MATH 8302: Manifolds & Topology
Homework 5

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April 22, 2019

The book referenced throughout is [1]. This solution is joint work with McCleary Philbin and Adrienne Sands.

1 6.45.

On the complex projective space $\mathbb{C}P^n$ there is a tautological line bundle S, called the *universal subbundle*; it is the subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ given by

$$S = \{(\ell, z) \ : \ z \in \ell\}.$$

Above each point ℓ in $\mathbb{C}P^n$, the fiber of S is the line represented by ℓ. Find the transition functions of the universal subbundle S of $\mathbb{C}P^1$ relative to the standard open cover and compute its Euler class.

Solution. The standard open cover for $\mathbb{C}P^1$ consists of two open sets: $U_0 = [1, \frac{z_1}{z_0}]$ whenever $z_0 \neq 0$, and $U_1 = [\frac{z_0}{z_1}, 1]$ whenever $z_1 \neq 0$. Now set $x = \frac{z_1}{z_0}$ and $u = \frac{z_0}{z_1}$. Then $[1, x]$ and $[1, u]$ are new coordinates for U_0 and U_1, respectively. Note that therefore we may identify each of U_0, U_1 with $\mathbb{C}$. Since the fibers of S are the complex lines consisting of all those points in the equivalence class of a specific point in $\mathbb{C}P^n$, then the transition functions must verify

$$(u, 1) = g_{01}(1, x), \qquad (1, x) = g_{10}(u, 1),$$

for each coordinate x, u. This is achieved if and only if $g_{01} = u = \frac{1}{x} = \frac{z_0}{z_1}$, and $g_{10} = \frac{1}{u} = x = \frac{z_1}{z_0}$.

We now compute the Euler class. As in (6.38), we have that

$$e(S) = -\frac{1}{2\pi i} d(\rho_0 d \log g_{01}), \qquad \text{on } U_1,$$

where ρ_0 is 1 in a neighborhood of the origin, and 0 in a neighborhood of infinity in $U_0 \simeq \mathbb{C}$. In particular,

$$e(S) = -\frac{1}{2\pi i} d\Big(\rho_0 d \log \frac{1}{x}\Big), \qquad \text{on } U_0 \cap U_1.$$

We now proceed to compute $\int_{\mathbb{C}P^1} e(S)$, as in [1] pages 76-77. Fix a circle C in the complex plane with so large a radius that the support of ρ_0 is contained inside C. Let A_r be the annulus centered at the origin whose outer circle is c and whose inner circle B_r has radius r. As the boundary of A_r, the circle C is oriented counterclockwise while B_r is oriented clockwise. Observe the computation

$$\begin{aligned}
\int_{\mathbb{C}P^1} e(S) = -\frac{1}{2\pi i}\int_{\mathbb{C}} d\Big(\rho_0 d\log\frac{1}{x}\Big) &= \frac{1}{2\pi i}\int_{\mathbb{C}} d\Big(\rho_0 \frac{dx}{x}\Big) = \frac{1}{2\pi i}\lim_{r\searrow 0}\int_{A_r} d\Big(\rho_0\frac{dx}{x}\Big) \\
&= \frac{1}{2\pi i}\lim_{r\searrow 0}\Big[\int_C \rho_0\frac{dx}{x} + \int_{B_r}\rho_0\frac{dx}{x}\Big] = \frac{1}{2\pi i}\lim_{r\searrow 0}\int_{B_r}\frac{dx}{x} \\
&= \frac{1}{2\pi i}(-2\pi i) = -1,
\end{aligned}$$

where we used Stokes Theorem and the properties of ρ_0. To recap,

$$\int_{\mathbb{C}P^1} e(S) = -1.$$

□

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer.

[2] T. Lawson, *Topology: a geometric approach*, Oxford University Press, 2006