The Discrete Fourier Transform and Its Properties

We assume discrete signals in $\mathbb{C}^N$, which we index their elements by $\{x(k)\}_{k=0}^{N-1}$. We extend these signals to $\mathbb{C}^{\mathbb{Z}}$ as N-periodic signals. That is,

$$x(k) = x(k+N) \text{ for all } k \in \mathbb{Z}\,.$$

We use the standard inner product on $\mathbb{C}^N$:

$$< u, v >= \sum_{i=0}^{N-1} u_i \bar{v}_i, \quad \text{for all } u, v \in \mathbb{C}^N\,,$$

so that

$$\|u\|_2 = \sqrt{< u, u >}, \quad \text{for all } u \in \mathbb{C}^N\,.$$

The DFT of $x = (x(0), \ldots, x(N-1)) \in \mathbb{C}^N$ is the following signal in $\mathbb{C}^N$:

$$\hat{x}(n) = \sum_{k=0}^{N-1} x(k) e^{-\frac{2\pi i k n}{N}}$$

The Inverse DFT (IDFT) is obtained as follows

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} \hat{x}(n) e^{\frac{2\pi i n k}{N}}$$

We recall that the convolution of N-periodic vectors x and y has the form

$$(x * y)(m) = \sum_{k=0}^{N-1} x(m-k) y(k) \text{ for all } m \in \mathbb{Z}\,,$$

and that $x * y$ is also N-periodic.

We proved the IDFT formula in the following way. We defined the N vectors in $\mathbb{C}^N$, $\{e_\ell\}_{\ell=0}^{N-1}$, by

$$e_\ell(k) = e^{\frac{2\pi i \ell k}{N}}\,, \quad k = 0, \ldots, N-1\,,$$

and expressed the DFT as follows

$$\hat{x}(n) =< x, e_n >\,.$$

We showed that $\left\{\frac{e_\ell}{\sqrt{N}}\right\}_{\ell=0}^{N-1}$ is an orthonormal basis for $\mathbb{C}^N$. Therefore,

$$x = \sum_{n=0}^{N-1} < x, \frac{e_n}{\sqrt{N}} > \frac{e_n}{\sqrt{N}} = \frac{1}{N}\sum_{n=0}^{N-1} < x, e_n > e_n = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n)\, e_n\,,$$

and

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n)\, e_n(k) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n)\, e^{\frac{2\pi i n k}{N}}\,, \quad k = 0, 1, \ldots, N-1\,.$$

This proved the IDFT formula.

You will also prove in your homework that

$$\sum_{k=0}^{N-1} x(k)\bar{y}(k) = \frac{1}{N}\sum_{n=0}^{N-1} \hat{x}(n)\bar{\hat{y}}(n)\,,$$

and thus

$$\|x\|_2 = \frac{1}{\sqrt{N}}\|\hat{x}\|_2\,.$$

Here are some properties of the DFT

Property	Signal	*DFT*
Linearity	$\alpha_1 x_1 + \alpha_2 x_2$	$\alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2$
Modulation	$x(k)e^{\frac{2\pi i k n_0}{N}}$	$\hat{x}(n - n_0)$
Specific modulation	$x(k)(-1)^k$	$\hat{x}(n - \frac{N}{2})$
Translation	$x(k - k_0)$	$\hat{x}(n)e^{\frac{-2\pi i n k_0}{N}}$
Translation to center	$x(k - \frac{N}{2})$	$\hat{x}(n)(-1)^n$
Convolution	$x * y(k)$	$\hat{x}(n)\hat{y}(n)$
Multiplication	$x(k)y(k)$	$\hat{x} * \hat{y}(n)$
Correlation	$x \diamond y(k)$	$\hat{x}(n)\bar{\hat{y}}(n)$
Multiplication by conjugate	$x(k)\bar{y}(k)$	$x \diamond y(n)$
Discrete unit impulse	$\delta(k)$	1
Constant impulse	1	$\delta(n)$

Property	Signal	DFT
Symmetries	x is real	$Re(\hat{x})$ is even $Im(\hat{x})$ is odd
	x is imaginary	$Re(\hat{x})$ is odd $Im(\hat{x})$ is even
	x is real and even	$\hat{x}$ is real and even
	x is real and odd	$\hat{x}$ is imagninary and odd
	x is imaginary and even	$\hat{x}$ is imaginary and even
	x is imaginary and odd	$\hat{x}$ is real and odd