Practice Problems 5

Math 5467

1. Elementary Properties of Haar Wavelets

Throughout this exercise ϕ is the Haar scaling function and ψ is the Haar wavelet function, i.e., $\phi = \chi_{[0,1)}$ and $\psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$.

Remark: Some of the following properties have been proved (or will be proved) for general wavelets and MRAs, but here you need to prove them directly for the Haar wavelets and MRA.

a) Show that

$$\phi_{j,k}(x) = \frac{1}{\sqrt{2}}(\phi_{j+1,2k}(x) + \phi_{j+1,2k+1}(x))$$

and

$$\psi_{j,k}(x) = \frac{1}{\sqrt{2}}(\phi_{j+1,2k}(x) - \phi_{j+1,2k+1}(x)).$$

b) If $c_0 = (c_0(0), \ldots, c_0(N-1))$ is a signal of length $N = 2^n$, we associate with it the function

$$f(x) = \sum_{k=0}^{N-1} c_0(k)\phi_{n,k}(x),$$

so that $c_0(k) =< f, \phi_{n,k} >$, for $k = 0, \ldots, N-1$. We define for $j = 1, \ldots, n$

$$c_j(k) =< f, \phi_{n-j,k} >$$

and

$$d_j(k) =< f, \psi_{n-j,k} > .$$

Show that for $j = 1, \ldots, n$:

$$\begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix}. \tag{2}$$

c) We assume an arbitrary signal of length $N = 4$ ($n = 2$) with given coefficients vector

$$(c_0(0), c_0(1), c_0(2), c_0(3))'.$$

Write down a matrix which when multiplying this vector gives $(c_1(0), c_1(1), d_1(0), d_1(1))$ and another matrix that when multiplying the same original vector gives the Haar DWT $(c_2(0), d_2(0), d_1(0), d_1(1))$.

d) We assume an arbitrary signal of length 8 as above ($n = 3$) with the coefficients vector

$$(c_0(0), c_0(1), \ldots, c_0(7))'.$$

Write down an 8×8 matrix which when multiplying this vector gives $(c_1(0), \ldots, c_1(3), d_1(0), \ldots, d_1(3))$ and another 8×8 matrix that when multiplying the same original vector gives the Haar DWT

$$(c_3(0), d_3(0), d_2(0), d_2(1), d_1(0), \ldots, d_1(3)).$$

e) Assume a general signal of length $N = 2^n$, where $n \in \mathbb{N}$. Show that the kth row ($k = 0, \ldots, N-1$) of the $N \times N$ matrix representing the Haar DWT is given by $h_k(z)$ for $z = 0/N$, $1/N$, $2/N$, $\ldots$, $(N-1)/N$, where $h_0(z) = 1/\sqrt{N}$ for all $z \in [0, 1]$ and for $k = 1, \ldots, N-1$ we write $k = 2^p + q - 1$, where $0 \le p \le n-1$, $1 \le q \le 2^p$, and for $z \in [0, 1]$:

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2}, & \text{if } (q-1)/2^p \le z < (q-0.5)/2^p; \\ -2^{p/2}, & \text{if } (q-0.5)/2^p \le z < q/2^p; \\ 0, & \text{otherwise.} \end{cases}$$

f) Show that the compuation of the Haar DWT of a signal of length N takes at most $4 \cdot N$ operations.

2. Another example of a wavelet

Hint: It will be useful to verify the following properties in the Fourier domain.

a) Let $\phi(x) = \mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. Prove that for each $j \in \mathbb{Z}$ the space V_j (corresponding to the scaling function ϕ) is the space of all band limited functions in L_2 with band $[-2^{j-1}, 2^{j-1}]$. That is, the space of functions whose fourier transforms are in L_2 and supported within the interval $[-2^{j-1}, 2^{j-1}]$.

b) Show that ϕ is a good scaling function. That is, it gives rise to an MRA, which is called Shannon's MRA.

c) Let ψ be the function such that

$$\hat{\psi}(\xi) = -\mathrm{e}^{-\pi i \xi} \chi_I, \quad \text{where} \quad I = \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$$

(one can show that $\psi(x) = \mathrm{sinc}(x)(1 - 2\sin(\pi x))$, but this is not part of this exercise). Show that

$$V_1 = V_0 \bigoplus W_0 \,.$$

Note that you need to show both that $V_1 = V_0 + W_0$ and that the space W_0 is orthogonal to the space V_0.

d) Write the MRA and the wavelet equations for the Shannon wavelet and scaling function. That is, find the scaling and wavelet coefficients h and g respectively.

Hint: Use Shannon's sampling Theorem. Note that the corresponding filters (represented by h and g) have infinite support.

3. Bonus Problem: Another Sampling Theorem

Prove that if $M \in \mathbb{N}$, $N = 2M + 1$ and

$$f(t) = \sum_{|n| \leq M} c_n e^{2\pi i t n} ,$$

then

$$f(t) = \frac{\sin(\pi t N)}{N} \sum_{m=0}^{N-1} f\left(\frac{m}{N}\right) \frac{(-1)^m}{\sin\left(\pi\left(t - \frac{m}{N}\right)\right)} .$$

4. Bonus Problem: Revisiting the MRA definition

Show that the third property defining MRA, i.e., $V_{-\infty} = \{0\}$, follows from the rest of the properties.