Quantifying Curvelike Structures of Measures by Using L_2 Jones Quantities

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Abstract

We study the curvelike structure of special measures on $\mathbb{R}^n$ in a multiscale fashion. More precisely, we consider the existence and construction of a sufficiently short curve with a sufficiently large measure. Our main tool is an L_2 variant of Jones' β numbers, which measure the scaled deviations of the given measure from a best approximating line at different scales and locations. The Jones function is formed by adding the squares of the L_2 Jones numbers at different scales and the same location. Using a special L_2 Jones function, we construct a sufficiently short curve with a sufficiently large measure. The length and measure estimates of the underlying curve are expressed in terms of the size of this Jones function. © 2003 Wiley Periodicals, Inc.

1 Introduction

We explore the curvelike structure of special measures on $\mathbb{R}^n$ by using multiscale geometric analysis and following a work of Jones [14]. The translation of our objective into a mathematical language is not immediate. Before doing it, we would like to describe our motivation.

The main problem we have in mind is that of "curve learning" (and, more generally, "manifold learning"; see, e.g., [1, 6, 13, 28, 30]). Given a data set in $\mathbb{R}^n$, it is natural to ask whether this set or some of its subsets are "well approximated" by sufficiently short curves. If the answer is positive, then we would like to study the underlying curves or analyze their smoothness properties.

In this paper we transcribe the above problem to the study of the curvelike structure of special measures. The manifoldlike structures of measures will be investigated in a joint work with Jones [16]. Numerical results and applications to data sets will be presented in [18] (see also [19]).

The most general (and abstract) notion of curvelike structure of measures is that of "weak" one-dimensional rectifiability, that is, the existence of a countable union of rectifiable curves of full measure (see Section 2 for further explanation). Note that this definition does not take into account any quantitative estimates like the lengths of the curves. The latter estimates are the scope of *quantitative rectifiability*.

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The theory of quantitative rectifiability was created by Peter Jones [14] in order to solve a continuous version of the traveling salesman problem. This version examines the existence and construction of a curve (a Lipschitz image of a compact interval) with nearly minimal length containing a given compact set in $\mathbb{R}^n$. Jones' theory was further extended by different authors. Kate Okikiolu [22] generalized [14, first part of theorem 1] to $\mathbb{R}^n$ (instead of $\mathbb{R}^2$). Guy David and Stephen Semmes [9, 10] developed a theory for embedding a d-dimensional Ahlfors-regular set, lying in $\mathbb{R}^n$, in a d-dimensional "regular" manifold with a small d-volume (d an integer, $1 \leq d < n$; for the definition of Ahlfors regularity, see, e.g., [10, definition 1.13] or Section 2 where $d = 1$). They formulated the notion of uniform rectifiability and showed its relation with different quantitative estimates (see, e.g., [10, section 1.4]; their notes [11] are written for the nonspecialist). Xiang Fang [12], Hervé Pajot [23, 24, 25], and J. C. Léger [17] expanded the theory further. Applications of quantitative rectifiability to various problems in geometric measure theory and potential theory can be found in [2, 3, 4, 5, 7, 8, 15].

One of the attractive features of this theory is the use of multiscale analysis. Jones [14] introduced the multiscale β_∞ numbers, which record normalized L_∞ approximation errors of a set by lines at different scales and different locations. David and Semmes [9] used L_p approximations instead of the L_∞ ones for similar multiscale analysis of d-dimensional Ahlfors-regular sets (or possibly Ahlfors-regular measures).

In this paper we extend the above theory to a wide class $\mathcal{J}^n$ of one-dimensional rectifiable measures. Given a measure μ in $\mathcal{J}^n$, we fit a sufficiently short curve with a sufficiently large measure μ. This problem is different from that of fitting a sufficiently short curve to the whole given set in the theory mentioned above.

We use L_2 variants of Jones' β_∞ numbers. These β_2 numbers generalize the ones of David and Semmes [10] for any locally finite Borel measure. Following Bishop and Jones [4], we form the "square function" J_2 by adding the squares of β_2 numbers from different scales and the same location. In order to avoid some technical difficulties in proving the relevant theorems, we modify our initial β_2 numbers. The corresponding modified square function is denoted by $\hat{J}_2$. For a given measure μ, the two functions J_2 and $\hat{J}_2$ are comparable in $L_1(\mu)$. By applying a technique from [16], one can use this comparability to replace the function $\hat{J}_2$ by J_2 in the theory presented here.

A main idea of this paper is the relation between the size of $\hat{J}_2$ of a given measure μ in $\mathcal{J}^n$ and one-dimensional quantitative estimates of μ. In Theorems 4.8 and 4.10 we show that if $\hat{J}_2$ is "well controlled," then there exists a sufficiently short curve with a sufficiently large measure. Moreover, the "optimal" length and measure of such a curve can be estimated by using the "size" of $\hat{J}_2$. In Section 5 we construct a measure for which these estimates are indeed optimal.

Due to the general setting, some of the estimates and constants in this paper are not practical for various subclasses of measures. For example, in Section 5 we show that the estimates of Theorems 4.8 and 4.10 are far from optimal when applied to one-dimensional Ahlfors-regular measures. Nonetheless, following [14, 22, 24] we can still show in this special case how to use the size of $\hat{J}_2$ to obtain the optimal estimates.

It would be appealing for us to find out the sharp dependence of the different constants in Theorems 4.8 and 4.10 on the dimension n (we ask the same question when replacing $\hat{J}_2$ by J_2 in these theorems). We are also interested in the following two related tasks: First, find large subclasses of rectifiable measures with constants depending weakly on n (see, e.g., Talagrand [29] and Pestov [26] for typical obstacles with large n). Second, replace the rigid dyadic grids by flexible grids adapted to the given measure so that the constants are reduced substantially and so that such grids can be easily used in numerical algorithms for data sets (see [18, 19] for the kind of numerical algorithms we have in mind).

The paper is organized as follows. In Section 2 we list some general definitions and notation. More specialized definitions appear in the relevant sections. In Section 3 we describe the "extended dyadic grids" and verify their main attractive property. In Section 4 we introduce L_2 versions of Jones quantities for analyzing locally finite Borel measures on $\mathbb{R}^n$ and define the class of measures $\mathcal{J}^n$. We then formulate the two theorems of this paper. The estimates of the theorems versus the optimal estimates are exemplified in Section 5. In Section 6 we verify a special case of Theorem 4.8. Based on this result, we conclude both Theorems 4.8 and 4.10 in Section 7.

2 Some Basic Notation and Definitions

We shall work in the Euclidean n-space $\mathbb{R}^n$. We assume that the space is endowed with a locally finite Borel measure μ. Here are the basic notation and definitions used throughout this paper.

Let A be a subset of $\mathbb{R}^n$. We denote by $\bar{A}$ the closure of A, by A^{c} the complement of A, and by ∂A the boundary of A. If u is any vector in $\mathbb{R}^n$, then $A + u = \{v + u \mid v \in A\}$. The restriction of the measure μ to the set A is designated by $\mu|_A$.

We use $\mathcal{H}^1$ to denote the one-dimensional Hausdorff measure and $\mathrm{dist}_{\mathcal{H}}$ to denote the Hausdorff distance (see, e.g., Mattila [20]). We denote by $B(x, t)$ a ball with center at x and radius t and by $Q(x, l)$ a cube with sides parallel to the axes, center at x, and side length $2 \cdot l$.

If Q is a cube in $\mathbb{R}^n$, then $l(Q)$ denotes its length and $C \cdot Q$ denotes the cube with the same center as Q, side length $C \cdot l(Q)$, and sides parallel to the sides of Q. Assume next that $\mu(Q) > 0$. The *center of mass* of the cube Q is denoted by z_Q and defined by the formula

$$z_Q := \frac{\int_Q z \, \mathrm{d}\mu(z)}{\mu(Q)} .$$

A *best* L_2 *line* in the cube Q is a line L_Q that attains the minimum of the integral $\int_Q \mathrm{dist}^2(z, L_Q)\mathrm{d}\mu(z)$ among all lines in $\mathbb{R}^n$. Note that a best L_2 line for a given cube is not necessarily unique. Denote by $\mathcal{H}_Q$ the set of all best L_2 lines for the cube Q.

A function $f : [a, b] \to \mathbb{R}^n$ is called a *Lipschitz function* if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C \cdot |x - y| \quad \text{for all } x, y \in [a, b] .$$

Denote by $\|f\|_{\mathrm{Lip}[a,b]}$ the infimum of all such possible constants C. A set Γ in $\mathbb{R}^n$ is a *rectifiable curve* if there exist a segment $[a, b]$ and a Lipschitz function $f : [a, b] \to \mathbb{R}^n$ such that $\Gamma = f([a, b])$. Equivalently, a rectifiable curve is a compact, connected set with finite one-dimensional Hausdorff measure (for a proof of the equivalence, see, e.g., [10, theorem 1.8]). The length of a rectifiable curve Γ, denoted by $l(\Gamma)$, is defined to be $\mathcal{H}^1(\Gamma)$. We refer to the measure $\mathcal{H}^1|_\Gamma$ as the arc length measure on Γ.

We use the notation supp, dist, and diam as short for "support," "distance," and "diameter," respectively.

We designate absolute constants larger than 1 by capital letters (mainly C) and very small absolute constants by ε and δ. We may use the same letter with the same subscript to denote different constants at different places in the text. We denote by $C(n)$ constants that depend on the dimension n. When using a constant often, we might omit its dependence on n in order to simplify notation.

We say that A is *approximately less than* B, or $A \lessapprox B$, if $A \leq C \cdot B$, where C is an absolute constant that may vary from line to line. Similarly, we say that A and B are *comparable*, or $A \approx B$, if $A \lessapprox B$ and $B \lessapprox A$. Let f and g be two functions defined on a domain D. We say that f is approximately less than g everywhere, or $f \lessapprox g$, if $f(x) \lessapprox g(x)$ for all $x \in D$. Similarly, f and g are *comparable everywhere*, or $f \approx g$, if $f(x) \approx g(x)$ for all $x \in D$. We use the notation $\lessapprox_n$, $\gtrapprox_n$, and $\approx_n$ whenever the constants of comparability depend (or might depend) on the dimension n.

We say that a measure μ in $\mathbb{R}^n$ is "*weakly*" *one-dimensional rectifiable* if and only if there exists a countable union of rectifiable curves in $\mathbb{R}^n$ of full measure. If, in addition, μ is absolutely continuous with respect to $\mathcal{H}^1$, then we omit the word *weak* (see Preiss [27] for a different definition and a useful criterion). A measure μ is *one-dimensional Ahlfors-regular* if and only if there exists a constant $C = C(\mu)$ such that for any cube (or ball) $Q = Q(x, r)$ in $\mathbb{R}^n$, where $x \in \underline{\mathrm{supp}(\mu)}$ and $r \leq \mathrm{diam}(\mathrm{supp}(\mu))$, the following inequality is satisfied: $C^{-1} \cdot l(Q) \leq \mu(\bar{Q}) \leq C \cdot l(Q)$. A set A is one-dimensional rectifiable if and only if the measure $\mathcal{H}^1|_A$ is one-dimensional rectifiable. Similarly, it is one-dimensional Ahlfors-regular if and only if the measure $\mathcal{H}^1|_{\bar{A}}$ is one-dimensional Ahlfors-regular.

3 Definitions and Properties of Grids

Throughout the paper we will use dyadic cubes and grids extensively. We say that $Q \subseteq \mathbb{R}^n$ is a *dyadic cube with side length* 2^{-j} if

$$Q = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_n}{2^j}, \frac{k_n+1}{2^j}\right),$$

where $k_1, \dots, k_n$ and j are arbitrary integers. Denote by $\mathcal{D}_j$ the *dyadic grid at the scale* 2^{-j}, that is, the collection of all dyadic cubes whose side length $l(Q)$ equals 2^{-j}.

Let Q be a half-closed, half-open cube in $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the affine transformation that maps $[0,1)^n$ onto Q. We define the *normalized dyadic grid with respect to* Q by $\mathcal{D}(Q) = T(\bigcup_{j=0}^{\infty} \mathcal{D}_j)$.

In order to avoid some edge effects of dyadic grids, we use an extended grid. It is the union of some shifted dyadic grids. This extended grid was introduced to the theory of quantitative rectifiability by Kate Okikiolu [22].

DEFINITION 3.1 (Extended normalized grids $\tilde{\mathcal{D}}(Q_0)$ and $\tilde{\mathcal{D}}_j(Q_0)$ w.r.t. Q_0) If Q_0 is a cube in $\mathbb{R}^n$, then

$$\tilde{\mathcal{D}}(Q_0) = \bigcup_{e \in \{0,1\}^n} \mathcal{D}\left(Q_0 + \frac{l(Q_0)}{3} \cdot e\right)$$

and

$$\tilde{\mathcal{D}}_j(Q_0) = \left\{Q : Q \in \tilde{\mathcal{D}}(Q_0) \text{ and } l(Q) = \frac{l(Q_0)}{2^j}\right\}.$$

The new collection of cubes, $\tilde{\mathcal{D}}(Q_0)$, has the property that any point in $\mathbb{R}^n$ is the "center" of a cube in $\tilde{\mathcal{D}}(Q_0)$ with a given length. This property can be formulated more precisely as follows:

PROPOSITION 3.2 *Let Q_0 be a cube in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $j \geq 0$, then there exists a cube Q in $\tilde{\mathcal{D}}_j(Q_0)$ such that $x \in \frac{2}{3} \cdot Q$.*

PROOF: We first prove the claim for $n = 1$, so that both Q and Q_0 are intervals. We distinguish between two cases:

Case 1. j is an even number, $j \geq 0$.

In this case $\frac{2^j}{3} \equiv 1 \pmod 2$ and thus

$$\tilde{\mathcal{D}}_j(Q_0) = \mathcal{D}_j(Q_0) \cup \left(\mathcal{D}_j(Q_0) + \frac{1}{3} \cdot l(Q_0)\right), \quad j = 0, 2, 4, \dots. \tag{3.1}$$

Consequently, by using appropriate shifting and scaling, it is sufficient to consider the special case where $Q_0 = [0,1)$, $j = 0$, and $x \in [0,1)$. Assume this reduced case and note that the following cube Q satisfies the desired proposition:

$$Q = \begin{cases} [-\frac{2}{3}, \frac{1}{3}) & \text{if } 0 \le x < \frac{1}{6} \\ [0, 1) & \text{if } \frac{1}{6} \le x < \frac{5}{6} \\ [\frac{1}{3}, \frac{4}{3}) & \text{if } \frac{5}{6} \le x < 1. \end{cases}$$

Case 2. j is an odd number, $j \ge 1$.

In this case $\frac{2^j}{3} \equiv 2 \pmod 2$ and thus

$$\tilde{\mathcal{D}}_j(Q_0) = \mathcal{D}_j(Q_0) \cup \left(\mathcal{D}_j(Q_0) + \frac{2}{3} \cdot l(Q_0)\right), \quad j = 1, 3, \dots. \tag{3.2}$$

Consequently, by using appropriate shifting and scaling, it is sufficient to consider the special case where $Q_0 = [0, 1)$, $j = 1$, and $x \in [0, \frac{1}{2})$. Assume this reduced case and note that the following cube Q satisfies the desired proposition:

$$Q = \begin{cases} [-\frac{1}{6}, \frac{1}{3}) & \text{if } 0 \le x < \frac{1}{12} \\ [0, \frac{1}{2}) & \text{if } \frac{1}{12} \le x < \frac{5}{12} \\ [\frac{1}{3}, \frac{5}{6}) & \text{if } \frac{5}{12} \le x < \frac{1}{2}. \end{cases}$$

The above two cases conclude the proof for $n = 1$.

Assume next that $n > 1$. Fix $x \in \mathbb{R}^n$ and $j \ge 0$. Denote $Q_0 = I_1^0 \times \cdots \times I_n^0$ and $x = (x_1, \dots, x_n)$. For each i, $i = 1, \dots, n$, let I_i be an interval in $\tilde{\mathcal{D}}_j(I_i^0)$ such that $x_i \in \frac{2}{3} \cdot I_i$. Note that $Q := I_1 \times \cdots \times I_n$ is a cube in $\tilde{\mathcal{D}}_j(Q_0)$ such that $x \in \frac{2}{3} \cdot Q$. □

We next extend the grid $\tilde{\mathcal{D}}(Q_0)$ to include cubes with side length larger than $l(Q_0)$. The idea is due to Peter Jones.

DEFINITION 3.3 (Cubes $Q^L(Q)$ and $Q^R(Q)$) If I is the interval $I = [a, b)$, then

$$I^L(I) = \left[a - \tfrac{2}{3} \cdot (b-a), a + \tfrac{4}{3} \cdot (b-a)\right)$$

and

$$I^R(I) = \left[a - \tfrac{1}{3} \cdot (b-a), a + \tfrac{5}{3} \cdot (b-a)\right).$$

If Q is a half-closed, half-open cube in $\mathbb{R}^n$ of the form $Q = I_1 \times \cdots \times I_n$, then

$$Q^L = I^L(I_1) \times \cdots \times I^L(I_n) \quad \text{and} \quad Q^R = I^R(I_1) \times \cdots \times I^R(I_n).$$

DEFINITION 3.4 (Cubes $Q_{+j}(Q)$, $j \ge 0$) If Q is a cube in $\mathbb{R}^n$, then the cubes $Q_{+j}(Q)$, $j \ge 0$, are formed recursively as follows:

$$Q_{+j}(Q) = \begin{cases} Q & \text{if } j = 0 \\ Q^L(Q_{+(j-1)}(Q)) & \text{if } j = 1, 3, 5, \dots \\ Q^R(Q_{+(j-1)}(Q)) & \text{if } j = 2, 4, 6, \dots. \end{cases}$$

The new grids, $\tilde{\mathcal{D}}^{+j}_{Q_0}$, $j \geq 0$, are defined as follows:

$$\tilde{\mathcal{D}}^{+j}_{Q_0} := \tilde{\mathcal{D}}(Q_{+j}(Q_0)), \quad j \geq 0. \tag{3.3}$$

Observe that

$$\tilde{\mathcal{D}}^{+j}_{Q_0} \subseteq \tilde{\mathcal{D}}^{+(j+1)}_{Q_0}, \quad j \geq 0.$$

Consequently, define the extended dyadic grid $\tilde{\mathcal{D}}^{+\infty}_{Q_0}$ that contains cubes from all scales by the formula

$$\tilde{\mathcal{D}}^{+\infty}_{Q_0} = \bigcup_{j \geq 0} \tilde{\mathcal{D}}^{+j}_{Q_0}.$$

In this paper we fix a cube Q_0 and frequently use the cubes in the grid $\tilde{\mathcal{D}}^{+3}_{Q_0}$. For simplicity of notation, we designate this grid by $\tilde{\mathcal{D}}$.

4 L_2 Theory of Quantitative Rectifiability

In this section we present an L_2 theory of one-dimensional quantitative rectifiability for a certain class $\mathcal{J}^n$ of locally finite Borel measures on $\mathbb{R}^n$. This theory originates from the L_∞ theory of quantitative rectifiability [4, 14].

We fix a cube Q_0 in $\mathbb{R}^n$ throughout the rest of the paper (in Section 6 we denote the underlying cube by Q_1). Let μ be a locally finite Borel measure on $\mathbb{R}^n$. Following [4, 14] and [9, 10], we study the one-dimensional properties of μ "around" Q_0 in a multiscale fashion. We start by defining the β_2 numbers that record normalized L_2 approximation errors of μ by lines at different scales and locations. Scales and locations are given by cubes in the grid $\tilde{\mathcal{D}} \equiv \tilde{\mathcal{D}}^{+3}_{Q_0}$. The Jones function J_2 is formed by adding the squares of the β_2 numbers at different scales and at the same location.

We employ a variant of J_2 : $\hat{J}_2$. The two functions J_2 and $\hat{J}_2$ are comparable in $L_1(\mu)$. We define a class $\mathcal{J}^n$ of measures whose Jones functions $\hat{J}_2$ are "well controlled" around Q_0 (see equation 4.5). We show that if μ is a measure in $\mathcal{J}^n$, then there exists a curve (in a neighborhood of the cube Q_0) with a sufficiently short length and a sufficiently large measure μ. We estimate the short length and large measure in terms of the size of the function $\hat{J}_2$. These ideas are formulated precisely in Theorems 4.8 and 4.10. A concrete construction of the underlying curve is described in their proof (see Sections 6 and 7). The $L_1(\mu)$ comparability of J_2 and $\hat{J}_2$ together with an observation from [16] imply that Theorems 4.8 and 4.10 are also valid when replacing $\hat{J}_2$ by J_2.

4.1 L_2 Jones Quantities and Related Definitions

We first define a one-dimensional L_2 version of Jones β_∞ numbers (see also [19] for a d-dimensional version). For Ahlfors-regular measures, these β_2 numbers are comparable to the ones of David and Semmes [10].

DEFINITION 4.1 (Jones β_2 numbers) If μ is a locally finite Borel measure on $\mathbb{R}^n$ and Q is a cube in $\mathbb{R}^n$, then

$$\beta_2(Q) \equiv \beta_2^{\mu}(Q) = \begin{cases} \min\limits_{\text{lines } L} \left(\frac{1}{\mu(Q)} \int\limits_Q \left(\frac{\operatorname{dist}(z, L)}{l(Q)} \right)^2 \mathrm{d}\mu(z) \right)^{1/2} & \text{if } \mu(Q) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that Q_0 is a fixed cube in $\mathbb{R}^n$ and that $\tilde{\mathcal{D}}$ denotes the grid $\tilde{\mathcal{D}}_{Q_0}^{+3}$ (see equation (3.3)). Following [4], we define the L_2 Jones function with respect to Q_0 as follows:

DEFINITION 4.2 (L_2 Jones function w.r.t. the cube Q_0) If μ is a locally finite Borel measure on $\mathbb{R}^n$ and Q_0 is a fixed cube in $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$

$$J_2(x) \equiv J_2^{Q_0,\mu}(x) = \sum_{Q \in \tilde{\mathcal{D}}} \beta_2^2(Q) \cdot \chi_Q(x) \,, \tag{4.1}$$

where $\chi_Q(x)$ denotes the indicator function of the cube Q.

We next modify the L_2 Jones numbers and function defined above. Recall that a best L_2 line in a cube Q is a line L_Q that attains the minimum of the integral $\int_Q \operatorname{dist}^2(z, L_Q)\mathrm{d}\mu(z)$ among all lines. A best L_2 line is not necessarily unique. We define the L_2 Jones number of a cube with respect to another cube as follows:

DEFINITION 4.3 (L_2 Jones number $\beta_2(Q_1, Q_2)$) If μ is a locally finite Borel measure on $\mathbb{R}^n$, and Q_1 and Q_2 are two cubes in $\mathbb{R}^n$ such that $Q_1 \subseteq Q_2$ and $\mathcal{H}_{Q_2}$ is the set of all best L_2 lines in Q_2, then

$$\beta_2(Q_1, Q_2) \equiv \beta_2^{\mu}(Q_1, Q_2) = \tag{4.2}$$
$$\begin{cases} \left(\sup\limits_{L_{Q_2} \in \mathcal{H}_{Q_2}} \int\limits_{Q_1} \left(\frac{\operatorname{dist}(z, L_{Q_2})}{l(Q_1)} \right)^2 \frac{\mathrm{d}\mu(z)}{\mu(Q_1)} \right)^{1/2} & \text{if } \mu(Q_1) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions use two integers j_0^* and $j_1^* = j_1^*(n)$ where $2 \le j_0^* \le j_1^*$. We fix these constants in the beginning of Section 6.

DEFINITION 4.4 (The set $\mathcal{P}(Q)$) If Q_0 is a cube in $\mathbb{R}^n$ and Q is a cube in $\tilde{\mathcal{D}}_{Q_0}^{+\infty}$, then $\mathcal{P}(Q) \equiv \mathcal{P}^{Q_0}(Q)$ is the set of all cubes Q' in $\tilde{\mathcal{D}}_{Q_0}^{+\infty}$ that contain Q and satisfy the length estimate

$$2^{j_1^*} \ge \frac{l(Q')}{l(Q)} \ge 2^{j_0^*} \,.$$

The modified Jones numbers $\hat{\beta}_2$ and function $\hat{J}_2$ are defined as follows:

DEFINITION 4.5 (Modified Jones numbers $\hat{\beta}_2$) If μ is a locally finite Borel measure on $\mathbb{R}^n$, Q_0 is a cube in $\mathbb{R}^n$, and Q is a cube in $\tilde{\mathcal{D}}$, then

$$\hat{\beta}_2(Q) \equiv \hat{\beta}_2^{Q_0,\mu}(Q) = \sup_{\hat{Q}\in\mathcal{P}(Q)} \beta_2(Q, \hat{Q})\,.$$

DEFINITION 4.6 (Modified Jones function $\hat{J}_2$ w.r.t. the cube Q_0) If μ is a locally finite Borel measure on $\mathbb{R}^n$, then

$$\hat{J}_2(x) \equiv \hat{J}_2^{Q_0,\mu}(x) = \sum_{Q\in\tilde{\mathcal{D}}} \hat{\beta}_2^2(Q)\cdot\chi_Q(x)\,. \tag{4.3}$$

REMARK 4.7 We did not worry about efficiency when defining the set $\mathcal{P}(Q)$. It is possible to replace this set by a subset containing fewer cubes, so that the techniques of Sections 6 and 7 also apply to the corresponding modified Jones function.

4.2 Theory

We explain here how to use the function $\hat{J}_2$ of a measure μ and a cube Q_0 to determine the solution of the following problem: Find a sufficiently short curve with a sufficiently large measure μ in a neighborhood containing the cube Q_0. We also discuss related optimization problems. Our emphasis is on the ability to use the function $\hat{J}_2$ to solve the above problem. We do not care here about the optimal estimates of particular cases or about the best constants.

The main result of this paper is stated in Theorem 4.10. A weaker version of this result is formulated as follows:

THEOREM 4.8 *There exist positive constants $C_1 = C_1(n)$, $C_2 = C_2(n)$, C_3, $C_4 = C_4(n)$, and C_5 such that the following property is satisfied: If μ is a locally finite Borel measure, Q_0 a fixed cube in $\mathbb{R}^n$, $\hat{J}_2 \equiv \hat{J}_2^{Q_0,\mu}$, and M is a positive number such that*

$$\hat{J}_2(x) \le M \quad \textit{for all } x \in \operatorname{supp}(\mu)\cap C_5\cdot Q_0\,,$$

then there exists a curve Γ, $\Gamma \subseteq C_5\cdot Q_0$, with length at most

$$C_1\cdot e^{C_2\cdot M}\cdot l(Q_0)$$

and measure at least

$$C_3^{-1}\cdot e^{-C_4\cdot M}\cdot\mu(Q_0)\,.$$

These two exponential bounds are both achieved by certain measures (up to logarithmic comparability).

REMARK 4.9 We are usually interested in the case where the support of μ is contained in Q_0.

The proof of Theorem 4.8 contains two parts. The first part validates the theorem when $M = \delta$, a sufficiently small constant (compare with Léger [17, proposition 1.2]). It is described in Section 6. The second part uses stopping-time arguments and the first part in order to prove the theorem for any value of M. It is described in Section 7.

The theorem provides estimates for some optimization problems. For example, fix a measure μ on $\mathbb{R}^n$, a cube Q_0 in $\mathbb{R}^n$, and a positive integer L. Among all curves contained in $C_5 \cdot Q_0$ and with length L (or less than or equal to L), find the ones with maximum measure. In view of the above theorem, if $\hat{J}_2 \leq M$ and $L \leq C_1 \cdot e^{C_2 \cdot M}$, then

$$r_{\max}(\mu, L) := \max_{\substack{l(\Gamma)=L \\ \Gamma \subseteq C_5 \cdot Q_0}} \frac{\mu(\Gamma)}{\mu(Q_0)} \cdot \frac{l(Q_0)}{L} \geq (C_1 \cdot C_3)^{-1} \cdot e^{-(C_2+C_4)\cdot M} . \tag{4.4}$$

A special case of this problem is the *bank robber problem* for atomic measures (see, e.g., [21]).

The dual problem is obtained by fixing a constant m, $0 < m \leq \mu(Q_0)$, and looking for the curve of shortest length among all curves contained in $C_5 \cdot Q_0$ with measure m. Discrete analogues of this problem are the *k-MST* and the *quota-driven salesman problems* (see, e.g., [21]).

A further optimization can be applied to the above problems. The corresponding version for the first problem goes as follows: Find the largest value of L among all values maximizing (or nearly maximizing) $r_{\max}(\mu, L)$. Denote this value by L_{opt}. Also denote by Γ_{opt} any curve such that $\Gamma_{\text{opt}} \subseteq C_5 \cdot Q_0$ and $l(\Gamma_{\text{opt}}) = L_{\text{opt}}$.

We remark that the ratio $r_{\max}(\mu, L)$ is a decreasing function of L. Therefore the last optimization problem is interesting only when $r_{\max}(\mu, L)$ is constant around $L = 0$ (or slowly varying around $L = 0$ when using the nearly optimal version). In this case the problem has the interpretation of finding the longest curve with maximum "uniform" linear density.

The exponential estimates implied by Theorem 4.8 for the above optimization problems are usually not satisfactory for particular examples. However, there exist measures for which these bounds are achieved (but not necessarily with the constants C_1, C_2, C_3, and C_4). In Section 5 we provide three examples where the lower exponential bound of $r_{\max}$ in equation (4.4) is "obtained." In one of these examples the unique curve Γ_{opt} "achieves" the exponential length and measure estimates of Theorem 4.8.

The next theorem slightly generalizes Theorem 4.8. We verify it in Section 7.6 by slightly modifying the proof of the above theorem.

