Diagonalization, Direct Summands, and Resolutions of the Diagonal

A dissertation
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by

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Abstract

This thesis concerns the interplay of algebraic geometry and multigraded commutative algebra, particularly in the setting of toric geometry. Recent years have seen a flourishing of new conjectures and techniques relating algebraic invariants like multigraded syzygies and the geometry of toric varieties, with new ideas originating from homological mirror symmetry to applied algebraic geometry.

An active program in commutative algebra seeks to construct *virtual resolutions* of ideals and module over multigraded polynomial rings known as *Cox rings* in order to study algebraic geometry over toric varieties. We solve several problems in different aspects of this program:

- Chapter 3: Uniqueness of virtual resolutions on products of projective spaces;
- Chapter 4: Existence of short virtual resolutions and Orlov's conjecture in Picard rank 2;
- Chapter 5: Horrocks' splitting criterion for vector bundles on smooth projective toric varieties;
- Chapter 6: Castelnuovo–Mumford regularity and truncations of multigraded modules;
- Chapter 7: Bounds on Castelnuovo–Mumford regularity of modules and powers of ideals;
- Chapter 8: Computing direct summand decompositions of multigraded modules and sheaves.

Background for the key concepts used in carrying out the goals above is given in Chapter 2.

Introduction for the casual reader

A defining step in understanding any complex structure, whether in mathematics or in nature, is to identify the simplest components which do not split any further. Discovering the primary colors, musical notes, or the elementary particles are all different expressions of this same instinct. Then comes our imagination of possible compositions, and the question becomes: are there any rules governing how these simple components can be meaningfully assembled?

Consider the following mathematical theorem: given a spherical object with hair growing from every point on its surface, it is impossible to comb the hair flat without at least one spot left uncombed. For instance, if strands of hair on a globe represent the direction of wind on Earth at any given location, then the theorem guarantees that at any given time there is at least one spot somewhere on Earth where the air is still. Crucially, if the surface of Earth was instead the shape of a doughnut, then it *would* be possible for rotating air flow to cover the entire surface. The wind direction here is a basic example of a *line bundle* on a surface—also known as a vector field in calculus courses. Line bundles are simple structures assembled into higher-dimensional structures called *vector bundles*. This example demonstrates how to learn *homological* information (e.g. the number of holes) about the underlying geometric object (in this case the surface of a planet) through studying the rules governing which vector bundles may be assembled on it.

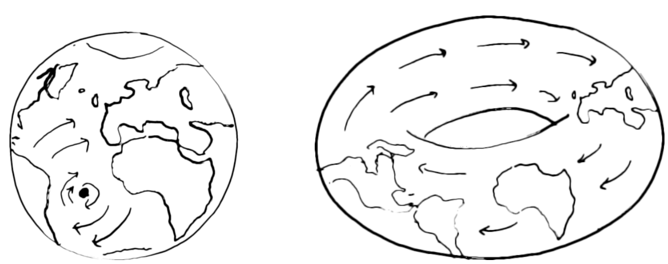


Figure 1: Unlike our planet, cyclones on a toroidal Earth may not have an eye of the storm.

In this thesis, the underlying geometric objects are sets of solutions to systems of polynomial equations which we call *varieties*. For example, the variety defined by the solutions of the equation $x^2 + y^2 + z^2 = 1$ is a sphere with radius one. A *toric variety* is a special type of variety which enjoys a kind of symmetry frequently found in applications within mathematics and across the sciences. An early source of this understanding dates back to Renaissance artists and architects who used perspective in their paintings and sketches, essentially discovering the *projective plane*, denoted $\mathbb{P}^2$.

From an algebraic perspective, vector bundles are also described by systems of polynomial functions. In case of $\mathbb{P}^2$, these functions are *homogeneous* polynomials in 3 variables with a *standard grading*. For instance, if x, y, and z have degree 1 then the functions $yz - x^2$ and $x^2y + xyz + yz^2$ are homogeneous because all added terms have the same total degree.

For more general toric varieties, the variables in the polynomials may be *multigraded*: for a toric variety known as the *Hirzebruch surface of type 1*, we have $\text{degree}(x) = \text{degree}(y) = (1, 0)$ while $\text{degree}(z) = (0, 1)$ and $\text{degree}(w) = (-1, 1)$, so that the polynomial $x^2w + yz$ will be homogeneous of degree $(1, 1)$. This introduces a wide range of new possibilities for geometric structures.

The results in this thesis are aimed at extending the classical, standard graded theory over $\mathbb{P}^n$ to the multigraded world of toric varieties by finding the rules for building vector bundles on them. Are there criteria for checking whether a vector bundle on a toric variety can be split as a *direct sum* of line bundles? See Chapters 4 and 5. Can a computer find the components, or *direct summands*, of the vector bundle? See Chapter 8. Most importantly, am I able to explain an explicit application of my thesis to a non-expert?

Consider a second mathematical theorem from linear algebra: an $n \times n$ matrix A is *diagonalizable* (i.e. can be written as $A = PDP^{-1}$ with D a diagonal matrix) if and only if it has n linearly independent eigenvectors. For example, the following is a diagonalization of a 5×5 matrix:

$$\begin{pmatrix} 2 & -3 & 3 & -3 & 3 \\ -3 & 8 & -3 & 10 & 0 \\ 7 & 3 & 2 & -1 & -3 \\ 1 & -3 & 3 & -2 & 3 \\ 13 & -5 & 5 & -13 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -3 & -3 & 0 & 0 \\ 4+2i & 4-2i & 3 & -1 & 1 \\ 3 & 3 & 3 & 1 & 1 \\ -3 & -3 & -3 & 1 & 0 \\ -2-i & -2+i & 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2+3i & 0 & 0 & 0 & 0 \\ 0 & 2-3i & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \cdot P^{-1}$$

Finding diagonalizations of numerical matrices has applications in almost any field where matrices are used, particularly physics and computer science.

What if we instead considered matrices of homogeneous polynomials? Such matrices represent systems of polynomial functions. The following matrix, for example, represents the system of polynomial functions that describes a particular vector bundle on a surface[1]. What does it mean to diagonalize it?

$$
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & x_1y_0y_1 & 0 & 0 & x_1y_0y_2 & 0 & 0 & 0 & x_0y_1y_2 & 0 & 0 \\
-x_0y_1+x_1y_1-x_1y_2 & -x_0y_2 & 0 & 0 & 0 & x_1y_2 & x_1y_2 & x_1y_2 & -x_1y_2 & -x_0y_1+x_1y_1-x_1y_2 & -x_1y_2 & 0 & x_1y_2 & x_1y_2 & x_1y_2 \\
0 & 0 & 0 & x_1y_1 & 0 & 0 & 0 & 0 & -x_1y_2 & 0 & -x_1y_2 & 0 & -x_0y_1-x_1y_2 & -x_0y_1 & -x_1y_2 \\
0 & 0 & -x_1y_2 & 0 & x_0y_0-x_1y_0-x_0y_2 & x_1y_2 & x_1y_2 & -x_1y_2 & -x_1y_2 & 0 & x_0y_2-x_1y_2 & x_0y_2 & -x_0y_2+x_1y_2 & x_0y_2+x_1y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_1y_2 & -x_1y_2 & x_1y_2 & x_1y_2 & x_0y_1-x_1y_1+x_1y_2 & x_1y_2 & 0 & -x_1y_2 & -x_1y_2 & -x_1y_2 \\
0 & 0 & x_1y_1 & 0 & x_1y_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0y_1 & x_1y_2 \\
0 & 0 & 0 & 0 & 0 & -x_1y_0 & 0 & -x_1y_0 & -x_1y_0 & 0 & x_0y_2 & x_0y_0 & 0 & 0 & 0 \\
x_0y_1 & x_0y_0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & 0 & -x_1 & -x_1 & -x_1 & -x_1 & 0 & x_0-x_1 & x_0 & x_0 & -x_0-x_1 & 0 \\
y_1y_2 & y_0y_2 & 0 & 0 & 0 & -y_0y_1+y_1y_2 & -y_0y_1+y_1y_2 & 0 & 0 & -y_1y_2 & 0 & 0 & -y_1y_2 & y_1y_2 & 0 \\
0 & 0 & y_1y_2 & 0 & y_0y_2 & 0 & y_0y_1 & 0 & y_0y_2 & -y_1y_2 & y_0y_2 & -y_0y_2 & -y_0y_2 & -y_0y_2 & -y_0y_2 \\
-y_1 & -y_0 & 0 & 0 & 0 & 0 & 0 & 0 & y_0-y_2 & y_1 & y_0-y_2 & 0 & -y_0-y_2 & -y_0 & -y_0+y_2 \\
0 & 0 & 0 & -y_1 & 0 & -y_1 & y_1 & 0 & y_2 & 0 & 0 & 0 & y_1+y_2 & y_1 & -y_2 \\
0 & 0 & -y_1 & y_1 & -y_0 & y_1 & y_1 & 0 & y_2 & 0 & y_2 & 0 & -y_1+y_2 & y_1 & -y_2 \\
0 & 0 & 0 & 0 & 0 & -y_0+y_2 & -y_0+y_2 & y_0-y_2 & y_0-y_2 & -y_2 & -y_2 & -y_0 & y_2 & y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_0 & -y_0 & y_0 & y_0 & y_0-y_2
\end{pmatrix}
$$

An explicit application of this thesis is an algorithm for finding invertible matrices P, Q with numerical entries such that $A = PBQ$ and B is the following matrix with blocks along the diagonal:

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1y_2 & x_0y_1-x_1y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_0+y_2 & y_0-y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_0y_1 & x_1y_1-x_1y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_0 & y_0-y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_0+x_1 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & y_0y_2-y_1y_2 & y_0y_1-y_1y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & x_0y_0-x_0y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_1+y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0y_1-x_1y_1+x_1y_2 & x_0y_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_1 & -y_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_2 & 0 & x_0y_2 & 0 & x_0y_0-x_1y_0-x_0y_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_1 & 0 & x_0y_1 & 0 & -x_1y_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1y_0 & -x_0y_2 & x_0y_0-x_0y_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1y_2 & y_0y_1 & 0 & y_0y_1-y_0y_2 & -y_0y_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_1 & y_1-y_2 & y_1-y_2 & -y_0
\end{pmatrix}
$$

Mathematically, this block diagonalization is equivalent to splitting the vector bundle into its simplest components. This example demonstrates that while six of the seven[2] blocks correspond to line bundles, the rightmost block is an *indecomposable* vector bundle on the surface. Learning what governs the existence of such vector bundles, while harder to visualize with a cartoon, is important for studying the geometry of the underlying varieties, and a natural motivation for my research.

