Things You Need to Know for the Final Exam

Math 5467 (Spring 2015)

The exam is cumulative and covers all material taught till and including the last class of instruction. Below are topics you need to know.

1. All questions given in homework assignments (5 worksheets).

2. Everything you needed to know for exams 1 and 2 (see previous 2 files).

3. a) Define an MRA for $L_2(\mathbb{R})$.

b) Specify the Haar scaling function ϕ, the corresponding functions $\phi_{j,k}(x)$ (know how to plot them for any j and k) and the corresponding spaces $\{V_j\}_{j\in\mathbb{Z}}$. Describe in words the Haar spaces V_j, $j \in \mathbb{Z}$ (make sure to characterize them precisely).

c) What is a scaling equation (equivalently an MRA equation)? Explain why it exists in an MRA.

d) Derive the coefficients of the Haar MRA equation.

4. a) What do we mean by a wavelet function corresponding to an MRA? (clearly state the theorem that guarantees the existence of a wavelet function).

b) Specify the Haar wavelet function ψ, the corresponding functions $\psi_{j,k}(x)$ (know how to plot them for any j and k) and the corresponding spaces $\{W_j\}_{j\in\mathbb{Z}}$.

c) What is a wavelet equation? Explain why it exists.

d) Derive the coefficients of the Haar wavelet equation.

5. a) What are the collection of dyadic intervals? and dyadic intervals of scale 2^{-j}?

b) Show directly (using properties of dyadic intervals) that if ψ_I is the Haar wavelet associated with the dyadic interval I, then $\{\psi_I\}_{I\in\mathcal{D}}$ is an orthonormal system in $L_2(\mathbb{R})$, where $\mathcal{D}$ denotes the collection of all dyadic intervals.

c) Let ϕ_I denotes the Haar scaling function associated with the dyadic interval I. Show that

$$\{\psi_I\}_{I\in\cup_{j=j_0}^{\infty}\mathcal{D}_j} \cup \{\phi_I\}_{I\in\mathcal{D}_{j_0}}$$

is an orthonormal system in $L_2(\mathbb{R})$.

6. Define the spaces W_j, where $j \in \mathbb{Z}$, and indicate the relation between the spaces $\{V_j\}_{j\in\mathbb{Z}}$ and $\{W_j\}_{j\in\mathbb{Z}}$.

7. Assume an MRA with a scaling function ϕ, wavelet function ψ, and corresponding spaces $\{V_j\}_{j\in\mathbb{Z}}$ and $\{W_j\}_{j\in\mathbb{Z}}$.

a) How do we express the space $L_2(\mathbb{R})$ by the spaces W_j, $j \in \mathbb{Z}$, and similarly any $f \in L_2(\mathbb{R})$ as an infinite linear combination of $\{\psi_{j,k}\}_{\substack{k\in\mathbb{Z}\\ j\in\mathbb{Z}}}$? (specify a formula for the coefficients)

b) How do we express $f \in L_2(\mathbb{R})$ by $\{\phi_{j_0,k}\}_{k\in\mathbb{Z}}$ and $\{\psi_{j,k}\}_{\substack{k\in\mathbb{Z}\\ j\geq j_0}}$? (specify the coefficients). Also, how do we express the function space $L_2(\mathbb{R})$ by MRA and wavelet subspaces in the same way of the function representation you have just described.

c) How do we express the MRA subspace V_l, for some $l \in \mathbb{Z}$, by the space V_{j_0}, for some $j_0 \in \mathbb{Z}$, where $j_0 < l$, and elements of $\{W_j\}_{j\in\mathbb{Z}}$? Also, write the representation of $f \in V_l$ by shifts and scales of ϕ and ψ in the same way as the subspace representation you have just specified.

8. Using the scaling equation and the wavelet equation prove that

$$\phi_{j,k}(x) = \sum_{m\in\mathbb{Z}} h(m-2k)\phi_{j+1,m}\,,$$

and

$$\psi_{j,k}(x) = \sum_{m\in\mathbb{Z}} g(m-2k)\phi_{j+1,m}\,.$$

9. a) Define the discrete wavelet transform. Make sure to distinguish between the DWT of stage j and the total DWT.

b) Prove that the DWT can be written in the form

$$c_{j+1}(k) = c_j * f_0(2k),$$
$$d_{j+1}(k) = c_j * f_1(2k).$$

Express f_0 and f_1 in terms of the scaling and wavelet filters h and g. Also, draw a diagram corresponding to those formulas.

c) Write down the formula for the discrete Haar transform (expressing $c_{j+1}(k)$ and $d_{j+1}(k)$ in terms of coefficients of the form $c_j(k')$). Explain how it is obtained as a special case of the above equation.

13. Assuming that the number of coefficients for $\{h\}$ is K and similarly the number of coefficients for $\{g\}$ is K. What is the number of operations for computing the first stage of the discrete wavelet

transform of a signal of length $M = 2^N$? What is the number of coefficients of the total discrete wavelet transform of a signal of length $M = 2^N$? Prove your claim.

14. a) Using ϕ and ψ, scaling and wavelet functions respectively for $L_2(\mathbb{R})$, describe two different orthonormal bases for $L_2(\mathbb{R}^2)$ (one of them uses all scales $j \in \mathbb{Z}$ and only the wavelet function ψ, but with two different variables, and the other one uses all scales $j \in \mathbb{Z}$ such that $j \geq J$ for a fixed $J \in \mathbb{Z}$ and both ϕ and ψ, but with two different variables).

b) Explain what is the 2D discrete wavelet transform and what are the three types of detail coefficients.

c) Given an $M \times N$ image, where $M = 2^m$ and $N = 2^n$, explain how the 2D discrete wavelet transform is organized as an $M \times N$ image.

d) Explain how the 2D discrete Haar transform is computed.