Let's start with linear algebra.

$$\begin{pmatrix} .37 & .28 & .23 & .26 & .36 \\ .87 & .76 & .11 & .37 & .86 \\ .52 & .82 & .76 & .26 & .12 \\ .91 & .91 & .79 & .82 & .48 \\ .44 & .13 & .28 & .37 & .84 \end{pmatrix} = \begin{pmatrix} -.25 & .03+.05\,\mathbf{i} & .03-.05\,\mathbf{i} & .76 & -.04 \\ -.46 & .3-.48\,\mathbf{i} & .3+.48\,\mathbf{i} & -.48 & -.03 \\ -.41 & -.57 & -.57 & .15 & .44 \\ -.67 & -.35+.3\,\mathbf{i} & -.35-.3\,\mathbf{i} & -.33 & -.83 \\ -.32 & .33+.19\,\mathbf{i} & .33-.19\,\mathbf{i} & -.23 & .35 \end{pmatrix} \cdot \begin{pmatrix} 2.5 & 0 & 0 & 0 & 0 \\ 0 & .39+.47\,\mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & .39-.47\,\mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & .02 & 0 \\ 0 & 0 & 0 & 0 & .27 \end{pmatrix} \cdot P^{-1}$$

We know conditions for diagonalization of linear transformations.

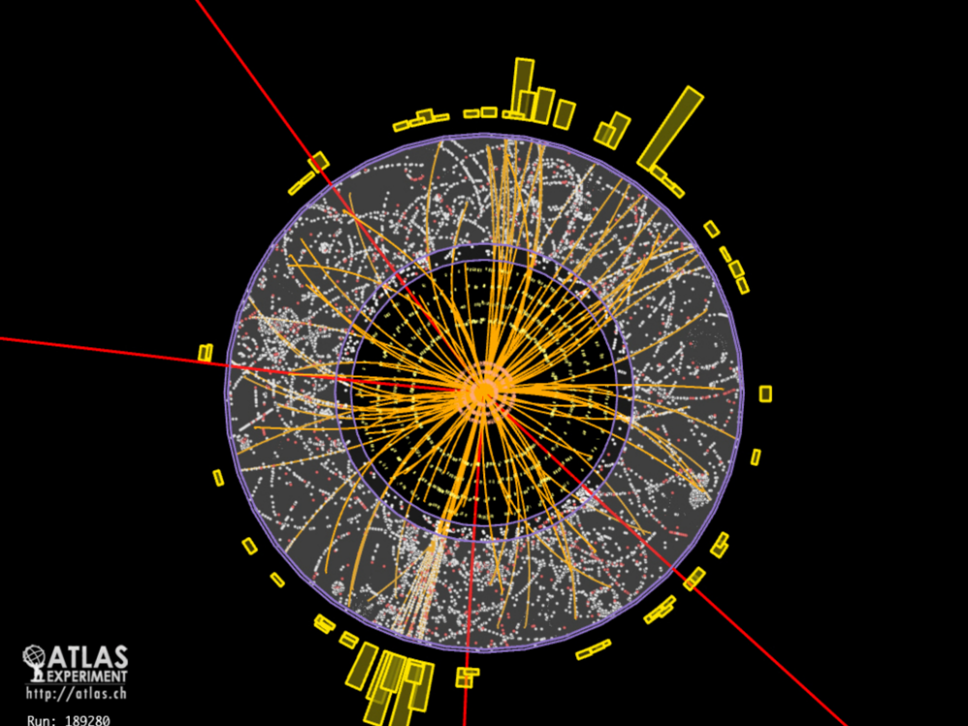
What if we worked over a polynomial ring?

