Practical Example of Convolution and Correlation

We have defined the convolution of two functions by

$$f * g = \int_{-\infty}^{\infty} f(x - y) \cdot g(y) \, dy \, ,$$

and their correlation by

$$f \diamond g = \int_{-\infty}^{\infty} f(x + y) \cdot g(y) \, dy \, .$$

Similarly for two vectors in $\ell_1(\mathbb{Z})$:

$$(v * w)_j = \sum v_{j-i} \cdot w_i \, ,$$

and

$$(v \diamond w)_j = \sum v_{j+i} \cdot w_i \, .$$

The purpose of these notes is to explain what we see in Figure 3.13 of Digital Image Processing using Matlab of Gonzalez, Woods and Eddins (see material from Feb. 9th in http://www.math.umn.edu/∼lerman/math5467/supp.shtml).

We are given two finite vectors $f = [0, 0, 0, 1, 0, 0, 0, 0]$ and $w = [1, 2, 3, 2, 0]$. We may think of w and f as vectors in $\ell_1(\mathbb{Z})$, where we are only given the coordinates at indices $0, 1, \ldots, 4$ for w and $0, 1, \ldots, 7$ for f, i.e. $w(0) = 1$, $w(1) = 2$, $\ldots w(5) = 0$, $\ldots, f(3) = 1$, etc. We padded f with zeros so

$$f := [0, 0, 0, 0, f, 0, 0, 0, 0] = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0].$$

Note that the previous indices of f have not changed, i.e. $f(3) = 1$.

On the left hand side of the figure we correlate f with w. We can think of the figure shown there in two ways. The first one is to shift f to the left (i.e. we look at $f(i+j)$ for $j >= 0$) and take dot products with w for each shift. That is, we compute the correlation $f \diamond w$. In part (c) there is no shift (j=0), the result of the dot product is zero which is written in the output (i.e. in $g(0)$), in (d) you see a shift by 1 ($j = 1$), the result is also 0, in (e) $j = 4$ the result of the dot product is 2 and in (f) $j = 11$, the result is 0. The other way to think of it is by shifting w to the right (i.e. we look at $w(i - j)$ for $j >= 0$) and taking dot products with f each time, so we actually convolve $\tilde{w}$ (so now instead of $w(i - j)$ we have $w(j - i)$)with f. This can also be done in Matlab by typing *conv(fliplr(w),f)*. Note that the equivalence of the two methods is reflected by the identity

$$f * \tilde{g} = f \diamond g \, .$$

In the right hand part of the figure we perform convolution via correlation. We recall that

$$f * g = f \diamond \tilde{g} \, . \tag{1}$$

We thus first take the mirror image of the vector w, i.e. we apply $\tilde{w}$ (to have it a mirror image we may need to index all vectors differently than before, but it causes no problem). We then correlate f with $\tilde{w}$ ($f \diamond \tilde{w}$) and get the vector 01232000. Equation 1 implies that this result is also the required convolution.