THEOREM 4.10 *There exist positive constants* $C'_0 = C'_0(n)$, $C'_1 = C'_1(n)$, C'_3, *and* C'_5 *such that the following property is satisfied*: *If* μ *is a locally finite Borel measure,* Q_0 *a fixed cube in* $\mathbb{R}^n$, $\hat{J}_2 \equiv \hat{J}_2^{Q_0,\mu}$, *and* A *is a positive number such that*

$$\int_{C'_5 \cdot Q_0} e^{C'_0 \cdot \hat{J}_2(x)} \, \mathrm{d}\mu(x) \leq A \cdot \mu(Q_0) ,$$

then there exists a curve Γ, $\Gamma \subseteq C'_5 \cdot Q_0$, *with length at most* $C'_1 \cdot A \cdot l(Q_0)$ *and measure at least* $C_3'^{-1} \cdot A^{-1} \cdot \mu(Q_0)$.

REMARK 4.11 Theorem 4.8 is a special case of the above theorem. Indeed, if $\hat{J}_2(x) \le M$ for all $x \in \text{supp}(\mu) \cap C_5' \cdot Q_0$, then $\hat{J}_2(x) \le M$ for all $x \in C_5' \cdot Q_0$. Now set $A := e^{C_0' \cdot M}$ so that the condition of the latter theorem is satisfied. Theorem 4.8 is then concluded with the following constants: $C_2 \equiv C_4 := C_0'$, $C_1 = C_1'$, $C_3 = C_3'$, and $C_5 = C_5'$.

REMARK 4.12 In the very special case of one-dimensional Ahlfors-regular measures, the function $e^{C_0' \cdot \hat{J}_2}$ in the above integral can be replaced by $1 + C_0' \cdot \hat{J}_2$, or equivalently $1 + \hat{J}_2$ (see Theorem 5.1 for a stronger statement).

REMARK 4.13 In general, it is impossible to replace the function $e^{C_0' \cdot \hat{J}_2}$ in the above integral by a polynomial function of $\hat{J}_2$. We verify this proposition in example 4 of Section 5.

We next define the most general class of measures (with respect to the fixed cube Q_0) for which Theorem 4.10 applies. Denote

$$I_C(\mu) \equiv I_C^{Q_0}(\mu) := \int_{C_5 \cdot Q_0} e^{C \cdot \hat{J}_2(x)} \, \mathrm{d}\mu(x) \,,$$

where $\hat{J}_2 \equiv \hat{J}_2^{Q_0, \mu}$. Fix C_0^*, the minimal (or nearly minimal) constant among all constants C_0' for which Theorem 4.10 is satisfied. Define the class of measures $\mathcal{J}^n$ (with respect to the cube Q_0) as follows:

$$\mathcal{J}^n \equiv \mathcal{J}^{n, Q_0} = \left\{ \mu : \mu \text{ is a locally finite Borel measure on } \mathbb{R}^n \text{ and } I_{C_0^*}^{Q_0}(\mu) < \infty \right\} . \tag{4.5}$$

REMARK 4.14 It is possible to elaborate on the proof of the above theorem to obtain that all measures in $\mathcal{J}^n$ are weakly one-dimensional rectifiable inside the cube Q_0 (see [16] for a more general result).

We end this section by showing that the L_1 norms of J_2 and $\hat{J}_2$ are comparable. This fact can be used to replace $\hat{J}_2$ by J_2 in the above two theorems (the way of doing it follows from [16]).

LEMMA 4.15 *If μ is a locally finite Borel measure on $\mathbb{R}^n$, Q_0 is a cube in $\mathbb{R}^n$, and J_2 and $\hat{J}_2$ are defined with respect to Q_0, then*

$$\int J_2(x) \mathrm{d}\mu(x) \approx_n \int \hat{J}_2(x) \mathrm{d}\mu(x) \,.$$

PROOF: We first develop two simple expressions for the integrals $\int J_2(x)\mathrm{d}\mu(x)$ and $\int \hat{J}_2(x)\mathrm{d}\mu(x)$.

Let Q be a cube in $\tilde{\mathcal{D}}$. Recall that $\mathcal{H}_Q$ denotes the set of all best L_2 lines for Q. Note that if $L_Q \in \mathcal{H}_Q$, then

$$\beta_2^2(Q) = \begin{cases} \frac{1}{\mu(Q)} \int_Q \left(\frac{\operatorname{dist}(z, L_Q)}{l(Q)}\right)^2 d\mu(z) & \text{if } \mu(Q) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Combine the above equation with equation (4.1) to obtain that if $\{L_Q\}_{Q\in\tilde{\mathcal{D}}}$ is any collection of best L_2 lines in $\{\mathcal{H}_Q\}_{Q\in\tilde{\mathcal{D}}}$, then

$$\int J_2(x) d\mu(x) = \int \sum_{Q\in\tilde{\mathcal{D}}} \int \frac{1}{\mu(Q)} \left(\frac{\operatorname{dist}(y, L_Q)}{l(Q)}\right)^2 \chi_Q(y) d\mu(y) \chi_Q(x) d\mu(x) . \tag{4.6}$$

The application of Fubini's theorem to equation (4.6) results in the following simple expression for $\int J_2(x)d\mu(x)$:

$$\int J_2(x) d\mu(x) = \sum_{Q\in\tilde{\mathcal{D}}} \int_Q \left(\frac{\operatorname{dist}(y, L_Q)}{l(Q)}\right)^2 d\mu(y) . \tag{4.7}$$

A similar estimate for $\int \hat{J}_2(x)d\mu(x)$ is derived as follows: If Q is a cube in $\tilde{\mathcal{D}}$, fix a cube $\hat{Q}(Q)$ in $\mathcal{P}(Q)$ and a best L_2 line $\hat{P}_Q$ for $\hat{Q}(Q)$ so that the following equation is satisfied:

$$\hat{\beta}_2^2(Q) = \begin{cases} \frac{1}{\mu(Q)} \int_Q \left(\frac{\operatorname{dist}(z, \hat{P}_Q)}{l(Q)}\right)^2 d\mu(z) & \text{if } \mu(Q) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By combining the above equation with equation (4.3) and Fubini's theorem, we obtain the following expression for $\int \hat{J}_2(x)d\mu(x)$:

$$\int \hat{J}_2(x) d\mu(x) = \sum_{Q\in\tilde{\mathcal{D}}} \int_Q \left(\frac{\operatorname{dist}(y, \hat{P}_Q)}{l(Q)}\right)^2 d\mu(y) . \tag{4.8}$$

Note that $\beta_2(Q) \le \hat{\beta}_2(Q)$ for any cube Q in $\tilde{\mathcal{D}}$. Thus in order to prove the lemma, it is sufficient to control $\int \hat{J}_2(x)d\mu(x)$ by $\int J_2(x)d\mu(x)$. The relevant inequality follows from two observations. First,

$$\int_Q \left(\frac{\operatorname{dist}(z, \hat{P}_Q)}{l(Q)}\right)^2 d\mu(z) \le 4^{j_1^*} \cdot \int_{\hat{Q}(Q)} \left(\frac{\operatorname{dist}(z, \hat{P}_Q)}{l(Q)}\right)^2 d\mu(z) .$$

Second, if Q is a cube in $\tilde{\mathcal{D}}$, then

$$\#\{Q' : \hat{Q}(Q') = \hat{Q}(Q)\} \leq C(n)$$

for some large constant $C(n)$. By combining the above two equations with equations (4.7) and (4.8), we obtain that

$$\int \hat{J}_2(x)\mathrm{d}\mu(x) \leq 4^{j_1^*} \cdot C(n) \cdot \int J_2(x)\mathrm{d}\mu(x)$$

and thus prove the lemma. □

5 Examples of Sets and Their L_2 Jones Quantities

In this section we introduce five examples where we clarify some of the definitions of the previous section and also Theorems 4.8 and 4.10. In each one of the examples there is a special measure or a class of measures on $\mathbb{R}^n$ and a fixed cube Q_0 in $\mathbb{R}^n$. We estimate the corresponding Jones function $\hat{J}_2 = \hat{J}_2^{Q_0,\mu}$ and also evaluate the sharp estimates for the optimization problems discussed in the previous section. We then compare these estimates with the ones implied by Theorem 4.8 (if applicable).

In the first three examples the lower bound in equation (4.4) is optimal up to logarithmic comparability. Moreover, in the third example both exponential estimates of Theorem 4.8 are sharp for the curve Γ_{opt}. The fourth example implies that the conditions controlling the size of $\hat{J}_2$ in Theorems 4.8 and 4.10 cannot be replaced with L_p bounds of $\hat{J}_2$. Finally, we discuss the special class of one-dimensional Ahlfors-regular measures. In this example, one has a linear bound in M for the length of a curve with full measure.

5.1 Example 1

This example is practically the same as the optimal example of [4]. Fix $\varepsilon > 0$ and assume for simplicity that $n = 2$. Construct the curves $\{\Gamma_i\}_{i \geq 0}$ as follows. The curve Γ_0 is a fixed line segment in $\mathbb{R}^2$. Assume that the curve Γ_{i-1}, $i \geq 1$, has been constructed and that it is piecewise linear. Form Γ_i by replacing each line segment I composing Γ_{i-1} with the two "opposite" sides of an isosceles triangle with base I and height $\varepsilon \cdot l(I)$ (by "opposite" we mean a negative scalar product between the current height vector and the previous-stage height vector that was used in forming the line segment I). Fix a sufficiently large integer N, $N \gg 1/\varepsilon^2$, and denote by μ_N the arc length probability measure of Γ_N. That is,

$$\mathrm{d}\mu_N = \frac{\mathrm{d}\mathcal{H}^1\big|_{\Gamma_N}}{l(\Gamma_N)}\,.$$

Let Q_0 be a cube containing the curve Γ_N such that $l(Q_0) \approx l(\Gamma_0)$.

We make the following observations: First,

$$\frac{l(\Gamma_N)}{l(\Gamma_0)} = (1+\varepsilon^2)^{\frac{N}{2}} \quad \text{and consequently} \quad \log\frac{l(\Gamma_N)}{l(\Gamma_0)} \approx \frac{N\cdot\varepsilon^2}{2}\,.$$

Second, there exists a constant C such that if Γ' is any curve in $\mathbb{R}^2$, then

$$\frac{\mu_N(\Gamma')}{\mu_N(Q_0)} \cdot \frac{l(\Gamma_0)}{l(\Gamma_N)} \lessapprox e^{-\frac{N\cdot\varepsilon^2}{C}} . \tag{5.1}$$

Third,

$$M' := \max_{x \in Q_0} \hat{J}_2 \lessapprox N \cdot \varepsilon^2 . \tag{5.2}$$

Indeed, if Q is a cube in $\tilde{\mathcal{D}}$, then $\hat{\beta}_2(Q) \lessapprox \varepsilon$. Also, if $Q \cap \Gamma_N = \varnothing$, then $\hat{\beta}_2(Q) = 0$.

Equations (5.1) and (5.2) imply that the lower bound in equation (4.4) is optimal (up to logarithmic comparability) if $L \leq l(\Gamma_N)$ and $M := M'$. Note that there exist curves that obtain both exponential bounds of Theorem 4.8 up to logarithmic comparability (take, e.g., Γ' to be a subcurve of Γ_N with length $(1 + \varepsilon^2)^{N/4}$). However, the unique curve $\Gamma_{\text{opt}} \equiv \Gamma_N$ only obtains the exponential bound of the length estimate but not the one of the measure estimate ($\mu_N(\Gamma_{\text{opt}}) = 1$).

We remark that it is possible to construct an analogous n-dimensional example where $\hat{J}_2$ is proportional to the dimension n. The corresponding sequence of curves $\{\Gamma'_i\}_{i\geq 0}$ is formed by replacing recursively each edge of Γ'_{i-1}, $i \geq 1$, with n smaller intervals of different directions in $\mathbb{R}^n$.

5.2 Example 2

Fix $\varepsilon > 0$ and a large integer N, $N \ll \frac{1}{\varepsilon} \cdot \log \frac{1}{\varepsilon}$. Assume for simplicity that $n = 2$. Define

$$I_i = [0, 1] \times \left\{ \frac{1}{8^i} \right\}, \quad 0 \leq i \leq N , \qquad I_{N+1} = [0, 1] \times \{0\} ,$$

and

$$\mathrm{d}\mu_N = \sum_{i=0}^{N} \varepsilon \cdot (1 - \varepsilon)^i \cdot \mathrm{d}\mathcal{H}^1\big|_{I_i} + (1 - \varepsilon)^{N+1} \, \mathrm{d}\mathcal{H}^1\big|_{I_{N+1}} .$$

Let Q_0 be a cube in $\mathbb{R}^2$ containing the support of μ_N and with side length 2. Note that

$$\mu_N(\mathbb{R}^2) = 1 , \quad \mu_N(I_{N+1}) = (1 - \varepsilon)^{N+1} , \quad \mu_N(I_i) \leq \varepsilon , \quad i = 0, \ldots, N.$$

The assumption $N \ll \varepsilon^{-1} \log \varepsilon^{-1}$ implies that $\mu_N(I_i) \ll \mu_N(I_{N+1})$, $i = 0, \ldots, N$. Therefore the ratio $\mu_N(\Gamma)/l(\Gamma)$ is optimized for curves Γ that are line segments contained in $[0, 1] \times \{0\}$. Moreover,

$$r_{\max}(\mu_N, L) = 2 \cdot \mu_N(I_{N+1}) = 2 \cdot (1 - \varepsilon)^{N+1} \lessapprox 2 \cdot e^{-\frac{N\cdot\varepsilon}{C}} , \quad 0 < L \leq 1 , \tag{5.3}$$

and

$$\Gamma_{\text{opt}} \equiv [0, 1] \times \{0\} .$$

We next compare the above upper bound for $r_{\max}(\mu_N, L)$, $0 < L \leq 1$, with the lower bound given in equation (4.4). Note that

$$\hat{J}_2(x) \approx i \cdot \varepsilon + C \quad \text{for any } x \in I_i , \quad i = 0, \ldots, N + 1.$$

Therefore,

$$M' := \max_{x \in Q_0} \hat{J}_2(x) \approx N \cdot \varepsilon \,. \tag{5.4}$$

Equations (5.3) and (5.4) imply that the lower bound in equation (4.4) is optimal (up to logarithmic comparability) when using $M \equiv M'$. Also note that the measure of Γ_{opt} decays exponentially in M like the estimate of Theorem 4.8; however, $l(\Gamma_{\text{opt}}) = 1$.

5.3 Example 3

We combine the previous two examples to obtain a measure with a unique curve Γ_{opt} such that Γ_{opt} achieves both the length and measure estimates of Theorem 4.8. Fix $\varepsilon > 0$ and $\frac{1}{\varepsilon} \ll N \ll \frac{1}{\varepsilon} \cdot \log \frac{1}{\varepsilon}$. Let

$$\Gamma_N = \bigcup_{i=0}^{N+1} \gamma_i \,,$$

where γ_i is the i^{th} iteration of the von Koch–type curve described in example 1 with parameter $\sqrt{\varepsilon}$ and with the initial interval $\gamma_0 := [0, 1] \times \{\frac{1}{8^i}\}$. Define

$$\mathrm{d}\mu_N = \sum_{i=0}^{N} \varepsilon \cdot (1-\varepsilon)^i \cdot \frac{\mathrm{d}\mathcal{H}^1|_{\gamma_i}}{l(\gamma_i)} + (1-\varepsilon)^{N+1} \cdot \frac{\mathrm{d}\mathcal{H}^1|_{\gamma_{N+1}}}{l(\gamma_{N+1})} \,.$$

Let Q_0 be a cube in $\mathbb{R}^2$ containing the support of μ_N with side length comparable to 1. Note that

$$\Gamma_{\text{opt}} = \gamma_{N+1} \,, \quad \mu_N(\Gamma_{\text{opt}}) = (1-\varepsilon)^{N+1} \lessapprox e^{-\frac{N\cdot\varepsilon}{C}} \,, \quad l(\Gamma_{\text{opt}}) = (1+\varepsilon)^{\frac{N}{2}} \gtrapprox e^{\frac{N\cdot\varepsilon}{C}} \,.$$

Also note that

$$M' := \max_{x \in Q_0} \hat{J}_2 \approx N \cdot \varepsilon \,.$$

Therefore by taking $M = M'$, the two exponential bounds of Theorem 4.8 are obtained by the curve Γ_{opt} but with different constants.

5.4 Example 4

This example can be thought of as a limit of example 2 as $N \to \infty$. It shows that the integral condition in Theorem 4.10 cannot be replaced with an L_p bound of J_2, where $0 < p < \infty$.

Fix $0 < \varepsilon \ll 1$ and set

$$I_i := [0, 1] \times \left\{ \frac{1}{8^i} \right\}, \quad i = 0, 1, \ldots .$$

Define the probability measure μ as follows:

$$\mathrm{d}\mu = \sum_{i=0}^{\infty} \varepsilon \cdot (1-\varepsilon)^i \cdot \mathrm{d}\mathcal{H}^1\big|_{I_i} \,.$$

Fix Q_0 a cube in $\mathbb{R}^2$ containing the support of μ with side length 2. Note that $\hat{J}_2(x) \approx i \cdot \varepsilon + C$ for any $x \in I_i$. Consequently,

$$\|\hat{J}_2\|_{L_\infty(\mu)} = \infty\,, \qquad \int e^{C_0' \cdot \hat{J}_2(x)}\,\mathrm{d}\mu(x) = \infty\,, \qquad \|\hat{J}_2\|_{\mathrm{BMO}(\mu)} < \infty\,,$$

and

$$\|\hat{J}_2\|_{L_p(\mu)} = O(p+1)\,, \quad 0 < p < \infty\,.$$

Also note that

$$r_{\max}(\mu, L) < 2 \cdot \varepsilon \quad \text{for any } L \geq 0\,.$$

On the other hand, taking ε to zero does not affect the finite values of $\|\hat{J}_2\|_{L_p(\mu)}$, $0 < p < \infty$, and $\|\hat{J}_2\|_{\mathrm{BMO}(\mu)}$. We thus conclude that it is impossible to replace the uniform bound of $\hat{J}_2$ in Theorem 4.8 or the exponential integral bound in Theorem 4.10 by an L_p or BMO norm of $\hat{J}_2$.

5.5 Example 5

Fix a cube Q_0 in $\mathbb{R}^n$. Assume that μ is a one-dimensional Ahlfors-regular probability measure on $\mathbb{R}^n$ with compact support. Furthermore, assume that

$$\text{supp}(\mu) \subseteq Q_0\,, \quad \text{diam}(\text{supp}(\mu)) \approx_n l(Q_0) = 1\,, \tag{5.5}$$

and

$$M := \|\hat{J}_2\|_{L_\infty(\mu)} < \infty \quad \text{where } \hat{J}_2 \equiv \hat{J}_2^{Q_0,\mu}.$$

Note that in this special case $\beta_2(Q) \approx_n \hat{\beta}_2(Q)$ for all cubes Q in $\tilde{\mathcal{D}}$ ($\tilde{\mathcal{D}} \equiv \tilde{\mathcal{D}}^{+3}_{Q_0}$) and consequently $\hat{J}_2 \approx_n J_2$. Moreover,

$$\sum_{Q \in \tilde{\mathcal{D}}} \beta_2^2(Q) \cdot l(Q) \approx_n \sum_{Q \in \tilde{\mathcal{D}}} \beta_2^2(Q) \cdot \mu(Q) = \int J_2(x)\mathrm{d}\mu(x) \approx_n \int \hat{J}_2(x)\mathrm{d}\mu(x) \leq M\,. \tag{5.6}$$

Combine equations (5.5) and (5.6) with the length estimate in [23] (this estimate can be adapted for Ahlfors-regular measures instead of sets) to obtain

$$\inf_{\substack{\Gamma \text{ rectifiable,} \\ \mu(\Gamma)=1}} l(\Gamma) \lesssim_n M + 1\,;$$

consequently,

$$r_{\max}(\mu, L) \gtrsim_n \frac{1}{1+M}\,, \quad 0 < L \leq \inf_{\substack{\Gamma \text{ rectifiable,} \\ \mu(\Gamma)=1}} l(\Gamma)\,.$$

In this case, the estimate in equation (4.4) is far from optimal if $M \gg 1$.

In this example, the optimal characterization of curvelike structure by Jones quantities follows from [14, theorem 1], [22] and [24, theorem 1.3]. By applying equation (5.6), we formulate this characterization in terms of the Jones function $\hat{J}_2(x)$ as follows:

THEOREM 5.1 [14, 22, 24] *If Q_0 is a cube in $\mathbb{R}^n$, μ a one-dimensional Ahlfors-regular probability measure on $\mathbb{R}^n$ with compact support satisfying equation* (5.5), *and $\hat{J}_2 \equiv \hat{J}_2^{Q_0,\mu}$, then there exists a rectifiable curve with full measure if and only if $\int \hat{J}_2 \,\mathrm{d}\mu(x) < \infty$. Moreover,*

$$\inf_{\substack{\Gamma \text{ rectifiable}, \\ \mu(\Gamma)=1}} l(\Gamma) \approx_n \int (1 + \hat{J}_2)\mathrm{d}\mu(x)\,.$$

6 The δ Theorem

In this section we formulate and prove Theorem 6.3. It is a "special case" (see Remark 6.2) of Theorem 4.8, where $M = \delta$, a small constant.

NOTATION AND DEFINITIONS 6.1 Throughout this section we fix the cube Q_1 instead of Q_0. This notation is consistent with that of Section 7, where Q_1 is used for the δ-construction. We also redefine here the grid $\tilde{\mathcal{D}}$ as follows: $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}^{+0}_{Q_1} \equiv \tilde{\mathcal{D}}(Q_1)$, whereas in other sections $\tilde{\mathcal{D}} \equiv \tilde{\mathcal{D}}^{+3}_{Q_0}$. Note that the Jones quantities ($\hat{\beta}_2$ and $\hat{J}_2$) are redefined here with respect to the cube Q_1 and the grid $\tilde{\mathcal{D}}^{+0}_{Q_1}$ (instead of Q_0 and $\tilde{\mathcal{D}}^{+3}_{Q_0}$).

REMARK 6.2 The redefinition of the grid $\tilde{\mathcal{D}}$ (see above notation and definitions) makes the estimates of the following theorem sharper than those of the special case of Theorem 4.8, where $M = \delta$.

THEOREM 6.3 *There exist positive constants $C_1 = C_1(n)$, $C_2 = C_2(n)$, $C_4 = C_4(n)$, $\varepsilon_1 = \varepsilon_1(n)$, and $\delta_0 = \delta_0(n)$ such that the following proposition is satisfied: If μ is a locally finite Borel measure on $\mathbb{R}^n$, Q_1 is a cube in $\mathbb{R}^n$, $\delta \leq \delta_0$, and*

$$\hat{J}_2(x) \equiv \hat{J}_2^{Q_1,\mu}(x) \leq \delta \quad \textit{for any } x \in \operatorname{supp}(\mu) \cap (1+\varepsilon_1)\cdot Q_1\,, \tag{6.1}$$

then there exists a curve Γ such that

$$l(\Gamma) \leq C_1 \cdot e^{C_2\cdot\delta} \cdot l(Q_1) \quad \textit{and} \quad \mu(\Gamma \cap (1+\varepsilon_1)\cdot Q_1) \geq e^{-C_4\cdot\delta}\cdot \mu(Q_1)\,.$$

We build the curve Γ suggested by the above theorem in a multiscale fashion. At each stage l of the basic construction we form a piecewise linear curve Γ_l, a stopping-time region S_l, and a "strip" E'_l around $\Gamma_l \cap S^{\mathrm{c}}_l$. The length of edges in Γ_l and the "thickness" of E'_l are of order $l(Q_1)\cdot C_L^{-l}$. The general construction is obtained by restarting the basic construction repeatedly at stopping-time cubes contained in S_l. It results in curves $\tilde{\Gamma}_N$, $N \geq 1$, and strips around it $\tilde{E}_N \cup \tilde{S}_N$, $N \geq 1$, with thickness of order $l(Q_1)\cdot C_L^{-N}$.

The estimates of the length and measure of Γ are based on simple geometric arguments together with martingale-type and stopping-time techniques. The geometric estimates follow mainly from three elementary theorems: Pythagoras' theorem, Jensen's inequality, and Chebyshev's inequality.

Jensen's and Chebyshev's inequalities are used in verifying the following local properties:

LEMMA 6.4 *If μ is a locally finite Borel measure on $\mathbb{R}^n$, Q and $\hat{Q}$ are cubes in $\mathbb{R}^n$ such that $Q \cap \operatorname{supp}(\mu) \subseteq \hat{Q} \cap \operatorname{supp}(\mu)$, $L_{\hat{Q}}$ is a best L_2 line in $\hat{Q}$, and z_Q denotes the center of mass of the cube Q, then*

$$\frac{\operatorname{dist}(z_Q, L_{\hat{Q}})}{l(Q)} \leq \beta_2(Q, \hat{Q}) .$$

PROOF: This is a simple application of Jensen's inequality as follows:

$$\begin{aligned}\operatorname{dist}^2(z_Q, L_{\hat{Q}}) &= \operatorname{dist}^2\left(\int_Q z\frac{\mathrm{d}\mu(z)}{\mu(Q)}, L_{\hat{Q}}\right) \\ &\leq \int_Q \operatorname{dist}^2(z, L_{\hat{Q}})\frac{\mathrm{d}\mu(z)}{\mu(Q)} \leq \beta_2^2(Q, \hat{Q}) \cdot l(Q)^2 .\end{aligned}$$

□

LEMMA 6.5 *If μ is a locally finite Borel measure on $\mathbb{R}^n$, Q and $\hat{Q}$ are cubes in $\mathbb{R}^n$ such that $Q \cap \operatorname{supp}(\mu) \subseteq \hat{Q} \cap \operatorname{supp}(\mu)$, and $L_{\hat{Q}}$ is a best L_2 line for $\hat{Q}$, then*

$$\mu\{x \in Q : \operatorname{dist}(x, L_{\hat{Q}}) \geq \varepsilon \cdot l(Q)\} \leq \frac{1}{\varepsilon^2} \cdot \beta_2^2(Q, \hat{Q}) \cdot \mu(Q) .$$

PROOF: This is a straightforward application of Chebyshev's inequality. Indeed,

$$\begin{aligned}\mu\{x \in Q : \operatorname{dist}(x, L_{\hat{Q}}) \geq \varepsilon \cdot l(Q)\} &\leq \frac{1}{\varepsilon^2} \cdot \int_Q \frac{\operatorname{dist}^2(x, L_{\hat{Q}})}{l(Q)^2}\frac{\mathrm{d}\mu(x)}{\mu(Q)} \cdot \mu(Q) \\ &\leq \frac{1}{\varepsilon^2} \cdot \beta_2^2(Q, \hat{Q}) \cdot \mu(Q) .\end{aligned}$$

□

The martingale-type idea of the proof is used in the basic construction and is described heuristically as follows. Following Bishop and Jones [4], we form a sequence of functions $\{F_l\}_{l \geq 0}$ such that $0 = F_0 \leq F_1 \leq \cdots \leq F_l \leq \cdots \leq \delta$. We show that there exists a sufficiently large constant $C_2 = C_2(n)$ such that the sequence $e^{-C_2 \cdot F_l}$, $l \geq 0$, is a "supermartingale" in the following sense:

$$\int_{\Gamma_l} e^{-C_2 \cdot F_l(x)}\, \mathrm{d}s(x) \leq \int_{\Gamma_{l-1}} e^{-C_2 \cdot F_{l-1}(x)}\, \mathrm{d}s(x) , \quad l \geq 1 .$$

This estimate follows mainly from Lemma 6.4 and Pythagoras' theorem (see Section 6.3). By applying it repeatedly, we obtain the following uniform bound on the lengths of the approximating curves:

$$l(\Gamma_l) \leq e^{C_2 \cdot \delta} \cdot l(\Gamma_0) .$$

Similarly, we show that there exists a sufficiently large constant $C_4 = C_4(n)$ such that the sequence $e^{C_4 \cdot F_l}$, $l \geq 0$, is a submartingale in the following sense:

$$\int_{E'_l \cup S_l} e^{C_4 \cdot F_l(x)} \, \mathrm{d}\mu(x) \geq \int_{E'_{l-1} \cup S_{l-1}} e^{C_4 \cdot F_{l-1}(x)} \, \mathrm{d}\mu(x) , \quad l \geq 1 .$$

This estimate follows mainly from Lemma 6.5 (see Section 6.4). By applying it repeatedly, we obtain the following uniform lower bound on the measures of the approximating curves:

$$\mu(E'_l \cup S_l) \geq e^{-C_4 \cdot \delta} \cdot \mu(Q_1) .$$

If there are no stopping-time cubes ($S_l = \varnothing, l \geq 0$), then a limit argument concludes the length and measure estimates of the curve Γ itself from the above estimates. Otherwise, we normalize the above estimates of the basic construction at each stopping-time cube and control the sum of the lengths of the stopping-time cubes to conclude the length and measure estimates for Γ.

We remark that condition (6.1) in Theorem 6.3 implies that

$$\hat{\beta}_2(Q) < \sqrt{\delta} \quad \text{for any } Q \in \tilde{\mathcal{D}} . \tag{6.2}$$

The above weak condition is used in our proof, but it is not sufficient to imply the theorem. Indeed, let K be a generalized von Koch arc in the plane with angles of size $\sqrt{\delta}$, and let s be the Hausdorff dimension of K (see, e.g., example 2 in Section 5 for constructing a sequence of curves converging to K; in that example $\varepsilon = \sqrt{\delta}$). Construct an s-dimensional Frostman's measure μ with support K (see, e.g., [20, proof of theorem 8.8]). This measure satisfies equation (6.2). However, any subset of the plane with positive such measure has an infinite length.

Different constants appear in the proof, sometimes with the same notation. They might depend on the dimension of the ambient space n. The following large constants are used extensively: A_0, $A_1 = A_1(n)$, $A_2 = A_2(n)$, $C_L = C_L(n)$, j_0^*, and $j_1^* = j_1^*(n)$. We have initialized them as follows: $A_0 = 5$, $A_1 = A_1(n)$ is the smallest integer greater than $6480 \cdot \sqrt{n} \cdot e$ such that $\log_2 A_1$ is an integer, $A_2 = 128 \cdot A_1$, $C_L = A_1$, $j_0^* = 2$, $j_1^* = 16 \cdot \log_2 A_1$, and $\varepsilon_1 = \varepsilon_1(n) = 2/(3 \cdot C_L - 2)$. We also initialize the constant $\delta_0 = \delta_0(n)$ so that $\delta_0 = 1/(6336 \cdot e \cdot C_L^3)$. We fix throughout this section a constant δ such that $\delta \leq \delta_0$. Our choice of constants is not optimal. We do not worry here about the most desirable numbers. We recall that the definitions of the set $\mathcal{P}(Q)$ and the Jones quantities $\hat{\beta}_2$ and $\hat{J}_2$ depend on the fixed values of the constants j_0^* and j_1^*.

In Section 6.1 we list properties of the l^{th} level basic construction and present the zeroth-level construction. In Section 6.2 we describe the basic inductive construction at stage l, $l \geq 1$. We verify the relevant length and measure estimates in Sections 6.3 and 6.4, respectively. In Section 6.5 we formulate the general construction by restarting the basic construction at stopping-time cubes. Finally, in

Section 6.6 we validate the length and measure estimates of the general construction and consequently conclude Theorem 6.3. Some of the highly technical but elementary computations of this section appear in the appendix.

6.1 Properties of the Basic Construction

In this section we describe the zeroth-level basic construction and list some properties of the l^{th}-level construction. In Section 6.2 we build the sets of level l, $l \geq 1$, while assuming that the sets of level $l-1$, $l \geq 1$, possess the properties given here. The construction implies that the same properties are then satisfied at level l.

The following sets appear at stage l of the basic construction: Γ_l, $\mathcal{M}_l$, E_l, $\mathcal{L}_l$, $\mathcal{M}'_l$, E'_l, $\mathcal{S}_l$, and S_l. The curve Γ_l approximates the ultimate curve Γ outside S_{l-1} at scale of order $\operatorname{diam}(K) \cdot C_L^{-l}$. The set $\mathcal{M}_l$, $l \geq 1$, is a collection of cubes surrounding the curve Γ_{l-1} outside S_{l-1}. The union of all cubes in $\mathcal{M}_l$ is the set E_l. The set $\mathcal{L}_l$ is composed of centers of masses of cubes in $\mathcal{M}_l$. We use the points in $\mathcal{L}_l$ as vertices of the curve Γ_l. The set $\mathcal{S}_l$ is a collection of disjoint stopping-time cubes. The union of these cubes is denoted by S_l and is referred to as the l^{th}-level stopping-time region. The sets $\mathcal{M}'_l$ and E'_l are obtained from $\mathcal{M}_l$ and E_l by excluding the stopping-time cubes in $\mathcal{S}_l$.

NOTATION 6.6 Denote

$$l_j = \frac{l(Q_1)}{C_L^j}, \quad j \geq 0 .$$

The letter l is used both for denoting length and for indexing levels. This convention results in the following notation: l_{l-1} and l_l.

DEFINITION 6.7 ("Phantom" point $\tilde{x}$ with respect to x and y) If x and y are two different points in $\mathbb{R}^n$, then a point $\tilde{x}$ in $\mathbb{R}^n$ is a *phantom point* with respect to x and y if and only if the following equations are satisfied:

$$\tilde{x} \in \partial Q(x, 4 \cdot A_0 \cdot l_{l-1}) \quad \text{and} \quad [x, y] \subseteq [\tilde{x}, y] .$$

DEFINITION 6.8 (Trivial and nontrivial components of $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$) A connected component γ of $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$ is trivial if and only if $\gamma \cap \mathcal{L}_{l-1} = \varnothing$. Similarly, γ is nontrivial if and only if $\gamma \cap \mathcal{L}_{l-1} \neq \varnothing$.

The zeroth-stage construction goes as follows: Fix a cube $\hat{Q}_1$ in $\mathcal{P}(Q_1)$ such that $l(\hat{Q}_1) = 2^{j_0^*} \cdot l(Q_1)$ and $\operatorname{dist}(\partial Q_1, \partial \hat{Q}_1) \geq \frac{1}{C_L} \cdot l(Q_1)$. The existence of such a cube follows from Proposition 3.2. Fix a best L_2 line for $\hat{Q}_1$ and denote it by $L_{\hat{Q}_1}$.

Define

$$\Gamma_0 = L_{\hat{Q}_1} \cap (9 \cdot A_0) \cdot Q_1 , \quad \mathcal{M}_0 = \varnothing , \quad \mathcal{M}'_0 = \{Q_1\} , \quad E_0 = E'_0 = Q_1 , \tag{6.3}$$

$$\mathcal{L}_0 = A_0 \cdot Q_1 \cap L_{\hat{Q}_1} \equiv \{z_1, z_2\} , \quad \text{and} \quad \mathcal{S}_0 = S_0 = \varnothing . \tag{6.4}$$

We remark that some of the above definitions are artificial. They are chosen so that we do not need to distinguish between the first-level construction and the ones at higher levels. In particular, there is no special meaning to the fact that $\mathcal{M}_0$ is empty.

The properties of the sets of level $l-1$, $l \geq 1$, are listed as follows. All of them are satisfied when $l = 1$.

(1) If $x \in \mathcal{L}_{l-1}$, then

$$Q(x, A_0 \cdot l_{l-1}) \cap \mathcal{L}_{l-1} \setminus \{x\} = \varnothing\,. \tag{6.5}$$

(2) If $l > 1$, $Q \in \mathcal{M}_{l-1}$, and z_Q is its center of mass, then there exists a point $x \in \mathcal{L}_{l-1} \cap S^{\mathrm{c}}_{l-1}$ such that

$$z_Q \in Q(x, A_0 \cdot l_{l-1})\,.$$

(3) If $l > 1$ and $x \in \mathcal{L}_{l-1} \cap S^{\mathrm{c}}_{l-1}$, then there exists a cube Q in $\mathcal{M}'_{l-1}$ such that x is its center of mass.

(4) The cubes in $\mathcal{S}_{l-1}$ are disjoint.

(5) The connected components of the set $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$ are piecewise linear curves with obtuse angles between neighboring edges.

(6) If γ is a nontrivial connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$, then there are at least two points in $\gamma \cap \mathcal{L}_{l-1}$.

(7) If γ is a nontrivial, connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$ and $z_1, \dots, z_m$, $m \geq 2$, are the points in $\gamma \cap \mathcal{L}_{l-1}$ indexed so that the union $\bigcup_{j=2}^{m}(z_{j-1}, z_j)$ is disjoint and contained in Γ_{l-1}, then

$$z_{j-1}, z_{j+1} \in Q(z_j, 4 \cdot A_1 \cdot l_{l-1}) \quad \text{for all } 2 \leq j \leq m-1\,. \tag{6.6}$$

(8) If γ is a nontrivial connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$, the points $z_1, \dots,$ z_m, $m \geq 2$, are defined as in property 7, z_0 is the phantom point with respect to z_1 and z_2, and z_{m+1} is the phantom point with respect to z_m and z_{m-1}. Then

$$\gamma \supseteq \bigcup_{j=1}^{m+1} [z_{j-1}, z_j]\,.$$

(9) If γ is a nontrivial connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$, the points $z_0,$ $\dots, z_m, z_{m+1}$, $m \geq 2$, are defined as in property 8, and Q is a cube in $\tilde{\mathcal{D}}$ with side length l_l such that

$$Q \cap \left(\gamma \setminus \bigcup_{j=1}^{m+1} [z_{j-1}, z_j]\right) \neq \varnothing\,,$$

then $Q \cap E'_{l-1} = \varnothing$.

(10) If γ is a trivial connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$ and Q is a cube in $\tilde{\mathcal{D}}$ with side length l_l such that $Q \cap \gamma \neq \varnothing$, then $Q \cap E'_{l-1} = \varnothing$.

(11) If γ is a nontrivial connected component of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$ and the points $z_0, \dots, z_m, z_{m+1}$, $m \geq 2$, are defined as in property 8, then for any point x in

$\bigcup_{j=1}^{m+1}[z_{j-1}, z_j]$ and a different nontrivial connected component γ' of $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$,

$$Q(x, 4 \cdot A_1 \cdot l_{l-1}) \cap \gamma' = \varnothing .$$

6.2 The Basic Construction for Stage *l*

In this section we construct the sets $\mathcal{M}_l$, E_l, $\mathcal{L}_l$, Γ_l, $\mathcal{S}_l$, S_l, $\mathcal{M}'_l$, and E'_l, $l \geq 1$. We assume that $\mathcal{M}_{l-1}$, E_{l-1}, $\mathcal{L}_{l-1}$, Γ_{l-1}, $\mathcal{S}_{l-1}$, S_{l-1}, $\mathcal{M}'_{l-1}$, and E'_{l-1} have been formed. The idea is to build the l^{th}-level sets locally around each point in $\mathcal{L}_{l-1}$ and then patch up the local parts to obtain the whole sets.

We define the sets $\mathcal{M}_l$ and E_l as follows:

$$\mathcal{M}_l = \left\{ \begin{array}{l} Q : Q \in \tilde{\mathcal{D}},\ l(Q) = l_l,\ \mu(Q) > 0,\ Q \cap E'_{l-1} \neq \varnothing, \\ \text{and there exists } x \in \Gamma_{l-1} \text{ such that } x \in \frac{2}{3} \cdot Q \end{array} \right\} \tag{6.7}$$

and

$$E_l = \bigcup_{Q \in \mathcal{M}_l} Q . \tag{6.8}$$

Before defining the set $\mathcal{L}_l$, we present some notation and formulate some properties of the curve Γ_{l-1} (Lemma 6.10). These properties are needed in order to properly define the set $\mathcal{L}_l$.

NOTATION AND DEFINITIONS 6.9 We refer to the nontrivial connected components of the set $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$ as its *segments*. Throughout the rest of this section, we fix a segment γ of $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$. We also fix the corresponding points $z_0, \ldots, z_m, z_{m+1}$, $m \geq 2$, defined in properties 7 and 8 of Section 6.1.

If $l > 1$ and $2 \leq j \leq m$, set a cube $\widehat{Q}_j$ in $\tilde{\mathcal{D}}$ with side length $A_2 \cdot l_{l-1}$ satisfying the equation

$$\widehat{Q}_j \supseteq Q\left(\frac{z_{j-1} + z_j}{2}, \frac{A_2}{6} \cdot l_{l-1}\right). \tag{6.9}$$

Fix a best L_2 line for $\widehat{Q}_j$ and denote it by $L_{\widehat{Q}_j}$. If $l = 1$, then $m = 2$ and $\hat{Q}_2$ is the fixed cube $\hat{Q}_1$ in $\mathcal{P}(Q_1)$ (see Section 6.1). The line $L_{\hat{Q}_2}$ is the best L_2 line $L_{\hat{Q}_1}$ in $\hat{Q}_1$ such that $\Gamma_0 \subseteq L_{\hat{Q}_1}$.

Let H_j, $j = 1, \ldots, m$, be the hyperplane containing z_j and bisecting the angle between the line segments $[z_{j-1}, z_j]$ and $[z_j, z_{j+1}]$. Let H_0 be the hyperplane containing z_0 and parallel to H_1, and let H_{m+1} be the hyperplane containing z_{m+1} and parallel to H_m.

Denote by $B_{j-1,j}$, $1 \leq j \leq m+1$, the closed region bounded between the hyperplanes H_{j-1} and H_{j+1}. Also denote by $I_{j-1,j}$ and $\tilde{I}_{j-1,j}$, $1 \leq j \leq m+1$, the regions

$$I_{j-1,j} = B_{j-1,j} \cap \overline{Q}\left(\frac{z_{j-1} + z_j}{2}, 4 \cdot A_1 \cdot l_{l-1}\right) \setminus H_{j-1}$$

and

$$\tilde{I}_{j-1,j} = \bigcup_{i=\max(j-3,1)}^{\min(j+3,m+1)} I_{i-1,i}\,.$$

Define the set $\mathcal{T}_{\widehat{Q}_j}$ as follows:

$$\mathcal{T}_{\widehat{Q}_j} = \left\{ Q : Q \in \mathcal{M}_{l-1} \cup \mathcal{M}_l,\ Q \cap \tilde{I}_{j-1,j} \neq \varnothing, \text{ and } \widehat{Q}_j \in \mathcal{P}(Q) \right\}.$$

Fix a cube $Q^j_{\max}$ in $\mathcal{T}_{\widehat{Q}_j}$ that satisfies the equality

$$\beta_2\big(Q^j_{\max}, \widehat{Q}_j\big) = \max_{Q \in \mathcal{T}_{\widehat{Q}_j}} \beta_2\big(Q, \hat{Q}_j\big)\,.$$

Observe that

$$\beta_2\big(Q^j_{\max}, \hat{Q}_j\big) \le \hat{\beta}_2\big(Q^j_{\max}\big)\,. \tag{6.10}$$

We next formulate some properties of the fixed segment γ with vertices $z_0, \ldots, z_{m+1}$ (see Notation and Definitions 6.9).

LEMMA 6.10 *The following properties are satisfied*:

(i) *If* $2 \le j \le m$ *and* $\max(1, j-2) \le p \le \min(m, j+2)$, *then*

$$\frac{\operatorname{dist}(z_p, L_{\hat{Q}_j})}{l_{l-1}} \le \beta_2\big(Q^j_{\max}, \hat{Q}_j\big)\,. \tag{6.11}$$

Also,

$$\frac{\operatorname{dist}(z_0, L_{\hat{Q}_2})}{l_{l-1}} \le 9 \cdot \beta_2\big(Q^2_{\max}, \hat{Q}_2\big) \tag{6.12}$$

and

$$\frac{\operatorname{dist}(z_{m+1}, L_{\hat{Q}_m})}{l_{l-1}} \le 9 \cdot \beta_2\big(Q^m_{\max}, \hat{Q}_m\big)\,. \tag{6.13}$$

(ii) *If* $0 \le j \le m+1$ *and* $\max(1, j-1) \le p \le \min(m+1, j+1)$, *then*

$$\cos(\operatorname{ang}([z_{p-1}, z_p], H_j)) \le \frac{3}{A_0} \cdot \beta_2\big(Q^j_{\max}, \hat{Q}_j\big)\,. \tag{6.14}$$

PROOF: (i) If $l = 1$, then this property is trivial due to the fact that $z_1, z_2 \in \Gamma_0 \subseteq L_{\hat{Q}_2}$ (see equation (6.4)). If, on the other hand, $l > 1$, then equation (6.11) follows mainly from Lemma 6.4. Indeed, fix integers p and j such that $1 \le j \le m$ and $\max(1, j-2) \le p \le \min(m, j+2)$. Recall that $z_p \in \mathcal{L}_{l-1} \cap S^{\mathrm{c}}_{l-1}$. Apply repeatedly equation (6.6) and obtain that

$$z_p \in Q\left(\frac{z_{j-1} + z_j}{2}, 10 \cdot A_1 \cdot l_{l-1}\right).$$

Let Q^* be a cube in $\mathcal{M}'_{l-1}$ such that z_p is its center of mass (see property 3 in Section 6.1). The above equation implies that

$$Q^* \subseteq \hat{Q}_j\,.$$

Apply Lemma 6.4 to the cubes Q^* and $\hat{Q}_j$ and conclude equation (6.11).

We next verify equation (6.12); equation (6.13) is derived similarly.

Note that

$$\text{dist}(z_0, L_{\hat{Q}_2}) \le \text{dist}(z_1, L_{\hat{Q}_2}) + \text{dist}(z_0, z_1) \cdot \sin\theta\,, \tag{6.15}$$

where θ is the angle between the line segment $[z_1, z_2]$ (or equivalently $[z_0, z_1]$) and the line $L_{\hat{Q}_2}$. Equation (6.11) implies the following estimate:

$$\sin\theta \le \frac{2 \cdot \beta_2(Q^1_{\max}, \hat{Q}_1) \cdot l_{l-1}}{\text{dist}(z_1, z_2)}\,. \tag{6.16}$$

Recall that

$$z_0 \in \partial Q(z_1, 4 \cdot A_0 \cdot l_{l-1}) \tag{6.17}$$

and that

$$z_2 \notin Q(z_1, A_0 \cdot l_{l-1})\,. \tag{6.18}$$

By applying equations (6.11) and (6.16) through (6.18) to equation (6.15), we conclude equation (6.12).

(ii) Equation (6.11) and the separation properties of the points $\{z_i\}_{i=1}^m$ (see equations (6.5) and (6.6)) imply that

$$\sin(\text{ang}([z_{p-1}, z_p], L_{\hat{Q}_j})) \le \frac{2}{A_0} \cdot \beta_2\big(Q^j_{\max}, \hat{Q}_j\big)\,, \tag{6.19}$$

where $1 \le j \le m$ and $\max(1, j-1) \le p \le \min(m, j+1)$; consequently,

$$\sin(\text{ang}([z_{p-1}, z_p], [z_p, z_{p+1}])) \le \frac{4}{A_0} \cdot \beta_2\big(Q^j_{\max}, \hat{Q}_j\big)\,, \quad 2 \le p \le m-1.$$

We prove (6.14) from the above equation and the fact that $\cos(2 \cdot \sqrt{\delta}/A_0) \le \frac{2}{3}$. □

We next construct the sets $\mathcal{L}_l(\gamma)$ for each segment γ of $\Gamma_{l-1} \cap S^{\mathsf{c}}_{l-1}$ (see Notation and Definitions 6.9). The set $\mathcal{L}_l$ is formed as the union of the sets $\mathcal{L}_l(\gamma)$ over all segments γ of $\Gamma_{l-1} \cap S^{\mathsf{c}}_{l-1}$.

Let $\mathcal{A}_l(\gamma)$ be the following set of centers of masses:

$$\mathcal{A}_l(\gamma) = \left\{ z_Q : Q \in \mathcal{M}_l,\, Q \cap \left(\bigcup_{j=1}^{m+1} I_{j-1,j} \right) \ne \varnothing \right\}.$$

Let $\mathcal{L}_l(\gamma)$ be a subset of $\mathcal{A}_l(\gamma)$ that is maximally separated by l_∞ distances $A_0 \cdot l_l$. That is,

$$Q(x, A_0 \cdot l_l) \cap \mathcal{L}_l(\gamma) \setminus \{x\} = \varnothing \quad \text{for all } x \in \mathcal{L}_l(\gamma) \tag{6.20}$$

and

$$\bigcup_{x \in \mathcal{L}_l(\gamma)} Q(x, A_0 \cdot l_l) \supseteq \mathcal{A}_l(\gamma)\,. \tag{6.21}$$

Note that if $x \in \mathcal{L}_l(\gamma)$, then there exists a unique j, $1 \le j \le m+1$, such that $x \in I_{j-1,j}$. This observation follows from Lemmata 6.4 and 6.10.

The curve Γ_l is built around each segment γ of $\Gamma_{l-1} \cap S^{\mathsf{c}}_{l-1}$ by using the set $\mathcal{L}_l(\gamma)$ as vertices. Before presenting the construction, we introduce some technical definitions.

NOTATION AND DEFINITIONS 6.11 Denote by $L_{j-1,j}$ the line containing the segment $[z_{j-1}, z_j]$ and by $P^{j-1,j}$ the projection onto $L_{j-1,j}$. The linear ordering $\le$ on $\mathcal{L}_l(\gamma)$ is defined as follows. Let $x_1, x_2 \in \mathcal{L}_l(\gamma)$ and assume that $x_1 \in I_{j_1-1,j_1}$ and $x_2 \in I_{j_2-1,j_2}$. If $j_1 < j_2$, then $x_1 \le x_2$. If $j_1 = j_2$, then we order x_1 and x_2 according to their projections on $L_{j-1,j}$; that is, let t_1 and t_2 be defined by the equation $P^{j_1-1,j_1} x_i = z_{j_1-1} + t_i \cdot (z_{j_1} - z_{j_1-1})$, $i = 1, 2$. We say that $x_1 \le x_2$ if and only if $t_1 \le t_2$.

We order the points $v_1^l, \dots, v_k^l$ in $\mathcal{L}_l(\gamma)$, where $k = k(\gamma)$, so that $v_1^l \le \cdots \le v_k^l$. We also define the points v_0^l and v_{k+1}^l as follows: $v_0^l = z_0$ and $v_{k+1}^l = z_{m+1}$.

The points in $\mathcal{L}_l(\gamma)$ are classified into three different types. We say that v_i^l, $i = 1, \dots, k$, is a type 1 point if

$$v_{i-1}^l, v_{i+1}^l \notin Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) . \tag{6.22}$$

We say that v_i^l, $i = 1, \dots, m$, is a type 2 point if either one of the following conditions is satisfied:

$$v_{i+1}^l \in Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) \quad \text{and} \quad v_{i-1}^l \notin Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) \tag{6.23}$$

or

$$v_{i-1}^l \in Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) \quad \text{and} \quad v_{i+1}^l \notin Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) . \tag{6.24}$$

We say that v_i^l, $i = 1, \dots, m$, is a type 3 point if

$$v_{i-1}^l, v_{i+1}^l \in Q\big(v_i^l, 4 \cdot A_1 \cdot l_l\big) .$$

We next extend the sequence $\{v_i^l\}_{i=0}^{k+1}$ to a larger sequence $\{v_i^l\}_{i=0}^{N+1}$ of vertices of Γ_l around the corresponding segment γ of $\Gamma_{l-1} \cap S^{\mathsf{c}}_{l-1}$. If v_i^l, $1 \le i \le k$, is a type 2 point and if it satisfies equation (6.23), then denote by $\tilde{v}_i^l$ the phantom point with respect to v_i^l and v_{i+1}^l and modify the whole sequence as follows:

$$v_i^l := \tilde{v}_i^l , \quad v_{i+p}^l := v_{i+p-1}^l , \quad 1 \le p \le k+1-i.$$

If v_i^l, $1 \le i \le k$, is a type 2 point that satisfies equation (6.24), then denote by $\tilde{v}_i^l$ the phantom point with respect to v_i^l and v_{i-1}^l and modify the whole sequence as follows:

$$v_{i+1}^l := \tilde{v}_i^l , \quad v_{i+p}^l := v_{i+p-1}^l , \quad 1 < p \le k+1-i.$$

If $v = v_i^l$, $i = 0, \dots, N$, define $v^+ = v_{i+1}^l$. Similarly, if $v = v_i^l$, $i = 1, \dots, N+1$, define $v^- = v_{i-1}^l$.

Form the curve $\Gamma_l(\gamma)$ as follows:

$$\Gamma_l(\gamma) = \bigcup_{i=0}^{N} \left[v_i^l, v_{i+1}^l\right].$$

Let $\Psi_{l-1}(\gamma)$ be the following part of the curve Γ_{l-1}:

$$\Psi_{l-1}(\gamma) = \bigcup_{i=0}^{m} [z_i, z_{i+1}].$$

Define the curve Γ_l by the formula

$$\Gamma_l = \Bigg(\Gamma_{l-1} \setminus \bigcup_{\substack{\gamma:\text{segment of}\\ \Gamma_{l-1}\cap S_{l-1}^{\mathsf{c}}}} \Psi_{l-1}(\gamma)\Bigg) \bigcup_{\substack{\gamma:\text{segment of}\\ \Gamma_{l-1}\cap S_{l-1}^{\mathsf{c}}}} \Gamma_l(\gamma).$$

We next establish some properties of the curve $\Gamma_l(\gamma)$ where γ is the fixed segment of $\Gamma_{l-1} \cap S_{l-1}^{\mathsf{c}}$. Most of these properties are used in the length and measure estimates of Sections 6.3 and 6.4.

LEMMA 6.12 *The following properties are satisfied*:

(i) *Fix j, $2 \le j \le m$, and $v = v_i^l$, $1 \le i \le N$. If $v \in I_{p-1,p} \cap \mathcal{L}_l(\gamma)$, where* $\max(1, j-3) \le p \le \min(m+1, j+3)$, *then*

$$\frac{\operatorname{dist}(v, L_{\hat{Q}_j})}{l_l} \le \beta_2\big(Q_{\max}^j, \hat{Q}_j\big). \tag{6.25}$$

If $v \notin \mathcal{L}_l(\gamma)$ and $v \in I_{p-1,p}$, where $\max(1, j-2) \le p \le \min(m+1, j+2)$, *then*

$$\frac{\operatorname{dist}(v, L_{\hat{Q}_j})}{l_l} \le 9 \cdot \beta_2\big(Q_{\max}^j, \hat{Q}_j\big). \tag{6.26}$$

(ii) *If $v = v_i^l$, $i = 1, \ldots, N$, and $v \in I_{p-1,p}$, $1 \le p \le m+1$, then*

$$\theta_v \le \frac{2}{A_0} \cdot \beta_2\big(Q_{\max}^p, \hat{Q}_p\big), \tag{6.27}$$

where θ_v is the small angle between $[v^-, v]$ *and* $[v, v^+]$.

(iii) *If v is a type 1 point in $\mathcal{L}_l(\gamma)$, then*

$$Q(v, 4 \cdot A_1 \cdot l_l) \cap \mathcal{L}_l = \varnothing. \tag{6.28}$$

(iv) *If* $\max(1, j-2) \le p \le \min(m+1, j+2)$ *and* $1 \le j \le m+1$, *then*

$$\operatorname{dist}_H(\Psi_{l-1}(\gamma) \cap I_{p-1,p}, \Gamma_l(\gamma) \cap I_{p-1,p}) \le 9 \cdot \sqrt{\delta} \cdot (1 + C_L) \cdot l_l, \tag{6.29}$$

where dist_H *denotes the Hausdorff distance.*

The verification of the above properties is based mainly on elementary geometric estimates but involves numerous details. Some of the arguments are similar to the ones in the proof of Lemma 6.10. We thus present the proof in Appendix A.1.

We next construct the stopping-time cubes and the corresponding stopping-time region S_l. If x is a type 1 point in $\mathcal{L}_l$, fix a cube Q_x in $\tilde{\mathcal{D}}$ such that $x \in \frac{2}{3} \cdot Q_x$ and with the following side length restriction:

$$6 \cdot (A_0 + 1) \cdot l_l \leq l(Q_x) < 12 \cdot (A_0 + 1) \cdot l_l \,. \tag{6.30}$$

The existence of this cube is guaranteed by Proposition 3.2. Note that

$$Q(x, (A_0 + 1) \cdot l_l) \subseteq Q_x \subseteq Q(x, 10 \cdot (A_0 + 1) \cdot l_l) \,. \tag{6.31}$$

The stopping-time collection $\mathcal{S}_l$ and the stopping-time region S_l are defined as follows:

$$\mathcal{S}_l = \{Q_x : x \text{ is a type 1 point in } \mathcal{L}_l\} \quad \text{and} \quad S_l = \bigcup_{Q_x \in \mathcal{S}_l} Q_x \,.$$

The construction of the curves Γ_i, $i > l$, is not modified in Q_x. Moreover, it is not modified "around" Q_x. Indeed, it follows from equations (6.20), (6.21), (6.22), and (6.28) that if x is a type 1 point, then

$$E_l \cap Q(x, (4 \cdot A_1 - A_0 - 1) \cdot l_l) \setminus Q(x, (A_0 + 1) \cdot l_l) = \varnothing \,.$$

Similarly, as in Jones [14], we define F_x, a free part of Γ_l around x, as follows:

$$F_x = \Gamma_{l-1} \cap \left(Q\left(x, \left(2 \cdot A_1 - \frac{A_0}{2} - \frac{1}{2}\right) \cdot l_l\right) \setminus Q\left(x, \frac{A_1}{2} \cdot l_l\right)\right). \tag{6.32}$$

Note that

$$l(F_x) \geq A_1 \cdot l_l \,. \tag{6.33}$$

Equation (6.32) implies that F_x will not be changed in the next stages of the construction. We use the free parts in the length estimate of the general construction (see Lemma 6.24).

Finally, define the sets $\mathcal{M}'_l$ and E'_l as follows:

$$\mathcal{M}'_l = \left\{Q : Q \in \mathcal{M}_l \text{ and } Q \cap S_l^{\mathrm{c}} \neq \varnothing\right\} \quad \text{and} \quad E'_l = \bigcup_{Q \in \mathcal{M}'_l} Q \,.$$

The l^{th}-level construction of the sets $\mathcal{M}_l$, E_l, $\mathcal{L}_l$, $\mathcal{S}_l$, S_l, $\mathcal{M}'_l$, and E'_l has been completed. Note that these sets satisfy the assumptions stated in Section 6.1. This observation follows from basic definitions and assumptions presented in this section, together with the properties stated in Lemma 6.10.

6.3 The Length Estimate for the Basic Construction

In this section we show that the lengths of the curves Γ_l, $l \geq 1$, are uniformly bounded. We establish this result through several technical lemmas. Before presenting the proof, we explain the basic ideas.