$$A = \left(\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & x_1y_0y_1 & 0 & 0 & 0 & x_1y_0y_2 & 0 & 0 & 0 & x_0y_1y_2 & 0 & 0 \\
-x_0y_1+x_1y_1-x_1y_2 & -x_0y_2 & 0 & 0 & 0 & x_1y_2 & x_1y_2 & x_1y_2 & x_1y_2 & -x_1y_2 & -x_0y_1+x_1y_1-x_1y_2 & -x_1y_2 & 0 & x_1y_2 & x_1y_2 & x_1y_2 \\
0 & 0 & 0 & x_1y_1 & 0 & 0 & 0 & 0 & 0 & -x_1y_2 & 0 & -x_1y_2 & 0 & -x_0y_1-x_1y_2 & -x_0y_1 & -x_1y_2 \\
0 & 0 & -x_1y_2 & 0 & x_0y_0-x_1y_0-x_0y_2 & x_1y_2 & x_1y_2 & -x_1y_2 & -x_1y_2 & -x_1y_2 & 0 & x_0y_2-x_1y_2 & x_0y_2 & -x_0y_2+x_1y_2 & x_0y_2+x_1y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_1y_2 & -x_1y_2 & x_1y_2 & x_1y_2 & x_1y_2 & x_0y_1-x_1y_1+x_1y_2 & x_1y_2 & 0 & -x_1y_2 & -x_1y_2 & -x_1y_2 \\
0 & 0 & x_1y_1 & 0 & x_1y_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0y_1 & x_1y_2 \\
0 & 0 & 0 & 0 & 0 & -x_1y_0 & 0 & -x_1y_0 & -x_1y_0 & -x_1y_0 & 0 & x_0y_2 & x_0y_0 & 0 & 0 & 0 \\
x_0y_1 & x_0y_0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & x_1y_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & 0 & -x_1 & -x_1 & -x_1 & -x_1 & -x_1 & 0 & x_0-x_1 & x_0 & x_0 & -x_0-x_1 & 0 \\
y_1y_2 & y_0y_2 & 0 & 0 & 0 & -y_0y_1+y_1y_2 & -y_0y_1+y_1y_2 & 0 & 0 & 0 & -y_1y_2 & 0 & 0 & -y_1y_2 & y_1y_2 & 0 \\
0 & 0 & y_1y_2 & 0 & y_0y_2 & 0 & y_0y_1 & 0 & 0 & y_0y_2 & -y_1y_2 & y_0y_2 & -y_0y_2 & -y_0y_2 & -y_0y_2 & -y_0y_2 \\
-y_1 & -y_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_0-y_2 & y_1 & y_0-y_2 & 0 & -y_0-y_2 & -y_0 & -y_0+y_2 \\
0 & 0 & 0 & -y_1 & 0 & -y_1 & y_1 & 0 & 0 & y_2 & 0 & 0 & 0 & y_1+y_2 & y_1 & -y_2 \\
0 & 0 & -y_1 & y_1 & -y_0 & y_1 & y_1 & 0 & 0 & y_2 & 0 & y_2 & 0 & -y_1+y_2 & y_1 & -y_2 \\
0 & 0 & 0 & 0 & 0 & -y_0+y_2 & -y_0+y_2 & y_0-y_2 & y_0-y_2 & y_0-y_2 & -y_2 & -y_2 & -y_0 & y_2 & y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_0 & -y_0 & y_0 & y_0 & y_0-y_2
\end{array}\right)$$

$$B = \left(\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1y_2 & x_0y_1-x_1y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_0+y_2 & y_0-y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_0y_1 & x_1y_1-x_1y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_0 & y_0-y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_0+x_1 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y_0y_2-y_1y_2 & y_0y_1-y_1y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & x_0y_0-x_0y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_1+y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0y_1-x_1y_1+x_1y_2 & x_0y_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_2 & 0 & x_0y_2 & 0 & x_0y_0-x_1y_0-x_0y_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_1 & 0 & x_0y_1 & 0 & -x_1y_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & -x_0y_2 & x_0y_0-x_0y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1y_2 & y_0y_1 & 0 & y_0y_1-y_0y_2 & -y_0y_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_1 & y_1-y_2 & y_1-y_2 & -y_0
\end{array}\right)$$

This means $\operatorname{coker} A \cong \operatorname{coker} B$ splits as a direct sum of 7 modules.

A is diagonalizable $\iff$ $\operatorname{coker} A$ splits as a sum of twists.


