MATH 8302 HW 1 Solutions and comments

February 24, 2020

Questions 3,4,9 are not graded, so solutions are given. For graded questions, only comments are given. Questions are ordered as on the website.

Question 1: Let $p_s, p_n : S^d \to \mathbb{R}^d$ be the stereographic projections. One need to show $p_s \circ p_n^{-1}$ and $p_n \circ p_s^{-1}$ are smooth and it is true because the domains of these two functions don't include $0 \in \mathbb{R}^d$. Also, one cannot say p_s or p_n is differentiable since we have not established S^d to be a differentiable manifold yet.

Question 2: $\mathcal{F} \neq \mathcal{F}_1$ because $\phi : t \mapsto \sqrt[3]{t}$ is not in $\mathcal{F}$. But ϕ defines a diffeomorphism between the two.

Question 3: (a) For $(x, v) \in \mathbb{R}^{2d}$, we have $\tilde{\psi} \circ \tilde{\varphi}^{-1}(x, v) = (\psi \circ \varphi^{-1}(x), d\psi \circ d\varphi^{-1}(v))$ is smooth.

(b) The collection is clearly a cover. To show it is a basis, let W_1, W_2 be open subset of $\mathbb{R}^{2d}$ and $(U, \varphi), (V, \psi)$ two coordinate charts such that $\tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$ is nonempty. Take $v \in \tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$. Then there exists chart (A, ϕ) such that $\pi(v) \in A \subset U \cap V$ and open set $W \subset \mathbb{R}^{2d}$ such that $v \in \tilde{\phi}^{-1}(W) \subset \tilde{\varphi}^{-1}(W_1) \cap \tilde{\psi}^{-1}(W_2)$.

(c) We already know $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth. $\{\pi^{-1}(U)\}$ is obviously an open cover.

Question 4: (a) Take the product chart $(U \times V, \psi \times \varphi)$ on $M \times N$ and a chart (W, f) on $\tilde{M}$. Clearly $(\psi \times \varphi) \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^{m+n}$ is smooth if and only if $\psi \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^m$ and $\varphi \circ \alpha \circ f^{-1} : \mathbb{R}^p \to \mathbb{R}^n$ are smooth.

(b) It is easy to see $v \to (d\pi_1(v), d\pi_2(v))$ is a linear map between two linear spaces of the same dimension. Suppose $d\pi_1(v) = d\pi_2(v) = 0$. Let $\gamma : (-\varepsilon, \varepsilon) \to M \times N$ be an integral curve with $\dot{\gamma}(0) = v$. Note that $\gamma = (\gamma_1, \gamma_2)$ and thus $v = \dot{\gamma} = (\dot{\gamma_1}, \dot{\gamma_2}) = (d\pi_1(v), d\pi_2(v)) = 0$. So this map is injective and thus isomorphism.

(d) Let $\gamma, \gamma_1, \gamma_2$ be as in (b), but at point (m_0, n_0). Then $v(f) = \left.\dfrac{d}{dt}\right|_{t=0} f(\gamma(t)) = \dfrac{\partial f}{\partial m}(m_0, n_0)\dfrac{d\gamma_1}{dt}(0) + \dfrac{\partial f}{\partial n}(m_0, n_0)\dfrac{d\gamma_2}{dt}(0)$. Let $f_1 = f \circ i_{n_0} \in C^\infty(M)$ and $f_2 = f \circ i_{m_0} \in$

$C^\infty(N)$. Then $v_1(f_1) + v_2(f_2) = \left.\frac{d}{dt}\right|_{t=0} f_1(\gamma_1(t)) + \left.\frac{d}{dt}\right|_{t=0} f_2(\gamma_2(t)) = \frac{\partial f}{\partial m}(m_0, n_0)\frac{d\gamma_1}{dt}(0) + \frac{\partial f}{\partial n}(m_0, n_0)\frac{d\gamma_2}{dt}(0)$.

Question 5: For diffeomorphism $f : M \to N$, the differential df_m is isomorphism between tangent spaces. So M, N must have equal dimension.

Question 6: Common mistake is to define $f : TS^1 \to S^1 \times \mathbb{R}$ by $(x, y\frac{d}{dx}) \mapsto (x, y)$, since we don't have a global coordinate on TS^1. One need to take the two charts U_1, U_2 on S^1 and define f on the two trivializations $U_i \times \mathbb{R}$. Note then f would be $(x, y\frac{d}{dx}) \mapsto (x, y)$ and $(x, y\frac{d}{dx}) \mapsto (x, -y)$ respectively on the two charts so that they can be patched together.

The proof using $TS^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ is a bit cheating but is also accepted.

Question 7 and 10: Practically the same question. If f is everywhere non-singular, then it is local diffeomorphism and thus an open map. So the image of f has to be open and compact, contradicting Heine-Borel.

Question 8: $f(x, y) = (e^x cosy, e^x siny)$ would work.

Question 9: Compute the differential $df = \begin{pmatrix} 3x^2 + y & x + 3y^2 \end{pmatrix}$

Note that it is singular at $(0, 0)$ and $(-1/3, -1/3)$. So not a submanifold there.

For $p = (1/3, 1/3)$, we have df nonsingular. Also $f(1/3, 1/3) \neq f(0, 0) \neq f(-1/3, -1/3)$, so df is nonsingular at every point of $f^{-1}(f(p))$. So a submanifold.