Following Bishop and Jones [4], we form an increasing sequence of positive functions $\{G_l\}_{l\geq 0}$ that are uniformly bounded by δ. The precise definition of these functions appears in equation (6.38). We show that there exists a constant $C_2 = C_2(n)$ such that $\{e^{-C_2 \cdot G_l}\}_{l\geq 0}$ is a supermartingale in the following sense:

$$\int_{\Gamma_l} e^{-C_2 \cdot G_l(x)} \, \mathrm{d}s(x) \leq \int_{\Gamma_{l-1}} e^{-C_2 \cdot G_{l-1}(x)} \, \mathrm{d}s(x) \,, \quad l \geq 1 \,. \tag{6.34}$$

A recursive application of this inequality together with the uniform bound on $\{G_l\}_{l\geq 0}$ suggests the following uniform bound on the lengths of the curves $\{\Gamma_l\}_{l\geq 0}$:

$$l(\Gamma_l) \leq l(\Gamma_0) \cdot e^{C_2 \cdot \delta} \,, \quad l \geq 0 \,.$$

We facilitate the understanding of equation (6.34) by observing the special case $l = 1$. It follows from equation (6.38) that

$$G_0 = 0$$

and

$$G_1 \geq \hat{\beta}_2^2(Q_{\max}) \cdot \chi_{(1-\frac{1}{C_L})\cdot Q_{\max}} \,, \tag{6.35}$$

where $Q_{\max}$ is a cube in $\mathcal{M}_1$ that maximizes $\hat{\beta}_2(Q)$ among all cubes Q in $\mathcal{M}_1$. Therefore, in order to verify equation (6.34) for the case $l = 1$, it is sufficient to show that

$$l\left(\Gamma_1 \setminus \left(1 - \frac{1}{C_L}\right) \cdot Q_{\max}\right) + e^{-C_2 \cdot \hat{\beta}_2^2(Q_{\max})} \cdot l\left(\Gamma_1 \cap \left(1 - \frac{1}{C_L}\right) \cdot Q_{\max}\right) \leq l(\Gamma_0) \,.$$

The above inequality follows from the following two different geometric estimates, which are instances of the Pythagorean theorem:

$$l(\Gamma_1) - l(\Gamma_0) \lessapprox \hat{\beta}_2^2(Q_{\max}) \cdot l(\Gamma_0) \tag{6.36}$$

and

$$l(\Gamma_0) \lessapprox_n l\left(\Gamma_1 \cap \left(1 - \frac{1}{C_L}\right) \cdot Q_{\max}\right) . \tag{6.37}$$

We will prove a more general form of the above inequalities later in this section.

In order to prove equation (6.34) for any level l, we form a sequence of functions $\{G_l^*\}_{l\geq 0}$ (see equation (6.39)) with the following two properties. First, there exists a collection of disjoint regions $\{J_i\}$ surrounding the curve Γ_{l-1} such that on each region J_i the following inequality is satisfied: $-G_{l-1}^* \leq C(J_i) \leq -G_{l-1}, l \geq$

1, where $C(J_i)$ is a given constant depending on J_i. The precise formulation of this property appears in equation (6.47). Second, the functions $G_l - G^*_{l-1}$, $l \geq 1$, can be controlled from below in a similar way as the function G_1 (or equivalently $G_1 - G^*_0$) in equation (6.35). The precise formulation of this property appears in equations (6.65) and (6.69). The point is that by writing $-G_l = -G^*_{l-1} - (G_l - G^*_{l-1})$ and using the above two properties, the proof of the supermartingale inequality for general l and a region J_i reduces to the proof of the special case $l = 1$. That is, one needs to prove two geometric estimates that are analogous to the ones in equations (6.36) and (6.37).

REMARK 6.13 When deriving equation (6.34) for level $l = 1$, it is possible to use more cubes of $\mathcal{M}_1$ (in addition to $Q_{\max}$) and consequently reduce the different constants. Similarly, in the subsequent proof of equation (6.34) for higher levels, one can use a larger collection of cubes in $\mathcal{M}_l$ and reduce the dependence of the constants C_2 and δ_0 on n.

REMARK 6.14 As noted before, the martingale-type idea of the length estimate is taken from Bishop and Jones [4]. However, the techniques here are more complicated than the original ones. For example, compare the definitions below of the functions G_l and G^*_l, $l \geq 0$, with the simpler definition of the single sequence $\{f_n\}$ in [4]. The extra complication is necessary. Indeed, Bishop and Jones use in their proof indicator functions of the form χ_{3Q} that do not apply for our purposes (we use the indicator functions $\{\chi_Q\}_{Q\in\tilde{\mathcal{D}}}$ in defining the Jones function $\hat{J}_2$).

NOTATION AND DEFINITIONS 6.15 Define the functions G_l and G^*_l, $l \geq 0$, as follows:

$$G_l(x) = \sum_{k=1}^{l} \sum_{Q\in\mathcal{M}_{k-1}\cup\mathcal{M}_k} \hat{\beta}_2^2(Q)\cdot \chi_{(1-(\frac{1}{C_L})^{l-k+1})Q}(x) \tag{6.38}$$

and

$$G^*_l(x) = \sum_{k=1}^{l} \sum_{Q\in\mathcal{M}_{k-1}\cup\mathcal{M}_k} \hat{\beta}_2^2(Q)\cdot \chi_{(1-(\frac{1}{C_L})^{l-k+2})Q}(x)\,. \tag{6.39}$$

We fix a segment (nontrivial connected component) γ of $\Gamma_{l-1}\cap S^{\mathrm{c}}_{l-1}$. The notation and definitions of the previous section also apply here. Recall that each region $I_{j-1,j} \equiv I_{j-1,j}(\gamma)$, $1 \leq j \leq m+1$, lies between two corresponding hyperplanes H_{j-1} and H_j that satisfy the angle estimate of equation (6.14).

Divide the regions $I_{j-1,j}$, $1 \leq j \leq m+1$, into finer disjoint regions $J^p_{j-1,j}$, $p = 1,\dots,N_j$, that satisfy the following properties:

$$I_{j-1,j} = \bigcup_{p=1}^{N_j} J^p_{j-1,j}\,,$$

$$\frac{1}{4}\cdot l_l < l\big(J^p_{j-1,j}\cap\Psi_{l-1}(\gamma)\big) \leq \frac{1}{2}\cdot l_l\,, \tag{6.40}$$

each region $J^p_{j-1,j}$ lies between two hyperplanes denoted by H^-_p and H^+_p, and for any $1 \le p \le N_j - 1$,

$$\cos\big(\text{ang}(H^-_p, [z_{j-1}, z_j])\big) \le \frac{3}{A_0} \cdot \beta_2\big(Q^j_{\max}, \hat{Q}_j\big)\,. \tag{6.41}$$

The following large constants $B_0 = B_0(n)$, $B_1 = B_1(n)$, $C_2 = C_2(n)$, and M_0 appear in the length estimate. The constant C_2 is obtained as follows:

$$C_2 = 2 \cdot B_0 \cdot B_1 \cdot M_0\,. \tag{6.42}$$

The constants δ and δ_0 were chosen so that

$$C_2 \cdot \delta \le C_2 \cdot \delta_0 < 1\,. \tag{6.43}$$

Equation (6.43) implies the following inequality:

$$e^{\frac{-C_2 \cdot t}{M_0}} \le 1 - \frac{C_2}{2 \cdot M_0} \cdot t\,, \quad 0 \le t \le \delta\,. \tag{6.44}$$

We follow by describing some properties of G_l and G^*_l. Note that if $x \in \mathbb{R}^n$, then

$$G_{l-1}(x) \le G_l(x) \le \hat{J}_2(x) \quad \text{and} \quad G^*_{l-1}(x) \le G^*_l(x) \le \hat{J}_2(x)\,. \tag{6.45}$$

A crucial proposition of these functions is formulated as follows:

LEMMA 6.16 *The following two properties are satisfied*:

(i) *If* $x, y \in \mathbb{R}^n$, $l \ge 1$, *and*

$$\text{dist}(x, y) < \left(1 - \frac{1}{C_L}\right) \cdot l_l\,, \tag{6.46}$$

then

$$G_{l-1}(x) \le G^*_{l-1}(y)\,.$$

(ii) *For any fixed segment* γ *of* Γ_{l-1} *and a corresponding fixed region* $J^p_{j-1,j}$,

$$\sup_{x \in \Gamma_l(\gamma) \cap J^p_{j-1,j}} e^{-C_2 \cdot G^*_{l-1}(x)} \le \inf_{x \in \Psi_{l-1}(\gamma) \cap j^p_{j-1,j}} e^{-C_2 \cdot G_{l-1}(x)}\,. \tag{6.47}$$

PROOF: (i) This property is implied by the following observation: If $1 \le k \le l-1$, Q is a cube in $\mathcal{M}_{k-1} \cup \mathcal{M}_k$, and x and y are in $\mathbb{R}^n$ satisfying equation (6.46), then

$$\chi_{(1-(\frac{1}{C_L})^{l-k})Q}(x) < \chi_{(1-(\frac{1}{C_L})^{l-k+1})Q}(y)\,.$$

We verify this observation as follows: If $x \in (1 - (\frac{1}{C_L})^{l-k})Q$, then

$$\text{dist}(x, \partial Q) \ge \left(\frac{1}{C_L}\right)^{l-k} \cdot l(Q) \quad \text{where } l(Q) = l_{k-1}, l_k\,. \tag{6.48}$$

If $y \in \mathbb{R}^n$ satisfies condition (6.46), then

$$\text{dist}(x, y) < \left(1 - \frac{1}{C_L}\right) \cdot \left(\frac{1}{C_L}\right)^{l-k} \cdot l_k\,. \tag{6.49}$$

By combining equations (6.48) and (6.49), we obtain that

$$\operatorname{dist}(y, \partial Q) \geq \left(\frac{1}{C_L}\right)^{l-k} \cdot (l(Q) - l_k + l_{k+1}) \quad \text{where } l(Q) = l_{k-1}, l_k\,.$$

This equation implies that $y \in (1 - (1/C_L)^{l-k+1})Q$.

(ii) The current property follows from the above property together with the following observation: If $x \in \Gamma_l(\gamma) \cap J^p_{j-1,j}$ and $y \in \psi_{l-1}(\gamma) \cap J^p_{j-1,j}$, then x and y satisfy equation (6.46). The proof of the last observation involves elementary geometric estimates. Due to the technical but rudimentary nature of the proof, we present it in Appendix B.

□

We fix γ, a segment of $\Gamma_{l-1} \cap S^{\mathrm{c}}_{l-1}$, where $\gamma \cap \mathcal{L}_l = \{z_1, \ldots, z_m\}$ and verify two preliminary length estimates. These estimates generalize equations (6.36) and (6.37) to any level l.

LEMMA 6.17 *If* $1 \leq j \leq m+1$ *and* $l \geq 1$, *then*

$$\int_{\Gamma_l(\gamma)\cap I_{j-1,j}} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x) \leq \big(1 + B_0 \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j)\big) \cdot \int_{\Psi_{l-1}(\gamma)\cap I_{j-1,j}} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x)\,. \tag{6.50}$$

PROOF: In order to verify equation (6.50), it is sufficient to prove the following geometric estimate:

$$l\big(J^p_{j-1,j} \cap \Gamma_l(\gamma)\big) \leq \big(1 + B_0 \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j)\big) \cdot l\big(J^p_{j-1,j} \cap \Psi_{l-1}(\gamma)\big)\,, \quad 1 \leq p \leq N_j\,. \tag{6.51}$$

Now, equations (6.47) and (6.51) imply that

$$\int_{\Gamma_l(\gamma)\cap J^p_{j-1,j}} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x) \leq \big(1 + B_0 \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j)\big) \cdot \int_{\Psi_{l-1}(\gamma)\cap J^p_{j-1,j}} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x) \tag{6.52}$$

and thus lead to equation (6.50).

Equation (6.51) practically follows from the Pythagorean theorem. Indeed, equations (6.25), (6.26), (6.41), and the Pythagorean theorem imply that

$$l\big(\hat{J}^p_{j-1,j} \cap \Gamma_l(\gamma)\big) \leq \big(l(\hat{J}^p_{j-1,j} \cap L_{\hat{Q}_j})^2 + 81 \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j) \cdot l_l^2\big)^{1/2}\,. \tag{6.53}$$

Moreover, the following two estimates are also satisfied (for a proof, see appendices C and D):

$$l\big(J^p_{j-1,j} \cap \Gamma_l(\gamma)\big) \le l\big(\hat{J}^p_{j-1,j} \cap \Gamma_l(\gamma)\big) + \frac{252}{A_0} \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j) \cdot l_l \tag{6.54}$$

and

$$l\big(\hat{J}^p_{j-1,j} \cap L_{\hat{Q}_j}\big) \le l\big(J^p_{j-1,j} \cap \Psi_{l-1}(\gamma)\big) + \frac{108 \cdot C_L}{A_0} \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j) \cdot l_l\,. \tag{6.55}$$

Finally, combine equations (6.40) and (6.53) through (6.55) and conclude equation (6.51) and consequently equation (6.50), where an upper bound for B_0 is $88 \cdot C_L$. □

LEMMA 6.18 *For* $l \ge 1$,

$$\int_{\Psi_{l-1}(\gamma)\cap I_{j-1,j}} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x) \le B_1 \cdot \int_{\Gamma_l(\gamma)\cap(1-\frac{1}{C_L})\cdot Q^j_{\max}} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x)\,. \tag{6.56}$$

PROOF: We first prove that

$$l(\Psi_{l-1}(\gamma) \cap I_{j-1,j}) \le B_2 \cdot l\left(\Gamma_l(\gamma) \cap \left(1 - \frac{1}{C_L}\right) \cdot Q^j_{\max}\right), \tag{6.57}$$

where $B_2 = 48 \cdot A_1 \cdot C_L$.

Note that there exists a point x in $\Psi_{l-1}(\gamma) \cap Q^j_{\max}$ that satisfies the equation

$$Q\left(x, \frac{1}{12} \cdot l_l\right) \subseteq \left(1 - \frac{1}{C_L}\right) \cdot Q^j_{\max}\,. \tag{6.58}$$

Indeed, if $Q^j_{\max} \in \mathcal{M}_l$, then the above equation follows from the definition of the set $\mathcal{M}_l$. If $Q^j_{\max} \in \mathcal{M}_{l-1}$, $l \ge 2$, then there exists a point $y \in \Gamma_{l-2}$ such that $y \in \frac{2}{3} \cdot Q^j_{\max}$. Recall that $24 \cdot 9 \cdot \sqrt{\delta} \cdot (1 + C_L^{-1}) < 1$. Let x be a point in Γ_{l-1} within distance $\frac{1}{24} \cdot l_{l-1}$ to y (see equation (6.29)); then x satisfies equation (6.58).

It is thus sufficient to show that

$$l(\Psi_{l-1}(\gamma) \cap I_{j-1,j}) \le B_2 \cdot l\left(\Gamma_l(\gamma) \cap Q\left(x, \frac{1}{12} \cdot l_l\right)\right).$$

It follows from equations (6.11) through (6.13) and (6.25) through (6.27) that

$$\frac{1}{12} \cdot l_l \le l\left(\Gamma_l(\gamma) \cap Q\left(x, \frac{1}{12} \cdot l_l\right)\right). \tag{6.59}$$

Recall that

$$l(\Psi_{l-1}(\gamma) \cap I_{j-1,j}) = \operatorname{dist}(z_{j-1}, z_j) \le 4 \cdot A_1 \cdot l_{l-1}\,. \tag{6.60}$$

Combine equations (6.58) through (6.60) to conclude equation (6.57), where $48 \cdot A_1 \cdot C_L$ is an upper bound on B_2. Equation (6.56) follows from equations (6.1), (6.43), (6.45), and (6.57), where $B_1 = B_2 \cdot e$. □

Lemmata 6.17 and 6.18 imply the length estimate for stage l, which is formulated as follows:

LEMMA 6.19

$$\int_{\Gamma_l} e^{-C_2 \cdot G_l(x)}\, \mathrm{d}s(x) \le \int_{\Gamma_{l-1}} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x)\,. \tag{6.61}$$

PROOF: It is sufficient to verify the equation

$$\int_{\Gamma_l(\gamma)} e^{-C_2 \cdot G_l(x)}\, \mathrm{d}s(x) \le \int_{\Psi_{l-1}(\gamma)} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x)\,, \tag{6.62}$$

where γ is a segment of $\Gamma_{l-1} \cap S^c_{l-1}$. The lemma then follows from the "separation" of the different segments of Γ_{l-1} (see property 11 in Section 6.1) and the monotonicity of the sequence $\{G_l\}_{l\ge 0}$.

The combination of inequalities (6.50) and (6.56) results in the new inequality

$$\int_{\Gamma_l(\gamma)\cap I_{j-1,j}} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x) - \int_{\Psi_{l-1}(\gamma)\cap I_{j-1,j}} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x) \le B_0 \cdot B_1 \cdot \beta_2^2(Q^j_{\max}, \hat{Q}_j) \cdot \int_{\Gamma_l(\gamma)\cap(1-\frac{1}{C_L})\cdot Q^j_{\max}} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x)\,. \tag{6.63}$$

Let $\{Q^k_{\max}\}_{k=1}^{m+1}$ be the collection of all cubes $Q_{\max}$ that arise in the construction of $\Gamma_l(\gamma)$. Inequality (6.63) can be extended to the curves $\Psi_{l-1}(\gamma)$ and $\Gamma_l(\gamma)$ as follows:

$$\int_{\Gamma_{l-1}(\gamma)} e^{-C_2 \cdot G^*_{l-1}(x)}\, \mathrm{d}s(x) - \int_{\Psi_{l-1}(\gamma)} e^{-C_2 \cdot G_{l-1}(x)}\, \mathrm{d}s(x) \le B_0 \cdot B_1 \cdot \int_{\Gamma_l(\gamma)} e^{-C_2 \cdot G^*_{l-1}(x)} \cdot \hat{J}^{\max}_l(x) \mathrm{d}s(x)\,, \tag{6.64}$$

where

$$\hat{J}^{\max}_l = \sum_{k=1}^{m+1} \beta_2^2\big(Q^k_{\max}, \hat{Q}_k\big) \cdot \chi_{\left(\frac{C_L-1}{C_L}\right)\cdot Q^k_{\max}}(x)\,. \tag{6.65}$$

Note that for a fixed j, $1 \le j \le m+1$,

$$\#\big\{Q^k_{\max} = Q^j_{\max} : 1 \le k \le N_l\big\} \le M_0\,, \tag{6.66}$$

where $M_0 = 9$. Equations (6.1) and (6.66) imply that

$$\hat{J}^{\max}_l(x) \le M_0 \cdot \delta \quad \text{for any } x \in \mathbb{R}^n. \tag{6.67}$$

The application of equations (6.42), (6.44), and (6.67) to inequality (6.64) results in the following inequality:

$$\int_{\Gamma_l(\gamma)} e^{-\left(C_2\cdot G^*_{l-1}(x)+\frac{C_2}{M_0}\hat{J}^{\max}_l(x)\right)}\,\mathrm{d}s(x) \le \int_{\Psi_{l-1}(\gamma)} e^{-C_2\cdot G_{l-1}(x)}\,\mathrm{d}s(x)\,. \tag{6.68}$$

Equations (6.10), (6.38), (6.39), (6.65), and (6.66) imply that

$$\hat{J}^{\max}_l(x) \le M_0\cdot (G_l(x) - G^*_{l-1}(x))\,. \tag{6.69}$$

Equation (6.62) and thus also the lemma result from equations (6.68) and (6.69). □

Finally, we conclude with a uniform bound for the lengths of the curves Γ_l, $l \ge 0$.

LEMMA 6.20

$$l(\Gamma_l) \le l(\Gamma_0)\cdot e^{C_2\cdot\delta}\,. \tag{6.70}$$

PROOF: By applying equation (6.61) recursively, we obtain that

$$\int_{\Gamma_l} e^{-C_2\cdot G_l(x)}\,\mathrm{d}s(x) \le \int_{\Gamma_0} e^{-C_2\cdot G_0(x)}\,\mathrm{d}s(x)\,. \tag{6.71}$$

By applying equations (6.1) and (6.45) to equation (6.71), we conclude equation (6.70). □

6.4 The Measure Estimate for the Basic Construction

In this section we show that the measures of the sets $E'_l \cup S_l$, $l \ge 0$, are bounded uniformly from below by a universal constant times the measure of Q_1. The proof is based on a martingale-type inequality that is analogous to that of Section 6.3 (see equation (6.34)).

We form an increasing sequence of positive functions $\{F_l\}_{l\ge 0}$ that are uniformly bounded by δ as follows:

$$F_l(x) = \sum_{k=1}^{l} \sum_{Q\in\mathcal{M}'_{k-1}} \hat{\beta}^2_2(Q)\cdot\chi_Q(x)\,,\quad l\ge 0\,.$$

We show that there exists a constant C_4 such that $\{e^{C_4\cdot F_l}\}_{l\ge 0}$ is a submartingale in the following sense:

$$\int_{E'_{l-1}\cup S_{l-1}} e^{C_4\cdot F_{l-1}(x)}\,\mathrm{d}\mu(x) \le \int_{E'_l\cup S_l} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x)\,,\quad l\ge 1\,. \tag{6.72}$$

A repetitive application of this inequality results in a uniform lower bound for the measures of $E'_l \cup S_l$, $l \ge 1$.

The above submartingale inequality practically follows from Chebyshev's inequality. In order to illustrate this claim, let us observe the case where $l = 1$. Here

$$F_0 = 0 \quad \text{and} \quad F_1 = \hat{\beta}_2(Q_1) \cdot \chi_{Q_1}\,.$$

Equation (6.72) thus obtains the form

$$\mu(Q_1) \leq e^{C_4 \cdot \hat{\beta}_2(Q_1)} \cdot \mu\big((E'_1 \cup S_1) \cap Q_1\big)\,. \tag{6.73}$$

This equation is verified as follows. Recall that $\Gamma_1 \subseteq L_{\hat{Q}_1}$ and note that

$$\mu\left(\left\{x : x \in Q_1 \text{ and } \operatorname{dist}(x, L_{\hat{Q}_1}) < \frac{l(Q_1)}{6 \cdot C_L}\right\}\right) \leq \mu\big((E'_1 \cup S_1) \cap Q_1\big)\,. \tag{6.74}$$

Combine this inequality with Lemma 6.5 to obtain the following estimate:

$$\mu(Q_1) \leq \big(1 - 36 \cdot C_L^2 \cdot \hat{\beta}_2^2(Q_1)\big)^{-1} \cdot \mu\big((E'_1 \cup S_1) \cap Q_1\big)\,.$$

Recall that $\delta < 1/(72 \cdot C_L^2)$ and conclude equation (6.73) where $C_4 \geq 72 \cdot C_L^2$.

In order to prove the submartingale inequality for $l > 1$, we apply at each cube in $\mathcal{M}'_{l-1}$ similar estimates as in equation (6.74) and combine them. Such local estimates are formulated as follows:

LEMMA 6.21 *If Q is a cube in $\mathcal{M}'_{l-1}$, $l \geq 1$, then there exists a cube $\hat{Q} \in \mathcal{P}(Q)$ and a best L_2 line $L_{\hat{Q}}$ such that*

$$\mu\left(Q \cap \left\{x : \operatorname{dist}(x, L_{\hat{Q}}) < \frac{l_l}{24}\right\}\right) \leq \mu\big(Q \cap (E'_l \cup S_l)\big)\,. \tag{6.75}$$

PROOF: Let Q be a cube in $\mathcal{M}'_{l-1}$. There exists a nontrivial connected component γ of $\Gamma_{l-1} \cap S^{\mathsf{c}}_{l-1}$ and a region $I_{j-1,j}$, $1 \leq j \leq m+1$, associated with γ such that

$$Q \cap I_{j-1,j} \neq \varnothing\,. \tag{6.76}$$

Let $\hat{Q}$ be the cube $\hat{Q}_j \in \mathcal{P}(Q)$ defined in equation (6.9). If $l = 1$, then we set $L_{\hat{Q}}$ to be the line containing Γ_0 ($L_{\hat{Q}} \equiv L_{\hat{Q}_1}$). If $l > 1$, then $L_{\hat{Q}}$ can be any best L_2 line for $\hat{Q}$.

We verify equation (6.75) by showing that if x is a point in Q such that

$$\operatorname{dist}(x, L_{\hat{Q}}) < \frac{l_l}{24}\,,$$

then there exists a cube $\tilde{Q}$ such that $x \in \tilde{Q}$ and either $\tilde{Q} \subseteq E'_l \cup S_l$ or $\mu(\tilde{Q}) = 0$.

Fix x in Q and let $y_{L_{\hat{Q}}}$ be a point in $L_{\hat{Q}}$ such that

$$\operatorname{dist}(x, y_{L_{\hat{Q}}}) < \frac{1}{24} \cdot l_l\,. \tag{6.77}$$

Recall that $24 \cdot 9 \cdot \sqrt{\delta} \cdot C_L < 1$. Combine equations (6.11) through (6.13) and (6.76) to conclude that there exists a point $s_{\Gamma_{l-1}} \in \Gamma_{l-1}$ that satisfies the inequality

$$\text{dist}(s_{\Gamma_{l-1}}, y_{L_{\hat{Q}}}) < \frac{1}{24} \cdot l_l \,. \tag{6.78}$$

Equations (6.77) and (6.78) imply that

$$\text{dist}(x, s_{\Gamma_{l-1}}) < \frac{1}{12} \cdot l_l \,. \tag{6.79}$$

Apply Proposition 3.2 to set a cube $\tilde{Q}$ in $\tilde{\mathcal{D}}$ such that $l(\tilde{Q}) = l_l$ and

$$s_{\Gamma_{l-1}} \in \frac{2}{3} \cdot \tilde{Q} \,. \tag{6.80}$$

It follows from equations (6.79) and (6.80) that

$$x \in \tilde{Q} \,. \tag{6.81}$$

Assume that $\mu(\tilde{Q}) > 0$. It thus follows from equations (6.7) and (6.80) that $\tilde{Q} \in \mathcal{M}_l$. Therefore

$$\tilde{Q} \subseteq E'_l \cup S_l \,. \tag{6.82}$$

The lemma is proven by combining equations (6.81) and (6.82). □

We are now ready to prove the submartingale inequality. The constant $C_4 = C_4(n)$ is set as follows:

$$C_4 = 2 \cdot e \cdot (24 \cdot C_L)^2 \,. \tag{6.83}$$

The constants δ and δ_0 are sufficiently small so that

$$C_4 \cdot \delta \leq C_4 \cdot \delta_0 < 1 \,. \tag{6.84}$$

The above inequality implies the following inequality:

$$e^{-C_4 \cdot t} \leq 1 - \frac{C_4}{2} \cdot t \,, \quad 0 \leq t \leq \delta. \tag{6.85}$$

LEMMA 6.22

$$\int_{E'_{l-1} \cup S_{l-1}} e^{C_4 \cdot F_{l-1}(x)} \, \mathrm{d}\mu(x) \leq \int_{E'_l \cup S_l} e^{C_4 \cdot F_l(x)} \, \mathrm{d}\mu(x) \,. \tag{6.86}$$

PROOF: Note that the following two inequalities are satisfied:

$$\int_{E'_{l-1} \cup S_{l-1}} e^{C_4 \cdot F_l(x)} \, \mathrm{d}\mu(x) - \int_{(E'_{l-1} \cap E'_l) \cup S_l} e^{C_4 \cdot F_l(x)} \, \mathrm{d}\mu(x) \leq \int_{E'_{l-1} \cap E'^{c}_l \cap S^{\mathrm{c}}_l} e^{C_4 \cdot F_l(x)} \, \mathrm{d}\mu(x) \tag{6.87}$$

and

$$F_l \leq \hat{J}\,. \tag{6.88}$$

Combine equations (6.1), (6.87), and (6.88) to obtain the inequality

$$\int_{E'_{l-1}\cup S_{l-1}} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x) - \int_{(E'_{l-1}\cap E'_l)\cup S_l} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x) \leq e^{C_4\cdot\delta}\cdot\mu\big(E'_{l-1}\cap E'^{c}_l\cap S^{\mathrm{c}}_l\big)\,. \tag{6.89}$$

We next bound from above $\mu(E'_{l-1}\cap E'^{\mathrm{c}}_l\cap S^{\mathrm{c}}_l)$. If $Q\in\mathcal{M}'_{l-1}$, then it follows from equation (6.75) that

$$\mu\big(Q\cap E'^{\mathrm{c}}_l\cap S^{\mathrm{c}}_l\big) \leq \mu\left(Q\cap\left\{x:\mathrm{dist}(x,L_{\hat{Q}})\geq\frac{l_l}{24}\right\}\right), \tag{6.90}$$

where $L_{\hat{Q}}$ is a best L_2 line for some cube $\hat{Q}$ in $P(Q)$. By a similar application of Chebyshev's inequality as in Lemma 6.5, inequality (6.90) assumes the weaker form

$$\begin{aligned}\mu\big(Q\cap E'^{\mathrm{c}}_l\cap S^{\mathrm{c}}_l\big) &\leq (24\cdot C_L)^2\cdot\beta_2^2(Q,\hat{Q})\cdot\mu(Q)\\ &\leq (24\cdot C_L)^2\cdot\hat{\beta}_2^2(Q)\cdot\mu(Q)\,.\end{aligned} \tag{6.91}$$

Combine equations (6.83), (6.84), (6.89), and (6.91) and get that

$$\int_{E'_{l-1}\cup S_{l-1}} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x) - \int_{(E'_{l-1}\cap E'_l)\cup S_l} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x) \leq \frac{C_4}{2}\cdot\sum_{Q\in\mathcal{M}'_{l-1}}\hat{\beta}_2^2(Q)\cdot\mu(Q)\,. \tag{6.92}$$

Define the function $\hat{J}_l$ by the formula

$$\hat{J}_l(x) = F_l(x) - F_{l-1}(x) \tag{6.93}$$

and note that

$$\int_{E'_{l-1}}\hat{J}_l(x)\mathrm{d}\mu(x) = \sum_{Q\in\mathcal{M}'_{l-1}}\hat{\beta}_2^2(Q)\cdot\mu(Q)\,.$$

Consequently, inequality (6.92) assumes the weaker form

$$\int_{E'_{l-1}\cup S_{l-1}} e^{C_4\cdot F_l(x)}\cdot\left(1-\frac{C_4}{2}\cdot\hat{J}_l(x)\right)\mathrm{d}\mu(x) \leq \int_{(E'_{l-1}\cap E'_l)\cup S_l} e^{C_4\cdot F_l(x)}\,\mathrm{d}\mu(x)\,. \tag{6.94}$$

The lemma is proven by combining equations (6.1), (6.85), (6.93), and (6.94). □

The submartingale inequality implies the following measure estimate:

LEMMA 6.23

$$\mu(E'_l \cup S_l) \geq e^{-C_4 \cdot \delta} \cdot \mu(Q_1)\,. \tag{6.95}$$

PROOF: By recursive application of equation (6.86) we obtain that

$$\int_{E'_0} e^{C_4 \cdot F_0(x)}\,\mathrm{d}\mu \leq \int_{E'_l \cup S_l} e^{C_4 \cdot F_l(x)}\,\mathrm{d}\mu(x)\,. \tag{6.96}$$

By combining equations (6.1), (6.88), and (6.96), we conclude equation (6.95). □

6.5 The General Construction

In this section we describe the general construction. The N^{th}-level sets of this construction are the approximating curve $\tilde{\Gamma}_N$ and the strips around this curve, $\tilde{E}_N$ and $\tilde{S}_N$. We form these sets by applying repeatedly the basic construction of Section 6.2 at stopping-time cubes. For each stage j of the recursive repetition, we construct the N^{th}-level sets Γ_N^j, $\mathcal{S}_N^j$, $\mathcal{M}'^j_N$, and $\tilde{\mathcal{S}}_N^j$. This process terminates at some stage $j_0 = j_0(N)$, where $\mathcal{S}_N^{j_0} = \varnothing$. The sets of the general construction are then obtained as follows: $\tilde{\Gamma}_N := \Gamma_N^{j_0}$, whereas $\tilde{E}_N$ and $\tilde{S}_N$ are the unions of all cubes in $\mathcal{M}_N^{j_0}$ and $\mathcal{S}_N^{j_0}$, respectively.