ATLAS
EXPERIMENT
http://atlas.ch
Run: 189280

Falnameh, 16th century

Chludov Psalter, 9th century

Diagonalization, Direct Summands, and Splitting of Vector Bundles

Mahrud Sayrafi

Dissertation Defense

May 7, 2024

Warm-up: splitting of vector bundles on $\mathbb{P}^n$

Let $S = \Bbbk[x_0, \ldots, x_n]$ be a polynomial ring with $\deg x_i = 1$.
Let E be a vector bundle of rank r on $\mathbb{P}^n = \operatorname{Proj} S$.

Fact: vector bundles on $\mathbb{P}^n \iff$ projective modules over S.

Classical question

When does E *split* as a direct sum of line bundles?

- Grothendieck: any vector bundle on $\mathbb{P}^1$ splits as $E = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$.
- There are indecomposable bundles of rank $n-1$ on $\mathbb{P}^n$, $n \geq 3$.
- Horrocks–Mumford: an indecomposable rank 2 bundle on $\mathbb{P}^4$ from

$$\mathcal{O}_{\mathbb{P}^4}^5 \leftarrow \Omega^2_{\mathbb{P}^4}(2) \oplus \Omega^2_{\mathbb{P}^4}(2) \leftarrow \mathcal{O}_{\mathbb{P}^4}(-1)^5.$$

- Hartshorne's conjecture: there are no rank 2 bundles on $\mathbb{P}^n$, $n \geq 7$.

Warm-up: a splitting criterion for vector bundles on $\mathbb{P}^n$

Theorem (Horrocks '64)

If for all twists $d \in \mathbb{Z}$ and $i \geq 0$, $H^i(\mathbb{P}^n, E \otimes \mathcal{O}_{\mathbb{P}^n}(d))$ is equal to the cohomology of positive sums of line bundles, then E splits.

"If it walks like a duck and quacks like a duck, then it's a duck."

- Dolgachev '82: Weighted projective spaces
- Ottaviani '89: $\mathrm{Gr}(k, n)$ and quadric hypersurfaces in $\mathbb{P}^n$, $n \geq 4$
- Fulger–Marchitan '11: rank 2 bundles on Hirzebruch surfaces

Goal

Prove a similar splitting criterion over toric varieties.

What makes a splitting criterion strong?

When $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$:

Theorem (Eisenbud–Erman–Schreyer '15)

Let E be a vector bundle on X whose sheaf cohomology matches that of $\sum_{i=1}^{r} \mathcal{O}(\underline{d}_i)^{e_i}$, for $\underline{d}_i \in \mathbb{Z}^r$. If $\underline{d}_r \leq \cdots \leq \underline{d}_1$, then E splits.

- Holds in arbitrary dimension and Picard rank.
- But only when Nef $X = \mathrm{Eff}(X) =$ the positive quadrant.

When $X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_s)) \to \mathbb{P}^t$:

Theorem (Brown–S. '22)

Let E be a vector bundle on X whose sheaf cohomology matches that of $\sum_{i=1}^{r} \mathcal{O}(b_i, c_i)^{e_i}$. If $(b_r, c_r) \leq \cdots \leq (b_1, c_1)$, then E splits.

- Holds in arbitrary dimension, but always Picard rank $= 2$.
- Now Nef $X =$ the positive quadrant, but $\mathrm{Eff}(X)$ is larger.

Example: the Hirzebruch surface

Let $\mathcal{H}_t = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ be a smooth projective toric variety.
Let $S = \Bbbk[x_0, x_1, x_2, x_3]$ with $B = \langle x_0, x_2 \rangle \cap \langle x_1, x_3 \rangle$ and

degrees: $\begin{matrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}$

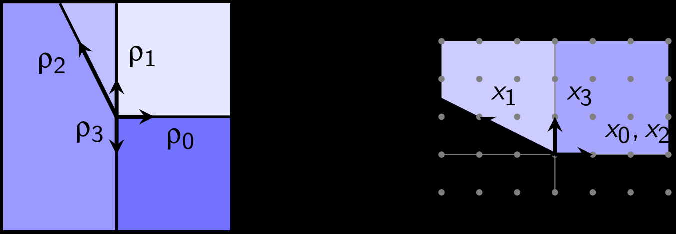


Figure: Left: fan of $\mathcal{H}_2$. Right: the cones $\operatorname{Nef}\mathcal{H}_2$ (dark blue) and $\operatorname{Eff}(\mathcal{H})_2$.