[1] This is not an introduction for the experts, but if you are one: the sheafification of the cokernel of A over $\mathbb{Z}/3$ is the Frobenius pushforward of the structure sheaf of the blow-up of $\mathbb{P}^2$ in four points as a variety embedded in $\mathbb{P}^1 \times \mathbb{P}^2$.

[2] For those who only count six blocks: if you look carefully, there is a 0×1 block hiding in the top left corner!

Contents

1 Introduction

Minimal free resolutions, or syzygies, of ideals over a standard graded polynomial ring capture subtle geometric properties of subvarieties of projective space, such as dimension, degrees of defining equations, and their deformations [GLP83, Gre84, EL93, Eis05]. However, free resolutions often fail to faithfully reflect the geometry of varieties embedded in other spaces.

1.1 Construction and Uniqueness of Short Virtual Resolutions

Introduced by Berkesch, Erman, and Smith, virtual resolutions are the homological substitutes for free resolutions over toric varieties, which are a class of varieties ubiquitous in birational geometry [Rei83, BMSZ18], combinatorics [Sta80, AHK18], and other fields.

In order to construct virtual resolutions over a toric variety X, we use the toric algebra-geometry dictionary to translate between sheaves on X and multigraded modules over the Cox ring S [Cox95]. Here, S is a polynomial ring graded by $\operatorname{Pic} X$, the group of isomorphism classes of line bundles on X.

Berkesch–Erman–Smith conjectured a version of Hilbert's Syzygy Theorem for toric varieties, asking whether any finitely generated, graded S-module M admits a virtual resolution of length at most $\dim(X)$, and proved it for a products of projective spaces $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ [BES20].

In Chapter 3, taken from [BCHS21], we establish the following uniqueness theorem for minimal virtual resolutions with terms in a full strong exceptional collection for $\mathbb{P}^{\mathbf{n}}$.

Theorem A ([BCHS21]; also Theorem 3.5.4)**.** *Let S be the Cox ring of a product of projective spaces $\mathbb{P}^{\mathbf{n}}$ and suppose $F_\bullet$ and $G_\bullet$ are minimal virtual resolutions of an S-module M. If every term is a direct sum of $S(-\mathbf{a})$ for $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$, then $G_\bullet$ and $F_\bullet$ are isomorphic.*

Our proof of Theorem A reduces the problem to the uniqueness of minimal projective resolutions over certain graded associative algebras, using established techniques from derived algebraic geometry and the representation theory of finite-dimensional algebras.

1.2 Orlov's Conjecture and Horrocks' Splitting Criterion

Beilinson's resolution of the diagonal over a projective space is a powerful tool in algebraic geometry [Beĭ78a]. This classical construction, which appears in many results from both derived algebraic geometry and commutative algebra [GLP83, Kap88, AO89], may be used to prove strong results about the bounded derived category of coherent sheaves $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$.

Consider the sheaf $\mathcal{O}_\Delta$ which cuts out the diagonal subvariety $\Delta \subset X \times X$. By a *resolution of the diagonal*, we mean a locally free resolution $\mathcal{K}$ of $\mathcal{O}_\Delta$. Taking a *Fourier–Mukai transform* with kernel given by Beilinson's resolution yields a representation of any object in $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ as a complex of vector bundles, called a *Beilinson monad*, which has been used to great effect in computational algebraic geometry [ES03, ES09].

In Chapter 4, taken from [BS22], we construct explicit resolutions of the diagonal for smooth projective toric varieties of Picard rank 2 in order to prove existence of short virtual resolutions. This class includes all smooth projective varieties whose Cox ring is a bigraded polynomial ring. By a result of Kleinschmidt [Kle88], every such toric variety is the projectivization of a sum of line bundle on a projective space $\mathbb{P}^r$, which is to say $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^r}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^r}(a_s))$ for $a_i \in \mathbb{Z}$.

Theorem B ([BS22]; also Theorem 4.1.1)**.** *Let X be a smooth toric variety of Picard rank 2, then any finitely generated, graded S-module M admits a virtual resolution of length at most* $\dim(X)$*.*

In 2023, Hanlon–Hicks–Lazarev and Brown–Erman proved the conjecture for all smooth toric stacks [HHL23, BE23b], with the former drawing ideas from homological mirror symmetry. Each of these results also established new cases of Orlov's conjecture, illustrating the usefulness of multigraded commutative algebra in derived algebraic geometry (c.f. [Rou08, BFK19, FH23]).

Conjecture 1.2.1 ([Orl09, Conj. 10])**.** *Let X be a smooth projective scheme. The Rouquier dimension of the bounded derived category of coherent sheaves $\mathcal{D}^{\mathrm{b}}(X)$ is equal to* $\dim(X)$*.*

Our method is analogous to Weyman's "geometric technique" for building free resolutions as pushforwards from certain fibrations [Wey03, §5]. Moreover, the explicit structure of our resolutions opens the door to other applications in algebraic geometry, such as the study of vector bundles on toric varieties.

A Splitting Criterion for Vector Bundles

In 1964, Horrocks gave criteria under which a vector bundle on $\mathbb{P}^n$ is a sum of line bundles [Hor64]. The importance of this result is revealed by its relationship with Hartshorne's conjecture that every rank 2 vector bundle on $\mathbb{P}^7$ splits, which implies that any codimension 2 subvariety is a complete intersection [Har74]. Horrocks' splitting criterion has been generalized to products of projective spaces [CM05, EES15, Sch22], Grassmannians and quadrics [Ott89], and Hirzebruch surfaces [Buc87, AM11, FM11, Yas15], among others.

We use our construction to prove a the following toric splitting criterion.

Theorem C ([BS22])**.** *Let X be an arbitrary smooth projective toric variety of Picard rank 2. Suppose $\mathcal{E}$ and $\mathcal{E}' = \bigoplus_{i=1}^t \mathcal{O}(a_i, b_i)^{m_i}$ are vector bundles on X with $(a_i, b_i) \leq (a_{i-1}, b_{i-1})$ for all i. If $H^p(X, \mathcal{E}(c, d)) = H^p(X, \mathcal{E}'(c, d))$ for all $c, d \in \mathbb{Z}$ and $p \geq 0$, then $\mathcal{E} \cong \mathcal{E}'$.*

The results of [EES15] hold in any Picard rank, but only for products of projective spaces, where all effective divisors are nef. Theorem C, however, holds true even when the nef cone is a strict subset of the effective cone in $\operatorname{Pic} X \simeq \mathbb{Z}^2$.

In Chapter 5, taken from [Say24], we further extend this theorem to all smooth projective toric varieties using the construction of Hanlon–Hicks–Lazarev.

Theorem D ([Say24]; also Theorem 5.1.1)**.** *Suppose $\mathcal{E}$ and $\mathcal{E}' = \bigoplus_{i=1}^t \mathcal{O}(D_i)^{m_i}$ are vector bundles on a smooth projective toric variety X such that $D_{i+1} - D_i$ is ample for all i. If $H^p(X, \mathcal{E} \otimes \mathcal{L}) = H^p(X, \mathcal{E}' \otimes \mathcal{L})$ for all $\mathcal{L} \in \operatorname{Pic} X$ and $p \geq 0$, then $\mathcal{E} \cong \mathcal{E}'$.*

As noted, the main ingredient in each of the above results is the construction of a resolution of the diagonal $\mathcal{K}$ for the toric variety. The corresponding *Fourier–Mukai transform* is a functor defined as the composition of derived functors

$$\Phi_{\mathcal{K}} \colon \mathcal{D}^{\mathrm{b}}(X) \xrightarrow{\pi_1^*} \mathcal{D}^{\mathrm{b}}(X \times X) \xrightarrow{-\otimes \mathcal{K}} \mathcal{D}^{\mathrm{b}}(X \times X) \xrightarrow{\mathbf{R}\pi_{2*}} \mathcal{D}^{\mathrm{b}}(X),$$

where π_i are the projections from $X \times X$ onto the factors. Importantly, the functor $\Phi_{\mathcal{K}}$ is simply the identity functor in the derived category $\mathcal{D}^{\mathrm{b}}(X)$, which is to say that it produces highly structured quasi-isomorphisms encapsulating the sheaf cohomology of a vector bundle and its twists [Huy06].

Therefore the key to proving Conjecture 3.1.2 is constructing a resolution of the diagonal $\mathcal{K}$ consisting of direct sums of line bundles whose length equals $\dim(X)$. In particular, $\mathcal{K}$ can be presented as the sheafification of a complex of free modules over the Cox ring of $X \times X$, making this a construction concerning virtual resolutions over $S \otimes_{\Bbbk} S$.

A Splitting Algorithm for Sheaves and Multigraded Modules

More generally, the problem of determining the indecomposable summands of a sheaf or module or finding isomorphism classes of indecomposable modules with certain properties is ubiquitous in commutative algebra, group theory, representation theory, and other fields.

