Things You Need to Know for Exam II

Math 5467

The exam will cover all we have done in class before starting the topic of wavelets. Here are things you need to know well.

1. Solutions to all questions in homework assignments 3, 4.

2. a) Define the Fourier transform of a function in $L_1(\mathbb{R})$ and the Fourier transform of a function in $L_1(\mathbb{R}^2)$.

(b) Define the inverse Fourier transform of a function in $L_1(\mathbb{R})$ and similarly of a function in $L_1(\mathbb{R}^2)$.

(c) Define the convolution of functions in $L_1(\mathbb{R})$ and $L_1(\mathbb{R}^2)$.

(d) State the convolution theorem for functions in $L_1(\mathbb{R})$ and also for functions in $L_1(\mathbb{R}^2)$. Prove the convolution theorem for functions in $L_1(\mathbb{R})$.

3. a) Define the DFT of an N-periodic signal and DFT of an $N \times M$-periodic image.

b) Define the inverse DFT of an N-periodic signal and inverse DFT of an $N \times M$-periodic image.

c) Prove the inversion formula for N-periodic signals (you may use the result of problem 6 below).

d) Define the convolution of N-periodic signals and also of $N \times M$ periodic images.

e) State the convolution theorems for N-periodic signals and $N \times M$ periodic images.

4. a) Establish a recursive formula expressing the Fourier coefficients of a $2 \cdot M$-periodic signal by the Fourier coefficients of the M-periodic signals sampled at the even and odd locations of the original signal.

b) Let $\#M$ denote the number of operations required to compute the DFT of an M-periodic signal. Assuming that the number $w_{2M} = e^{-\frac{2\pi i}{2M}}$ is given to you, prove that $\#(2M) \leq 2 \cdot \#M + 4M$.

c) Assume that you have an N-periodic signal, where $N = 2^n$ and n is an integer. Assume further that the number $e^{-\frac{2\pi i}{N}}$ is given to you. Show that it is possible to compute the Fourier coefficients of the signal with at most $2N \log_2 N$ operations.

5. a) Define the general notion of an inner product on a vector space over $\mathbb{R}$ or $\mathbb{C}$ (distinguish between the two cases).

b) What is the norm induced by an inner product?

c) Provide specific examples of inner products and induced norms for the following vectors spaces: $\mathbb{R}^n$, $\mathbb{C}^n$, real-valued functions in $L_2(\mathbb{R})$, and complex-valued functions in $L_2(\mathbb{R})$.

6. For each $0 \leq \ell \leq N-1$, we define the N-periodic signal e_ℓ as follows:

$$e_\ell(k) = e^{\frac{2\pi i k \ell}{N}}, \quad k = 0, \ldots, N-1.$$

Show that $\left\{ \frac{e_\ell}{\sqrt{N}} \right\}_{\ell=0}^{N-1}$ is an orthonormal basis for the vector space of N-periodic complex valued signal.

7. a) Assume that V is a finite dimensional vector space of dimension N with an inner product and that $\{v_i\}_{i=1}^N$ is an orthonormal basis of V . What are the coefficients expressing a vector $v \in V$ by $\{v_i\}_{i=1}^N$?

b) Maintaining the same assumptions as above, express $||v||$ by the coefficients you have found.

8. a) Write the Fourier series of functions in the space of complex valued functions $L_2([0,1])$, which we view as periodic functions on $\mathbb{R}$. Specify the coefficients of the expansion and also express them as inner products in that space.

b) Which of the above is analogous to the Fourier transform and which to the inverse Fourier transform?

c) Define the periodic convolution of functions in $L_2([0,1])$. State the related convolution theorem.

9. a) State Shannon sampling theorem (also get familiar with the relevant definitions).

b) State Poisson's summation formula (two equivalent formulations). Prove one of the formulations and show that the other is equivalent to it.

c) Prove Shannon's sampling theorem in two different ways.

d) Explain what is aliasing (when it occurs and why). Understand its explanation via the second proof of Shannon's sampling theorem.

e) Describe a method for anti-aliasing. Explain why it provides best L_2 approximation to a function along all relevant discrete sample (state and prove the relevant theorem).

f) Explain how Shannon's sampling theorem can be improved to the case of oversampling (sampling at a frequency higher than Nyquist).