Cox: every coherent sheaf on $\mathcal{H}_t$ corresponds to a B-saturated finitely generated $\mathbb{Z}^2$-graded S-module.

Interlude: Fourier–Mukai Transforms

Consider the diagram:

$$\begin{array}{ccccc} & & X \times X & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ X & & & & X \end{array}$$

Let $\mathcal{K}$ be a *resolution of the diagonal* $\Delta = \mathrm{im}(X \to X \times X)$

$$\mathcal{K} \colon 0 \leftarrow \mathcal{O}_\Delta \leftarrow \mathcal{K}_1 \leftarrow \mathcal{K}_2 \leftarrow \cdots \leftarrow \mathcal{K}_n \leftarrow 0$$

Definition

The *Fourier–Mukai transform* with kernel $\mathcal{K}$ is the functor

$$\Phi_{\mathcal{K}} : \mathcal{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{D}^{\mathrm{b}}(X),$$

given by $\quad E \longmapsto \mathbf{R}\pi_{2*}(\pi_1^* E \otimes \mathcal{K}).$

- identity functor on $\mathcal{D}^{\mathrm{b}}(X)$ produces quasi-isomorphisms.
- only finitely many line bundles may appear in this complex.

Toy Example: $X = \mathbb{P}^2$

Compare Beilinson's resolution ('78):

$$\mathcal{O} \leftarrow \mathcal{O}(-1) \boxtimes \Omega(1) \leftarrow \mathcal{O}(-2) \boxtimes \Omega^2(2)$$

With resolutions by King ('97) and Canonaco–Karp ('07):

$$\mathcal{O}(-2,2) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O} \leftarrow \mathcal{O}(-2,1)^3 \oplus \mathcal{O}(-1,0)^3 \leftarrow \mathcal{O}(-2,0)^3$$

And by Hicks–Hanlon–Lazarev, Favero–Huang, Brown–Erman ('23):

$$\mathcal{O} \leftarrow \mathcal{O}(-1,-1)^3 \leftarrow \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2)$$

"Reimagined" proof of Horrocks' criterion on $\mathbb{P}^2$

The Beilinson spectral sequence has E_1 page:

$$\begin{array}{ccccc}
\mathbb{R}^2\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_0) & \longleftarrow & \mathbb{R}^2\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_1) & \longleftarrow & \mathbb{R}^2\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_2) \\
\mathbb{R}^1\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_0) & \longleftarrow & \mathbb{R}^1\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_1) & \longleftarrow & \mathbb{R}^1\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_2) \\
\pi_{2*}(\pi_1^*E\otimes\mathcal{K}_0) & \longleftarrow & \pi_{2*}(\pi_1^*E\otimes\mathcal{K}_1) & \longleftarrow & \pi_{2*}(\pi_1^*E\otimes\mathcal{K}_2) \\
k=0 & & k=1 & & k=2
\end{array}$$

But this is all we need:

$$\begin{array}{ccccc}
E_1^{-0,2} & \longleftarrow & E_1^{-1,2} & \longleftarrow & E_1^{-2,2} \\
E_1^{-0,1} & \longleftarrow & E_1^{-1,1} & \longleftarrow & E_1^{-2,1} \\
E_1^{-0,0} & \longleftarrow & E_1^{-1,0} & \longleftarrow & E_1^{-2,0} \\
k=0 & & k=1 & & k=2
\end{array}$$

- There is a complex with terms $\mathrm{Tot}(E_1)$ quasi-isomorphic to E.
- The terms are bundles $E_1^{-i,j} \cong H^j(\mathbb{P}^n, E(-i)) \otimes \Omega^i(i)$.