Within commutative algebra, for instance, the classification of Cohen–Macaulay local rings R for which there are only finitely many indecomposable maximal Cohen–Macaulay R-modules (the *finite CM-type* property), or determining whether iterated Frobenius pushforwards of a Noetherian ring in positive characteristic have finitely many isomorphism classes of indecomposable summands (the *finite F-representation type* property) are two well-established research problems. For both these problems, and many others, making and testing conjectures depends on finding summands of modules and verifying their indecomposability.

In Chapter 8, taken from forthcoming work with Mallory, we give an explicit method for computing the indecomposable direct summands of multigraded modules and sheaves, which we also implement in the computer algebra software Macaulay2 [M2].

1.3 Castelnuovo–Mumford Regularity and Truncations

Serre's vanishing, an important theorem in algebraic geometry, states that all cohomological subtleties of a coherent sheaf $\mathscr{F}$ on the projective space $\mathbb{P}^n$ disappear after tensoring by a high enough power of the line bundle $\mathcal{O}(1)$. Castelnuovo–Mumford regularity measures the smallest power where this happens, and hence controls the algebraic complexity, or positivity, of coherent sheaves [Mum66]. Remarkably, if M is a graded module corresponding to $\mathscr{F}$, Eisenbud and Goto proved that the regularity of $\mathscr{F}$ can be computed in terms of (1) degrees of syzygies, (2) local cohomology, and (3) resolutions of truncations of M [EG84].

Motivated by toric geometry, Maclagan and Smith leveraged the correspondence between quasi-coherent sheaves on a simplicial toric variety X and graded modules over the multigraded Cox ring S to define and relate multigraded regularity for both, in essence generalizing Eisenbud–Goto's condition (2) [MS04]. In this setting, regularity is a subset $\operatorname{reg} M \subset \operatorname{Pic} X$.

When $X = \mathbb{P}^n$, the minimal element of multigraded regularity recovers the classical Castelnuovo–Mumford regularity. However, when the Picard rank is more than one, discerning an equivalent algebraic condition on truncations and syzygies which translates to Maclagan and Smith's definition has been an open problem [HW04, SVTW06, Hà07, CM07, BC17, CN20].

In Chapters 6 and 7, taken from [BCHS21, BCHS22], we use techniques from derived algebraic geometry to prove several theorems about the relationship between multigraded regularity, truncations, and syzygies of M, as well as asymptotic behavior of regularity of powers of ideals.

Generalizing Eisenbud–Goto's condition (3) for a product of projective spaces X, we show that under a mild saturation hypothesis, multigraded Castelnuovo–Mumford regularity is determined by a different linearity condition, which we call *quasilinearity* (see Definition 6.3.3).

Theorem E ([BCHS21]; also Theorem 6.3.6)**.** *Let M be a finitely generated, graded S-module with $H^0_B(M) = 0$. Then $\mathbf{d} \in \operatorname{reg} M$ if and only if the free resolution of the truncation $M_{\geq \mathbf{d}}$ is quasilinear.*

We also show that the graded Betti numbers of M at best bound its multigraded regularity, which is emblematic of nuances of commutative algebra over multigraded Cox rings. As an application, we found the regularity of all complete intersections of ample divisors.

A recurring technique in my research involves resolutions of multigraded truncations. For a finitely generated, $\operatorname{Pic}(X)$-graded S-module M with free presentation $0 \leftarrow M \leftarrow F \leftarrow F'$ we define the (nef-)truncation $M_{\geq \mathbf{d}}$ at a (multi)degree $\mathbf{d}$ to be the S-module with presentation

$$0 \leftarrow M_{\geq \mathbf{d}} \leftarrow F_{\geq \mathbf{d}} \leftarrow F'_{\geq \mathbf{d}},$$

where $F_{\geq \mathbf{d}}$ is the submodule generated by graded pieces $F_{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{d} + \operatorname{Nef} X$, and similarly for $F'_{\geq \mathbf{d}}$ (c.f. [MS04, Def. 5.1]). The significance of truncations lies in the fact that a free resolution of $M_{\geq \mathbf{d}}$ is a virtual resolution for M. Syzygies of nef-truncations are also tied to Oda's conjecture that smooth projective toric varieties are projectively normal [Oda97, Mac07].

To show Theorem E, we completely describe the twists that occur in minimal free resolutions of

$M_{\geq \mathbf{d}}$ when M saturated and $\mathbf{d}$-regular. In particular, these twists from a *full strong exceptional collection of line bundles* for the ambient product of projective spaces.

Working over an arbitrary smooth projective toric variety, we also use truncations of modules with respect to the net cone to expand the body of literature by Smith–Swanson, Cutkosky–Herzog–Trung, Kodiyalam, and others [BEL91, CEL01, Swa97, SS97, CHT99, Kod00, CK11] on the asymptotic behavior of regularity for powers of an ideal $I \subset S$ by bounding $\operatorname{reg}(I^n)$ between two linearly translating regions.

Theorem F ([BCHS22]; also Theorem 7.4.1)**.** *There exist degrees $\mathbf{d}_1, \mathbf{d}_2, \mathbf{a} \in \operatorname{Pic} X$ so that for all $n > 0$ we have:*

$$n \cdot \mathbf{d}_1 + \mathbf{a} + \operatorname{Nef} X \subseteq \operatorname{reg}(I^n) \subseteq n \cdot \mathbf{d}_2 + \operatorname{Nef} X.$$

It is worth emphasizing that this result holds in arbitrary toric multigradings, where the cones of nef and effective divisors may differ. In this case, we find finitely generated modules with torsion whose regularity has infinitely many minimal elements. This illustrates another subtle but deep divergence from the standard graded case, requiring new techniques.

The key to addressing the above divergences is relating the syzygies of truncated modules $M_{\geq \mathbf{d}}$ to the sheaf cohomology of $\widetilde{M}$ using Beilinson's resolution of the diagonal for products of projective spaces and the corresponding Fourier–Mukai transform [BES20].

Remarkably, we show that when M is saturated and $\mathbf{d}$-regular, the minimal free resolution of $M_{\geq \mathbf{d}}$ is isomorphic to the Fourier–Mukai transform of M (see [BCHS21, Sec. 3]. Specifically, we introduce a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the sheaf cohomology of $\widetilde{M}$ tensored with truncated Koszul complexes, generalizing the machinery of Koszul homology introduced by Green [Gre84].

2 Background

Some of the material in this chapter originally appeared in the author's Master's thesis.

2.1 Simplicial Toric Varieties

Toric varieties are algebraic varieties with an intrinsic combinatorial structure, making them suitable test subjects for conjectures in algebraic geometry. At the same time, there are interesting problems in algebraic combinatorics whose solutions involve toric varieties.

In this section we briefly recall the terminology for normal toric varieties and give a number of key running examples of toric varieties in the end. For an in depth exposition see [CLS11] and [Ful93].

Definition 2.1.1. A *toric variety* of dimension n over $\mathbb{C}$ is a variety X containing the torus $\mathbb{T} = (\mathbb{C}^*)^n$ as a dense open subset, with an action of $\mathbb{T}$ on X that extends the action of $\mathbb{T}$ on itself.

The structure of an *affine toric variety* can be characterized by a *strongly convex rational cone*. Concretely, let N be a lattice in $\mathbb{Z}^n$ and σ a cone in N, then the dual cone $\sigma^\vee$ in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the set of vectors in $M \otimes_{\mathbb{Z}} \mathbb{R}$ with non-negative inner product on σ. This gives a commutative semi-group $S_\sigma = \sigma^\vee \cap M = \{\eta \in M : \eta(\nu) \geq 0 \text{ for all } \nu \in \sigma\}$, and an open affine toric variety $U_\sigma = \mathrm{Spec}\ \mathbb{C}[S_\sigma]$ corresponding to its group algebra. The gluing data for a general toric variety comes from a *fan* of cones.

Definition 2.1.2. A *fan* Σ of strongly convex polyhedral cones in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a set of rational strongly convex polyhedral cones $\sigma \in N_{\mathbb{R}}$ such that:

1. each face of a cone in Σ is also a cone in Σ;

2. the intersection of two cones in Σ is a face of each of the cones.

(2.1.3) $\mathbb{P}^1$	(2.1.4) $\mathbb{P}^2$	(2.1.5) $\mathbb{P}^1 \times \mathbb{P}^1$	(2.1.6) $\mathbb{F}_1$	$\mathrm{Bl}_x(\mathbb{F}_1)$	dP6

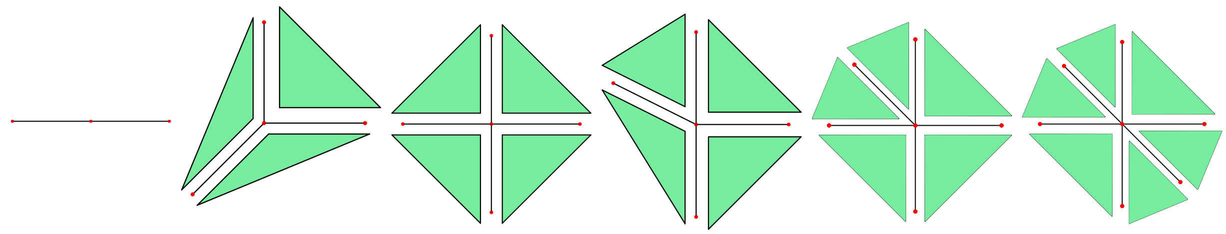


Table 2.1: Fans of $\mathbb{P}^1$ and the five smooth toric Fano surfaces

With this information in hand, a normal toric variety X is fully characterized as follows: given a fan Σ, $X(\Sigma)$ is assembled by gluing the affine toric subvarieties U_σ for each $\sigma \in \Sigma$. The toric variety X is smooth (resp. simplicial) if every cone $\sigma \in \Sigma$ is generated by a subset of a $\mathbb{Z}$-basis (resp. $\mathbb{R}$-basis) of N, and it is projective if the fan Σ is complete; i.e. all points are in a maximal cone.

