigma\in\Omega}{\oplus}H^\bullet_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},W_{\sigma,s}}/\mathsf{S}_{W_{\sigma,s}},\underline{V}^\vee_{[\mu],W_{\sigma,s}})\Big)\cong H^\bullet_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},K^{\mathrm{ac}}},{}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_s}).$$

Proof. This follows from Proposition 9.3 and [16, Thm. 5.3]. □

Corollary 9.6 (cf. [42, Prop. 5.8])**.** *With the assumptions on μ and p above, if $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1},\underline{V}^\vee_{[\mu],k_1})=0$ (resp. $H^i_{\mathrm{dR,c}}(\mathsf{M}_{\mathcal{H},1},\underline{V}^\vee_{[\mu],k_1})=0$) for some integer i, then $H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}},{}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})=0$ (resp. $H^i_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}},{}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})=0$) for the same i.*

Definition 9.7. *We set $|\mu|'_{\mathrm{comp}}:=1+d+n=1+d+|\mu|_{\mathrm{L}}$, called the* **comparison size** *of μ.*

Remark 9.8 (cf. [42, Rem. 5.10]). If $|\mu|'_{\text{comp}} \leq p-2$, then the assumption $d+n < p-2$ in Proposition 9.5 is satisfied. In general, the condition $|\mu|'_{\text{comp}} \leq p-2$ is (weaker and hence) *better* than the condition $|\mu|_{\text{comp}} = 2d+n \leq p-2$ in [42, §5.2].

Remark 9.9 (cf. [57, §5]). If $j \leq p-2$, then by Tsuji [67] and Breuil [6] (in a much more general context, following earlier works of Fontaine, Messing, and Kato), one can compare the log de Rham cohomology $H^j_{\log\text{ dR}}(\mathsf{N}^{\text{tor}}_{m,\kappa,W_s}/W_s) := H^j(\mathsf{N}^{\text{tor}}_{m,\kappa,W_s}, \overline{\Omega}^\bullet_{\mathsf{N}^{\text{tor}}_{m,\kappa,W_s}/W_s})$ with the p-adic étale cohomology $H^j_{\text{ét}}(\mathsf{N}_{m,\kappa,K^{\text{ac}}}, \mathbb{Z}/p^s\mathbb{Z})$. (See [57, §5.1].) Moreover, the comparison is functorial in the log smooth total schemes (such as $\mathsf{N}^{\text{tor}}_{m,\kappa,W_s}$), and is compatible with the formation of the log Chern classes of line bundles and with the cup product structures. (In [57, §5.2], these are credited to Tsuji [67] and Yamashita [69].) Therefore, by the same argument as in [42, §5.2] and by the strong geometric plethysm provided by Proposition 5.14, if μ and p satisfy the stronger condition $|\mu|_{\text{comp}} = 2d+n \leq p-2$, then one can compare $\underset{\sigma\in\Omega}{\oplus} H^\bullet_{\text{dR}}(\mathsf{M}_{\mathcal{H},W_{\sigma,s}}/\mathsf{S}_{W_{\sigma,s}}, \underline{V}^\vee_{[\mu],W_{\sigma,s}})$ with $H^\bullet_{\text{ét}}(\mathsf{M}_{\mathcal{H},K^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_s})$ without using the (stronger) relative comparison theorem in Proposition 9.1. (This is essentially the same argument as in [57]; but since the hypotheses in [57, §1.2.2] are still *not* verified yet, we cannot apply the result in *loc. cit.* literally. Moreover, as explained in [42, §3.6], we do not need Poincaré duality in our strong geometric plethysm.)

10. Main results: part II

10.1. **Main results for étale and Betti cohomology.** Let Λ be an integral domain, finite flat over the p-adic completion of R_1 (and hence finite flat over $\mathbb{Z}_p$). (See the second paragraph of [42, §4.3].) Let $\Lambda_1 = \Lambda/p\Lambda$ (as in Section 9.2).