“Reimagined” proof of Grothendieck’s theorem or $\mathbb{P}^1$

The Beilinson spectral sequence on $\mathbb{P}^1$ has E_1 page:

$$\begin{array}{ccc} E_1^{0,1} & \longleftarrow & E_1^{-1,1} \\ E_1^{0,0} & \longleftarrow & E_1^{-1,0} \end{array} \quad \Longrightarrow \quad \begin{array}{cccc} E_1^{0,1} & \leftarrow & E_1^{-1,-1} & \\ & & \oplus & \\ & & E_1^{0,0} & \leftarrow E_1^{-1,0} \end{array}$$

Claim: we can always twist the vector bundle E until:

- $E_1^{0,0} = H^0(\mathbb{P}^1, E) \otimes \mathcal{O} \neq 0$
- $E_1^{-1,0} = H^0(\mathbb{P}^1, E(-1)) \otimes \Omega(1) = 0$

Then $E_1^{0,0}$ is a summand of E.

Translating the proof for Picard rank 2

Theorem (Brown–S. '22)

Let E be a vector bundle on X whose sheaf cohomology matches that of $\sum_{i=1}^{r} \mathcal{O}(b_i, c_i)^{e_i}$. If $(b_r, c_r) \leq \cdots \leq (b_1, c_1)$, then E splits.

We again have a Beilinson-"type" spectral sequence with first page:

$$
\begin{array}{ccccc}
E_1^{-0,2} & \longleftarrow & E_1^{-1,2} & \longleftarrow & E_1^{-2,2} \\
E_1^{-0,1} & \longleftarrow & E_1^{-1,1} & \longleftarrow & E_1^{-2,1} \\
E_1^{-0,0} & \longleftarrow & E_1^{-1,0} & \longleftarrow & E_1^{-2,0}
\end{array}
$$

$k=0$ $\quad k=1$ $\quad k=2$

- There is a complex with terms $\mathrm{Tot}(E_1)$ quasi-isomorphic to E.
- The terms are direct sums of line bundles.
- Twist E until the generators have degrees 0 and **"above"**.
- For all $a, b \leq 0$, if $E = \mathcal{O}_X(a, b)$ this page is "upper triangular".

Ingredient list for other toric varieties

When X is an arbitrary smooth projective toric variety:

Theorem (S. '24)

Let E be a vector bundle on X whose sheaf cohomology matches that of $\sum_1^r \mathcal{O}(D_i)^{e_i}$, $D_i \in \operatorname{Pic} X$. If $D_{i+1} - D_i$ is ample, then E splits.

- Holds in arbitrary dimension and Picard rank.
- Holds for arbitrary $\operatorname{Nef} X$ and $\operatorname{Eff}(X)$.
- But: the additional hypothesis on E is stronger ... probably!

Key step:

a resolution of the diagonal for X, such that we can prove for all D ample, if $E = \mathcal{O}(-D)$ the "page" is "upper triangular".

Candidate: Hanlon–Hicks–Lazarev's resolution of the diagonal.

New tool: vanishing theorems on line bundles and $\mathbb{Q}$-divisors.

What if E doesn't totally split?

Questions:

- can we explicitly deduce (or compute) summands of E?
- can we "block diagonalize" the presentation $E \leftarrow F \xleftarrow{f} G$? **Yes!**

[Mallory–S. '24]

Input: finitely generated graded module M over an algebra.

1. Compute $\mathrm{Hom}(M, M)$;
   Note: the degree zero part only, so **multigrading helps!**
2. Find element h corresponding to an idempotent;
   - Hope that generators of Hom are idempotents;
   - In finite characteristic, use a general endomorphism.
3. Split $M = \mathrm{im}\, h \oplus \mathrm{coker}\, h$;

And repeat!

[EES15] Eisenbud, Erman, Schreyer, *Tate Resolutions for Products of Projective Spaces*.

[BS22] Brown, Sayrafi, *A short resolution of the diagonal for smooth projective toric varieties of Picard rank 2*.

[HHL23] Hanlon, Hicks, Lazarev, *Resolutions of toric subvarieties by line bundles and applications*.

[FH23] Favero, Huang, *Rouquier dimension is Krull dimension for normal toric varieties*.

[BE23] Brown, Erman, *A short proof of the Hanlon-Hicks-Lazarev Theorem*.