Let $\Sigma(i)$ denote the set of i-dimensional cones in Σ. A 1-dimensional cone $\rho \in \Sigma(1)$ is referred to as a *ray* and corresponds to an irreducible $\mathbb{T}$-invariant *Weil divisor* D_ρ on X. Hence we identify $\mathrm{Div}(X) = \mathbb{Z}^{\Sigma(1)}$ and define $\mathrm{CDiv}(X)$ to be the subgroup of $\mathbb{T}$-invariant *Cartier divisors* on X. To any Cartier divisor D on a normal variety X we can associate an *invertible sheaf* $\mathcal{L} = \mathcal{O}_X(D)$ which is a locally free sheaf of sections of a *line bundle* $L \to X$. Isomorphism classes[1] of such line bundles on X define the *Picard group* $\operatorname{Pic} X$, with tensor product as the group operation.

The relationship between these groups is captured in a commutative diagram of $\mathbb{Z}$-modules

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\mathrm{div}"] \arrow[d, equal] & \mathrm{CDiv}(X) \arrow[r] \arrow[d, hook] & \operatorname{Pic} X \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathrm{Div}(X) \arrow[r] & A_{n-1}(X) \arrow[r] & 0,
\end{tikzcd} \tag{2.1.1}$$

where $\mathrm{div}(m) = \sum_\rho \langle m, n_\rho \rangle D_\rho$ with $n_\rho \in N$ the generator of $\rho \subset N_\mathbb{R}$ computes the orders of poles and zeros along the divisors D_ρ. See [Cox95] or [Har77, Ch.II §6] for further context.

Key examples

Example 2.1.3. The Projective Line $\mathbb{P}^1$

Let V be a 2-dimensional $\Bbbk$-vector space with dual space $W = V^*$. Classically, the projective space $\mathbb{P}^1 = \mathbb{P}V$ is defined to be the set of 1-dimensional subspaces in V.

[1] Two divisors are *linearly equivalent*, denoted $D \sim D'$, if $D - D'$ is a *principal divisor*.

As a projective variety, $\mathbb{P}^1$ is the simplest case of the Proj construction: let $S = \operatorname{Sym} W$ be the symmetric algebra on W and $E = \bigwedge V$ the exterior algebra on V; the set $\mathbb{P}^1 = \mathbb{P}(W) = \operatorname{Proj} S$ is the set of all homogeneous prime ideals that are strictly contained in the irrelevant ideal $\mathfrak{m} = \langle x_0, x_1 \rangle$. Note that S is a polynomial ring with generators corresponding to a set of coordinates on W, while E is a skew symmetric algebra with generators corresponding to the dual basis. In particular, observe that $H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big) = W$ and $H^n\big(\mathbb{P}^1, \omega_{\mathbb{P}^1}\big) = V$. Crucially, since $\operatorname{Hom}_{\Bbbk}(E, \Bbbk) \simeq E$, the exterior algebra is Gorenstein and has finite dimension over the field.

The projective line $\mathbb{P}^1$ is the simplest non-affine example. Let Σ be a fan with cones $\mathbb{R}_{\geq 0}$, $\{0\}$, and $\mathbb{R}_{\leq 0}$, corresponding to the affine toric varieties $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{C}$, respectively, which are glued via

$$\mathbb{C}\left[\frac{x_0}{x_1}\right] \hookrightarrow \mathbb{C}\left[\frac{x_0}{x_1}, \frac{x_1}{x_0}\right] \hookleftarrow \mathbb{C}\left[\frac{x_1}{x_0}\right].$$

Generalizing this definition yields $\mathbb{P}^n$ as the set of lines in a vector space of dimension $n+1$.

Example 2.1.4. The Projective Plane $\mathbb{P}^2$

A second way of constructing a toric variety is through the geometric quotient construction: let $\mathbb{C}^*$ act on $\mathbb{C}^3 \setminus \{0\}$ by scalar multiplication; since the $\mathbb{C}^*$-orbits are closed, $\mathbb{P}^2 = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$ is a geometric quotient. This construction extends to arbitrary n as $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. See [Cox95, Thm. 2.1] or [CLS11, Ch.5] for the quotient construction of toric varieties.

Example 2.1.5. Product of Projective Spaces $\mathbb{P}^1 \times \mathbb{P}^1$

The Proj construction of a projective variety can be further extended to the relative case: let $\mathscr{S} = \mathcal{O}_{\mathbb{P}^1}[y_0, y_1]$ be a sheaf of graded algebras over $\mathbb{P}^1$ with $\deg y_i = 1$; the projectivization $\mathbb{P}^1 \times \mathbb{P}^1 = \operatorname{Proj} \mathscr{S}$ is a rational ruled surface which also carries the structure of a toric variety induced by the equivariant morphism $\pi_1 \colon \operatorname{Proj} \mathscr{S} \to \mathbb{P}^1$. In particular, the pullback $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ yields the invertible sheaf $\mathcal{O}(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Seen through this lens, the definition of $\mathbb{P}^1$ as the Proj of a graded ring is a relative Proj of the sheaf $\mathcal{O}_{\operatorname{Spec}\mathbb{C}}[x_0, x_1]$. Repeating this construction yields products of projective spaces with higher Picard rank, such as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 =: \mathbb{P}^{(1,1,1)}$, while increasing the number of generators in $\mathscr{S}$ increases the dimension of each projective space factor.

Recall that a vector bundle on a variety X is a locally free sheaf on X. In the language of modules over the coordinate ring, vector bundles correspond to modules which are free after localization at a sufficiently small neighborhood of every point. An important class of varieties, called *projective bundles*, arise as projectivizations of vector bundles.

Example 2.1.6. The Hirzebruch Surface $\mathbb{F}_a$

Another construction is the projective bundle $\mathbb{P}(\mathscr{E})$ associated with a locally free sheaf $\mathscr{E}$: let $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}$ be a locally free coherent sheaf on $\mathbb{P}^1$, and consider the sheaf of graded algebras $\mathscr{S} = \bigoplus_{m \geq 0} \operatorname{Sym}^m \mathscr{E}^\vee$. The projective bundle $\mathbb{F}_a = \mathbb{P}(\mathscr{E}) = \operatorname{Proj} \mathscr{S}$ is the Hirzebruch surface of type a. In particular, there is a projection $\pi\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ which induces a natural surjection $\pi^*\mathscr{E} \to \mathcal{O}_{\mathbb{P}^1}(1)$. See [Har77, Ch.II §7] for further context on this construction.

Remark 2.1.7. While projectivization of a toric vector bundle on a toric variety yields a variety with a $\mathbb{T}$-action, it is only a toric variety if the vector bundle splits as a sum of line bundles [Oda78, §7].

2.2 Cox Rings of Algebraic Varieties

The Cox ring of an algebraic variety X is an invariant ring that captures the birational geometry of X. Initially defined for a toric variety by Cox [Cox95], Hu and Keel [HK00] extended the definition for a smooth projective variety X with divisor class group $\operatorname{Cl}(X)$ to be

$$S \coloneqq \bigoplus_{[D] \in \operatorname{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

with multiplicative structure defined by a choice of divisor classes that generate $\operatorname{Cl}(X)$. In particular, S is a multigraded commutative ring with grading induced by $\operatorname{Cl}(X)$, and if S is finitely generated we call X a *Mori dream space* (MDS). This terminology was used by Hu and Keel, who studied the moduli spaces $\overline{M}_{0,n}$ from the perspective of the Mori program in birational geometry. Other examples include Grassmannians and flag varieties, projective toric varieties, square determinantal varieties, many complete intersection rings, and smooth Fano varieties over $\mathbb{C}$ [BCHM10].

A smooth projective variety with finitely generated and free Picard group is a Mori Dream Space precisely when its Cox ring is finitely generated as an algebra over the base field. This provides a strong link between the birational geometry of a Mori Dream Space and the birational geometry of an ambient toric variety.

Mori dream spaces generalize projective spaces in a natural way. Whereas $\mathbb{P}^n$ can be defined from the standard graded polynomial ring and homogeneous maximal ideal $\mathfrak{m}$ via the geometric quotient $(\operatorname{Spec}\mathbb{C}[x_0,\dots,x_n] \setminus \mathbb{V}(\mathfrak{m})) \mathbin{/\!\!/} \mathbb{C}^*$, so can a Mori dream space be constructed from the Cox ring S and some additional data via a GIT quotient by the group $G \coloneqq \operatorname{Hom}(\operatorname{Cl}(X), \mathbb{C}^*)$ [MFK94, AH09].

Through this lens, a projective toric variety X is a Mori dream space with S a polynomial ring. The rays $\Sigma_X(1)$ correspond to G and the multigrading on S via Gale duality [BFS90, OP91], and the additional data required to construct X is any of the following equivalent ingredients:

1. the choice of maximal cones of Σ_X;
2. the choice of an irrelevant ideal $B \subset S$ (which generalizes the maximal ideal $\mathfrak{m}$ for $\mathbb{P}^n$);
3. the choice of an ample class in $\operatorname{Cl}(X)$, which determines the cone $\operatorname{Nef} X$ of nef divisors.

While (1) is only defined for toric varieties, (2) and (3) can be used in general to define the algebraic operations *B-saturation* and *nef-truncation* for multigraded modules over arbitrary Cox rings.

