MATH 8302: Manifolds & Topology
Homework 3

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The book referenced throughout is [1].

Contents

1 2.1.1.

More generally show that if $\omega = \sum g_I du_I$, then $d\omega = \sum dg_I du_I$.

Solution. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that $u_i = x_i \circ f = f^*(x_i)$. Observe the calculation

$$d\omega = d\Big(\sum g_I(u)du_I\Big) = d\Big(\sum (g_I\circ f)(x)df^*(x_1)\cdots df^*(x_n)\Big)$$

$$= \sum_I \Big(d(g_I\circ f)(x)df^*(x_1)\cdots df^*(x_n) + \sum_{j=1}^n (g_I\circ f)(x)df^*(x_1)\cdots d^2f^*(x_j)\cdots df^*(x_n)\Big)$$

$$= \sum_I d(g_I\circ f)(x)df^*(x_1)\cdots df^*(x_n)$$

$$= \sum_I dg_I(u)du_I,$$

where in the third equality we used the linearity of d and the product rule, and in the fourth equality we used the fact that $d^2 = 0$. □

2 3.6.

Prove Stokes' Theorem for the upper half space.

Solution. Let $\omega = \sum_{j=1}^n f_n dx_1\cdots\widehat{dx_j}\cdots dx_n$ be an $(n-1)$-form with compact support. Observe the calculation

$$\int_{\mathbb{H}^n} d\omega = \sum_{j=1}^n (-1)^{j-1}\int_{\mathbb{H}^n}\frac{\partial f_j}{\partial x_j}dx_1\cdots dx_n$$

$$= (-1)^{n-1}\int_{\mathbb{H}^n}\frac{\partial f_n}{\partial x_n}dx_1\cdots dx_n$$

$$+\sum_{j=1}^{n-1}(-1)^{j-1}\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\Big(\int_{-\infty}^\infty\frac{\partial f_j}{\partial x_j}dx_j\Big)dx_1\cdots\widehat{dx_j}\cdots dx_n$$

$$= (-1)^n\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty f(x_1,...,x_{n-1},0)dx_1\cdots dx_{n-1}$$

$$= \int_{\partial\mathbb{H}^n}\omega,$$

where in the second equality we used Fubini's theorem, and in the fourth equality we used the Fundamental Theorem of Calculus as well as the fact that ω (hence f) has compact support in $\mathbb{H}^n$. □

3 4.3.1

Volume form on a sphere. Let $S^n(r)$ be the sphere of radius r

$$x_1^2 + \cdots + x_{n+1}^2 = r^2$$

in $\mathbb{R}^{n+1}$, and let

$$\omega = \frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}x_i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

(a) Write S^n for the unit sphere $S^n(1)$. Compute the integral $\int_{S^n}\omega$ and conclude that ω is not exact.

(b) Regarding r as a function on $\mathbb{R}^{n+1}\setminus\{0\}$, show that $(dr)\cdot\omega = dx_1\cdots dx_{n+1}$. Thus, ω is the Euclidean volume form on the sphere $S^n(r)$.

Solution. (a) Let $B^{n+1}(1)$ be the $(n+1)-$dimensional unit ball centered at the origin. Observe the calculation

$$\begin{aligned}\int_{S^n(1)}\omega = \int_{B^{n+1}(1)} d\omega = \int_{B^{n+1}(1)}\sum_{j=1}^{n+1}\frac{\partial \frac{x_j}{r}}{\partial x_j}(-1)^{2(j-1)}dx_1\cdots dx_{n+1} \\ = \int_{B^{n+1}(1)}\sum_{j=1}^{n+1}\frac{r^2-x_j^2}{r^3}dx_1\cdots dx_{n+1} = \int_{B^{n+1}(1)}\frac{n}{r}dx_1\cdots dx_{n+1} \\ = \int_{S^n(1)}\int_0^1 \frac{n}{r}r^{n+1-1}\,dr\,dz = \{\text{Surface area of } S^n(1)\}\end{aligned}$$

where in the first equality we used Stokes' Theorem, in the fifth equality we used Fubini's Theorem to change into spherical coordinates, which is not zero. In particular, we conclude that $d\omega \neq 0$, or equivalently, ω is not exact.

(b) Since r is a $0-$form, we may use Proposition 1.3 to easily see that

$$\begin{aligned}(dr)\cdot\omega = d(r\cdot\omega) - r\cdot d\omega = \sum_{j=1}^{n+1}\frac{\partial x_j}{\partial x_j}dx_1\cdots dx_{n+1} - r\cdot\frac{n}{r}dx_1\cdots dx_{n+1} \\ = dx_1\cdots dx_{n+1},\end{aligned}$$

as desired. □

4 4.5.

Show that $d\pi_* = \pi_* d$; in other words, $\pi_* : \Omega_c^*(M\times\mathbb{R}^1)\to\Omega_c^{*-1}(M)$ is a chain map.

