Group Testing:
How to find out what's important in life

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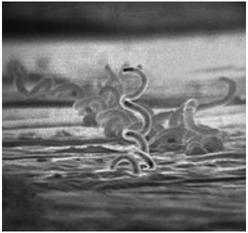
Institute for Mathematics and Its Applications

April 12, 2010

Outline

Group Testing - Overview

- Find a small number of interesting items hidden in a large set.
  - Syphilis Testing [Dorfman 1943]



  - Industrial Experiment Design [Sobel and Groll, 1959]
- Test Requirements...
  - Test large arbitrary groups of objects
  - Tests must be sensitive to defectives isolated with (many) other non-defective items

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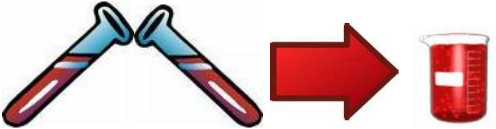

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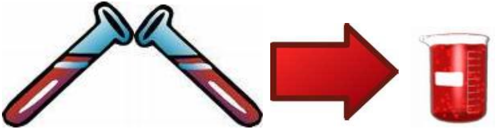

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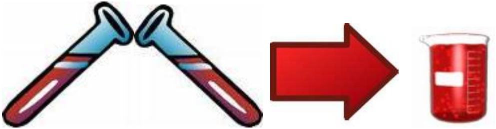

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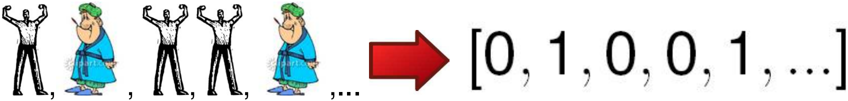
- Encode the problem in a binary array



- Find the nonzero entries by testing subsets of the array
  - Boolean $K \times N$ measurement matrix $\mathcal{M}$
  - Boolean signal $\mathbf{a} \in \{0, 1\}^N$ containing k ones
  - All arithmetic Boolean (+ = OR, * = AND)
  - Identify the location of k ones using $y = \mathcal{M}\mathbf{a}$ measurements
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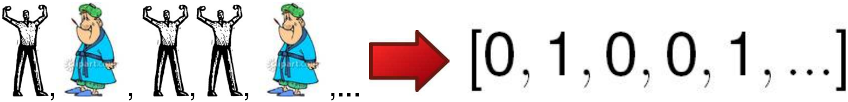

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$[0, 1, 0, 0, 1, \ldots]$

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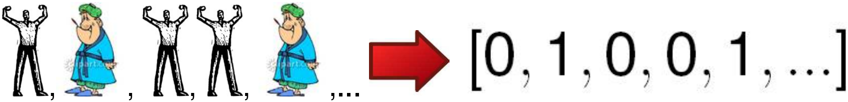

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Group Testing - Example

- $\mathcal{M}$ is 5×30, **a** contains 1 nonzero entry.

$$\begin{array}{l} 0^{\text{th}} \text{ bit} \\ 1^{\text{st}} \text{ bit} \\ 2^{\text{nd}} \text{ bit} \\ 3^{\text{rd}} \text{ bit} \\ 4^{\text{th}} \text{ bit} \end{array} \quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

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- Recovery is simple: The result is the position of 1 in binary.
- QUIZ: Can we do better if we let our measurement matrix contains arbitrarily large integers?

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$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = 3$$

- Recovery is simple: The result is the position of 1 in binary.
- QUIZ: Can we do better if we let our measurement matrix contains arbitrarily large integers?
- YES!!!

Group Testing - More Than One Sick Person

Measurement Matrix Construction

A binary matrix $\mathcal{M}$ is k-strongly selective if for any column, x, and subset of columns containing at most k elements, X, there exists a row in $\mathcal{M}$ with a 1 in column x and zeros in all of the other $X - \{x\}$ columns.

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0	1	1	0	0	1	0
0	1	0	1	0	1	0
0	0	1	0	0	1	0
1	0	0	1	0	0	1
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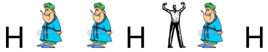

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- If there are at most k sick individuals, we will find them all!

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Theorem 1

Let $\mathbf{a} \in \{0,1\}^N$ be a binary vector containing k nonzero entries. Furthermore, let $\mathcal{M}$ be a k-strongly selective binary matrix. Then, the positions of all k nonzero entries in $\mathbf{a}$ can be recovered using only the result of $\mathcal{M}\mathbf{a}$.

Theorem 2

There exist explicitly constructible $\left(\min\{k^2 \cdot \log N, N\}\right) \times N$ k-strongly selective binary matrices. And, they are optimal in the number of rows.[a]

[a] See Porat and Rothschild's paper "Explicit Non-Adaptive Combinatorial Group Testing Schemes".

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Group Testing - Another example

Error Detection

Suppose we want to transmit a binary vector $\mathbf{a} \in \{0, 1\}^N$ through a noisy environment. How can we tell if we received the real message?

- Used for DVD, CD, and other media devices in your house!
- Basic Methods: Parity and Checksums
- A bit stronger: Use a strongly selective matrix!
  - Transmit (or read) both $\mathbf{a}$ and $\mathcal{M}\mathbf{a}$
  - The receiver gets (or reads) $\mathbf{a}' = \mathbf{a} + \epsilon$
  - Check to see if $\mathcal{M}\mathbf{a} = \mathcal{M}\mathbf{a}'$

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Group Testing - Adaptive Methods Let Us Do Better!

- We have seen strongly selective matrices with $O(k^2 \log N)$ rows allow us to find all k nonzero entries in $\mathbf{a} \in \{0,1\}^N$ when we are allowed to compute ONE round of sampling.
- What if we can adaptively sample $\mathbf{a} \in \{0,1\}^N$ several times?
- ANSWER: We can use at most $\log(N)$ matrices with at most $2k+1$ rows each! The total number of inner products is now only $O(k \log N)$!

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Group Testing - A Question

Suppose $\mathbf{a} \in \mathbf{R}^N$ contains k nonzero entries all of which are positive. Can we still identify all k locations of the nonzero entries using the group testing methods we have seen?

YES!

HOMEWORK: Write out the step of the adaptive group testing example just discussed in class to find the two nonzero entries in the vector

$$(0, 0, 0, 0, 0, .4, 0, 23, 0, 0, 0, 0, 0, 0, 0, 0).$$

How many inner products does it take? How many inner products would it have taken to use a strongly selective binary matrix (see Theorem 2)? Which method requires fewer tests?

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Compressed Sensing - Overview

- $K \times N$ measurement matrix $\mathcal{M}$ with complex entries
- An $N \times N$ complex matrix Ψ which induces sparsity in signals of interest.
- Signal $\mathbf{a} \in \mathbf{C}^N$ which is sparse under Ψ (i.e., $\Psi\mathbf{a}$ contains k nonzero entries).
- Identify the location of k ones using $y = \mathcal{M}\Psi\mathbf{a}$ measurements
- How small can we make K and still recover $\Psi\mathbf{a}$ – and therefore $\mathbf{a}$ – using only y?

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- How small can we make K and still recover $\Psi\mathbf{a}$ – and therefore $\mathbf{a}$ – using only y?

Bit Testing - Group Testing for Compressed Sensing

- $\mathcal{B}$ is 4×6, **a** contains 1 nonzero entry.

$$\begin{matrix} \text{All ones} \\ 0^{\text{th}} \text{ bit} \\ 1^{\text{st}} \text{ bit} \\ 2^{\text{nd}} \text{ bit} \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -3.5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Sparsity pattern gives nonzero location in binary.
- We get the entry value at least once.
- $\mathcal{B}$ is called a bit testing matrix.

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Compressed Sensing - More Than One Nonzero Value

- Suppose $\mathbf{a} \in \mathbf{C}^N$ contains k nonzero values.
- We will say that binary vector $\mathbf{m} \in \{0,1\}^N$ is a good mask for $\mathbf{a}$ if the component-wise multiple of $\mathbf{m}$ and $\mathbf{a}$ contains only one isolated nonzero value.
- Anytime this happens we can identify the isolated value!
- Their will be at least one row in $\mathcal{M}$ that contains each nonzero value in $\mathbf{a}$ isolated from all the others.
- Thus, $\mathcal{M}$ will contain a good mask for every nonzero value!

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Strongly Selective Matrices

A binary matrix $\mathcal{M}$ is k-strongly selective if for any column, x, and subset of columns containing at most k elements, X, there exists a row in $\mathcal{M}$ with a 1 in column x and zeros in all of the other $X - \{x\}$ columns.

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Theorem 3

Let $\mathbf{a} \in \mathbf{C}^N$ be a complex valued vector containing k nonzero entries. Furthermore, let $\mathcal{M}$ be a k-strongly selective binary matrix and $\mathcal{B}$ be a bit testing matrix. Then, all k nonzero entries in $\mathbf{a}$ can be recovered using only the result of $(\mathcal{M} \circledast \mathcal{B})\,\mathbf{a}$.

Theorem 4

We can explicitly construct binary measurement matrices, $\mathcal{M} \circledast \mathcal{B}$, of size $\left(\min\left\{k^2 \cdot \left(\log_2^2 N + \log_2 N\right), N\right\}\right) \times N$.

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