For a finitely generated, $\operatorname{Cl}(X)$-graded S-module M with free presentation $0 \leftarrow M \leftarrow F \leftarrow F'$ we define the nef-truncation $M_{\geq \mathbf{d}}$ at a (multi)degree $\mathbf{d}$ to be the S-module with presentation

$$0 \leftarrow M_{\geq \mathbf{d}} \leftarrow F_{\geq \mathbf{d}} \leftarrow F'_{\geq \mathbf{d}},$$

where $F_{\geq \mathbf{d}}$ is the submodule generated by graded pieces $F_{\mathbf{a}}$ with $\mathbf{a} \in \mathbf{d} + \operatorname{Nef} X$ (c.f. [MS04, Def. 5.1]).

The significance of nef-truncations lies in the fact that a free resolution of $M_{\geq \mathbf{d}}$ is a virtual resolution for M (see Definition 3.1.1). Syzygies of nef-truncations are also tied to Oda's conjecture that smooth projective toric varieties are projectively normal [Oda97, Mac07].

An additional, equivalent ingredient for constructing X from the information in the Cox ring S involves the Frobenius morphism:

4. the choice of higher extensions among the summands of the Frobenius pushforward of S.

In other words, the higher extensions of line bundles are a source of geometric information.

By working with ideals and modules over the multigraded Cox ring of X, many theorems about subvarieties of $\mathbb{P}^n$ and coherent sheaves on it can be generalized to geometric statements over X.

2.3 Representations of Bound Quivers

A *quiver* Q is a directed graph consisting of a finite set Q_0 of vertices and Q_1 of arrows. Notably, the category of representations of a quiver over a field $\Bbbk$ is equivalent to the category of finite-dimensional left modules over a $\Bbbk$-algebra. This equivalence is significant to the study of bounded derived categories of certain toric varieties, hence this section reviews the basic terminology for quivers and their representations. See Section 2.3 for examples of quivers corresponding to the running examples. See [Cra07, §1] for further details.

To each arrow $x \xrightarrow{\alpha} y$, the maps $t, h \colon Q_1 \to Q_0$ correspond a tail $t(\alpha) = x$ and head $h(\alpha) = y$ such that a sequence of arrows $p = \alpha_l \cdots \alpha_1$ form a *path* of length l whenever $h(\alpha_i) = t(\alpha_{i+1})$ for $1 \leq i < l$. By convention, for each vertex x there is a trivial path $x \xrightarrow{e_x} x$.

Definition 2.3.1. A *representation* W *of a quiver* Q *over a field* $\Bbbk$ consists of

- a $\Bbbk$-vector space W_x for each vertex $x \in Q_0$;
- a $\Bbbk$-linear map $w_\alpha \colon W_{t\alpha} \to W_{h\alpha}$ for each arrow $\alpha \in Q_1$.

A representation W is finite-dimensional if each W_i has finite dimension over $\Bbbk$. More generally, a representation may be defined over any ring, but here working over a field suffices.

A key object in this section is the *path algebra* associated to a quiver.

Definition 2.3.2. The *path algebra* $\Bbbk Q$ *of a quiver* Q is the graded $\Bbbk$-algebra where

- $(\Bbbk Q)_l$ is a $\Bbbk$-vector space with basis the set of paths of length l, and
- multiplication is defined by concatenation of paths, when possible, or zero otherwise.

The path algebra is an associative algebra with identity $\sum_{x \in Q_0} e_x$ and it is finite-dimensional when the quiver is acyclic. Moreover, the subring $(\Bbbk Q)_0$ generated by the trivial paths e_x is a semi-simple ring with e_x as orthogonal idempotents; that is, $e_x e_y = e_x$ when $x = y$ and zero otherwise.

Our interest in quiver representations stems from the construction of tilting algebras as the quotient of a path algebra with vertices corresponding to the exceptional objects and arrows corresponding to homomorphisms between them.

2.3.5 $\mathbb{P}^1$	2.3.6 $\mathbb{P}^2$	2.3.7 $\mathbb{P}^1 \times \mathbb{P}^1$	2.3.8 $\mathbb{F}_2$
$\begin{tikzcd} \bullet \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & \bullet \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[r, shift left=2, "\bar{x}"] \arrow[r, "\bar{y}"] \arrow[r, shift right=2, "\bar{z}"'] & \bullet \arrow[r, shift left=2, "x"] \arrow[r, "y"] \arrow[r, shift right=2, "z"'] & \bullet \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[r, shift left, "\bar{a}_0"] \arrow[r, shift right, "\bar{a}_1"'] \arrow[d, shift right, "\bar{b}_0"'] \arrow[d, shift left, "\bar{b}_1"] & \bullet \arrow[d, shift right, "b_0"'] \arrow[d, shift left, "b_1"] \\ \bullet \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \bullet \end{tikzcd}$	$\begin{tikzcd} \bullet \arrow[r, shift left, "\bar{a}_0"] \arrow[r, shift right, "\bar{a}_1"'] \arrow[dr, "\bar{b}"'] & \bullet \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_2"] \arrow[dr, "b"] & \\ & \bullet \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \bullet \end{tikzcd}$

Table 2.2: Examples of Beilinson or Bondal quivers of toric varieties

Definition 2.3.3. A *bound quiver* (Q, R) is a quiver Q together with a finite set of relations R, given as $\Bbbk$-linear combinations of paths of length at least 2 with the same head and tail. Note that R can be identified with an ideal in $\Bbbk Q$.

A representation of (Q, R) is a representation of Q where for each $p - p' \in R$ the homomorphisms associated to p and p' coincide; that is, $\operatorname{Hom}(W_{tp}, W_{hp}) = \operatorname{Hom}(W_{tp'}, W_{hp'})$. Note that for each quiver representation W we may associate a $\Bbbk Q/R$-module $\bigoplus_{x \in Q_0} W_x$. Conversely, for any left $\Bbbk Q/R$-module M we have a quiver representation given by the $\Bbbk$-vector spaces $W_x = e_x M$ for $x \in Q_0$ and maps $w_\alpha \colon W_{t\alpha} \to W_{h\alpha}$ given by $m \mapsto \alpha(m)$ for $\alpha \in Q_1$.

Proposition 2.3.4. *The category* $\operatorname{Rep}_{\Bbbk}(Q, R)$ *of representations of bound quivers is equivalent to the category of finitely generated left* $\Bbbk Q/R$*-modules.*

Observe that if $(\Bbbk Q)^{\mathrm{op}}$ is the opposite algebra with product $a \cdot b = ba$, then $(\Bbbk Q)^{\mathrm{op}} \simeq \Bbbk Q^{\mathrm{op}}$ where Q^{op} is the opposite quiver with arrows reversed. In particular, the proposition above gives an equivalence of $\operatorname{mod}(A^{\mathrm{op}})$ and $\operatorname{Rep}_{\Bbbk}(Q^{\mathrm{op}}, R)$.

Key examples

Example 2.3.5. Kronecker quiver

This is the unique acyclic quiver with two vertices and two nontrivial arrows. A representation W of this quiver consists of a pair of vector spaces (W_0, W_1) and maps $w, w' : W_0 \to W_1$.

Beilinson's exceptional collection for $\mathbb{P}^1$ proved the equivalence $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$. The importance of the Kronecker quiver, also known as the Beilinson quiver or Bondal quiver for $\mathbb{P}^1$, is due to the isomorphism $\Bbbk Q \simeq \operatorname{End}_{\Bbbk}(\mathcal{O} \oplus \mathcal{O}(1))$; that is, Q encodes the data of endomorphisms of the tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$. Writing $\mathbb{P}^1 = \operatorname{Proj} \Bbbk[x, y]$, the arrows of Q correspond to the maps $\mathcal{O}(1) \xrightarrow{\cdot x} \mathcal{O}$ and $\mathcal{O}(1) \xrightarrow{\cdot y} \mathcal{O}$.

Recall that a vector bundle E on $\mathbb{P}^1$ splits as a direct sum $E = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i)$ for $d_i \in \mathbb{Z}$, hence E has a trivial locally free resolution consisting only of twists $\mathcal{O}_{\mathbb{P}^1}(d_i)$. However, the existence of Beilinson's exceptional collection $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$ implies that there is also a locally free resolution consisting only of those two twists, though *a priori* this resolution may be longer. This observation points to the fact that the fixing an exceptional collection amounts to limiting permissible representations of objects in the derived category.

Example 2.3.6. Beilinson quiver for $\mathbb{P}^2$

This quiver is the first example of a Beilinson quiver for which the tilting algebra of interest is isomorphic to a quotient of the path algebra. Therefore we introduce an ideal of relations

$$R = \langle \bar{x}y - \bar{y}x, \bar{x}z - \bar{z}x, \bar{y}z - \bar{z}y \rangle.$$

We can then write $\Bbbk Q/R \cong \mathrm{End}_{\Bbbk}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$ as before.

Example 2.3.7. Quiver of sections for $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the collection of line bundles $\{\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)\}$, with $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ along the first and second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

The *quiver of sections* of this collection is the quiver with vertices corresponding to the bundles L_i and an arrow $i \to j$ for each indecomposable $\mathbb{T}$-invariant section

$$s \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, L_j \otimes L_i^{-1}).$$

A $\mathbb{T}$-invariant section is indecomposable if the divisor $div(s)$ does not split as a sum $div(s') + div(s'')$ for nonzero sections s', s'' of $L_j \otimes L_k^{-1}$ and $L_k \otimes L_i^{-1}$.

Example 2.3.8. Quiver of sections for $\mathbb{F}_2$

The path algebra of this quiver, modulo the ideal of relations, represents the homomorphisms among the bundles $\{\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_4), \mathcal{O}(D_1 + D_4)\}$, where D_i are the $\mathbb{T}$-invariant divisors of X. For more complicated examples, such as this one, the ideal of relations is easier to write as having generators coming from the composition rule; i.e., the relations that arise are of the form $p - p' \in R$.

2.4 The Derived Category of Coherent Sheaves

Let X be a smooth projective variety, and $\mathrm{Coh}(X)$ the Abelian category of coherent sheaves on X. Every morphism $f : X \to Y$ between such varieties induces two functors:

- the inverse image functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ (pullback), and
- the direct image functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ (pushforward).

However, these two functors are not exact, in the sense that exact sequences are not preserved. To preserve functoriality, Cartan and Eilenberg introduced the notion of derived functors.

The derived category of coherent sheaves on X, denoted $\mathcal{D}(X)$, contains geometric information about X. In some cases one can even recover X from $\mathcal{D}(X)$, but there are also examples of different varieties, for instance non-isomorphic K3 surfaces, with equivalent derived categories. This section provides an introduction to the derived category theory used in the rest of this thesis. See [Huy06] or [Orl03] for further context and topics.

Definition 2.4.1. We denote by $\mathrm{K}^b(X)$ the bounded *homotopy category* on $\mathrm{Coh}(X)$, where the objects are bounded chain complexes of coherent sheaves on X modulo the relation of homotopy and chain maps as morphisms. The *bounded derived category* on $\mathrm{Coh}(X)$, denoted $\mathcal{D}^{\mathrm{b}}(X)$, has the same objects as $\mathrm{K}^b(X)$, but each quasi-isomorphism is endowed with an inverse morphism. Explicitly, morphisms in the derived category can be expressed as roofs $A \leftarrow A' \to B$ where $A' \to A$ is a quasi-isomorphism. Note that caution is needed in order to check whether a morphism $A \to B$ in $\mathcal{D}^{\mathrm{b}}(X)$ can be lifted back to $\mathrm{K}^b(X)$ (e.g. see Lemma 2.4.7).

Unlike Abelian categories, short exact sequences do not exist in derived categories, and kernels and cokernels of morphisms are not defined. However, derived categories are endowed with the structure of a *triangulated category*, formalized by Verdier.

Definition 2.4.2. An additive category $\mathcal{D}$ is a *triangulated category* if for any morphism $f : A \to B$ in $\mathcal{D}$ there exists a *distinguished triangle* $A \to B \to C \to A[1]$, where $C = \mathrm{Cone}\, f$ is the cone of the morphism f, satisfying certain axioms.

A triangulated subcategory is a full subcategory $\mathcal{D}$ that is closed under the shift functor and taking the mapping cone of morphisms. In other words, if two objects of some triangle belong to a triangulated subcategory, then so does the third object. We say that $\mathcal{D}$ is *thick* (or epaisse) if it is further closed under isomorphisms and direct summands of objects. The thick *envelope* of an object E in $\mathcal{D}$ is the smallest thick triangulated subcategory of $\mathcal{D}$ containing E. When the thick envelope is equal to $\mathcal{D}$ we say that E generates $\mathcal{D}$.

Remark 2.4.3. The derived category $\mathcal{D}^{\mathrm{b}}(X)$ is a triangulated category, with the shift $E^\bullet \mapsto E^\bullet[1]$ given by $E^\bullet[1]^i = E^{i+1}$ and $d^i_{E[1]} = -d^{i+1}_E$, and the cone of morphism $f : E^\bullet \to F^\bullet$ given by $\operatorname{Cone} f^i = E^{i+1} \oplus F^i$.

There is a fully faithful functor $\mathrm{Coh}(X) \hookrightarrow \mathcal{D}^{\mathrm{b}}(X)$ that guarantees

$$\mathrm{Ext}^i(E^\bullet, F^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(X)}(E^\bullet, F^\bullet[i]),$$

hence we will identify the two going forward.

Derived categories of coherent sheaves appear in many other areas of algebraic geometry as well. For instance, the Homological Mirror Symmetry Conjecture states that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi–Yau variety and the derived Fukaya category of its mirror. While this is beyond the scope of this chapter, it is worth mentioning that a large number of Calabi–Yau manifolds are realized as subspaces of toric varieties, in particular weighted projective spaces.

In the next two sections the structure of $\mathcal{D}^{\mathrm{b}}(X)$ is determined using two related approaches: exceptional collections and tilting bundles.

2.4.1 Exceptional Collections

Studying the structure of the derived category is an important step towards studying the underlying scheme. Recall that an object E in a triangulated category $\mathcal{D}$ generates the category $\mathcal{D}$ if any thick triangulated subcategory containing it is equivalent to $\mathcal{D}$. In this section, this idea is generalized to that of a *full strong exceptional collection* for $\mathcal{D}^{\mathrm{b}}(X)$, the existence of which implies that $\mathcal{D}^{\mathrm{b}}(X)$ is freely and finitely generated. Refer to [Kuz14] for further details on semiorthogonal decompositions.

Consider a full triangulated subcategory $\mathcal{B}$ in a triangulated category $\mathcal{D}$. The *right (resp. left) orthogonal* to $\mathcal{B}$ is the full subcategory $\mathcal{B}^\perp \subset \mathcal{D}$ (resp. ${}^\perp\mathcal{B}$) consisting of the objects C such that

$\mathrm{Hom}(B, C) = 0$ (resp. $\mathrm{Hom}(C, B) = 0$) for all $B \in \mathcal{B}$. Both right and left orthogonal subcategories are also triangulated.

A sequence of triangulated subcategories $(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ in a triangulated category $\mathcal{D}$ is a *semiorthogonal sequence* if $\mathcal{B}_j \subset \mathcal{B}_i^{\perp}$ for all $0 \leq j < i \leq n$.

Definition 2.4.4. A *semiorthogonal decomposition* $\mathcal{D} = \langle \mathcal{B}_0, \ldots, \mathcal{B}_n \rangle$ is a semiorthogonal sequence that generates $\mathcal{D}$ as a triangulated category.

The first examples of semiorthogonal decompositions arise from full exceptional collections.

Definition 2.4.5. Let $\mathcal{D}$ be a $\Bbbk$-linear triangulated abelian category.

- An object $E \in \mathcal{D}$ is *exceptional* if

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0. \end{cases}$$

  Equivalently, using the notation of the derived functors, the condition states $\mathrm{Hom}(E, E) = \mathbb{C}$ and $R^{\ell}\,\mathrm{Hom}(E, E) = \mathrm{Ext}^{\ell}(E, E) = 0$ when $\ell \neq 0$.

- An *exceptional collection* is an ordered sequence $E_1, \ldots, E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.

  Equivalently, $R^{\bullet}\,\mathrm{Hom}(E_i, E_j) = \mathrm{Ext}^{\bullet}(E_i, E_j) = 0$ when $i > j$.

- An exceptional collection is *strong* if $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i \leq j$ and $\ell > 0$.

  Equivalently, $R^{\ell}\,\mathrm{Hom}(E_i, E_j) = \mathrm{Ext}^{\ell}(E_i, E_j) = 0$ when $i \leq j$ and $\ell > 0$.

- An exceptional collection is *full* if $\mathcal{D}$ is generated by $\{E_i\}$; that is, any full triangulated subcategory containing all objects E_i is equivalent to $\mathcal{D}$ via inclusion of E_i.

There is a mutative structure available on such sets, including an action by the braid group, which we will not need for our purposes.

Example 2.4.6 ([Beĭ78b]). The projective space $\mathbb{P}^n$ has a full strong exceptional collection consisting of twists of the structure sheaf,

$$\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) = \big\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n) \big\rangle,$$

as well as one consisting of exterior products of the cotangent bundle $\Omega^1(1)$,

$$\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) = \big\langle \mathcal{O}, \Omega^1(1), \Omega^2(2), \ldots, \Omega^n(n) \big\rangle.$$

Observe that while the first exceptional collection is comprised of symmetric products of $\mathcal{O}(1)$, any "window" of $n+1$ consecutive twists of $\mathcal{O}$ is a full strong exceptional collection for $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$.

The following lemma implies that given two complexes consisting of terms in a strong exceptional collection, any map between them in the derived category lifts to a map of complexes.

Lemma 2.4.7 ([Kap88]). *Let C and D be bounded complexes over an abelian category $\mathcal{A}$. Suppose that $\mathrm{Ext}^p_{\mathcal{A}}(C_i, D_j) = 0$ for $p > 0$ and all i, j. Then*

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathcal{A})}(C, D) = \mathrm{Hom}_{\mathrm{H}ot(\mathcal{A})}(C, D).$$

In particular, if $f\colon C \to D$ is a quasi-isomorphism, then the inverse in the derived category lifts to a map $g\colon D \to C$ with a chain homotopy $fg = \mathrm{id}_D + sd + ds$.

In the next section we will use full strong exceptional collections to construct equivalences of bounded derived categories.

2.4.2 Tilting Bundles

Bondal's work established an equivalence between the category of bounded complexes of coherent sheaves on a projective space and the category of finitely generated representations of a bound quiver (Q, R). See [Cra07] for the details and proofs of theorems cited.

Borrowing terminology from representation theory (cf. [Bae88]), the notion of a *tilting sheaf* on a scheme X aims to generalize Beilinson's result in the following sense: a tilting sheaf is a sheaf T of $\mathcal{O}_X$-modules that induces an equivalence of triangulated categories $\mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}})$ which sends

T to $A = \mathrm{End}_X(T)$, its endomorphism algebra.

Recall that the *global dimension* of an algebra A is defined to be the supremum of projective dimension of all right A-modules.

Definition 2.4.8. A sheaf of $\mathcal{O}_X$-modules is a *tilting sheaf* (resp. *bundle*) if the following hold:

(i) T has no higher self-extensions, that is, $\mathrm{Ext}^i_X(T,T) = 0$ for $i > 0$,

(ii) the endomorphism algebra $A = \mathrm{Hom}_X(T,T)$ has finite global dimension, and

(iii) T generates the bounded derived category $\mathcal{D}^{\mathrm{b}}(X)$.

More specifically, Bondal showed in [Bon90] that a triangulated category generated by a strong exceptional collection is equivalent to the derived category of right modules over the algebra of homomorphisms of the collection, which we represented as a bound quiver in Section 2.3.