Theorem 10.1 (cf. [42, Thm. 8.12]). *Suppose that $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ satisfies $|\mu|_{\text{re},+} < p$ and $|\mu|'_{\text{comp}} \leq p-2$ (see Definition 9.7). Then the following are true:*

1. *$H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) = 0$ for every $i < d$.*
2. *$H^i_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) = 0$ for every $i > d$.*
3. *$H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) = 0$ for every $i < d$.*
4. *$H^i_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) = 0$ for every $i > d$.*
5. *$H^d_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]})$ is a free Λ-module of finite rank.*
6. *The morphism $H^d_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \longrightarrow H^d_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})$ induced by the canonical morphism $\Lambda \longrightarrow \Lambda_1 = \Lambda/p\Lambda$ is surjective; in other words, every element of $H^d_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})$ is the reduction modulo p of some element in $H^d_{\text{ét,c}}(\mathsf{M}_{\mathcal{H},F_0^{\text{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]})$.*

The same are true if we base change the coefficient Λ to any Λ-algebra, except that we need the algebra to be flat over Λ for statement (5)*.*

Proof. As in the proof of Corollary 8.7, it suffices to prove (1) and (2). (The base change statement follows from the "universal coefficient theorem" for étale cohomology; cf. the proof of Theorem 8.2.) To do this, we may replace Λ with a domain finite flat over Λ and assume that the set $\Omega := \mathrm{Hom}_{\mathbb{Z}_p\text{-alg.}}(W,\Lambda)$ has cardinality $[F_0 : \mathbb{Q}]$, so that the results in Section 9.2 apply. By [42, Lem. 8.11 and 7.24], $|\mu|_{\text{re},+} < p$ implies that $2d < p$ and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Since $|\mu|'_{\text{comp}} \leq p-2$, Corollary 9.6 applies, and (1) and (2) follow from Theorem 8.1, as desired. $\square$

Corollary 10.2 (cf. [42, Thm. 8.12])**.** *Suppose that* $\mu \in \mathrm{X}_{\mathrm{G}_1}^{++,<p}$ *satisfies* $|\mu|_{\mathrm{re},+} < p$ *and* $|\mu|'_{\mathrm{comp}} \leq p-2$ (*see Definition* 9.7)*. Then the following are true:*

(1) $H^i_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) = 0$ *for every* $i \neq d$.
(2) $H^i_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) = 0$ *for every* $i \neq d$.
(3) $H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]})$ *is a free* Λ*-module of finite rank.*
(4) *The morphism* $H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \longrightarrow H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})$ *induced by the canonical morphism* $\Lambda \longrightarrow \Lambda_1 = \Lambda/p\Lambda$ *is surjective. In other words, every element of* $H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1})$ *is the reduction modulo* p *of some element in* $H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]})$.

The same are true if we base change the coefficient Λ *to any* Λ*-algebra, except that we need the algebra to be flat over* Λ *for statement* (3)*.*

Proof. These statements follow from corresponding statements in Theorem 10.1, for reasons similar to those in the proof of Corollary 8.8. □

Remark 10.3. We *do not* claim that the natural sequence

$$0 \longrightarrow H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \xrightarrow{[p]} H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \longrightarrow H^d_{\text{ét},\mathrm{int}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) \longrightarrow 0$$

is exact in the middle.

Corollary 10.4 (cf. [42, Cor. 8.13])**.** *Theorem* 10.1 *and Corollary* 10.2 *remain true with the étale cohomology replaced with the Betti cohomology* (*and with the base field* F_0^{ac} *replaced with* $\mathbb{C}$)*.*

Proof. This follows from [42, Prop. 4.14]. □

10.2. **Concluding remarks.**

Remark 10.5. The potential failure for the natural sequences in Remarks 8.9 and 10.3 to be exact has a meaning (in terms of the boundary cohomology). If they are indeed not exact, then (unlike in the compact case) the process of taking the interior cohomology (with assumptions such as the sufficient regularity, etc, in our results) in the middle degree no longer gives *exact functors*.

Remark 10.6 (cases beyond PEL-type). Although there are technical details more complicated than in the PEL-type cases, the arguments in [42] and this article should also hold in the context of non-PEL-type Shimura varieties, using the good integral models and their toroidal compactifications constructed in works of Vasiu, Kisin, and Madapusi Pera.

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