We form the following sets of the basic construction: Γ_N, $\mathcal{S}_N$, and $\mathcal{M}'_N$. We then initialize the zeroth-repetition sets as follows:

$$\Gamma_N^0 = \Gamma_N\,, \quad \mathcal{M}'^0_N = \mathcal{M}'_N\,, \quad \mathcal{S}_N^0 = \mathcal{S}_N\,, \quad \tilde{\mathcal{S}}_N^0 = \varnothing\,. \tag{6.97}$$

Assume that Γ_N^j, $\mathcal{M}'^j_N$, $\mathcal{S}_N^j$, and $\tilde{\mathcal{S}}_N^j$, $j \geq 0$, have been defined and that $\mathcal{S}_N^j \neq \varnothing$. Let Q_x be a stopping-time cube in $\mathcal{S}_N^j$ that was created around a point x. Let k_x be the integer that satisfies the following condition:

$$l_{N+1} < \frac{l(Q_x)}{C_L^{k_x}} \leq l_N\,.$$

Apply the basic construction in Q_x and form the following sets: $\Gamma_k(Q_x)$, $\mathcal{S}_k(Q_x)$, and $\mathcal{M}'_k(Q_x)$, where $0 \leq k \leq k_x$. If $\Gamma_N^j \cup \Gamma_{k_x}(Q_x)$ is not a connected set, construct a curve $\gamma(Q_x)$ such that

$$l(\gamma(Q_x)) \leq 10 \cdot \sqrt{\delta} \cdot l(Q_x) \tag{6.98}$$

and $\Gamma_N^j \cup \Gamma_{k_x}(Q_x) \cup \gamma(Q_x)$ is connected (see Appendix E on constructing such a curve). If $\Gamma_N^j \cup \Gamma_{k_x}(Q_x)$ is connected, the set $\gamma(Q_x) := \varnothing$.

The sets $\tilde{\mathcal{S}}_N^{j+1}$, Γ_N^{j+1}, $\mathcal{M}'^{j+1}_N$, and $\mathcal{S}_N^{j+1}$, $j \geq 0$, are defined recursively as follows:

$$\tilde{\mathcal{S}}_N^{j+1} = \tilde{\mathcal{S}}_N^j \cup \left\{ Q_x : Q_x \in \mathcal{S}_N^j \text{ and } l(Q_x) < l_{N-1} \right\}, \tag{6.99}$$

$$\Gamma_N^{j+1} = \Gamma_N^j \cup \bigcup_{Q_x \in \mathcal{S}_N^j \setminus \tilde{\mathcal{S}}_N^{j+1}} (\Gamma_{k_x}(Q_x) \cup \gamma(Q_x))\,, \tag{6.100}$$

$$\mathcal{M}'^{j+1}_N = \mathcal{M}'^{j}_N \cup \bigcup_{Q_x \in \mathcal{S}^j_N \setminus \tilde{\mathcal{S}}^{j+1}_N} \mathcal{M}'_{k_x}(Q_x)\,, \tag{6.101}$$

$$\mathcal{S}^{j+1}_N = \bigcup_{Q_x \in \mathcal{S}^j_N \setminus \tilde{\mathcal{S}}^{j+1}_N} \mathcal{S}_{k_x}(Q_x)\,. \tag{6.102}$$

Also define

$$E'^j_N = \bigcup_{Q \in \mathcal{M}'^j_N} Q \quad \text{and} \quad \tilde{S}^j_N = \bigcup_{Q \in \tilde{\mathcal{S}}^j_N} Q\,, \quad j \geq 0\,. \tag{6.103}$$

Let $j_0 = j_0(N)$ be the minimal integer such that $\mathcal{S}^{j_0}_N = \varnothing$. The recursive construction stops at j_0. Denote

$$\tilde{\Gamma}_N = \Gamma^{j_0}_N\,, \quad \tilde{E}_N = E'^{j_0}_N\,, \quad \tilde{S}_N = \tilde{S}^{j_0}_N\,.$$

Note that

$$\text{if} \quad Q \subseteq \tilde{E}_N \cup \tilde{S}_N, \quad \text{then} \quad l_{N+1} < l(Q) < l_{N-1}\,. \tag{6.104}$$

6.6 Estimates for the General Construction

In this section we bound from above the length of the curve $\tilde{\Gamma}_N$ and bound from below the measure of the set $\tilde{E}_N \cup \tilde{S}_N$. Theorem 6.3 is then concluded from these length and measure estimates together with an elementary limit argument.

The length of the curve $\tilde{\Gamma}_N$ is estimated in the following lemma:

LEMMA 6.24 *There exist constants $C_1 = C_1(n)$ and $C_2 = C_2(n)$ such that*

$$l(\tilde{\Gamma}_N) \leq C_1 \cdot e^{C_2 \cdot \delta} \cdot l(Q_1)\,. \tag{6.105}$$

PROOF: Define the set $\Omega^{Q_1}_\delta$ as follows:

$$\Omega^{Q_1}_\delta = \{\mu : \mu \text{ is a locally finite Borel measure satisfying equation (6.1)}\}\,.$$

Fix a measure μ in $\Omega^{Q_1}_\delta$ and a cube Q in $\tilde{\mathcal{D}}$ (recall that $\tilde{\mathcal{D}} \equiv \tilde{\mathcal{D}}(Q_1)$). Note that $J^{Q,\mu}_2(x) \leq \delta$ for all x in $\operatorname{supp}(\mu) \cap (1+\varepsilon_1) \cdot Q$. That is, $\mu \in \Omega^Q_\delta$. Denote by $\Gamma_N(Q,\mu)$, $\Gamma^j_N(Q,\mu)$, $0 \leq j \leq j_0(Q,\mu)$, and $\tilde{\Gamma}_N(Q,\mu)$ the corresponding curves of the basic and general constructions formed with respect to μ and Q. Define the number L_N ($L_N = L_N(n)$) as follows:

$$L_N = \sup_{\substack{Q \in \tilde{\mathcal{D}} \\ \mu \in \Omega^{Q_1}_\delta}} \frac{l(\tilde{\Gamma}_N(Q,\mu))}{l(Q)}\,. \tag{6.106}$$

We prove the lemma by showing that

$$L_N \lessapprox \sqrt{n} \cdot e^{C_2 \cdot \delta}\,.$$

We combine different length estimates to form a linear inequality in L_N that implies the above equation (this idea is similar to that of Jones [14, p. 12]).

The first length estimate follows from equation (6.100) and is formulated as follows:

$$l(\tilde{\Gamma}_N(Q,\mu)) \le l\big(\Gamma_N^0(Q,\mu)\big) + \sum_{Q_x \in \mathcal{S}_N^0 \setminus \tilde{\mathcal{S}}_N^1} l(\gamma(Q_x,\mu)) + l(\tilde{\Gamma}_N(Q_x,\mu)), \tag{6.107}$$

where $\mathcal{S}_N^0 = \mathcal{S}_N^0(Q,\mu)$ and $\tilde{\mathcal{S}}_N^1 = \tilde{\mathcal{S}}_N^1(Q,\mu)$. Apply equations (6.98) and (6.106) to equation (6.107) and obtain that

$$l(\tilde{\Gamma}_N(Q,\mu)) \le l\big(\Gamma_N^0(Q,\mu)\big) + (10\cdot\sqrt{\delta} + L_N)\cdot \sum_{Q_x \in \mathcal{S}_N^0 \setminus \tilde{\mathcal{S}}_N^1} l(Q_x). \tag{6.108}$$

Combine equations (6.30), (6.33), and (6.108) and form the inequality

$$l(\tilde{\Gamma}_N(Q,\mu)) \le l\big(\Gamma_N^0(Q,\mu)\big) + \frac{12\cdot(A_0+1)}{A_1}\cdot(10\cdot\sqrt{\delta} + L_N)\cdot \sum_{Q_x \in \mathcal{S}_N^0 \setminus \tilde{\mathcal{S}}_N^1} l(F_x(Q)). \tag{6.109}$$

Recall that the free parts F_x ($F_x \equiv F_x(Q_x) \equiv F_x(Q_x, Q)$) are disjoint and thus note that

$$\sum_{Q_x \in \mathcal{S}_N^0 \setminus \tilde{\mathcal{S}}_N^1} l(F_x(Q_x)) \le l\big(\Gamma_N^0\big). \tag{6.110}$$

The application of equations (6.3), (6.70), (6.106), and (6.110) to equation (6.109) results in the following inequality:

$$L_N \le 9\cdot A_0\cdot\sqrt{n}\cdot e^{C_2\cdot\delta} + \frac{12\cdot(A_0+1)}{A_1}\cdot(10\cdot\sqrt{\delta}+L_N)\cdot 9\cdot A_0\cdot\sqrt{n}\cdot e^{C_2\cdot\delta}. \tag{6.111}$$

Note that $A_1 \ge 24\cdot(A_0+1)\cdot 9\cdot A_0\cdot e\cdot\sqrt{n}$. By applying this fact and equation (6.43) into equation (6.111), we obtain the estimate

$$L_N \le 18\cdot(1+5\cdot\sqrt{\delta})\cdot A_0\cdot\sqrt{n}\cdot e^{C_2\cdot\delta}.$$

The lemma is thus concluded where $36\cdot A_0\cdot\sqrt{n}$ is an upper bound for $C_1(n)$ (it is possible to reduce this bound to $(1+\varepsilon_1)\cdot\sqrt{n}$ by modifying the construction slightly). An upper bound for $C_2(n)$ follows from equation (6.42) and the different bounds on B_0, B_1, and M_0 specified in Section 6.3. □

We next estimate the measure of the sets $\tilde{E}_N \cup \tilde{S}_N$, $N \ge 1$. In Lemmata 6.25 and 6.26 we establish some separation properties of the stopping-time cubes. In Lemma 6.27 we formulate a submartingale inequality for the general construction. This inequality follows from the submartingale inequality of the basic construction (see Section 6.4) and the separation properties formulated in Lemmata 6.25 and 6.26. A repetitive application of the general submartingale inequality results in the measure estimate of $\tilde{E}_N \cup \tilde{S}_N$ (see Lemma 6.28).

LEMMA 6.25 *If Q_x and Q_y are two stopping-time cubes in $\mathcal{S}_N^j$, $0 \leq j \leq j_0$, and $Q_x \neq Q_y$, then*

$$2Q_x \cap 2Q_y = \varnothing\,. \tag{6.112}$$

LEMMA 6.26 *If Q_x is a stopping-time cube in $\mathcal{S}_N^j$, $0 \leq j < j_0$, then*

$$2Q_x \cap E_N'^j = \varnothing\,. \tag{6.113}$$

These two lemmata follow by straightforward induction. We present the proofs in Appendices F and G.

Fix a stopping-time cube Q_x in $\mathcal{S}_N^{j-1}$. Define the functions F_{l,Q_x}, $l \geq 0$, as follows:

$$F_{l,Q_x}(x) = \sum_{k=1}^{l} \sum_{Q \in \mathcal{M}'_{k-1}(Q_x)} \hat{\beta}_2^2(Q) \cdot \chi_Q(x)\,. \tag{6.114}$$

The functions F_N^j, $0 \leq j \leq j_0$, are defined by the formula

$$F_N^j = F_{N,Q_1} + \sum_{k=0}^{j-1} \sum_{Q_x \in \mathcal{S}_N^k \setminus \tilde{\mathcal{S}}_N^{k+1}} F_{k_x,Q_x}\,. \tag{6.115}$$

The submartingale inequality for the general construction is formulated as follows:

LEMMA 6.27

$$\int_{E_N'^{j-1} \cup S_N^{j-1} \cup \tilde{S}_N^j} e^{C_4 \cdot F_N^{j-1}(x)}\, \mathrm{d}\mu(x) \leq \int_{E_N'^j \cup S_N^j \cup \tilde{S}_N^{j+1}} e^{C_4 \cdot F_N^j(x)}\, \mathrm{d}\mu(x)\,. \tag{6.116}$$

PROOF: It follows from Lemmata 6.25 and 6.26 that the sets $E_N'^{j-1} \cup \tilde{S}_N^j$ and $S_N^{j-1} \setminus \tilde{S}_N^j$ are disjoint and that the dilation by a factor of 2 of all stopping-time cubes in $\mathcal{S}_N^{j-1} \setminus \tilde{\mathcal{S}}_N^j$ are disjoint. Thus in order to prove the lemma, it is sufficient to verify the following two inequalities:

$$\int_{E_N'^{j-1} \cup \tilde{S}_N^j} e^{C_4 \cdot F_N^{j-1}(x)}\, \mathrm{d}\mu(x) \leq \int_{E_N'^{j-1} \cup \tilde{S}_N^j} e^{C_4 \cdot F_N^j(x)}\, \mathrm{d}\mu(x) \tag{6.117}$$

and

$$\int_{Q_x} e^{C_4 \cdot F_N^{j-1}(x)}\, \mathrm{d}\mu(x) \leq \int_{E'_{k_x}(Q_x) \cup S_{k_x}(Q_x)} e^{C_4 \cdot F_N^j(x)}\, \mathrm{d}\mu(x) \tag{6.118}$$

for any stopping-time cube Q_x in $\mathcal{S}_N^{j-1} \setminus \tilde{\mathcal{S}}_N^j$.

The proof of equation (6.117) is immediate. Indeed, it follows from equations (6.114) and (6.115) that

$$\operatorname{supp}(F_N^j - F_N^{j-1}) \subseteq \bigcup_{Q_x \in \mathcal{S}_N^{j-1} \setminus \tilde{\mathcal{S}}_N^j} 2Q_x \,.$$

This equation together with Lemmata 6.25 and 6.26 imply that

$$F_N^j(x) = F_N^{j-1}(x) \quad \text{for any } x \in E_N'^{j-1} \cup \tilde{S}_N^j. \tag{6.119}$$

Thus equation (6.117) is proven.

To prove equation (6.118), use the arguments in the proof of Lemma 6.22 to obtain that if $0 \le j \le j_0$, $Q_x \in \mathcal{S}_N^{j-1} \setminus \tilde{\mathcal{S}}_N^j$, and $0 \le l \le k_x$, then

$$\int\limits_{E'_{l-1}(Q_x) \cup S_{l-1}(Q_x)} e^{C_4 \cdot (F_N^{j-1}(x) + F_{l-1,Q_x}(x))} \, \mathrm{d}\mu(x) \le \int\limits_{E'_l(Q_x) \cup S_l(Q_x)} e^{C_4 \cdot (F_N^{j-1}(x) + F_{l,Q_x}(x))} \, \mathrm{d}\mu(x) \,.$$

Applying this inequality repeatedly for $0 \le l \le k_x$ results in the equation

$$\int\limits_{Q_x} e^{C_4 \cdot F_N^{j-1}(x)} \, \mathrm{d}\mu(x) \le \int\limits_{E'_{k_x}(Q_x) \cup S_{k_x}(Q_x)} e^{C_4 \cdot (F_N^{j-1}(x) + F_{k_x,Q_x}(x))} \, \mathrm{d}\mu(x) \,. \tag{6.120}$$

Equation (6.118) follows from equation (6.120) and Lemma 6.25. □

The next measure estimate follows immediately from the above lemma.

LEMMA 6.28

$$\mu(\tilde{E}_N \cup \tilde{S}_N) \ge e^{-C_4 \cdot \delta} \cdot \mu(Q_1) \,, \quad N \ge 1 \,. \tag{6.121}$$

PROOF: Apply repeatedly equation (6.116) and obtain that

$$\mu(Q_1) \le \int\limits_{\tilde{E}_N \cup \tilde{S}_N} e^{C_4 \cdot F_N^{j_0}(x)} \, \mathrm{d}\mu(x) \,. \tag{6.122}$$

Lemmata 6.25 and 6.26 imply that

$$F_N^{j_0}(x) \le \hat{J}_2(x) \quad \text{for any } x \in \operatorname{supp}(\mu) \,. \tag{6.123}$$

Equation (6.121) thus follows from equations (6.1), (6.122), and (6.123). □

Theorem 6.3 is proven as follows. For any fixed number N, $\tilde{\Gamma}_N$ is a curve that satisfies the length estimate (6.105). Thus there exists a Lipschitz function $f_N : [0,1] \to \mathbb{R}^n$ and a constant C independent of N such that $\|f_N\|_{\mathrm{Lip}[0,1]} \le C \cdot \sqrt{n} \cdot l(Q_1)$ and such that $\tilde{\Gamma}_N = f_N[0,1]$ (see, e.g., [10, theorem 1.8] for a proof of this fact). By the Arzelà-Ascoli theorem, there exists a Lipschitz function $f : [0,1] \to \mathbb{R}^n$ such that f is the uniform limit of a subsequence $\{f_{N_j}\}_{j=1}^{\infty}$ of $\{f_N\}_{N=1}^{\infty}$.

Let $\Gamma = f[0,1]$. Note that

$$l(\Gamma) \le C_1 \cdot e^{C_2 \cdot \delta} \cdot l(Q_1)\,.$$

It follows from the construction that for any given N and $x \in \tilde{S}_N \cup \tilde{E}_N$

$$\operatorname{dist}(x, \tilde{\Gamma}_N) \le \sqrt{n} \cdot C_L \cdot l_N\,.$$

Consequently,

$$\Gamma \supseteq \limsup_{j \to \infty} (\tilde{S}_{N_j} \cup \tilde{E}_{N_j})\,. \tag{6.124}$$

Recall that $\varepsilon_1 = 2/(3 \cdot C_L - 2)$ and thus note that

$$\tilde{E}_N \cup \tilde{S}_N \subseteq (1 + \varepsilon_1) \cdot Q_1\,. \tag{6.125}$$

Finally, combine equations (6.121), (6.124), and (6.125) and conclude that

$$\mu(\Gamma \cap (1 + \varepsilon_1) \cdot Q_1) \ge e^{-C_4 \cdot \delta} \cdot \mu(Q_1)\,.$$

7 Proof of Theorems 4.8 and 4.10

In this section we conclude Theorems 4.8 and 4.10 from Theorem 6.3 by the use of stopping-time arguments. We first prove Theorem 4.8 and then slightly modify the proof to conclude Theorem 4.10.

We set $C_5 := 15$ (see Remark 7.19). Let Q_0 be a fixed cube in $\mathbb{R}^n$, and let μ be a locally finite Borel measure that satisfies the main assumption of Theorem 4.8:

$$\hat{J}_2(x) \le M \quad \text{for any } x \in \operatorname{supp}(\mu) \cap C_5 \cdot Q_0. \tag{7.1}$$

We fix an integer N and construct a curve $\hat{\Gamma}_N$ and sets $\hat{E}_N$ and $\hat{S}_N$. The curve $\hat{\Gamma}_N$ is an approximation at scale of order $l(Q_0) \cdot C_L^{-N}$ of the curve Γ stated in Theorem 4.8. The sets $\hat{E}_N$ and $\hat{S}_N$ are unions of cubes with side length of order $l(Q_0) \cdot C_L^{-N}$ surrounding the curve $\hat{\Gamma}_N$. We estimate the length of $\hat{\Gamma}_N$ and the measure of $\hat{E}_N \cup \hat{S}_N$ in Lemmata 7.15 and 7.18, respectively. These estimates also apply to the length and measure of the limit curve Γ and thus prove Theorem 4.8.

The basic ingredient of the whole construction is the δ-construction. It is applied first at Q_0 and repeats recursively at stopping-time cubes (which are different from those in the previous section). The union of all curves from the various δ-constructions and of some additional segments results in the N^{th}-level curve, $\hat{\Gamma}_N$. The sets $\hat{E}_N$ and $\hat{S}_N$ are obtained similarly as unions of corresponding sets from the different δ-constructions.

A simplified idea of the δ-construction in a cube Q_1 is described as follows. Let δ_0 be the constant suggested by Theorem 4.8. Assume first that Q_1 is a cube in $\tilde{\mathcal{D}}$ whose $\hat{\beta}_2$ number is small with respect to $\sqrt{\delta_0}$. In this case the δ-construction is composed of two parts. One part is the formation of a stopping-time region inside the underlying cube Q_1. Outside this region, a certain Jones function (renormalized with respect to Q_1) is bounded by δ_0. The other part is the construction of an N^{th}-level curve outside the stopping-time region in a similar way as in the proof of

Theorem 6.3. If the cube Q_1 does not have a small $\hat{\beta}_2$ number, then it is partitioned into subcubes that are the stopping-time cubes of Q_1. A curve connecting these cubes is also formed at Q_1.

We recall that the whole construction applies similar δ-constructions at the stopping-time cubes of the δ-construction described above.

In Section 7.1 we introduce notation and definitions. In Section 7.2 we describe the δ-construction. The whole construction is described in Section 7.3. In Sections 7.4 and 7.5 we present the corresponding length and measure estimates and consequently deduce Theorem 4.8. We end with Section 7.6, where we prove Theorem 4.10.

7.1 Notation and Definitions

Recall that the constant C_L is fixed in the beginning of Section 6. The integer N denotes the level of the whole construction (which results in the sets $\hat{\Gamma}_N$, $\hat{E}_N$, and $\hat{S}_N$). Let δ_0 be the constant suggested by Theorem 6.3 (a lower bound for δ_0 is fixed in the beginning of Section 6).

We first define a collection of cubes $\mathcal{B}_j^+(Q)$, $j \geq 1$. The cubes in this collection substitute for the cubes $C \cdot Q$, where $Q \in \tilde{\mathcal{D}}$ and $C > 1$. Unlike the latter cubes, the former ones belong to $\tilde{\mathcal{D}}$.

DEFINITION 7.1 (Collection $\mathcal{B}_j^+(Q)$) If Q is a cube in $\tilde{\mathcal{D}}$ and c_Q is the center of the cube Q, then

$$\mathcal{B}_j^+(Q) = \left\{Q' : Q' \in \tilde{\mathcal{D}},\ l(Q') = 2^{j+1} \cdot l(Q),\ \text{and}\ c_Q \in \tfrac{2}{3} \cdot Q'\right\}, \quad j \geq 1\,.$$

Proposition 3.2 implies that the sets $\mathcal{B}_j^+(Q)$, $j \geq 1$, are not empty. Note that there are at most 2^n cubes in each one of these sets.

If $Q \in \tilde{\mathcal{D}}$ and $j \geq 1$, we assign to it a unique cube $\mathbf{Q}^{\mathbf{j}+}$ (or $\mathbf{Q}^{\mathbf{j}+}(\mathbf{Q})$) in $\mathcal{B}_j^+(Q)$. There are different possible assignment rules. The rule itself does not play any role in the subsequent analysis and can also be random. Denote $\mathbf{Q}^+ = \mathbf{Q}^{\mathbf{1}+}$ and $\mathbf{Q}^{++} = \mathbf{Q}^{\mathbf{2}+}$.

If Q_1 is a cube in $\tilde{\mathcal{D}}$, define $k_{Q_1} = k_{Q_1}(N)$ by the formula

$$k_{Q_1} = N - \left\lfloor \log_{C_L} \frac{l(Q_0)}{l(Q_1)} \right\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x.

The δ-construction at a cube Q_1 applies the sets $\mathcal{C}^e_{Q_1,l}$, $\mathcal{C}_{Q_1,l}$, and the function $J_{Q_1,l}$, where $1 \leq l \leq k_{Q_1}$. They are defined as follows:

DEFINITION 7.2 (Sets $\mathcal{C}^e_{Q_1,l}$ and $\mathcal{C}_{Q_1,l}$) If Q_1 is a cube in $\tilde{\mathcal{D}}$ and e is a vector in $\{0, 1\}^n$, then

$$\mathcal{C}^e_{Q_1,l} = \left\{Q : Q \in \mathcal{D}\left(Q_0 + \frac{1}{3} \cdot e\right),\ Q \subseteq Q_1^{++}\ \text{and}\ l(Q) \geq \frac{l(Q_1)}{C_L^l}\right\}$$

and

$$\mathcal{C}_{Q_1,l} = \bigcup_{e\in\{0,1\}^n} \mathcal{C}^e_{Q_1,l} .$$

DEFINITION 7.3 (Jones function $J_{Q_1,l}$) If Q_1 is a cube in $\tilde{\mathcal{D}}$, then the function $J_{Q_1,l}$ is defined as follows:

$$J_{Q_1,l}(x) = \sum_{Q\in\mathcal{C}_{Q_1,l}} \hat{\beta}_2^2(Q) \cdot \chi_Q(x) . \tag{7.2}$$

The stopping-time rule for the δ-construction is determined according to the following condition:

DEFINITION 7.4 (Small sum condition) If Q_1 is a cube in $\tilde{\mathcal{D}}$, e is a vector in $\{0, 1\}^n$, and Q' is a cube in $\mathcal{C}^e_{Q_1,l}$, $l \geq 0$, then Q' satisfies the small sum condition with respect to the cube Q_1 if and only if

$$\sum_{\substack{Q\in\mathcal{C}^e_{Q_1,l},\\ l(Q')\leq l(Q)}} \hat{\beta}_2^2(Q) \cdot \chi_Q(x) < \frac{\delta_0}{2^n} \quad \text{for any } x \in Q' .$$

The next condition is also used in the construction:

DEFINITION 7.5 (Small β condition) If Q is a cube in $\tilde{\mathcal{D}}$, then Q satisfies the small β condition if and only if

$$\hat{\beta}_2^2(Q) < \frac{\delta_0}{2^{n+1}} .$$

We next introduce a wide collection of "good" cubes that are used in defining both the stopping-time region and the δ-construction.

DEFINITION 7.6 ("Good" set of cubes $\mathcal{G}^\delta_{Q_1,l}$) If Q_1 is a cube in $\tilde{\mathcal{D}}$, then

$$\mathcal{G}^\delta_{Q_1,l} = \{Q : Q \in \mathcal{C}_{Q_1,l} \text{ and } Q \text{ satisfies the small sum condition w.r.t. } Q_1\} .$$

The next condition is necessary for the small β-construction (see Section 7.2).