Theorem 2.4.9 ([Bae88, Bon90])**.** *Let T be a tilting sheaf on a smooth projective variety X, with associated tilting algebra $A = \mathrm{End}_X(T)$. Then the functors*

$$\begin{aligned} \mathrm{Hom}_X(\ T\ \ ,\ -\ )\colon\ &\mathrm{Coh}(X) \longrightarrow \mathrm{mod}(A^{\mathrm{op}}) \quad \textit{and} \\ -\otimes_A T\ \ :\ &\mathrm{mod}(A^{\mathrm{op}}) \longrightarrow \mathrm{Coh}(X) \end{aligned}$$

induce derived equivalences of triangulated categories

$$\begin{aligned} \mathbf{R}\,\mathrm{Hom}_X(\ T\ \ ,\ -\ )\colon\ &\mathcal{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) \quad \textit{and} \\ -\otimes^{\mathbf{L}}_A T\ \ :\ &\mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) \longrightarrow \mathcal{D}^{\mathrm{b}}(X) \end{aligned}$$

which are quasi-inverse to each other.

Corollary 2.4.10. *Suppose T is a coherent sheaf on X satisfying (i) and (ii), then T satisfies (iii) if and only if for any $E \in \mathcal{D}^{\mathrm{b}}(X)$ we have $\mathbf{R}\,\mathrm{Hom}_X(T,E) \otimes^{\mathbf{L}}_A T \cong E$.*

Establishing that a sheaf is a tilting sheaf on X is essentially done in two steps: first find a strong exceptional collection on X, then show that the exceptional collection is full. With this information, the tilting sheaf can be constructed simply as a direct sum of the exceptional collection. When the exceptional collection contains only vector bundles, then T is a tilting bundle, and if it contains only line bundles, then T is a particularly useful invariant of the derived category.

Proposition 2.4.11 ([Cra07], Prop. 2.7)**.** *Let $T = \oplus_{i=0}^{n} E_i$ be a locally-free sheaf on X with each E_i a line bundle (in particular,* $\mathrm{Hom}_X(E_i, E_i) = 0$ *for all i). Then*

1. *If T satisfies (i) and (ii), then $(E_0, \ldots, E_n)$ forms a strong exceptional collection.*

2. *If T satisfies (iii), then $(E_0, \ldots, E_n)$ is a full strong exceptional collection.*

Conversely, every full strong exceptional collection defines a tilting sheaf.

The connection with Beilinson's work is apparent from the following theorem, which can be seen as a strengthening of Corollary 2.4.10.

Theorem 2.4.12 ([Kin97], Theorem 1.2)**.** *Let X be a smooth projective variety and T be a bundle satisfying conditions (i) and (ii). Then T is a tilting bundle if and only if the map $T^\vee \boxtimes_A^{\mathbf{L}} T \to \mathcal{O}_\Delta$ is an isomorphism in $\mathcal{D}^{\mathrm{b}}(X \times X)$.*

Once again, the structure of a resolution of $\mathcal{O}_\Delta$ as an object in $\mathcal{D}^{\mathrm{b}}(X \times X)$ is closely related to the derived category of X.

3 Uniqueness of Minimal Virtual Resolutions

The material in this chapter originally appeared in [BCHS21], with the exception of Theorem 3.5.4, which was stated as a conjecture in that paper.

3.1 Introduction

Let X be a smooth projective toric variety with $\mathrm{Pic}(X)$-graded Cox ring S and irrelevant ideal B. Consider a graded S-module M. While a minimal free resolution $F_\bullet$ of M can be easily computed using Gröbner methods, it does not always provide a faithful reflection of the geometry of X. For example, when the Picard rank of X is greater than one, the length of $F_\bullet$ may exceed $\dim X$. To bridge this gap, Berkesch, Erman, and Smith introduced *virtual resolutions* in [BES20] as the homological substitutes for free resolutions in the toric case.

Definition 3.1.1 ([BES20])**.** A $\mathrm{Pic}(X)$-graded complex of free S-modules $G_\bullet$ is a *virtual resolution* of M if the complex $\widetilde{G}_\bullet$ of locally free sheaves on X is a resolution of the sheaf $\widetilde{M}$.

Using this notion, one may expect a version of Hilbert's Syzygy Theorem to hold for other spaces.

Conjecture 3.1.2 ([BES20, Question 6.5])**.** *If S is the Cox ring of X and M is a finitely generated, $\mathrm{Cl}(X)$-graded S-module, then M admits a virtual resolution of length at most $\dim(X)$.*

This conjecture was proven for products of projective spaces in [BES20], for smooth projective toric varieties of Picard rank 2 by Brown and the author (Theorem B), and for all smooth toric stacks by Hanlon–Hicks–Lazarev and Brown–Erman [HHL23, BE23b], using independent ideas. These results established new cases of Orlov's conjecture below, which further illustrates the potential of multigraded commutative algebra techniques in derived algebraic geometry (c.f. [Rou08, BDM19, BFK19, FH23]).

Conjecture 3.1.3 ([Orl09, Conj. 10]). *Let X be a smooth quasi-projective scheme. The Rouquier dimension of the bounded derived category of coherent sheaves $\mathcal{D}^{\mathrm{b}}(X)$ is equal to* $\dim(X)$.

Despite more faithfully capturing the geometry of X, virtual resolutions are often less rigid than minimal free resolutions. For example, a module M generally has many non-isomorphic virtual resolutions. In this section we consider virtual resolutions containing no degree 0 maps, which we show to be subcomplexes of minimal free resolutions in certain situations. Remarkably, we prove uniqueness of virtual resolutions which consist only of certain twists.

This uniqueness is key to our proof of Theorem E. Inspired by the work of Berkesch, Erman, and Smith, we use a Fourier-Mukai transform to construct a virtual resolution of M whose Betti numbers are computable in terms of certain cohomology groups. In Section 6.3.1 we then prove that this virtual resolution is isomorphic to the minimal free resolution of $M_{\geq \mathbf{d}}$.

3.2 Minimal Virtual Resolutions

A complex of S-modules is *trivial* if it is a direct sum of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow S \xrightarrow{1} S \longrightarrow 0 \longrightarrow \cdots .$$

A free resolution of a finitely generated $\mathrm{Pic}(X)$-graded S-module M is isomorphic to the direct sum of the $\mathrm{Pic}(X)$-graded minimal free resolution of M and a trivial complex. With this in mind we introduce the following notion of a minimal virtual resolution.

Definition 3.2.1. A virtual resolution $F_\bullet$ is *minimal* if it is not isomorphic to a $\mathrm{Pic}(X)$-graded chain complex of the form $F'_\bullet \oplus F''_\bullet$ where $F''_\bullet$ is a trivial complex.

Note that, unlike in the case of ordinary free resolutions, minimal virtual resolutions are not unique, even up to isomorphism. Further, minimal virtual resolutions need not have the same length. That said, analogous to the case of minimal free resolutions, minimal virtual resolutions are characterized by having no constant entries in their differentials.

Lemma 3.2.2. *A virtual resolution of M is minimal if and only if its differentials have no degree 0 components.*

Proof. Since S is positively graded, a graded version of Nakayama's Lemma holds (see [MS05, pp. 155-156]). The statement follows from an argument similar to those in [Eis95, Thm. 20.2, Exc. 20.1]. $\square$

The following structure result shows that a minimal virtual resolution F of a module M satisfying certain conditions on the Betti numbers arises as a subcomplex of the minimal free resolution of $H_0(F_\bullet)$. Here we denote by $\operatorname{Eff} X$ the cone generated by the degrees of the variables of S in $\operatorname{Pic} X$.

Proposition 3.2.3. *Let $(F_\bullet, \varphi_\bullet)$ be a finite minimal virtual resolution and let $N = H_0(F_\bullet)$. Suppose*

1. $\dim_\Bbbk \operatorname{Tor}_i(F_\bullet, \Bbbk)_d \leq \dim_\Bbbk \operatorname{Tor}_i(N, \Bbbk)_d$ *for all d and all i;*
2. *whenever $c - d \in \operatorname{Eff} X$ and $\operatorname{Tor}_i(F_\bullet, \Bbbk)_c \neq 0$ then equality holds in* (1).

Then $F_\bullet$ is a subcomplex of the minimal free resolution of N.

Proof. First, we will inductively construct a resolution $(G_\bullet, \psi_\bullet)$ of N which contains $(F_\bullet, \varphi_\bullet)$ as a subcomplex. Let $G_0 = F_0$, $G_1 = F_1$, and $\psi_1 = \varphi_1$, so that $H_0(G_\bullet) = N$.

Suppose G_i has been defined for $0 \leq i \leq n-1$ so that $F_\bullet$ is a summand and $G_\bullet$ is exact for $0 < i < n-1$. Consider φ_n as a map $F_n \to G_{n-1}$ by composing with the inclusion $F_{n-1} \hookrightarrow G_{n-1}$. Choose $z_1, \ldots, z_s \in \ker \psi_{n-1}$ such that their images generate $\ker \psi_{n-1} / \operatorname{im} \varphi_n$. Let $G_n = F_n \oplus S(-\mathbf{a}_1) \oplus \cdots \oplus S(-\mathbf{a}_s)$ where $\deg z_j = \mathbf{a}_j$. Define ψ_n by $\psi_n|_{F_n} = \varphi_n$ and $\psi_n(g_j) = z_j$, where g_j is the generator of $S(-\mathbf{a}_j)$. Then $\operatorname{im} \psi_n = \ker \psi_{n-1}$, so that $G_\bullet$ is a complex and exact at $n-1$.

