Things You Need to Know for Exam I

Math 5467 (Spring 2015)

The exam will cover all class material that is relevant to homework assignments 1-2 (material on the topics: Introduction to imaging and Matlab; SVD and applications; Image enhancement in the spatial domain, I plan to finish teaching/reviewing this material by March 3rd). More specifically, I suggest you review carefully the following things:

1. All homework problems.
2. Describe what is the SVD of an $m \times n$ matrix $\boldsymbol{A}$, where $m \geq n$ (distinguish between regular, thin and compressed).
3. Prove the existence of SVD.
4. Define a norm on a vector space.
5. For any matrix $\boldsymbol{A}$ define $\|\boldsymbol{A}\|_2$ and $\|\boldsymbol{A}\|_F$.
6. How can you express range($\boldsymbol{A}$), Ker($\boldsymbol{A}$), rank($\boldsymbol{A}$), $\|\boldsymbol{A}\|_2$, $\|\boldsymbol{A}\|_F$, and the set $\{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n \text{ and } \|\boldsymbol{x}\|_2 = 1\}$ in terms of the SVD of $\boldsymbol{A}$? Explain your answer.
7. If $\boldsymbol{A}$ is an $m \times n$ matrix where $m \geq n$, rank($\boldsymbol{A}$) $= n$ and $\boldsymbol{b} \in \mathbb{R}^n$, define the least squares solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ as a solution of a minimization problem.
8. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ such that rank($\boldsymbol{A}$) $= n$ define the pseudo-inverse of $\boldsymbol{A}$ (If $\boldsymbol{A}$ is invertible then your definition should coincide with the common definition of the inverse, explain why).
9. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ such that rank($\boldsymbol{A}$) $= n$, show that the definition above is equivalent with two different formulas for the pseudo-inverse (one of them uses SVD and the other one operations on $\boldsymbol{A}$). Which one of these formulas is preferable for numerical computations? (explain).
10. Prove that if $m \geq n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ such that rank($\boldsymbol{A}$) $= n$ and $\boldsymbol{b} \in \mathbb{R}^m$, then $\boldsymbol{x}_*$ is a least squares solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$, where $\boldsymbol{A}^+$ is the pseudo-inverse of $\boldsymbol{A}$.

11. Given data points $\{\boldsymbol{z}_i\}_{i=1}^N = \{(x_{i,1}, x_{i,2}, \dots, x_{i,n})\}_{i=1}^N$ in $\mathbb{R}^n$, describe an algorithm to find a_0, $a_1, \dots, a_n$ that minimize the quantity

$$\frac{1}{N}\sum_{i=1}^{N}(x_{i,n} - (a_1 \cdot x_{i,1} + \ldots + a_n \cdot x_{i,n-1} + a_0))^2.$$

That is, describe an algorithm for finding the least squares regression $(n-1)$-dimensional plane of the form $x_{i,n} = a_1 \cdot x_{i,1} + \ldots a_n \cdot x_{i,n-1} + a_0$.

12. Given data points $\{\boldsymbol{x}_i\}_{i=1}^N = \{(x,y)\}_{i=1}^N$ in $\mathbb{R}^2$, describe an algorithm that expresses the y-coordinate as a polynomial $p(x)$ of the x-coordinate with degree at most d so that the following quantity is minimized

$$\frac{1}{N}\sum_{i=1}^{N}(y - p(x))^2.$$

13. Assume data points $\{\boldsymbol{x}_i\}_{i=1}^N$ in $\mathbb{R}^n$, $1 \le d < n$, describe an algorithm to find a d-dimensional plane P in $\mathbb{R}^n$ of the form $\boldsymbol{c} + \mathrm{Sp}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_d)$, where $\boldsymbol{c}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_d \in \mathbb{R}^n$, which minimizes among all other d-dimensional planes the quantity:

$$\frac{1}{N}\sum_{i=1}^{N}\mathrm{dist}^2(\boldsymbol{x}_i, P)\,.$$

That is, describe an algorithm for finding the orthogonal least squares d-dimensional plane.

14. Given data points $\{(x_{i,1}, x_{i,2}, \dots, x_{i,n})\}_{i=1}^N$ in $\mathbb{R}^n$, $1 \le d < n$, find coordinates $\{(x'_{i,1}, x'_{i,2}, \dots, x'_{i,d})\}_{i=1}^N$ describing the projection of the original points on the orthogonal least squares d-dimensional plane.

15. Given a continuous image with intensities in the range $[0, L-1]$ and pdf function $p_r(r)$. Develop (rigorously) a formula for a monotone transformation $S = T(r)$ of the intensities of the image, mapping $[0, L-1]$ onto $[0, L-1]$, and resulting in a new image with uniform pdf $p_s(s)$. What would you do if $p_s(s)$ is not uniform?

16. Given a discrete image having intensities $k = 0, \dots, L-1$ with probabilities $p_k(k)$, $k = 0, \dots, L-1$ describe an algorithm (based on your formula above) for monotonically transforming the intensities of that image onto $0, \dots, L-1$ so that the new image has approximately uniform distribution.

17. Define the spaces $\ell_p(\mathbb{Z})$, $\ell_p(\mathbb{Z}^2)$, for $1 \le p \le \infty$. Also define the convolution and correlation of vectors in $\ell_1(\mathbb{Z})$ and also in $\ell_1(\mathbb{Z}^2)$. Give examples of signals in $\ell_p(\mathbb{Z})$ and images in $\ell_p(\mathbb{Z}^2)$ for some $1 \le p \in \mathbb{R}$ and not in the corresponding $\ell_{p'}$ space for another $1 \le p' \le \mathbb{R}$. Prove the main properties of the convolution and correlation.

18. Define the spaces $L_p(\mathbb{R})$, $L_p(\mathbb{R}^2)$, for $1 \leq p < \infty$. Also define the convolution and correlation of vectors in $L_1(\mathbb{R})$ and also in $L_1(\mathbb{R}^2)$. Give examples of functions in $L_p(\mathbb{R})$, $L_p(\mathbb{R}^2)$ but not in the corresponding $L_{p'}$ space another $1 \leq p' \leq \mathbb{R}$ (similarly for $L_p([0,1])$ and $L_p([0,1]^2)$). Prove the main properties of the convolution and correlation.

19. Give an example of a smoothing filter (low-pass) and a "sharpening"/derivative-type filter (high-pass).

20. What are unsharp mask and high-boost masks?

21. Explain why any 3×3 Laplacian filter is an unsharp mask (or the negative of an unsharp mask).

22. You need to know how to find numerical derivatives (at class level) and their corresponding masks.

23. Review the common image filters taught in class