DEFINITION 7.7 A cube Q in $\tilde{\mathcal{D}}$ satisfies the *centered-line condition* if and only if Q satisfies the small β condition and for any cube $\hat{Q} \in \mathcal{P}(Q)$ with $l(\hat{Q}) = 2^{j_0^*} \cdot l(Q)$ and any best L_2 line $L_{\hat{Q}}$ in $\hat{Q}$

$$L_{\hat{Q}} \cap \left(\frac{\sqrt{\delta_0}}{4} + \frac{11}{12}\right) \cdot Q \neq \varnothing .$$

Denote

$$\tilde{\mathcal{D}}^S = \{Q : Q \in \tilde{\mathcal{D}} \text{ and } Q \text{ satisfies the small } \beta \text{ condition}\},$$
$$\tilde{\mathcal{D}}^L = \tilde{\mathcal{D}} \setminus \tilde{\mathcal{D}}^S ,$$
$$\tilde{\mathcal{D}}^{SC} = \{Q : Q \text{ satisfies the centered-line condition}\} ,$$

$$\begin{aligned}
\tilde{\mathcal{D}}^{LL} &= \{Q : Q, Q^+ \in \tilde{\mathcal{D}}^L\},\\
\tilde{\mathcal{D}}^{LSL} &= \{Q : Q, Q^{++} \in \tilde{\mathcal{D}}^L \text{ and } Q^+ \in \tilde{\mathcal{D}}^S \setminus \tilde{\mathcal{D}}^{SC}\},\\
\tilde{\mathcal{D}}^{L\beta} &= \tilde{\mathcal{D}}^{LL} \cup \tilde{\mathcal{D}}^{LSL},\\
\tilde{\mathcal{D}}^{SLL} &= \{Q : Q \in \tilde{\mathcal{D}} \setminus \tilde{\mathcal{D}}^{SC} \text{ and } Q^+, Q^{++} \in \tilde{\mathcal{D}}^L\},\\
\tilde{\mathcal{D}}^{+SC} &= \{Q : Q \in \tilde{\mathcal{D}} \setminus \tilde{\mathcal{D}}^{SC} \text{ and } Q^+ \in \tilde{\mathcal{D}}^{SC}\},\\
\tilde{\mathcal{D}}^{++SC} &= \{Q : Q, Q^+ \in \tilde{\mathcal{D}} \setminus \tilde{\mathcal{D}}^{SC}, Q^{++} \in \tilde{\mathcal{D}}^{SC}, \text{ and } Q \notin \tilde{\mathcal{D}}^{LL}\}.
\end{aligned}$$

PROPOSITION 7.8 (i) *If Q is a cube in $\tilde{\mathcal{D}}^S \setminus \tilde{\mathcal{D}}^{SC}$ and Q^+ (or Q^{++}) is in $\tilde{\mathcal{D}}^S$, then Q^+ (or Q^{++}) is in $\tilde{\mathcal{D}}^{SC}$.*

(ii) *The set $\tilde{\mathcal{D}}$ can be written as the following disjoint union*:

$$\tilde{\mathcal{D}} = \tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta} \cup \tilde{\mathcal{D}}^{+SC} \cup \tilde{\mathcal{D}}^{++SC} \cup \tilde{\mathcal{D}}^{SLL}.$$

PROOF: (i) We prove this property by contradiction. Assume without loss of generality that Q^+ does not belong to $\tilde{\mathcal{D}}^{SC}$, and let $L_{\hat{Q}^+}$ be a noncentered line for a cube $\hat{Q}^+$ in $\mathcal{P}(Q^+)$ with side length $2^{j_0^*} \cdot l(Q^+)$; that is, $(\frac{11}{12} + \frac{\sqrt{\delta_0}}{4}) \cdot Q^+ \cap L_{\hat{Q}^+} = \varnothing$. Then $\hat{\beta}_2(Q) \geq \beta_2(Q, \hat{Q}^+) > \delta_0$, which contradicts the assumption that $Q \in \tilde{\mathcal{D}}^S$.

(ii) This fact follows by exhausting all cases and using the first property. □

The above proposition shows that the following function $f^{\uparrow}$ is uniquely defined for all cubes in $\tilde{\mathcal{D}}$. Note that its image is the set $\tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$.

DEFINITION 7.9 (Function $f^{\uparrow}$) If Q is a cube in $\tilde{\mathcal{D}}$, then

$$f^{\uparrow}(Q) = \begin{cases} Q & \text{if } Q \in \tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta} \\ Q^+ & \text{if } Q \in \tilde{\mathcal{D}}^{+SC} \cup \tilde{\mathcal{D}}^{SLL} \\ Q^{++} & \text{if } Q \in \tilde{\mathcal{D}}^{++SC}. \end{cases}$$

For any cube Q in $\tilde{\mathcal{D}}^{SC}$, fix (arbitrarily) a cube $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}(\mathbf{Q})$ in $\mathcal{P}(Q)$ and a best L_2 line $\mathbf{L}_{\hat{\mathbf{Q}}}$ for $\hat{Q}$. Moreover, we assign to Q a set $\mathcal{U}_0(Q)$ of two *edge cubes* as follows: Set $\{z^1, z^2\} := Q \cap L_{\hat{Q}}$ and fix (arbitrarily) two cubes Q^i, $i = 1, 2$, of side length $\frac{1}{16} \cdot l(Q)$ such that $z^i \in \frac{2}{3} \cdot Q^i$. Define

$$\mathcal{U}_0(Q) = \{f^{\uparrow}(Q^1), f^{\uparrow}(Q^2)\}.$$

If Q is a cube in $\tilde{\mathcal{D}}^{L\beta}$, then it has no edge cubes. That is,

$$\mathcal{U}_0(Q) = \varnothing.$$

If Q is a cube in $\tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$, then the j^{th} *chain of edge cubes*, $\mathcal{U}_j(Q)$, is defined inductively as follows:

$$\mathcal{U}_j(Q) = \mathcal{U}_{j-1}(Q) \cup \bigcup_{Q' \in \mathcal{U}_{j-1}(Q)} \mathcal{U}_0(Q'), \quad j \geq 1. \tag{7.3}$$

7.2 The δ-Construction

In this section we explain how to form the δ-construction in a cube Q_1 in $\tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$. This construction results in the curve $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$, the "strip" of cubes around it $\tilde{E}^{\delta}_{k_{Q_1}}(Q_1)$, and the sets of stopping-time cubes $\tilde{\mathcal{S}}^{\delta}_{k_{Q_1}}(Q_1)$ and $\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$. We refer to the cubes in $\tilde{\mathcal{S}}^{\delta}_{k_{Q_1}}(Q_1)$ as type 1 stopping-time cubes, whereas the ones in $\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$ are type 2 stopping-time cubes. We distinguish between two cases: $Q_1 \in \tilde{\mathcal{D}}^{L\beta}$ and $Q_1 \in \tilde{\mathcal{D}}^{SC}$. In the former case the δ-construction is referred to as the large β-construction and is described in Section 7.2. In the latter case the δ-construction is referred to as the small β-construction and is described in Section 7.2.

The Large β-Construction

The large β-construction can be performed in any cube Q_1 in $\tilde{\mathcal{D}}^{L}$. In practice, we apply it whenever Q_1 is in $\tilde{\mathcal{D}}^{L\beta}$.

This construction goes as follows. Partition the cube Q_1 into 16^n disjoint subcubes in $\tilde{\mathcal{D}}$ with side length $l(Q_1)/16$. Denote by $\text{subs}(Q_1)$ the set of all such subcubes with positive measure. Recall that $k_{Q_1} = N - \lfloor \log_{C_L} \frac{l(Q_0)}{l(Q_1)} \rfloor$. Define

$$\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1) = \left\{ Q : Q \text{ is a maximal cube in } \bigcup_{Q \in \text{subs}(Q_1)} \{f^{\uparrow}(Q)\} \right\},$$

where maximality is defined according to ordering by inclusion. The set $\mathcal{R}^{\delta}_{k_{Q_1}}(Q_1)$ of type 2 stopping-time cubes is given by the formula

$$\mathcal{R}^{\delta}_{k_{Q_1}}(Q_1) = \left\{ Q : Q \text{ is a maximal cube in } \hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1) \cup \bigcup_{Q \in \hat{\mathcal{R}}^{\delta}_{l}(Q_1)} \mathcal{U}_{k_Q}(Q) \right\}, \tag{7.4}$$

where $\mathcal{U}_{k_Q}(Q)$ is the $k_Q{}^{\text{th}}$ chain of edge cubes (see equation (7.3)). The corresponding stopping-time region $R^{\delta}_{l}(Q_1)$ is set as follows:

$$R^{\delta}_{l}(Q_1) = \bigcup_{Q \in \mathcal{R}^{\delta}_{l}(Q_1)} Q .$$

Define $\mathcal{V} = \{z_{Q_1}\} \cup \{z_Q : Q \in \mathcal{R}^{\delta}_{k_{Q_1}}(Q_1)\}$. Let $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ be a rectifiable curve of shortest length containing $\mathcal{V}$. Also define

$$\tilde{E}^{\delta}_{k_{Q_1}}(Q_1) = \tilde{\mathcal{S}}^{\delta}_{k_{Q_1}}(Q_1) = \tilde{S}^{\delta}_{k_{Q_1}}(Q_1) = \varnothing ,$$

$$\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1) = \mathcal{R}^{\delta}_{k_{Q_1}}(Q_1) , \quad \tilde{R}^{\delta}_{k_{Q_1}}(Q_1) = R^{\delta}_{k_{Q_1}}(Q_1) .$$

The Small β-Construction

The small β-construction is performed in any cube Q_1 in $\tilde{\mathcal{D}}^{SC}$. It is similar to the construction performed in Section 6. But it also includes some extra stopping-time cubes. We refer to them as type 2 stopping-time cubes.

Fix a cube Q_1 in $\tilde{\mathcal{D}}^{SC}$. Recall that $\hat{Q}_1$ is a fixed cube in $\mathcal{P}(Q_1)$ and $L_{\hat{Q}_1}$ a fixed best L_2 line for $\hat{Q}_1$. The zeroth-stage sets of the δ-construction—$\mathcal{M}_0^\delta(Q_1)$, $E_0^\delta(Q_1)$, $\Gamma_0^\delta(Q_1)$, $\mathcal{S}_0^\delta(Q_1)$, $S_0^\delta(Q_1)$, $\mathcal{M}_0'^\delta(Q_1)$, and $E_0'^\delta(Q_1)$—are the same as the sets $\mathcal{M}_0$, E_0, Γ_0, $\mathcal{S}_0$, S_0, $\mathcal{M}_0'$, and E_0', which are defined (with respect to Q_1) in Section 6.2. The line $L_{\hat{Q}_1}$ determines the curve $\Gamma_0^\delta(Q_1)$ (see equation 6.3). We also define

$$\hat{\mathcal{M}}_0^\delta(Q_1) = \mathcal{M}_0^\delta(Q_1) \quad \text{and} \quad \mathcal{R}_0^\delta(Q_1) = R_0^\delta(Q_1) = \varnothing\,.$$

The construction proceeds inductively until stage k_{Q_1}, where $k_{Q_1} = N - \lfloor \log_{C_L} \frac{l(Q_0)}{l(Q_1)} \rfloor$. At a given stage l, $0 \le l \le k_{Q_1}$, the sets $\hat{\mathcal{M}}_l^\delta(Q_1)$, $\mathcal{M}_l^\delta(Q_1)$, and $E_l^\delta(Q_1)$ are defined as follows:

$$\hat{\mathcal{M}}_l^\delta(Q_1) = \left\{ \begin{array}{l} Q^{5+}(Q) : Q \in \mathcal{C}_{Q_{1,k_{Q_1}}},\ Q \cap E_{l-1}'^\delta(Q_1) \neq \varnothing, \\ \mu(Q) > 0,\ l(Q) = \frac{l(Q_1)}{C_L^l}, \text{and there exists a} \\ \text{point } x \in \Gamma_{l-1}^\delta(Q_1) \text{ such that } x \in \frac{2}{3}\cdot Q \end{array} \right\}, \tag{7.5}$$

$$\mathcal{M}_l^\delta(Q_1) = \hat{\mathcal{M}}_l^\delta(Q_1) \cap \mathcal{G}_{Q_1,l}^\delta\,, \quad \text{and} \quad E_l^\delta(Q_1) = \bigcup_{Q \in \mathcal{M}_l^\delta(Q_1)} Q\,.$$

REMARK 7.10 The cubes in $\hat{\mathcal{M}}_l^\delta(Q_1)$ cover the curve $\Gamma_{l-1}^\delta(Q_1)$ inefficiently. For example, there exists a point x in $\Gamma_{l-1}^\delta(Q_1)$ such that $\sum_{Q \in \hat{\mathcal{M}}_l^\delta(Q_1)} \chi_Q(x) = 2^n$. It is possible to redefine the set $\hat{\mathcal{M}}_l^\delta(Q_1)$ so that $\sum_{Q \in \hat{\mathcal{M}}_l^\delta(Q_1)} \chi_Q \lessapprox \sqrt{n}$. It is also feasible to use this hinted sparse covering together with other improvements of the construction and substantially decrease some of the subsequent constants. We will not pursue these improvements here.

If Q' is a cube in $\hat{\mathcal{M}}_l^\delta(Q_1)$, let $\mathbf{Q^{5-}(Q')}$ denote the set of all cubes Q used in equation (7.5) so that $Q' = Q^{5+}(Q)$.

The sets $\Gamma_l^\delta(Q_1)$, $\mathcal{S}_l^\delta(Q_1)$, $S_l^\delta(Q_1)$, $\mathcal{M}_l'^\delta(Q_1)$, and $E_l'^\delta(Q_1)$ are defined similarly to the sets Γ_l, $\mathcal{S}_l$, S_l, $\mathcal{M}_l'$, and E_l' (see Section 6.2). The only difference is in the underlying collection $\mathcal{M}_l^\delta(Q_1)$ as opposed to $\mathcal{M}_l(Q_1)$. The stopping-time cubes in $\mathcal{S}_l^\delta(Q_1)$, $0 \le l \le k_{Q_1}$, are referred to as type 1 stopping-time cubes.

Let $\hat{\mathcal{R}}_l^\delta(Q_1)$, $0 \le l \le k_{Q_1}$, be the set of all maximal cubes (with respect to ordering by inclusion) in the following collection:

$$\left\{ f^\uparrow(Q) : Q \in Q^{5-}(Q') \text{ and } Q' \text{ is a cube in } \bigcup_{k=0}^{l} \left(\hat{\mathcal{M}}_k^\delta(Q_1) \setminus \mathcal{M}_k^\delta(Q_1)\right) \right\}.$$

The set $\mathcal{R}_l^\delta(Q_1)$, $1 \leq l \leq k_{Q_1}$, of type 2 stopping-time cubes is given by the formula

$$(7.6) \qquad \mathcal{R}_l^\delta(Q_1) = \left\{ Q : Q \text{ is a maximal cube in } \hat{\mathcal{R}}_l^\delta(Q_1) \cup \bigcup_{Q \in \hat{\mathcal{R}}_l^\delta(Q_1)} \mathcal{U}_l(Q) \right\}.$$

We remark that in the above equation maximality is defined according to ordering by inclusion and $\mathcal{U}_l(Q)$ is the l^{th} chain of edge cubes (see equation (7.3)). The corresponding stopping-time region $R_l^\delta(Q_1)$ is given by the formula

$$R_l^\delta(Q_1) = \bigcup_{Q \in \mathcal{R}_l^\delta(Q_1)} Q\,.$$

Note that the maximality property of the cubes in $\hat{\mathcal{R}}_l^\delta(Q_1)$ and $\mathcal{R}_l^\delta(Q_1)$, $1 \leq l \leq k_{Q_1}$, and their belonging to $\tilde{\mathcal{D}}$ imply the following property:

$$(7.7) \qquad \sum_{Q \in \hat{\mathcal{R}}_l^\delta(Q_1)} \chi_Q \leq 2^n \quad \text{and} \quad \sum_{Q \in \mathcal{R}_l^\delta(Q_1)} \chi_Q \leq 2^n, \quad 1 \leq l \leq k_{Q_1}\,.$$

The construction is restarted at type 1 stopping-time cubes in a similar way as in Section 6.5. We recall that type 1 stopping-time cubes are cubes of the form Q_x, where x is a type 1 point. The sets $\Gamma_{k_{Q_1}}^{j,\delta}(Q_1)$, $E_{k_{Q_1}}^{\prime j,\delta}(Q_1)$, $\tilde{S}_{k_{Q_1}}^{j,\delta}(Q_1)$, and $S_{k_{Q_1}}^{j,\delta}(Q_1)$, $j \geq 0$, are defined recursively as in equations (6.97) and (6.99) through (6.103). The sets $\mathcal{R}_{k_{Q_1}}^{j,\delta}(Q_1)$, $j \geq 0$, are defined similarly as follows:

$$\mathcal{R}_{k_{Q_1}}^{0,\delta}(Q_1) = \mathcal{R}_{k_{Q_1}}^{\delta}(Q_1)\,,$$

$$(7.8) \qquad \mathcal{R}_{k_{Q_1}}^{j+1,\delta}(Q_1) = \mathcal{R}_{k_{Q_1}}^{j,\delta}(Q_1) \cup \bigcup_{Q_x \in \mathcal{S}_{k_{Q_1}}^{j,\delta}(Q_1) \setminus \tilde{\mathcal{S}}_{k_{Q_1}}^{j+1,\delta}(Q_1)} \mathcal{R}_{k_{Q_x}}^{\delta}(Q_x)\,, \quad j \geq 0.$$

Let j_{Q_1} be the smallest integer such that $S_{k_{Q_1}}^{j_{Q_1},\delta} = \varnothing$. Define

$$\tilde{\Gamma}_{k_{Q_1}}^{\delta}(Q_1) = \Gamma_{k_{Q_1}}^{j_{Q_1},\delta}(Q_1)\,,$$

$$\tilde{E}_{k_{Q_1}}^{\delta}(Q_1) = E_{k_{Q_1}}^{\prime j_{Q_1},\delta}(Q_1)\,,$$

$$\tilde{S}_{k_{Q_1}}^{\delta}(Q_1) = \tilde{S}_{k_{Q_1}}^{j_{Q_1},\delta}(Q_1)\,,$$

$$\tilde{\mathcal{R}}_{k_{Q_1}}^{\delta}(Q_1) = \mathcal{R}_{k_{Q_1}}^{j_{Q_1},\delta}(Q_1)\,, \; \tilde{R}_{k_{Q_1}}^{\delta}(Q_1) = \bigcup_{Q \in \tilde{\mathcal{R}}_{k_{Q_1}}^{\delta}(Q_1)} Q\,.$$

7.3 The Whole Construction

In this section we explain how to repeat recursively the δ-construction and obtain the sets $\hat{\Gamma}_N$, $\hat{E}_N$, and $\hat{S}_N$ for a given measure μ and cube Q_0. We start by defining generations of stopping-time cubes. The idea of the whole construction is to apply repeatedly the δ-construction at each generation.

If the underlying cube Q_0 is not in $\tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$, then modify it as follows: $Q_0 := f^{\uparrow}(Q_0)$ (however, do not change the grid $\tilde{\mathcal{D}}$). The zeroth-level generation is defined as follows:

$$\mathcal{I}_0 = \{Q_0\}\,.$$

Assume that the k^{th} generation has been defined. The $(k+1)^{\text{th}}$ generation is given by the formula

$$\mathcal{I}_{k+1} := \left\{ Q' : Q' \text{ is a maximal cube in } \bigcup_{Q \in \mathcal{I}_k} \tilde{\mathcal{R}}^{\delta}_{k_Q}(Q) \right\}.$$

Note that if Q_1 is a cube in $\bigcup_{k\geq 0} \mathcal{I}_k$, then we can define $\mathcal{I}_k(Q_1)$, $k \geq 0$, in an analogous way to $\mathcal{I}_k$ by replacing Q_0 with Q_1.

The following lemma implies that there are only a finite number of generations.

LEMMA 7.11 *If Q_1 is a cube in $\mathcal{I}_k \equiv \mathcal{I}_k(Q_1)$, $k \geq 0$, then*

$$J_{Q_1} \leq \sup J_{Q_0} - \frac{k \cdot \delta_0}{2^{n+1}}\,. \tag{7.9}$$

PROOF: We prove equation (7.9) by induction. The case $k = 0$ is immediate. Let Q_1 be a cube in $\mathcal{I}_k$, $k \geq 0$, and let Q_2 be a cube in $\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$. We show that

$$J_{Q_2} \leq \sup J_{Q_1} - \frac{\delta_0}{2^{n+1}}\,. \tag{7.10}$$

This equation implies the induction step and concludes the lemma.

Note that if $Q_1 \in \mathcal{I}_k$, $k \geq 0$, then $Q_1 \in \tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$. This fact follows from the use of the function $f^{\uparrow}$ throughout the construction.

Assume first that $Q_1 \in \tilde{\mathcal{D}}^{SC}$. Note that the constant C_L is sufficiently large so that $Q_2^{5+} \subsetneq Q_1^{++}$. Also observe that one of the cubes Q_2^{3+}, Q_2^{4+}, or Q_2^{5+} does not satisfy the small sum condition with respect to the cube Q_1. These observations imply equation (7.10) in the current case.

Assume next that $Q_1 \in \tilde{\mathcal{D}}^{L\beta}$. Note that $Q_2^{++} \subsetneq Q_1^{+}$ and that either Q_1^{+} or Q_1^{++} is in $\tilde{\mathcal{D}}^{L}$. These observations imply equation (7.10) in this last case. □

At last we form the sets $\hat{\Gamma}_N$, $\hat{E}_N$, and $\hat{S}_N$. Given a cube Q in $\bigcup_{k\geq 0} \mathcal{I}_k$, let $\gamma^{\delta}(Q)$ be a set of minimal one-dimensional Hausdorff measure such that the set

$$\tilde{\Gamma}^{\delta}_{k_Q}(Q) \cup \gamma^{\delta}(Q) \cup \bigcup_{Q' \in \tilde{\mathcal{R}}_{k_Q}(Q)} \tilde{\Gamma}^{\delta}_{k_{Q'}}(Q')$$

is connected. Define

$$\hat{\Gamma}_N = \bigcup_{k\geq 0} \bigcup_{Q \in \mathcal{I}_k} \left(\tilde{\Gamma}_{k_Q}(Q) \cup \gamma^{\delta}(Q) \right), \tag{7.11}$$

$$\hat{E}_N = \bigcup_{k\geq 0} \bigcup_{Q \in \mathcal{I}_k} \tilde{E}^{\delta}_{k_Q}(Q)\,, \tag{7.12}$$

and

$$\hat{S}_N = \bigcup_{k\geq 0} \bigcup_{Q\in\mathcal{I}_k} \tilde{S}^{\delta}_{k_Q}(Q)\,. \tag{7.13}$$

7.4 The Length Estimate

We first estimate the lengths of the curves $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ and $\gamma^{\delta}(Q_1)$, where $Q_1 \in \tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$. We then use these estimates in order to bound uniformly the lengths of the curves $\hat{\Gamma}_N$, $N \geq 0$.

If $Q_1 \in \tilde{\mathcal{D}}^{SC}$, then the length estimate of the curve $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ follows from the length estimates of Section 6.3. If $Q_1 \in \tilde{\mathcal{D}}^{L\beta}$, then the length estimate of $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ is straightforward. In order to estimate the length of the curve $\gamma^{\delta}(Q_1)$, we need to control the following sum: $\sum_{Q\in\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q)$. This is done in Lemma 7.13 by using the property described in Lemma 7.12. Lemma 7.14 concludes the length estimate of $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1) \cup \gamma^{\delta}(Q_1)$. Finally, Lemma 7.15 describes the length estimate of the curves $\hat{\Gamma}_N$, $N \geq 0$.

LEMMA 7.12 *If Q_1 is a cube in $\tilde{\mathcal{D}}^{SC}$, Q' is a cube in $\hat{\mathcal{M}}^{\delta}_l(Q_1)$, $0 \leq l \leq k_{Q_1}$, $Q'' \in Q^{5-}(Q')$, and $Q = f^{\uparrow}(Q'')$, then*

$$l\big(\Gamma^{\delta}_m(Q_1) \cap Q\big) \geq \frac{1}{12} \cdot l(Q)\,, \quad l-1 \leq m \leq k_{Q_1}\,. \tag{7.14}$$

PROOF: Let Q' be a cube in $\hat{\mathcal{M}}^{\delta}_l(Q_1)$, $0 \leq l \leq k_{Q_1}$, $Q'' \in Q^{5-}(Q')$, and $Q = f^{\uparrow}(Q'')$. It follows from the definition of $\hat{\mathcal{M}}^{\delta}_l(Q_1)$ that there exists a point $x^* \in \Gamma^{\delta}_{l-1}(Q_1) \cap Q$ such that

$$\operatorname{dist}(x^*, \partial Q) \geq \frac{1}{6} \cdot l(Q)\,. \tag{7.15}$$

In an analogous way to proving equation (6.29), we obtain that if $1 \leq i \leq k_{Q_1}$ and $x_{i-1} \in \Gamma^{\delta}_{i-1}(Q_1)$, then there exists a point $x_i \in \Gamma^{\delta}_i(Q_1)$ such that

$$\operatorname{dist}(x_{i-1}, x_i) < \frac{1}{24} \cdot \frac{l(Q_1)}{C_L^i}\,, \quad 1 \leq i \leq k_{Q_1}\,.$$

Consequently, there exists a point $x_m \in \Gamma^{\delta}_m(Q_1)$, $l-1 \leq m \leq k_{Q_1}$, such that

$$\operatorname{dist}(x^*, x_m) < \frac{l(Q_1)}{24} \cdot \sum_{i=l}^{m} \frac{1}{C_L^i} < \frac{l(Q)}{12}\,, \quad l-1 \leq m \leq k_{Q_1}\,. \tag{7.16}$$

The lemma is proven by combining equations (7.15) and (7.16). □

LEMMA 7.13 *There exist constants $C_6'(n)$ and $C_6(n)$ such that if Q_1 is a cube in $\bigcup_{j\geq 0}\mathcal{I}_j$, then the following inequalities are satisfied:*

$$\sum_{Q\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq C_6'(n)\cdot l(Q_1)\,, \tag{7.17}$$

$$\sum_{Q\in\mathcal{R}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq 2\cdot C_6'(n)\cdot l(Q_1)\,, \tag{7.18}$$

$$\sum_{Q\in\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq C_6(n)\cdot l(Q_1)\,. \tag{7.19}$$

PROOF OF EQUATION (7.17): If Q_1 is a cube in $\bigcup_{j\geq 0}\mathcal{I}_j$, then $Q_1 \in \tilde{\mathcal{D}}^{SC}\cup\tilde{\mathcal{D}}^{L\beta}$. Assume first that $Q_1\in\tilde{\mathcal{D}}^{SC}$. In this case, equations (7.7) and (7.14) imply the following estimate:

$$\sum_{Q\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq 12\cdot 2^n\cdot l\big(\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)\big)\,. \tag{7.20}$$

Note that the curve $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ satisfies the following length estimate:

$$l\big(\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)\big) \leq C_1(n)\cdot e\cdot l(Q_1)\,. \tag{7.21}$$

This estimate is obtained similarly as equation (6.105) (also use equation (6.43)). Combine the above two equations and obtain the following bound:

$$\sum_{Q\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq 12\cdot 2^n\cdot e\cdot C_1(n)\cdot l(Q_1)\,.$$

If $Q_1\in\widetilde{\mathcal{D}}^{L\beta}$, then there are at most 16^n cubes in $\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$ with side length at most $l(Q_1)/4$ and thus

$$\sum Q\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1) l(Q) \leq \frac{(16)^n}{4}\cdot l(Q_1)\,.$$

Equation (7.17) is thus verified, where $C_6'(n) = \max\{(16)^n/4,\, 12{\cdot}2^n{\cdot}e{\cdot}C_1(n)\}$. □

PROOF OF EQUATION (7.18): We show that if $Q_1\in\bigcup_{j\geq 0}\mathcal{I}_j$, then

$$\sum_{Q\in\mathcal{R}^{\delta}_{Q_1}(Q_1)} l(Q) \leq 2\cdot \sum_{Q\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q)\,.$$

Note that if $Q\in\mathcal{R}^{\delta}_{k_{Q_1}}(Q_1)\backslash\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$, then there exists a cube $Q'\in\hat{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$ such that $Q\in\mathcal{U}_{k_{Q'}}(Q')$. The above equation thus follows from the following fact:

$$\sum_{Q\in\mathcal{U}_j(Q')} l(Q) \leq l(Q') \quad \text{for any } Q'\in\tilde{\mathcal{D}}^{SC}\cup\tilde{\mathcal{D}}^{L\beta} \text{ and } j\geq 1\,.$$

This fact is clear once we note that there are at most 2^j cubes in $\mathcal{U}_j(Q') \setminus \mathcal{U}_{j-1}(Q')$ of side length at most $\frac{1}{4^j} \cdot l(Q_1)$. □

PROOF OF EQUATION (7.19): If $Q_1 \in \tilde{\mathcal{D}}^{L\beta}$, then equation (7.19) reduces to (7.18).

Assume that $Q_1 \in \tilde{\mathcal{D}}^{SC}$. Equations (7.8), (7.18) and the definition of the set $\tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)$ imply the following estimate:

$$\sum_{Q \in \tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq 2 \cdot C_6'(n) \cdot l(Q_1) + 2 \cdot C_6'(n) \cdot \sum_{Q_x \in \bigcup_{j=1}^{j_{Q_1}} \mathcal{S}^{j-1,\delta}_{k_{Q_1}}(Q_1) \setminus \tilde{\mathcal{S}}^{j,\delta}_{k_{Q_1}}(Q_1)} l(Q_x)\,.$$

Equation (7.14), Lemma 6.25, and the definition of the curve $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ yield that

$$\sum_{Q \in \tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)} l(Q) \leq 2 \cdot C_6'(n) \cdot l(Q_1) + 24 \cdot C_6'(n) \cdot l\big(\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)\big)\,.$$

Equation (7.19) follows from the above equation together with equation (7.21), where $C_6(n) = 2 \cdot C_6'(n) \cdot (1 + 12 \cdot e \cdot C_1(n))$. □

LEMMA 7.14 *There exists a constant $C_7(n)$ such that if Q_1 is a cube in $\bigcup_{j \geq 0} \mathcal{I}_j$, then*

$$l\big(\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1) \cup \gamma^{\delta}(Q_1)\big) \leq C_7(n) \cdot l(Q_1)\,. \tag{7.22}$$

The only nontrivial estimate needed to prove Lemma 7.14 has been stated in equation (7.21). Its proof is similar to that of equation (6.105). The other estimates are straightforward. We thus present the proof of Lemma 7.14 in Appendix H.