Solution. We use the definitions and notation immediately preceding Exercise 4.5 in [1]. Let ω be a form of type (I). Then note that $\pi_*\omega = 0$, so that $d\pi_*\omega = 0$. On the other hand,

$$\begin{aligned}
\pi_* d\omega = \pi_* d\Big(\pi^*\phi \cdot f\Big) &= \pi_*\Big((d\pi^*\phi)\cdot f(x,t)\Big) + \pi_*\Big((-1)^{\deg(\pi^*\phi)}\pi^*\phi \cdot df\Big) \\
&= (-1)^{\deg(\pi^*\phi)}\pi_*\Big(\pi^*\phi \cdot \frac{\partial f}{\partial t}dt + \sum \pi^*\phi \cdot \frac{\partial f}{\partial x_i}dx_i\Big) \\
&= (-1)^{\deg(\pi^*\phi)}\pi_*\Big(\pi^*\phi \cdot \frac{\partial f}{\partial t}dt + \sum \pi^*(\phi dx_i) \cdot \frac{\partial f}{\partial x_i}\Big) \\
&= (-1)^{\deg(\pi^*\phi)}\pi_*\Big(\pi^*\phi \cdot \frac{\partial f(x,t)}{\partial t}dt\Big) = (-1)^{\deg(\pi^*\phi)}\phi \int_{-\infty}^{\infty} \frac{\partial f(x,t)}{\partial t}dt \\
&= 0,
\end{aligned}$$

where first we used the product rule, then we used the chain rule, the fact that d and π^* commute, and the fact that π_* maps forms of type (I) to 0, and finally we used the fact that f has compact support.

Now let ω be a form of type (II). Then

$$\begin{aligned}
\pi_* d\omega = \pi_* d\Big(\pi^*\phi \cdot f dt\Big) & \\
&= \pi_*\Big((d\pi^*\phi)\cdot f dt\Big) + \pi_*\Big((-1)^{\deg(\pi^*\phi)}\pi^*\phi \cdot df dt\Big) \\
&= (d\phi)\int_{-\infty}^{\infty} f dt + (-1)^{\deg(\pi^*\phi)}\pi_*\Big(\sum \pi^*(\phi dx_i)\cdot \frac{\partial f}{\partial x_i}dt\Big) \\
&= (d\phi)\int_{-\infty}^{\infty} f dt + (-1)^{\deg(\pi^*\phi)}\sum \phi dx_i \int_{-\infty}^{\infty}\frac{\partial f(x,t)}{\partial x_i}dt \\
&= d\left(\phi\int_{-\infty}^{\infty} f(x,t)dt\right) \\
&= d\pi_*\Big(\pi^*\phi \cdot f(x,t)dt\Big) \\
&= d\pi_*\omega,
\end{aligned}$$

as claimed.

5 4.8.

Compute the cohomology groups $H^*(M)$ and $H_c^*(M)$ of the open Möbius strip M, i.e., the Möbius strip without the bounding edge.

Solution. We claim that the cohomology groups are

$$H^q(M) = \begin{cases} \mathbb{R} & q = 0, 1, \\ 0 & q > 1, \end{cases}$$

and

$$H_c^*(M) = 0, \quad \text{for each } q.$$

Recall that M deformation retracts onto its center circle. By Corollary 4.1.2.1, it follows that $H^q(M) \simeq H^q(S^1)$ for each q. Since

$$H^q(S^1) \simeq \begin{cases} \mathbb{R} & q = 0, 1, \\ 0 & q > 1, \end{cases},$$

due to Exercise 4.3, the first claim follows.

We now consider the compact support cohomology. Since M is a surface, it is clear that $H_c^q(M) = 0$ for each $q \geq 3$. Thus we have to study $H_c^q(M)$ for $q = 0, 1, 2$. Let U and V be open disks in M such that $U \cup V = M$, and $V \cap U$ is the disjoint union of two open disks D_1, D_2 (for instance, we may think of U as covering one half of the Möbius band with a bit of the other half, and V covers the other half with a bit of the first half, in such a way that the intersection has two connected components, each of which diffeomorphic to disks). Then U, V, D_1, D_2 are all orientable and diffeomorphic to $\mathbb{R}^2$, so the Poincaré Lemma for compact supports gives us that

$$H_c^q(U) \simeq H_c^q(V) \simeq H_c^q(D_1) \simeq H_c^q(D_2) \simeq \begin{cases} \mathbb{R} & q = 2, \\ 0 & q \neq 2 \end{cases},$$

whence

$$H_c^q(U \cap V) = H_c^q(D_1 \cup D_2) \simeq \begin{cases} \mathbb{R}^2 & q = 2 \\ 0 & q \neq 2 \end{cases}.$$

We now apply the Mayer-Vietoris sequence. For each q, we have the long exact sequence

$$\cdots \to H_c^q(U \cap V) \to H_c^q(U) \oplus H_c^q(V) \to H_c^q(M) \to H_c^{q+1}(U \cap V) \to \cdots.$$

In particular,

$$0 \simeq H_c^0(U) \oplus H_c^0(V) \to H_c^0(M) \to H_c^1(U \cap V) \simeq 0,$$

so that $H_c^0(M) = 0$. Moreover, we have the exact sequence

$$0 \to H_c^1(M) \to H_c^2(U \cap V) \xrightarrow{\delta} H_c^2(U) \oplus H_c^2(V) \to H_c^2(M) \to 0.$$

Via Stokes' Theorem, it is trivial to check that δ sends generators of $H_c^2(U \cap V)$ to elements with no zero-component of $H_c^2(U) \oplus H_c^2(V)$. Hence δ is a group isomorphism. By the exactness of the sequence, this observation forces that $H_c^1(M) \simeq H_c^2(M) \simeq 0$, as desired. $\square$

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer.