The Fourier Transform and Its Properties

If $f \in L_1(\mathbb{R})$, where $f : \mathbb{R} \to \mathbb{C}$, we defined its Fourier transform as follows

$$F(f) \equiv \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \,.$$

It is possible to extend it to other spaces of functions (different than $L_1(\mathbb{R})$).

If $f \in L_1(\mathbb{R})$ we denote

$$F^{-1}(f) \equiv \check{f}(\xi) = \hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \xi} dx \,.$$

We will show that if both $f \in L_1(\mathbb{R})$ and $\hat{f} \in L_1(\mathbb{R})$, then

$$\check{\hat{f}} = \hat{\check{f}} = f \,.$$

We recall the following definitions of convolution and correlation of functions f, $g : \mathbb{R} \to \mathbb{C}$ in $L_1(\mathbb{R})$:

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(x-y) g(y) dy \\ f \diamond g(x) &= \int_{-\infty}^{\infty} f(x+y) \overline{g(y)} dy \end{aligned}$$

Here are some basic properties of the Fourier transform:

Concept	**Change in f**	**Corresponding FT**
Linearity	$\alpha_1 f_1 + \alpha_2 f_2$	$\alpha_1 \hat{f}_1 + \alpha_2 \hat{f}_2$
Shift	$f(x - x_0)$	$e^{-2\pi i \xi x_0} \hat{f}(\xi)$
modulation	$f(x)\, e^{2\pi i \xi_o x}$	$\hat{f}(\xi - \xi_0)$
conjugation	$\overline{f(x)}$	$\overline{\hat{f}(-\xi)}$
reflection	$f(-x)$	$\hat{f}(-\xi)$
reflection + conjugation	$\overline{f(-x)}$	$\overline{\hat{f}(\xi)}$
convolution	$f * g$	$\hat{f}\, \hat{g}$
multiplication	$f\, g$	$\hat{f} * \hat{g}$
correlation	$f \diamond g$	$\hat{f}\, \bar{\hat{g}}$
multiplication with conjugate	$f\, \bar{g}$	$\hat{f} \diamond \hat{g}$
scaling with a $\in \mathbb{R}$	$f(ax)$	$\frac{1}{\lvert a \rvert} \hat{f}\left(\frac{\xi}{a}\right)$
Fourier of Fourier	$\hat{f}(x)$	$f(-\xi)$
derivative	$f'(x)$	$(2\pi i \xi) \hat{f}(\xi)$
n-th derivative	$f^{(n)}(x)$	$(2\pi i \xi)^n \hat{f}(\xi)$
multiplication by x	$x f(x)$	$\frac{i}{2\pi} \frac{d\hat{f}(\xi)}{d\xi}$
multiplication by x^n	$x^n f(x)$	$\left(\frac{i}{2\pi}\right)^n \frac{d^n \hat{f}(\xi)}{d\xi^n}$

Concept	**Property of f**	**Corresponding FT**
Fourier of real	f is real	$\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$ real part even, imaginary part odd
Fourier of imaginary	f is imaginary	$\hat{f}(-\xi) = -\overline{\hat{f}(\xi)}$ real part odd, imaginary part even
Even	f is even	$\hat{f}$ is even ($\hat{f}(\xi) = \hat{f}(-\xi)$)
Odd	f is odd	$\hat{f}$ is odd ($\hat{f}(\xi) = -\hat{f}(-\xi)$)
Real + even	f is real and even	$\hat{f}$ is real and even
Real + odd	f is real and odd	$\hat{f}$ is imaginary and odd
Imaginary + odd	f is imaginary and odd	$\hat{f}$ is real and odd
Imaginary + even	f is imaginary and even	$\hat{f}$ is imaginary and even

If $f \in L_2(\mathbb{R})$ we can write

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x) e^{-2\pi i x \xi} dx.$$

We have Parseval's Theorem:

$$\int f(x)\bar{g}(x)dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

and in particular

$$\|f\|_2 = \int |f(x)|^2 dx = \int |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_2 .$$

Also,

$$\int f(x)\hat{g}(x)dx = \int \hat{f}(x)g(x)dx.$$

We defined the class of Schwartz functions as follows. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be rapidly decreasing if for every integer N, there exists a constant $C(N)$ such that

$$|f(x)| \leq \frac{C(N)}{|x|^N} \text{ for all } x \in \mathbb{R}.$$

The Schwartz class $\mathcal{S}$ is the set of all functions $f \in C^\infty(\mathbb{R})$ such that f and all of its derivatives are rapidly decreasing. The Fourier transform is an invertible mapping from $\mathcal{S}$ onto $\mathcal{S}$ and an isometry in the L_2 norm on $\mathcal{S}$.

Heisenberg's Uncertainty Principle: If $f \in L_2(\mathbb{R})$ and $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$, then for every $x_0, \xi_0 \in \mathbb{R}$:

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} (\xi - \xi_0)^2 |\hat{f(\xi)}|^2 d\xi \right) \geq \frac{1}{16\pi^2} ,$$

and equality holds if and only if $f(x) = \sqrt{\frac{2B}{\pi}}\, e^{-Bx^2}$ for any $B > 0$.