The length estimate for the curve $\hat{\Gamma}_N$ is proven as follows:

LEMMA 7.15 *There exist constants $\tilde{C}_1(n)$ and $\tilde{C}_2(n)$ such that*

$$l(\hat{\Gamma}_N) \leq \tilde{C}_1(n) \cdot e^{\tilde{C}_2(n) \cdot M} \cdot l(Q_0)\,.$$

PROOF: If $Q_1 \in \bigcup_{k \geq 0} \mathcal{I}_k$, define

$$\hat{\Gamma}^{k}_{k_{Q_1}}(Q_1) = \bigcup_{0 \leq j \leq k} \bigcup_{Q \in \mathcal{I}_j(Q_1)} \tilde{\Gamma}^{\delta}_{k_Q}(Q) \cup \gamma^{\delta}(Q)\,, \quad k \geq 0.$$

Equations (7.1), (7.11), and Lemma 7.11 imply that there exists a constant $C(n)$ independent of M and N and an integer $k^* = C(n) \cdot M$ such that

$$\hat{\Gamma}_N = \hat{\Gamma}^{k^*}_N(Q_0)\,.$$

Therefore, in order to prove the lemma it is sufficient to show that if $Q_1 \in \bigcup_{j \geq 0} \mathcal{I}_j$, then for any $k \geq 0$

$$l\big(\hat{\Gamma}^{k}_{k_{Q_1}}(Q_1)\big) \leq C_7(n) \cdot C_6(n)^k \cdot l(Q_1)\,. \tag{7.23}$$

We prove the above equation by induction. The case $k = 0$ follows from equation (7.22). Assume that the above equation is true for a given k. Note that

$$\hat{\Gamma}^{k+1}_{k_{Q_1}}(Q_1) \subseteq \bigcup_{Q \in \mathcal{I}_1(Q_1)} \hat{\Gamma}^{k}_{k_Q}(Q)\,. \tag{7.24}$$

By combining equations (7.19) and (7.24) and the induction assumption, we obtain that equation (7.23) is satisfied when $k := k + 1$ and thus prove the lemma. $\square$

7.5 The Measure Estimate

The main part of this section is Lemma 7.18, where we bound uniformly from below the measures of the sets $\hat{E}_N \cup \hat{S}_N$, $N \geq 0$. This result proves Theorem 4.8. The proof of the lemma is based on the measure estimate of the δ-construction (see equation (7.25)) and on estimating the number of joint intersections of some δ-construction regions (see Lemma 7.17, which follows from Lemma 7.16).

Fix a cube Q_1 in $\tilde{\mathcal{D}}^{SC} \cup \tilde{\mathcal{D}}^{L\beta}$ and denote

$$\tilde{T}^{\delta}_{k_{Q_1}}(Q_1) = \tilde{E}^{\delta}_{k_{Q_1}}(Q_1) \cup \tilde{S}^{\delta}_{k_{Q_1}}(Q_1) \cup \tilde{\mathcal{R}}^{\delta}_{k_{Q_1}}(Q_1)\,.$$

The measure estimate of the δ-construction is formulated as follows:

$$\mu\big(\tilde{T}^{\delta}_{k_{Q_1}}(Q_1)\big) \geq \frac{1}{e} \cdot \mu(Q_1)\,. \tag{7.25}$$

If $Q_1 \in \mathcal{D}^{L\beta}$, then the above estimate is trivial because no measure was thrown away. If $Q_1 \in \mathcal{D}^{SC}$, then the above estimate is proven similarly to equation (6.121) (also use equation (6.84)). Indeed, all the measure estimates of Sections 6.4 and 6.6 leading to equation (6.121) are valid when replacing the sets S_l and $\tilde{S}_l$ by $S^{\delta}_l \cup R^{\delta}_l$ and $\tilde{S}^{\delta}_l \cup \tilde{R}^{\delta}_l$, respectively.

Our main obstacle in estimating the measure of the set $\hat{E}_N \cup \hat{S}_N$ is the following fact: If Q_1 is a cube in $\tilde{\mathcal{D}}$, then the region $\tilde{T}^{\delta}_{k_{Q_1}}(Q_1)$ is not necessarily contained in Q_1. We overcome this obstacle by bounding the number of joint intersections of the different regions $\tilde{T}^{\delta}_{k_{Q'_1}}(Q'_1)$ of cubes Q'_1 in $\mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC}$. We formulate this bound precisely in Lemma 7.17, which is an immediate consequence of the following lemma.

LEMMA 7.16 *If Q_2 and Q'_2 are two cubes in $\mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC}$, $k \geq 1$, such that*

$$\tilde{T}^{\delta}_{k_{Q_2}}(Q_2) \cap \tilde{T}^{\delta}_{k_{Q'_2}}(Q'_2) \neq \varnothing\,,$$

then

$$\frac{l(Q_2)}{128} \leq l(Q'_2) \leq 128 \cdot l(Q_2)\,.$$

PROOF: We prove the required property by contradiction. Assume that Q_2 and Q'_2 are two cubes in $\mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC}$, $k \geq 1$, such that $l(Q'_2) \leq l(Q_2)/256$. We show that $\tilde{T}^{\delta}_{k_{Q_2}}(Q_2) \cap \tilde{T}^{\delta}_{k_{Q'_2}}(Q'_2) = \varnothing$.

Denote by Q_2^1 and Q_2^2 the two edge cubes of Q_2 in $\mathcal{U}_0(Q_2)$. Recall that

$$l\big(Q_2^1\big) = l\big(Q_2^2\big) \geq \frac{1}{16} \cdot l(Q_2)\,. \tag{7.26}$$

It follows from the maximality of the cubes in $\mathcal{I}_k(Q_1)$ and the properties of the grid $\tilde{\mathcal{D}}$ (see, e.g., equations (3.1) and (3.2)) that

$$Q_2' \cap \frac{191}{192} \cdot Q_2 = \varnothing \quad \text{and} \quad Q_2' \cap \frac{11}{12} \cdot Q_2^i = \varnothing\,, \quad i = 1, 2. \tag{7.27}$$

The above numbers are obtained as follows:

$$\frac{191}{192} = 1 - 2 \cdot \frac{1}{256} \cdot \frac{2}{3}$$

and

$$\frac{11}{12} = 1 - 2 \cdot \frac{16}{256} \cdot \frac{2}{3} \leq 1 - \frac{4}{3 \cdot 256} \cdot \frac{l(Q_2)}{l(Q_2^i)}\,, \quad i = 1, 2.$$

Equation (7.27) implies that

$$\frac{448}{192} \cdot Q_2' \cap \frac{190}{192} \cdot Q_2 = \varnothing \quad \text{and} \quad \frac{448}{192} \cdot Q_2' \cap \frac{5}{6} \cdot Q_2^i = \varnothing\,, \quad i = 1, 2. \tag{7.28}$$

The numbers above were set as follows:

$$\frac{448}{192} = 1 + \frac{256}{192} \leq 1 + \frac{1}{192} \cdot \frac{l(Q_2)}{l(Q_2')}\,, \qquad \frac{190}{192} = \frac{191}{192} - \frac{1}{192}\,,$$

and

$$\frac{5}{6} = \frac{11}{12} - \frac{16}{192} \leq \frac{11}{12} - \frac{1}{192} \cdot \frac{l(Q_2)}{l(Q_2^i)}\,.$$

Note that the constant C_L is sufficiently large so that the following equation is satisfied:

$$\tilde{T}^{\delta}_{k_{Q_2'}}(Q_2') \subseteq \frac{448}{192} \cdot Q_2'\,. \tag{7.29}$$

Recall that the edge cubes Q_2^i, $i = 1, 2$, were set so that

$$\partial Q_2 \cap \frac{2}{3} \cdot Q_2^i \neq \varnothing\,, \quad i = 1, 2.$$

Reformulate equation (7.26) in the following way:

$$\begin{aligned} l\left(\frac{5}{6} \cdot Q_2^i\right) - l\left(\frac{2}{3} \cdot Q_2^i\right) &= \frac{1}{6} \cdot l(Q_2^i) \geq \frac{1}{96} \cdot l(Q_2) \\ &= l(Q_2) - l\left(\frac{190}{192} \cdot Q_2\right), \quad i = 1, 2. \end{aligned}$$

The above two equations and the choice of the constant C_L imply that

$$\tilde{T}^{\delta}_{k_{Q_2}}(Q_2) \subseteq \left(\frac{190}{192} \cdot Q_2\right) \cup \bigcup_{i=1}^{2} \left(\frac{5}{6} \cdot Q_2^i\right), \quad i = 1, 2. \tag{7.30}$$

Combine equations (7.28) through (7.30) and conclude that

$$\tilde{T}^{\delta}_{k_{Q_2}}(Q_2) \cap \tilde{T}^{\delta}_{k_{Q'_2}}(Q'_2) = \varnothing .$$

□

LEMMA 7.17 *There exists a constant* $C_9(n)$ (*independent of* k) *such that*

$$\sum_{Q \in \mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC}} \chi_{\tilde{T}^{\delta}_{k_Q}(Q)} \leq C_9(n) , \quad k \geq 1. \tag{7.31}$$

PROOF: Fix a cube $Q_2 \in \mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC}$ and an integer i, $-7 \leq i \leq 7$. Denote

$$\mathcal{C}_i = \mathcal{C}_i(Q_2) = \left\{ Q'_2 : Q'_2 \in \mathcal{I}_k(Q_1) \cap \tilde{\mathcal{D}}^{SC},\ l(Q'_2) = \frac{l(Q_2)}{2^i},\ \tilde{T}^{\delta}_{k_{Q_2}}(Q_2) \cap \tilde{T}^{\delta}_{k_{Q'_2}}(Q'_2) \neq \varnothing \right\}.$$

Note that if $Q'_2 \in \mathcal{C}_i$, then $Q'_2 \subseteq (1 + l(Q'_2)/l(Q_2)) \cdot Q_2$, and moreover Q'_2 is not contained in Q_2. Therefore there exists a constant $C^i_9(n)$ (depending only on n) such that

$$\#\mathcal{C}_i \leq C^i_9(n) , \tag{7.32}$$

where $\#\mathcal{C}_i$ denotes the number of points in $\mathcal{C}_i$.

Combine Lemma 7.16 with equation (7.32) and obtain equation (7.31), where the constant $C_9(n)$ can be set as follows:

$$C_9(n) = \sum_{i=-5}^{5} C^i_9(n) .$$

□

Finally, we estimate the measure of the set $\hat{E}_N \cup \hat{S}_N$.

LEMMA 7.18 *There exist constants* $\tilde{C}_3$ *and* $\tilde{C}_4(n)$ *such that*

$$\mu(\hat{E}_N \cup \hat{S}_N) \geq \tilde{C}_3 \cdot e^{-\tilde{C}_4(n) \cdot M} \cdot \mu(Q_0) .$$

PROOF: If $Q_1 \in \bigcup_{k \geq 0} \mathcal{I}_k$, define for $k \geq 0$,

$$\hat{E}^k_{k_{Q_1}}(Q_1) = \bigcup_{0 \leq j \leq k} \bigcup_{Q \in \mathcal{I}_j(Q_1)} \tilde{E}^{\delta}_{k_Q}(Q) ,$$

$$\hat{S}^k_{k_{Q_1}}(Q_1) = \bigcup_{0 \leq j \leq k} \bigcup_{Q \in \mathcal{I}_j(Q_1)} \tilde{S}^{\delta}_{k_Q}(Q) ,$$

$$\hat{R}^k_{k_{Q_1}}(Q_1) = \bigcup_{Q \in \mathcal{I}_{k+1}(Q_1)} Q ,$$

$$\hat{T}^k_{k_{Q_1}}(Q_1) = \hat{E}^k_{k_{Q_1}}(Q_1) \cup \hat{S}^k_{k_{Q_1}}(Q_1) \cup \hat{\mathcal{R}}^k_{k_Q}(Q) .$$

Similarly, as in the proof of Lemma 7.15, let $k^* = C(n) \cdot M$ be an integer such that

$$\hat{E}_N = \hat{E}_N^{k^*}(Q_0), \quad \hat{S}_N = \hat{S}_N^{k^*}(Q_0), \quad \text{and} \quad \hat{R}_N^{k^*}(Q_0) = \varnothing .$$

In order to prove the lemma, it is sufficient to show that there exists a constant $C_8(n)$ such that if $Q_1 \in \bigcup_{j \geq 0} \mathcal{I}_j$, then for any $k \geq 0$ the following equation is satisfied:

$$\mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) \geq \frac{1}{e} \cdot \frac{1}{C_8(n)^k} \cdot \mu(Q_1) . \tag{7.33}$$

We prove the above equation by induction. The initial case $k = 0$ follows from equation (7.25). The induction step follows from the equation

$$\mu\big(\hat{T}^{k+1}_{k_{Q_1}}(Q_1)\big) \geq \frac{1}{C_8(n)} \cdot \mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) , \quad k \geq 0. \tag{7.34}$$

We verify this equation for three different cases.

The first case occurs when

$$\mu\big(\hat{R}^k_{k_{Q_1}}(Q_1)\big) < \left(1 - \frac{1}{C_8(n)}\right) \cdot \mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) .$$

Note that this condition implies that

$$\mu\big(\hat{E}^k_{k_{Q_1}}(Q_1) \cup \hat{S}^k_{k_{Q_1}}(Q_1)\big) \geq \frac{1}{C_8(n)} \cdot \mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) .$$

Also note that

$$\hat{T}^{k+1}_{k_{Q_1}}(Q_1) \supseteq \hat{E}^k_{k_{Q_1}}(Q_1) \cup \hat{S}^k_{k_{Q_1}}(Q_1) .$$

The above two equations imply the induction step (equation (7.34)) in that case.

The second case occurs when the following two conditions are satisfied:

$$\mu\big(\hat{R}^k_{k_{Q_1}}(Q_1)\big) \geq \left(1 - \frac{1}{C_8(n)}\right) \cdot \mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) \tag{7.35}$$

and

$$\mu\left(\bigcup_{Q \in \mathcal{I}_{k+1}(Q_1) \cap \tilde{\mathcal{D}}^{L\beta}} Q\right) \geq \frac{1}{C_8(n) - 1} \cdot \mu\big(\hat{R}^k_{k_{Q_1}}(Q_1)\big) . \tag{7.36}$$

Combine these two conditions to obtain that

$$\mu\left(\bigcup_{Q \in \mathcal{I}_{k+1}(Q_1) \cap \tilde{\mathcal{D}}^{L\beta}} Q\right) \geq \frac{1}{C_8(n)} \cdot \mu\big(\hat{T}^k_{k_{Q_1}}(Q_1)\big) .$$

Note that

$$\mu\big(\hat{T}^{k+1}_{k_{Q_1}}(Q_1)\big) \geq \mu\left(\bigcup_{Q \in \mathcal{I}_{k+1}(Q_1) \cap \tilde{\mathcal{D}}^{L\beta}} \tilde{\mathcal{R}}^{\delta}_{k_Q}(Q)\right) \geq \mu\left(\bigcup_{Q \in \mathcal{I}_{k+1}(Q_1) \cap \tilde{\mathcal{D}}^{L\beta}} Q\right) .$$

The above two equations imply the induction step (equation (7.34)) in the current case.

The last case occurs whenever equation (7.35) is satisfied together with the following condition:

$$\mu\Bigg(\bigcup_{Q\in\mathcal{I}_{k+1}(Q_1)\cap\tilde{\mathcal{D}}^{SC}} Q\Bigg) \geq \frac{C_8(n)-2}{C_8(n)-1}\cdot\mu\big(\hat{\mathcal{R}}^k_{k_{Q_1}}(Q_1)\big). \tag{7.37}$$

Note that conditions (7.36) and (7.37) are all-inclusive and therefore the three different cases are all-inclusive.

The induction step in this case is verified as follows. Observe that

$$\mu\big(\hat{T}^{k+1}(Q_1)\big) \geq \mu\Bigg(\bigcup_{Q\in\mathcal{I}_{k+1}(Q_1)\cap\tilde{\mathcal{D}}^{SC}} \tilde{T}^{\delta}_{k_{Q_1}}(Q)\Bigg). \tag{7.38}$$

Combine equations (7.31) and (7.38) to obtain that

$$\mu\big(\hat{T}^{k+1}(Q_1)\big) \geq \frac{1}{C_9(n)}\cdot\sum_{Q\in\mathcal{I}_{k+1}(Q_1)\cap\tilde{\mathcal{D}}^{SC}} \mu\big(\tilde{T}^{\delta}_{k_Q}(Q)\big). \tag{7.39}$$

Recall that the cubes in $\mathcal{I}_{k+1}(Q_1)\cap\tilde{\mathcal{D}}^{SC}$ are maximal and therefore

$$\sum_{Q\in\mathcal{I}_{k+1}(Q_1)\cap\tilde{\mathcal{D}}^{SC}} \chi_Q \leq 2^n. \tag{7.40}$$

The application of equations (7.25) and (7.40) to equation (7.39) results in the estimate

$$\mu\big(\hat{T}^{k+1}(Q)\big) \geq \frac{1}{2^n\cdot C_9(n)\cdot e}\cdot\mu\Bigg(\bigcup_{Q\in\mathcal{I}_{k+1}(Q)\cap\tilde{\mathcal{D}}^{SC}} Q\Bigg). \tag{7.41}$$

Adjust the constant $C_8(n)$ so that

$$C_8(n) \geq 2^n\cdot e\cdot C_9(n)\cdot\left(1-\frac{2}{C_8(n)}\right)^{-1}. \tag{7.42}$$

The induction step in the last case thus follows from equations (7.35), (7.37), (7.41), and (7.42). □

We deduce Theorem 4.8 from Lemmata 7.15 and 7.18 similarly as in the proof of Theorem 6.3. That is, the curve Γ is parametrized by a Lipschitz function that is a uniform limit of Lipschitz functions that parametrize a subsequence $\{\hat{\Gamma}_{N_j}\}_{j=1}^{\infty}$ of the curves $\{\hat{\Gamma}_N\}_{N\geq 1}$.

REMARK 7.19 Our choice $C_5 = 15$ complies with the initialization step $Q_0 := f^{\uparrow}(Q_0)$. It is possible to change the construction slightly so that Q_0 does not need to be updated and consequently reduce the value of the constant C_5.

7.6 Proof of Theorem 4.10

We fix a locally finite Borel measure μ on $\mathbb{R}^n$ and a cube Q_0 in $\mathbb{R}^n$. For any integer N, we suggest a construction of a curve $\hat{\Gamma}_N$ and sets surrounding it $\hat{E}_N$, $\hat{S}_N$, and $\hat{B}_N$. The elements $\hat{\Gamma}_N$, $\hat{E}_N$, and $\hat{S}_N$ are identical to the ones of the whole construction described in Section 7.3. The set $\hat{B}_N$ is a union of type 3 stopping-time cubes on which the Jones function is large. Unlike the other kinds of stopping-time cubes, it makes no sense to restart the construction in these cubes.

For certain measures, for example Lebesgue measure on $\mathbb{R}^n$, the following construction is not interesting because $\mu(\hat{B}_N) \gg \mu(\hat{E}_N \cup \hat{S}_N)$. However, we show that there exists a constant C_0' such that if $I_{C_0'}(\mu) \equiv I_{C_0'}^{Q_0}(\mu) < \infty$, then $\mu(\hat{B}_N) \ll \mu(\hat{E}_N \cup \hat{S}_N)$. In this case

$$l(\hat{\Gamma}_N) \lessapprox_n I_{C_0'}(\mu) \cdot l(Q_0) \quad \text{and} \quad \mu(\hat{E}_N \cup \hat{S}_N) \gtrapprox (I_{C_0'}(\mu))^{-1} \cdot \mu(Q_0)\,.$$

These estimates conclude Theorem 4.10.

We start by defining the new stopping-time condition:

DEFINITION 7.20 (Large sum condition with constant M) If $M > 0$, Q_1 is a cube in $\tilde{\mathcal{D}}$, e is a vector in $\{0, 1\}^n$, and Q' is a cube in $\mathcal{C}^e_{Q_1,l}$, $l \geq 0$, then the large sum condition with constant M in the cube Q' with respect to Q_1 is formulated as follows:

$$\sum_{\substack{Q \in \mathcal{C}^e_{Q_1,l},\\ l(Q) \geq l(Q')}} \hat{\beta}_2^2(Q) \cdot \chi_Q(x) \geq M \quad \text{for any } x \in Q'.$$

Denote

$$\tilde{\mathcal{D}}^M = \{Q : Q \text{ satisfies the large sum condition with } M \text{ and w.r.t. } f^{\uparrow}(Q_0)\}\,.$$

The construction of $\hat{\Gamma}_N$, $\hat{E}_N$, $\hat{S}_N$, and $\hat{B}_N$ for a fixed integer N and a fixed positive constant M proceeds as follows: We redefine the cube Q_0 so that $Q_0 := f^{\uparrow}(Q_0)$. Define

$$\mathcal{I}_0 = \tilde{\mathcal{I}}_0 = \{Q_0\} \quad \text{and} \quad \mathcal{B}_0 = \varnothing.$$

Assume that the family of cubes $\mathcal{I}_k$ has been defined and set

$$\tilde{\mathcal{I}}_{k+1} = \left\{ Q' : Q' \text{ is a maximal cube in } \bigcup_{Q \in \mathcal{I}_k} \mathcal{R}^{\delta}_{k_Q}(Q) \right\},$$

$$\mathcal{B}_{k+1} = \tilde{\mathcal{I}}_{k+1} \cap \tilde{\mathcal{D}}^M\,, \quad \mathcal{I}_{k+1} = \tilde{\mathcal{I}}_{k+1} \setminus \tilde{\mathcal{D}}^M\,.$$

The sets $\hat{\Gamma}_N$, $\hat{E}_N$, and $\hat{S}_N$ are defined by equations (7.11) through (7.13). Also define

$$\hat{\mathcal{B}}_N = \bigcup_{k \geq 0} \mathcal{B}_k \quad \text{and} \quad \hat{B}_N = \bigcup_{Q \in \hat{\mathcal{B}}_N} Q\,.$$

Following the same arguments as in Sections 7.4 and 7.5, we obtain the following length and measure estimates (compare with equations (7.15) and (7.18)):

$$l(\hat{\Gamma}_N) \le \tilde{C}_1(n) \cdot e^{\tilde{C}_2(n)\cdot M} \cdot l(Q_0) \tag{7.43}$$

and

$$\mu(\hat{E}_N \cup \hat{S}_N \cup \hat{B}_N) \ge \frac{1}{e} \cdot e^{-\tilde{C}_4(n)\cdot M} \cdot \mu(Q_0)\,. \tag{7.44}$$

Set $C_5 = 15$. The measure of the set $\hat{B}_N$ is bounded as follows:

LEMMA 7.21 *If* $C > 0$, *then*

$$\mu(\hat{B}_N) \le e^{-C\cdot M} \cdot \int_{C_5\cdot Q_0} e^{C\cdot \hat{J}_2(x)}\, \mathrm{d}\mu(x)\,. \tag{7.45}$$

PROOF: Note that if $Q \in \mathcal{B}_N$, then $\hat{J}_2(x) \ge M$ for all $x \in Q$. Consequently, $\hat{J}_2(x) \ge M$ for all $x \in \hat{B}_N$ and thus

$$\hat{B}_N \subseteq \{x : e^{C\cdot \hat{J}_2} \ge e^{C\cdot M}\} \cap C_5 \cdot Q_0\,.$$

Apply Chebyshev's inequality to the above set containing $\hat{B}_N$ and conclude equation (7.45). □

Finally, set

$$C_0' \equiv C_0'(n) = \max\{2\cdot \tilde{C}_2(n), 2\cdot \tilde{C}_4(n)\}\,. \tag{7.46}$$

Let μ be a locally finite Borel measure satisfying the estimate

$$I_{C_0'}(\mu) \le A \cdot \mu(Q_0)\,. \tag{7.47}$$

Define M by the equation

$$A = \frac{1}{2\cdot e} \cdot e^{\frac{C_0'}{2}\cdot M}\,. \tag{7.48}$$

Set $C := C_0'$ in equation (7.45) and combine equations (7.44) through (7.48) to obtain that

$$\mu(\hat{E}_N \cup \hat{S}_N) \ge \frac{1}{2\cdot e} \cdot e^{-\frac{C_0'}{2}\cdot M} \cdot \mu(Q_0) \ge \frac{1}{4\cdot e^2} \cdot \frac{1}{A} \cdot \mu(Q_0)\,.$$

Apply equations (7.46) and (7.48) to equation (7.43) and obtain the inequality

$$l(\hat{\Gamma}_N) \le 2\cdot e \cdot \tilde{C}_1(n) \cdot A \cdot l(Q_0)\,.$$

The theorem follows from the above two estimates and the limit argument used before.

Appendix A: Proof of Lemma 6.12

A.1 Proof of (i)

Assume first that $v \in \mathcal{L}_l(\gamma) \cap I_{p-1,p,}$ where $\max(1, j-3) \le p \le \min(m+1, j+3)$. Note that there exists a cube Q' in $\mathcal{M}_l$ such that

(A.1) $$v = z_{Q'} \,.$$

If $l = 1$, then $j = 2$ and $Q' \subseteq \hat{Q}_2$. If $l > 1$, then the definition of the set $I_{p-1,p}$ implies that

$$v \in Q\left(\frac{z_{p-1} + z_p}{2}, 4 \cdot A_1 \cdot l_{l-1}\right)$$

and thus $Q' \subseteq \hat{Q}_p$. Join these facts together with Lemma 6.4 and get that

(A.2) $$\frac{\operatorname{dist}(v, L_{\hat{Q}_p})}{l_l} \le \beta_2(Q', \hat{Q}_p)\,;$$

that is, equation (6.25) is proven in the case $p = j$.

The proof of the same equation for other values of p goes as follows. Recall that $v \in I_{p-1,p}$ and $v = z_{Q'}$. Combine these facts together with equation (A.2) and Lemma 6.10 to obtain that

(A.3) $$v \in \bigcup_{\substack{z \in [z_{p-1}, z_p],\\ \mu(Q(z, l_{l-1})) > 0}} Q(z, l_{l-1}) \,.$$

The separation properties of the points $\{z_i\}_{i=1}^m$ (see equations (6.5) and (6.6)) together with the fact that z_0 and z_{m+1} are "phantom" points with respect to z_1, z_2 and z_{m-1}, z_m, respectively, suggest the following relation:

(A.4) $$\bigcup_{z \in [z_{p-1}, z_p]} Q(z, l_{l-1}) \subseteq Q\left(\frac{z_{j-1} + z_j}{2}, (14 \cdot A_1 + 1) \cdot l_{l-1}\right).$$

Equations (A.1), (A.3), and (A.4) and the choice of constants imply that $Q' \subseteq \hat{Q}_j$. Apply Lemma 6.4 with the cubes Q' and $\hat{Q}_j$ and conclude equation (6.25).

We next assume that $v \notin \mathcal{L}_l(\gamma)$ and $v \in I_{p-1,p}$, where $\max(1, j-2) \le p \le \min(m+1, j+2)$. In this case v is a phantom point, that is, $v = \tilde{u}$, where $u \in \mathcal{L}_l(\gamma)$. Assume without loss of generality that $v = u^+$.

We first show that

(A.5) $$u^-, u \in I_{p-2,p-1} \cup I_{p-1,p} \,,$$

so that we can apply equation (6.25) to the points u^- and u. Indeed, recall that

(A.6) $$v \in \partial Q(u, 4 \cdot A_0 \cdot l_l)\,, \quad u^- \in Q(u, 4 \cdot A_1 \cdot l_l)\,,$$

and that

(A.7) $$z_k \notin Q(z_{k-1}, A_0 \cdot C_L \cdot l_l)\,, \quad 1 \le k \le m+1.$$

Combine equations (A.6) and (A.7) with Lemma 6.10 and the fact that $4 \cdot (A_0 + A_1) \leq A_0 \cdot C_L$ and consequently validate equation (A.5).

We next verify the following estimate for $\operatorname{dist}(v, L_{\hat{Q}_j})$:

$$\operatorname{dist}(v, L_{\hat{Q}_j}) \leq \operatorname{dist}(u, L_{\hat{Q}_j}) + \operatorname{dist}(u, v) \cdot \sin\theta\,, \tag{A.8}$$

where θ is the angle between the line segment $[u^-, u]$ and the line $L_{\hat{Q}_j}$. Apply equation (6.25) to the points u and u^- and get the following estimate for $\sin\theta$:

$$\sin\theta \leq \frac{2 \cdot \beta_2(Q^j_{\max}, \hat{Q}_j) \cdot l_l}{\operatorname{dist}(u, u^-)}\,. \tag{A.9}$$

Recall that $u \notin Q(u^-, A_0 \cdot l_l)$ and that u^-, u, and v lie on the same line. Combine these two facts with equation (A.6) to conclude that

$$\operatorname{dist}(u, v) \leq 4 \cdot \operatorname{dist}(u, u^-)\,. \tag{A.10}$$

Equations (A.8) through (A.10), together with equation (6.25), applied to the point $u \in \mathcal{L}_l(\gamma)$ prove equation (6.26).

A.2 Proof of (ii)

Assume first that $v \in \mathcal{L}_l(\gamma) \cap I_{p-1,p}$, $1 \leq p \leq m+1$, and $v \neq v^l_1, v^l_N$. If v is a type 2 point, then $\theta_v = 0$. We can thus assume that v is either a type 1 or type 3 point.

