Practice Problems 3

Math 5467

1. Spatial Filters in Practice

Choose your favorite image and use the Matlab commands *fspecial*, *imfilter* and *imshow* to create the following enhancements:

a) Apply 3×3, 5×5, 10×10, and 20×20 averaging to the image.

b) Apply Gaussian filters, with various mask sizes and choices of σ.

c) Apply 3×3 Laplacian filters, with various choices of the parameter α.

d) Apply Laplacian of Gaussian filters with various mask sizes.

e) Apply the Sobel filter for differentiating both in the x and y direction. Then apply filters corresponding to both l_1 and l_2 norms of the gradient.

f) Choose one filter from the ones you used in part c) and one from part d). For each one of those filters apply a high-boost filter with two choices of the parameter k (note that $k = 1$ coincides with the filter *unsharp*).

2. Properties of the Fourier Transform

Assume that f is a function in $L_1(\mathbb{R})$ (or in the Schwartz space $\mathcal{S}(\mathbb{R})$ if you prefer). Establish the following properties.

a) $f\widehat{(x+h)} = \hat{f}(\xi)e^{2\pi i h \xi}$.

b) $f\widehat{(x)e^{-2\pi}}{}^{ixh} = \hat{f}(\xi + h)$.

c) $\widehat{f(\delta x)} = \delta^{-1}\hat{f}(\delta^{-1} \cdot \xi)$.

d) Formulate the analogs of the above properties for a function $f \in L_1(\mathbb{R}^2)$.

3. More Properties of the Fourier Transform

You may solve only 4 of the next following 5 subquestions.

Assume that f is a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ (or any reasonable space where you can prove the following identities).

a) Prove that $\widehat{f'(x)} = 2\pi i \xi \hat{f}(\xi)$.

b) Prove that $-2\widehat{\pi i x f}(x) = \frac{d\hat{f}(\xi)}{d\xi}$.

c) Formulate an analog of the above two properties when replacing $f'(x)$ by $f^{(n)}(x)$, the n-th derivative of $f(x)$ in part a) above and when replacing the derivative of $\hat{f}(\xi)$ by its n-th derivative in part b) above.

d) Show that if $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$. Hint: use your result in part c).

e) Formulate the analogs of the above properties for a function $f \in \mathcal{S}(\mathbb{R}^2)$

4. Computation of the Continuous Fourier Transform

Let a be a real positive number. Compute 3 of the Fourier transforms of the following 4 functions (in parts 2 and 4 you need to compute the Fourier transform in two different ways):

1. $f(x) = \chi_{[-a,a]}(x)$, where $\chi_{[-a,a]}$ is the indicator function obtaining the value 1 on the interval $[-a, a]$ and 0 outside it.
2. $f(x) = (1 - |x/a|)\chi_{[-a,a]}(x)$. Compute the Fourier transform of this function in two different ways. First, by direct calculation. Second, by expressing this function as $g * g$ for some function g (verify your claim) and then using properties of the Fourier transform (Hint: you may use a scaled version of the function in 1 above, or look at exercise 4.7 in the textbook).
3. $f(x) = e^{-a|x|}$.
4. $f(x) = a^{-1/2}e^{-\pi x^2/a}$. Compute the Fourier transform in two different ways. First, by direct calculation (see e.g., online solution of the textbook problem 4.31$\star$). Then calculate it in the following way: assume that $a = 1$ and show that in this case

$$\frac{d\hat{f}(\xi)}{d\xi} = -2\pi\xi\hat{f}(\xi)$$

and that $\hat{f}(0) = 1$. Conclude the form of $\hat{f}(\xi)$ when $a = 1$. Use properties of the Fourier transform to conclude the Fourier transform of f for any $a \in \mathbb{R}$.

5. Questions from the textbook Solve problems 4.1, 4.5, 4.14, 4.20

6. Bonus Problem: Fixed points of the Fourier transform

Suggest an infinite sequence of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ such that $\hat{f}_n = f_n$ and moreover no function in the sequence is a linear combination of other functions in the sequence. Explain your answer.

7. Bonus Problem: Demonstration of the "General Principle"

Assume that f is a function in $L_1(\mathbb{R})$ whose Fourier transform satisfies

$$\hat{f}(\xi) = O(\frac{1}{|\xi|^{1+\alpha}}) \text{ as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Hölder condition of order α, that is

$$|f(x+h) - f(x)| \leq M|h|^\alpha \text{ for some } M > 0 \text{ and all } x, h \in \mathbb{R}\,.$$