We first show that

$$v^+ \in I_{p-1,p} \cup I_{p,p+1} \cup I_{p+1,p+2}\,. \tag{A.11}$$

Similarly, it can be shown that

$$v^- \in I_{p-3,p-2} \cup I_{p-2,p-1} \cup I_{p-1,p}\,. \tag{A.12}$$

Recall that $C_L \geq 4 \cdot A_1/A_0$. If v is a type 3 point, then

$$v^+ \in Q(v, 4 \cdot A_1 \cdot l_l)\,. \tag{A.13}$$

It follows from Lemma 6.10 and equations (6.25), (A.7), and (A.13) that

$$v^+ \in I_{p-1,p} \cup I_{p,p+1}\,.$$

If v is a type 1 point, then it follows from Lemma 6.10, equations (6.25), (6.26), (A.7), and the definition of the set $\mathcal{L}_l$ that

$$v^+ \in \bigcup_{z \in \cup_{i=p-1}^{p}[z_i, z_i+1]} Q(z, l_{l-1})$$

and consequently equation (A.11) is proven.

Now we can apply equations (6.25) and (6.26) to the points v^-, v, and v^+ and obtain that

$$\frac{\operatorname{dist}(v, L_{\hat{Q}_j})}{l_l}, \frac{\operatorname{dist}(v^+, L_{\hat{Q}_j})}{l_l}, \frac{\operatorname{dist}(v^-, L_{\hat{Q}_j})}{l_l} \leq \beta_2\big(Q^p_{\max,} \hat{Q}_p\big)\,. \tag{A.14}$$

Note that the following separation property of v^-, v, and v^+ is satisfied:

$$\text{dist}(v^-, v), \text{dist}(v, v^+) \geq A_0 \cdot l_l \,. \tag{A.15}$$

By combining equations (A.14) and (A.15), obtain that

$$\text{ang}([v^-, v], L_{\hat{Q}_p}), \text{ang}([v, v^+], L_{\hat{Q}_p}) \leq \frac{\beta_2(Q^p_{\max}, \hat{Q}_p)}{A_0} \tag{A.16}$$

and thus conclude equation (6.27) for the case discussed above.

The above estimates are similar if $v = v_1^l, v_N^l$ and $v \in \mathcal{L}_l(\gamma) \cap I_{p-1,p}$, $1 \leq p \leq m+1$. Indeed, assume, for example, that $v = v_1^l$. Combine Lemma 6.10, equation (6.25), the definition of the set $\mathcal{L}_l(\gamma)$, and the fact that v_1^l is a minimal point in $\mathcal{L}_l(\gamma)$ (with respect to the previously defined linear ordering) and conclude that

$$v_1^l \in \bigcup_{z \in [z_1, z_2]} Q(z, l_l) \,. \tag{A.17}$$

This equation and the fact that $v_0^l \equiv z_0$ is a phantom point with respect to z_1 and z_2 imply that

$$v_0^l \notin Q\big(v_1^l, (4 \cdot A_0 \cdot C_L - 1) \cdot l_l\big) \,. \tag{A.18}$$

Thus equation (A.15) is satisfied in this case. Moreover, equations (6.12), (6.25), and (6.26) imply that equation (A.14) is also true in this case. Consequently, equations (A.16) and (6.27) are valid for $v = v_1^l$.

We next consider the case where $v \notin \mathcal{L}_l(\gamma)$ and $v \in I_{p-1,p}$, $1 \leq p \leq m+1$. Let u be a point in $u \in \mathcal{L}_l$ such that $v = \tilde{u}$ (that is, v is a phantom point with respect to u and an additional point). Assume without loss of generality that $v^- = u$ and $v^+ = \tilde{w}$, where $w \in \mathcal{L}_l$. We remark that other special cases are proved similarly. In particular, if $v^+ = v_N^l$, then also use the same arguments as in equations (A.17) and (A.18).

It follows from analogous estimates to the ones above that

$$v^- \in I_{p-2,p-1} \cup I_{p-1,p}$$

and consequently

$$\text{ang}([v^-, v], L_{\hat{Q}_p}) \leq \frac{\beta_2(Q^p_{\max}, \hat{Q}_p)}{A_0} \,. \tag{A.19}$$

Note that

$$v^- \notin Q(v^+, 4 \cdot (A_1 - A_0) \cdot l_{l-1}) \,. \tag{A.20}$$

Equations (6.25) and (A.20) imply that

$$\text{ang}([v, v^+], L_{\hat{Q}_p}) \leq \frac{9}{4 \cdot (A_1 - A_0)} \cdot \beta_2\big(Q^p_{\max,} \hat{Q}_p\big) \,. \tag{A.21}$$

By combining equations (A.19) and (A.21), we conclude that equation (6.27) is satisfied in this last case.

A.3 Proof of (iii)

Assume that $v = v_i^l$, $1 \leq i \leq N$, is a type 1 point in $\mathcal{L}(\gamma)$. We first show that

$$Q(v, 4 \cdot A_1 \cdot l_l) \cap \mathcal{L}_l(\gamma) = \varnothing\,. \tag{A.22}$$

We suppose on the contrary that there exists a point w such that

$$w \in Q(v, 4 \cdot A_1 \cdot l_l) \cap \mathcal{L}_l(\gamma) \setminus \{v\}\,. \tag{A.23}$$

Assume without loss of generality that $v \neq v_N^l$ and $v \leq w$ according to the linear ordering defined on $\mathcal{L}_l(w)$. We will verify that if x is a point in $\mathcal{L}_l(\gamma)$ such that $v \leq x \leq w$, then $x \in Q(v, 4 \cdot A_1 \cdot l_l)$. This observation implies the following contradiction: $v \leq w \leq v^+$.

Recall that there exists a cube Q^w in $\mathcal{M}_l'$ such that w is its center of mass (see property 3 in Section 6.1). Properties 9, 10, and 11 in Section 6.1 imply that $Q^w \cap \gamma \neq \varnothing$.

Fix an integer p, $1 \leq p \leq m+1$, such that $v \in I_{p-1,p}$. It follows from the separation properties of $\{z_i\}_{i=1}^m$ (see equations (6.5) and (6.6)), the nonempty intersection of Q^v and Q^w with γ, and the relation $A_0 \cdot C_L \geq 4 \cdot A_1 + 2$ that $w \in I_{p-1,p} \cup I_{p,p+1}$.

Assume that x is a point in $\mathcal{L}_l(\gamma)$ such that $v \leq x \leq w$. Apply equation (6.25) to the points v, x, and w to obtain that

$$\frac{\operatorname{dist}(x, L_{\hat{Q}_p})}{l_l}, \frac{\operatorname{dist}(v, L_{\hat{Q}_p})}{l_l}, \frac{\operatorname{dist}(w, L_{\hat{Q}_p})}{l_l} \leq \sqrt{\delta}\,.$$

This fact together with Lemma 6.10 imply the equation

$$\frac{\operatorname{dist}(x, L_{i-1,i})}{l_l}, \frac{\operatorname{dist}(v, L_{i-1,i})}{l_l}, \frac{\operatorname{dist}(w, L_{i-1,i})}{l_l} \leq 10 \cdot \sqrt{\delta}\,, \qquad i = p, p+1, \tag{A.24}$$

where $L_{i-1,i}$ is the line containing the segment $[z_{i-1}, z_i]$.

It follows from equations (6.20), (6.21), (A.23), and (A.24) and from Lemma 6.10 that $x \in Q(v, 4 \cdot A_1 \cdot l_l)$. Consequently, we obtain the contradiction $w \leq v^+$ and conclude equation (A.22). Equation (6.28) then follows from equation (A.22), property 11 in Section 6.1, and the nonempty intersection of Q^v with γ.

A.4 Proof of (iv)

This property follows from equations (6.11) through (6.14), (6.25), and (6.26).

Appendix B: Proof of Equation (6.47)

We show that if $x \in \Gamma_l(\gamma) \cap J_{j-1,j}^p$ and $y \in \psi_{l-1}(\gamma) \cap J_{j-1,j}^p$, then x and y satisfy equation (6.46) and therefore the current property follows from the above property.

NOTATION AND DEFINITIONS B.1 Fix p, $1 \le p \le N_j$. Denote

$$x_p^+ := H_p^+ \cap L_{\hat{Q}_j} \quad \text{and} \quad x_p^- := H_p^- \cap L_{\hat{Q}_j}, \quad p = 1, \dots, N_j.$$

Let $\hat{H}_p^+$ and $\hat{H}_p^-$ be the hyperplanes orthogonal to $L_{\hat{Q}_j}$ and passing through the points x_p^+ and x_p^-, respectively. Denote by $\hat{J}_{j-1,j}^p$ the intersection of $\bigcup_{j=1}^{m+1} I_{j-1,j}$ with the closed region bounded between the hyperplanes $\hat{H}_p^-$ and $\hat{H}_p^+$. Let $z_{j,p}^-$ and $z_{j,p}^+$ be the two points in $\psi_{l-1}(\gamma)$ such that $[z_{j,p}^-, z_{j,p}^+] = \psi_{l-1}(\gamma) \cap J_{j-1,j}^p$. Define

$$\hat{z}_{j,p}^- := \hat{H}_p^- \cap \psi_{l-1}(\gamma) \quad \text{and} \quad \hat{z}_{j,p}^+ = \hat{H}_p^+ \cap \psi_{l-1}(\gamma).$$

We first establish the following estimate: If x is any point in $\psi_{l-1}(\gamma) \cap J_{j-1,j}^p$, then

$$\text{dist}\left(x, [x_p^-, x_p^+]\right) \le 18 \cdot \sqrt{\delta} \cdot C_L \cdot l_l. \tag{B.1}$$

Equations (6.2) and (6.11) through (6.13) imply that if $x \in [\hat{z}_{j,p}^-, \hat{z}_{j,p}^+]$, then

$$\text{dist}\left(x, [x_p^-, x_p^+]\right) \le 9 \cdot \sqrt{\delta} \cdot C_L \cdot l_l. \tag{B.2}$$

In particular, equation (B.1) is satisfied for any $x \in [\hat{z}_{j,p}^-, \hat{z}_{j,p}^+]$.

Assuming that $[z_{j,p}^-, \hat{z}_{j,p}^+] \supseteq [\hat{z}_{j,p}^-, \hat{z}_{j,p}^+]$, we show that x satisfies equation (B.1) if x is a point in $[z_{j,p}^-, \hat{z}_{j,p}^-]$. The same result can be proven under the assumptions $[\hat{z}_{j,p}^-, z_{j,p}^+] \supseteq [\hat{z}_{j,p}^-, \hat{z}_{j,p}^+]$ and x is a point in $[\hat{z}_{j,p}^+, z_{j,p}]$. Note that

$$\cos\left(\text{ang}([\hat{z}_{j,p}^-, z_{j,p}^-], [z_{j,p}^-, x_p^-])\right) \le \cos\left(\text{ang}([\hat{z}_{j,p}^-, z_{j,p}^-], H_p^-)\right). \tag{B.3}$$

It follows from the above estimate together with the law of sines that

$$\text{dist}\left(z_{j,p}^-, x_p^-\right) \le \frac{\text{dist}(\hat{z}_{j,p}^-, x_p^-)}{\sin\left(\text{ang}([\hat{z}_{j,p}^-, z_{j,p}^-], H_p^-)\right)}.$$

Combine the above equation with equation (6.41) to conclude that

$$\text{dist}\left(z_{j,p}^-, x_p^-\right) \le 2 \cdot \text{dist}\left(\hat{z}_{j,p}^-, x_p^-\right). \tag{B.4}$$

Similarly, obtain the inequality

$$\text{dist}\left(z_{j,p}^+, x_p^+\right) \le 2 \cdot \text{dist}\left(\hat{z}_{j,p}^+, x_p^+\right). \tag{B.5}$$

This equation together with equation (B.2) imply that if $x \in [z_{j,p}^-, \hat{z}_{j,p}^+]$, then

$$\text{dist}\left(x, x_p^-\right) \le 18 \cdot \sqrt{\delta} \cdot C_L \cdot l_l.$$

Likewise, if $x \in [\hat{z}_{j,p}^+, z_{j,p}^+]$, then

$$\text{dist}\left(x, x_p^+\right) \le 18 \cdot \sqrt{\delta} \cdot C_L \cdot l_l.$$

The above two estimates together with equation (B.2) imply equation (B.1).

By applying Lemma 6.12, one can obtain similarly as above that if $x \in \Gamma_l(\gamma) \cap J^p_{j-1,j}$, then

$$\operatorname{dist}\left(x, [x_p^-, x_p^+]\right) \leq 18 \cdot \sqrt{\delta} \cdot l_l\,. \tag{B.6}$$

We next estimate the length of $[x_p^-, x_p^+]$. Equations (B.2), (B.4), and (B.5) imply that

$$\operatorname{dist}\left(z^-_{j,p}, \hat{z}^-_{j,p}\right), \operatorname{dist}\left(z^+_{j,p}, \hat{z}^+_{j,p}\right) \leq 9 \cdot \sqrt{\delta} \cdot C_L \cdot l_l\,.$$

Combine the above equation with equation (6.40) to obtain that

$$\operatorname{dist}\left(x_p^-, x_p^+\right) \leq \frac{l_l}{2} + 18 \cdot \sqrt{\delta} \cdot C_L \cdot l_l\,. \tag{B.7}$$

Equations (B.1), (B.6), and (B.7) and their derivation imply that if $x \in \Gamma_l(\gamma) \cap J^p_{j-1,j}$ and $y \in \psi_{l-1}(\gamma)$, then

$$\operatorname{dist}(x, y) \leq \left(\frac{1}{2} + 18 \cdot (C_L + 1) \cdot \sqrt{\delta}\right) \cdot l_l\,.$$

Note that δ is sufficiently small so that x and y satisfy equation (6.46). Consequently, equation (6.47) is proven.

Appendix C: Proof of Equation (6.54)

Denote

$$\hat{v}^-_{j,p} = \Gamma_l(\gamma) \cap \hat{H}_p^-\,, \quad \hat{v}^+_{j,p} = \Gamma_l(\gamma) \cap \hat{H}_p^+\,,$$
$$v^-_{j,p} = \Gamma_l(\gamma) \cap H_p^-\,, \quad v^+_{j,p} = \Gamma_l(\gamma) \cap H_p^+\,.$$

We verify the estimate

$$l\left([v^-_{j,p}, \hat{v}^-_{j,p}]\right) \leq \frac{126}{A_0} \cdot \beta_2^2\left(Q^2_{\max}, \hat{Q}_j\right) \cdot l_l\,. \tag{C.1}$$

A similar estimate holds for $l([\hat{v}^+_{j,p}, v^+_{j,p}])$. Equation (6.54) follows from these two estimates.

Observe the triangle formed by the vertices $v^-_{j,p}$, $\hat{v}^-_{j,p}$, and x_p^-. Denote by $\theta_{v^-_{j,p}}$, $\theta_{\hat{v}^-_{j,p}}$, and $\theta_{x_p^-}$ the angles at the vertices $v^-_{j,p}$, $\hat{v}^-_{j,p}$, and x_p^-, respectively. Note that

$$l\left([v^-_{j,p}, \hat{v}^-_{j,p}]\right) = \frac{l([\hat{v}^-_{j,p}, x_p^-])}{\sin \theta_{v^-_{j,p}}} \cdot \sin \theta_{x_p^-}\,. \tag{C.2}$$

We proceed by estimating $\cos \theta_{v^-_{j,p}}$ and $\sin \theta_{x_p^-}$. Recall the following notation, which was introduced in Appendix B: $z^-_{j,p}$ and $z^+_{j,p}$ are two points in $\Psi_{l-1}(\gamma)$ such that $[z^-_{j,p}, z^+_{j,p}] = \psi_{l-1}(\gamma) \cap J^p_{j-1,j}$ and $\hat{z}^-_{j,p}$ and $\hat{z}^+_{j,p}$ are defined by the formulae $\hat{z}^-_{j,p} := \hat{H}_p^- \cap \Psi_{l-1}(\gamma)$ and $\hat{z}^+_{j,p} := \hat{H}_p^+ \cap \Psi_{l-1}(\gamma)$. Note that

$$|\cos \theta_{v^-_{j,p}}| \leq \left|\cos\left(\operatorname{ang}([z^-_{j,p}, \hat{z}^-_{j,p}], [z^-_{j,p}, x_p^-])\right)\right| + \left|\sin\left(\operatorname{ang}([z^-_{j,p}, \hat{z}^-_{j,p}], [v^-_{j,p}])\right)\right|\,.$$

Apply equations (6.19), (A.16), (A.19), (6.41), and (B.3) to the above equation and obtain the estimate

$$|\cos\theta_{v^-_{j,p}}| \le \frac{6}{A_0}\cdot\beta_2\big(Q^j_{\max},\hat{Q}_j\big)\,. \tag{C.3}$$

Before estimating $\sin\theta_{x^-_p}$, we obtain a bound for $\cos\theta_{\hat{v}^-_{j,p}}$. Note that $\frac{\pi}{2}-\theta_{\hat{v}^-_{j,p}}$ is the angle between $[v^-_{j,p},\,\hat{v}^-_{j,p}]$, and $L_{\hat{Q}_j}$. Combine equations (A.16) and (A.19) to obtain the estimate

$$|\cos\theta_{\hat{v}^-_{j,p}}| \le \frac{\beta_2(Q^2_{\max},\hat{Q}_j)}{A_0}\,. \tag{C.4}$$

We evaluate $\sin\theta_{x^-_p}$ as follows: Note that

$$|\sin\theta_{x^-_p}| \le |\sin(\theta_{v^-_{j,p}}+\theta_{\hat{v}^-_{j,p}})| \le |\cos\theta_{v^-_{j,p}}| + |\cos\theta_{\hat{v}^-_{j,p}}|\,.$$

Apply equations (C.3) and (C.4) to the above equation to obtain that

$$|\sin\theta_{x^-_p}| \le \frac{7}{A_0}\cdot\beta_2\big(Q^j_{\max},\hat{Q}_j\big)\,. \tag{C.5}$$

Equations (C.2), (C.3), and (C.5) imply the estimate

$$l\big([v^-_{j,p},\hat{v}^-_{j,p}]\big) \le \frac{14}{A_0}\cdot\beta_2\big(Q^j_{\max},\hat{Q}_j\big)\cdot l\big([\hat{v}^-_{j,p},x^-_p]\big)\,. \tag{C.6}$$

Note that $l[v^-_{j,p},x^-_p] = l(\mathrm{dist}(v^-_{j,p},L_{\hat{Q}_j}))$. By combining this observation with equations (6.25), (6.26), and (C.6), we conclude equation (C.1) and consequently equation (6.54).

Appendix D: Proof of Equation (6.55)

Note that

$$l\big(\hat{J}^p_{j-1,j}\cap L_{\hat{Q}_j}\big) \le l\big(\hat{J}^p_{j-1,j}\cap\Psi_{l-1}(\gamma)\big)\,.$$

Thus, in order to verify equation (6.55), it is sufficient to show that

$$l\big([\hat{z}^-_{j,p},z^-_{j,p}]\big),\, l\big([z^+_{j,p},\hat{z}^+_{j,p}]\big) \le \frac{54\cdot C_L}{A_0}\cdot\beta_2\big(Q^j_{\max},\hat{Q}_j\big)\cdot l_l\,. \tag{D.1}$$

We show how to control the length of the segment $[\hat{z}^-_{j,p},z^-_{j,p}]$. The length of the segment $[\hat{z}^+_{j,p},z^+_{j,p}]$ is estimated similarly.

Observe the triangle with vertices x^-_p, $\hat{z}_{j,p}$, and $z^-_{j,p}$. Denote by $\theta_{x^-_p}$, $\theta_{z_{j,p}}$, and $\theta_{\hat{z}^-_{j,p}}$ the angles at the vertices x^-_p, $z^-_{j,p}$, and $\hat{z}^-_{j,p}$, respectively. Note that

$$l\big([\hat{z}^-_{j,p},z^-_{j,p}]\big) = \frac{l([\hat{z}^-_{j,p},x^-_p])\cdot\sin\theta_{x^-_p}}{\sin\theta_{z^-_{j,p}}} \quad\text{and}\quad \sin\theta_{x^-_p} \le \cos\theta_{z^-_{j,p}} + \cos\theta_{\hat{z}_{j,p}}\,.$$

It follows from equations (6.41) and (B.3) that

$$\cos\theta_{z^-_{j,p}} \le \frac{3}{A_0}\cdot\beta_2\big(Q^j_{\max},\hat{Q}_j\big)\,.$$

Also observe that

$$\cos\theta_{\hat{z}^-_{j,p}} = \sin\left(\operatorname{ang}(L_{\hat{Q}_j}, [\hat{z}^-_{j,p}, z^-_{j,p}])\right).$$

Combine this equation with equation (6.19) and obtain that

$$\cos\theta_{\hat{z}_{j,p}} \leq \frac{2}{A_0} \cdot \beta_2\left(Q^j_{\max}, \hat{Q}_j\right).$$

The above equations imply the following estimate:

$$l\left([\hat{z}^-_{j,p}, z^-_{j,p}]\right) \leq \frac{6}{A_0} \cdot \beta_2\left(Q^j_{\max}, \hat{Q}_j\right) \cdot l\left([\hat{z}^-_{j,p}, x^-_p]\right).$$

A further application of equations (6.11) through (6.13) results in equation (D.1) for the interval $[\hat{z}^-_{j,p}, z^-_{j,p}]$. The same estimate for the other interval is proven similarly. Equation (6.55) then follows from equation (D.1).

Appendix E: Construction and Length Estimate of the Curve $\gamma(Q_x)$

Let Q_x be a stopping-time cube in $\mathcal{S}^j_N$ and assume that the set $\Gamma^j_N \cup \Gamma_{k_x}(Q_x)$ is not connected. We construct here a curve $\gamma(Q_x)$ whose length satisfies equation (6.98) and such that $\Gamma^j_N \cup \Gamma_{k_x}(Q_x) \cup \gamma(Q_x)$ is connected. This curve appeared first in Section 6.5. We follow here the notation and definitions of that subsection.

Define the points $x_{-1}, x_0, \ldots, x_{k_x}$ recursively by the equations

$$\begin{aligned} x_{-1} &:= x\,, \\ x_i &:= \arg\min_{y \in \Gamma_i(Q_x)} \operatorname{dist}(y, x_{i-1})\,, \quad 0 \leq i \leq k_x. \end{aligned}$$

Form the curve $\gamma(Q_x)$ as follows:

$$\gamma(Q_x) := \bigcup_{i=0}^{k_x} [x_{i-1}, x_i].$$

The above definitions imply that the set $\Gamma^j_N \cup \Gamma_{k_x}(Q_x) \cup \gamma(Q_x)$ is connected. At last we show that the curve $\gamma(Q_x)$ satisfies the length estimate of equation (6.98).

Recall that x is the center of mass of a cube Q^* whose side length is comparable to that of Q_x (see equation (6.30)). Let $\hat{Q}_x$ be the cube with side length $2^{j_0^*} \cdot l(Q_x)$ that was fixed in the first step of the zeroth-level basic construction for Q_x (see the paragraph before equation (6.3)). Note that $12 \cdot (A_0 + 1) \geq 2^{j_0^*} \leq 2^{j_1^*}$. This inequality together with equation (6.30) and Definition 4.4 imply that

$$\hat{Q}_x \in \mathcal{P}(Q^*). \tag{E.1}$$

Apply Lemma 6.4 to the cubes Q^* and $\hat{Q}_x$. Combine the resulting estimate with equations (6.2) and (E.1) and obtain that

$$l([x_{-1}, x_0]) \leq \sqrt{\delta} \cdot l(Q).$$

Moreover, equation (6.29) implies that

$$l([x_{i-1}, x_i]) \le 9 \cdot \sqrt{\delta} \cdot (1 + C_L) \cdot \frac{l(Q_x)}{C_L^i}, \quad 2 \le i \le k_x.$$

Equation (6.98) follows from the above two equations.

Appendix F: Proof of Lemma 6.25

The lemma is proven by induction on the index j. The case $j = 0$ follows from equations (6.31) and (6.32). We assume that $j > 0$ and that the lemma is true for $j := j - 1$. Let Q_{px} be a cube in $\mathcal{S}_N^{j-1}$ such that Q_x is a stopping-time cube in $\mathcal{S}_{k_{px}}(Q_{px})$ and let Q_{py} be a cube in $\mathcal{S}_N^{k-1}$ such that Q_y is a stopping-time cube in $\mathcal{S}_{k_{py}}(Q_{py})$. If $Q_{py} = Q_{px}$, then equation (6.112) is proven in a similar way as in the case $j = 0$. If $Q_{py} \ne Q_{px}$, then equation (6.112) is a consequence of the following two observations: First, it follows from the induction assumption that $2Q_{px} \cap 2Q_{py} = \varnothing$. Second, the choice of the constants C_L and A_0 imply that $2Q_x \subseteq 2Q_{px}$ and $2Q_y \subseteq 2Q_{py}$ (see also equation (6.30) and note that $Q_x \cap Q_{px} \ne \varnothing$ and $Q_y \cap Q_{py} \ne \varnothing$).

Appendix G: Proof of Lemma 6.26

The lemma is proven by induction on the index j. If $j = 0$, then equation (6.113) follows from equations (6.31), (6.32), and (6.97). Assume that the lemma is true for $j := j - 1$. Let Q_{px} be a cube in $\mathcal{S}_N^{j-1}$ such that Q_x is a stopping-time cube in $\mathcal{S}_{k_{px}}(Q_{px})$. It follows from the induction assumption that $2Q_{px} \cap E_N'^{j-1} = \varnothing$ and consequently $2Q_x \cap E_N'^{j-1} = \varnothing$ (note that the choice of the constants C_L and A_0 imply that $2Q_x \subseteq 2Q_{px}$). Thus in order to conclude equation (6.113), it is sufficient to verify that if Q_{py} is any cube in $\mathcal{S}_N^{j-1} \setminus \tilde{\mathcal{S}}_N^j$, then

$$2Q_x \cap E'_{k_y}(Q_{py}) = \varnothing.$$

If $Q_{px} = Q_{py}$, then this equation follows from equations (6.31) and (6.32). If $Q_{px} \ne Q_{py}$, then note that the choice of the constants C_L and A_0 imply that $E'_{k_y}(Q_y) \subseteq 2Q_y$. Combine this fact together with Lemma 6.25 to conclude the above equation.

Appendix H: Proof of Lemma 7.14

PROOF: Assume first that $Q_1 \in \tilde{\mathcal{D}}^{SC}$. Recall that the curve $\tilde{\Gamma}^\delta_{k_{Q_1}}(Q_1)$ satisfies the length estimate of equation (7.21). Thus we only need to estimate the length of $\gamma^\delta(Q_1)$.

Note that if $Q_2 \in \tilde{\mathcal{R}}^\delta_{k_{Q_1}}(Q_1)$, then there exists a line segment $I(Q_2)$ such that the set $I(Q_2) \cup \tilde{\Gamma}^\delta_{k_{Q_2}}(Q_2) \cup \tilde{\Gamma}^\delta_{k_{Q_1}}(Q_1)$ is connected and such that

$$l(I(Q_2)) \le \sqrt{n} \cdot l(Q_2).$$

Let $\gamma'^{\delta}(Q_1)$ be the curve

$$\gamma'^{\delta}(Q_1) := \bigcup_{Q_2 \in \tilde{\mathcal{R}}_{k_{Q_1}}(Q_1)} I(Q_2)\,.$$

It follows from the definition of the curve $\gamma^{\delta}(Q_1)$ that

$$l\big(\gamma^{\delta}(Q_1)\big) \leq l(\gamma'^{\delta}(Q_1))\,.$$

Combine the above three equations with equation (7.19) to obtain the following length estimate for $\gamma^{\delta}(Q_1)$:

$$l\big(\gamma^{\delta}(Q_1)\big) \leq \sqrt{n} \cdot C_6(n) \cdot l(Q_1)\,. \tag{H.1}$$

The above equation together with equation (7.21) results in the length estimate

$$l\big(\gamma^{\delta}(Q_1) \cup \tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)\big) \leq (e \cdot C_2(n) + \sqrt{n} \cdot C_6(n)) \cdot l(Q_1)\,.$$

Finally, assume that $Q_1 \in \widetilde{\mathcal{D}}^{L\beta}$. The length estimate of $\gamma^{\delta}(Q_1)$ stated in equation (H.1) can also be used in this case. We estimate the length of $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ as follows. Recall that $\mathcal{V} = \{z_{Q_1}\} \cup \{z_Q : Q \in \mathcal{R}^{\delta}_{k_{Q_1}}(Q_1)\}$ and $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ is a rectifiable curve of shortest length containing $\mathcal{V}$. We form a curve Γ''_{Q_1} that contains the set $\mathcal{V}$ and whose length estimate is straightforward.

Let Γ'_{Q_1} be the shortest curve containing all centers of cubes in subs(Q_1). Note that

$$l\big(\Gamma'_{Q_1}\big) \leq (16)^{n-1} \cdot l(Q_1)\,.$$

Obtain the curve Γ''_{Q_1} from Γ'_{Q_1} by connecting with line segments each point in the set $\mathcal{V}$ to the center of the cube in subs(Q_1) containing it. Note that there are at most $16^n + 1$ points in $\mathcal{V}$ and that the length of each connecting line segment is at most $\frac{\sqrt{n}}{2} \cdot \frac{1}{16} \cdot l(Q_1)$. This observation implies the following length estimate for Γ''_{Q_1}:

$$l\big(\Gamma''_{Q_1}\big) \leq l\big(\Gamma'_{Q_1}\big) + \sqrt{n} \cdot (16)^{n-1} \cdot l(Q_1)\,.$$

By definition, the length of the curve $\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)$ is shorter than that of Γ''_{Q_1}. Therefore the above equations result in the following length estimate:

$$l\big(\tilde{\Gamma}^{\delta}_{k_{Q_1}}(Q_1)\big) \leq (\sqrt{n} + 1) \cdot 16^{n-1} \cdot l(Q_1)\,.$$

The above estimates for the two different cases imply equation (7.22). □

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