We will now show by induction that it is possible to prune $G_\bullet$ to a minimal free resolution of N that contains $F_\bullet$ as a subcomplex. At each step, take a nonminimal homogeneous relation among the images of generators of some G_i. Write it as

$$\psi_i\Big(\sum a_j f_j + \sum b_j g_j\Big) = 0,$$

where $f_j \in F_i$, $g_j \in G_i \setminus F_i$, and $a_j, b_j \neq 0$ for all j. As $F_\bullet$ is minimal, at least one g_j does appear. Since each G_i has only finitely many generators, it is possible to choose a relation whose degree c satisfies $c - d \notin \operatorname{Eff} X$ for all degrees $d \neq c$ of other available relations.

Assume by induction that no generator of $F_\bullet$ has been removed in a previous step. Since the chosen relation is nonminimal, at least one of its coefficients is a unit. If some b_j is a unit then we may remove the corresponding g_j and continue pruning.

Suppose instead that all unit coefficients appear among the a_j. In this case we must prune some f_k in order to remove the relation. Note that by homogeneity

$$\deg f_k = \deg a_k f_k = c = \deg b_j g_j = \deg b_j + \deg g_j$$

for all j. Thus $c - \deg g_j = \deg b_j \in \operatorname{Eff} X$, so equality holds in (1) for $d = \deg g_j$ by hypothesis. By choice of c we cannot remove anything of degree $\deg g_j$ in a subsequent step. Hence g_j appears in the minimal free resolution of N, so by the equality in (1) some generator f of F_i with degree d must be removed. However, it cannot have been removed before f_k by the induction hypothesis, and it cannot be removed after f_k by choice of c. This is a contradiction, so we are never required to prune a generator of $F_\bullet$, completing the proof. $\square$

In the language of [BES20], this proposition implies that a virtual resolution that appears to be a *virtual resolution of a pair* based only on its Betti numbers can indeed be produced by that construction. Note that the proposition is not true without conditions on the Betti numbers. For instance, [BC21, Ex. 1.2] gives a minimal virtual resolution which is not a subcomplex of the minimal free resolution of its cokernel.

3.3 The Fourier–Mukai Transform

The sheafification of a virtual resolution of M is a resolution of $\widetilde{M}$ by direct sums of line bundles. More generally, following [EES15, §8], we define a *free monad* of a coherent sheaf $\mathcal{F}$ to be a finite complex

$$\mathcal{L}\colon 0 \leftarrow \mathcal{L}_{-s} \leftarrow \cdots \leftarrow \mathcal{L}_{-1} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \mathcal{L}_t \leftarrow 0$$

whose terms are direct sums of line bundles and whose homology is $H_\bullet(\mathcal{L}) = H_0(\mathcal{L}) \simeq \mathcal{F}$.

In this section we introduce a type of geometric functor between derived categories known as a Fourier–Mukai transform. We will use a particular instance in Section 3.4 to prove that a complex constructed from the Beilinson spectral sequence is a free monad. See [Huy06, §5] for background and further details.

Let X and Y be smooth projective varieties and consider the two projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Y. \end{tikzcd}$$

A *Fourier–Mukai transform* is a functor

$$\Phi_{\mathcal{K}} \colon \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$$

between the derived categories of bounded complexes of coherent sheaves. It is represented by an object $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times Y)$ and constructed as a composition of derived functors

$$\mathcal{F} \mapsto \mathbf{R}p_*\big(\mathbf{L}q^*\mathcal{F} \otimes^{\mathbf{L}} \mathcal{K}\big).$$

Here $\mathbf{L}q^*$, $\mathbf{R}p_*$, and $- \otimes^{\mathbf{L}} \mathcal{K}$ are the derived functors induced by q^*, p_*, and $- \otimes \mathcal{K}$, respectively. Moreover, since q is flat $\mathbf{L}q^*$ is the usual pull-back, and if $\mathcal{K}$ is a complex of locally free sheaves $- \otimes^{\mathbf{L}} \mathcal{K}$ is the usual tensor product. In fact, all equivalences between $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ arise in this way.

A special case of the Fourier–Mukai transform occurs when $Y = X$ and $\mathcal{K} \in \mathcal{D}^{\mathrm{b}}(X \times X)$ is a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal subscheme $\iota\colon \Delta \to X \times X$. Such $\mathcal{K}$ is referred to as a *resolution of the diagonal*.

Using the projection formula, one can see that the Fourier–Mukai transform $\Phi_{\mathcal{O}_\Delta}$ is simply the identity in the derived category; that is to say, replacing $\mathcal{O}_\Delta$ with $\mathcal{K}$ produces quasi-isomorphisms. We will use this fact in the proof of Proposition 3.5.1.

3.4 The Beilinson Spectral Sequence

Returning to the case of products of projective spaces, we consider coherent sheaves on $X = \mathbb{P}^{\mathbf{n}}$. We construct a free monad for $\widetilde{M}$ from the Beilinson spectral sequence on $\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ and describe its Betti numbers. When M is $\mathbf{0}$-regular it is a minimal virtual resolution, which we will use in Sections 6.3. See [OSS80, §3.1] for a geometric exposition and [Huy06, §8.3] or [AO89, §3] for an algebraic exposition on a single projective space.

For sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{P}^{\mathbf{n}}$, denote $p^*\mathcal{F} \otimes q^*\mathcal{G}$ by $\mathcal{F} \boxtimes \mathcal{G}$. Consider the vector bundle

$$\mathcal{W} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{e}_i) \boxtimes \mathcal{T}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(-\mathbf{e}_i),$$

where $\mathcal{T}_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}$ is the pullback of the tangent bundle, as in the Euler sequence on the factor $\mathbb{P}^{n_i}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_i}}^{n_i+1}(\mathbf{e}_i) \longrightarrow \mathcal{T}_{\mathbb{P}^{n_i}} \longrightarrow 0. \tag{3.4.1}$$

There is a canonical section $s \in H^0(\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}, \mathcal{W})$ whose vanishing cuts out the diagonal subscheme $\Delta \subset \mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ (see [BES20, Lem. 2.1]), giving a Koszul resolution of $\mathcal{O}_\Delta$:

$$\mathcal{K}\colon 0 \longleftarrow \mathcal{O}_{\mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}} \longleftarrow \mathcal{W}^\vee \longleftarrow \bigwedge^2 \mathcal{W}^\vee \longleftarrow \cdots \longleftarrow \bigwedge^n \mathcal{W}^\vee \longleftarrow 0. \tag{3.4.2}$$

The terms of $\mathcal{K}$ can be written as

$$\mathcal{K}_j = \bigwedge^j \left(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{e}_i) \boxtimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{e}_i}(\mathbf{e}_i) \right) = \bigoplus_{|\mathbf{a}|=j} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{a}}(\mathbf{a}), \quad \text{for } 0 \le j \le |\mathbf{n}|. \tag{3.4.3}$$

As in Section 3.3, we are interested in the derived pushforward of $q^*\widetilde{M} \otimes \mathcal{K}$, which we will compute by resolving the second term of each box product with a Čech complex to obtain a spectral sequence. Since $\mathcal{K}$ is a resolution of the diagonal, the pushforward will be quasi-isomorphic to $\widetilde{M}$.

Consider the double complex

$$C^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \boxtimes \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{a}}(\mathbf{a})\Big),$$

with vertical maps from the Čech complexes and horizontal maps from $\mathcal{K}$. Since taking Čech complexes is functorial and exact we have $\mathrm{Tot}(C) \sim q^*\widetilde{M} \otimes \mathcal{K}$, which is a resolution of $q^*\widetilde{M} \otimes \mathcal{O}_\Delta$ because $\mathcal{K}$ is locally free. Moreover, since the first term of each box product in $q^*\widetilde{M} \otimes \mathcal{K}$ is locally free, the columns of C are p_*-acyclic (c.f. [Har66, Prop. 3.2], [AO89, Lem. 3.2]). Hence the pushforward

$$E_0^{-s,t} = p_*(C^{-s,t}) = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes \Gamma\Big(\mathbb{P}^{\mathbf{n}}, \check{C}^t\Big(\mathfrak{U}_B, \widetilde{M} \otimes \Omega_{\mathbb{P}^{\mathbf{n}}}^{\mathbf{a}}(\mathbf{a})\Big)\Big) \tag{3.4.4}$$

satisfies $\mathrm{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M}$. With this notation, the *Beilinson spectral sequence* is the spectral sequence of the double complex E_0, whose (vertical) first page has terms

$$E_1^{-s,t} = \bigoplus_{|\mathbf{a}|=s} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes H^t\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big) = \mathbf{R}^t p_*(q^* \widetilde{M} \otimes \mathcal{K}_s). \tag{3.4.5}$$

Beilinson's resolution of the diagonal and the associated spectral sequence are crucial ingredients in constructions of Beilinson monads, Tate resolutions, and virtual resolutions [EFS03, EES15, BES20]. Recently, Brown and Erman [BE21] expanded these constructions to toric varieties using a noncommutative analogue of a Fourier–Mukai transform. More generally, Costa and Miró-Roig [CM07] have introduced a Beilinson type spectral sequence for a smooth projective variety under certain conditions on its derived category.

3.5 Construction and Uniqueness

The main results of this chapter are Proposition 3.5.1 and Theorem 3.5.4, which describe the Betti numbers of a free monad constructed from the Beilinson spectral sequence (c.f. [BES20, Thm. 2.9]) and prove its uniqueness.

Proposition 3.5.1. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. There is a free monad $\mathcal{L}$ for $\widetilde{M}$ with terms*

$$\mathcal{L}_k = \bigoplus_{|\mathbf{a}|\geq k} \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a}) \otimes H^{|\mathbf{a}|-k}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$$

so that

1. *the free complex $G_\bullet = \Gamma_*(\mathcal{L})$ has Betti numbers $\beta_{k,\mathbf{a}}(G_\bullet) = h^{|\mathbf{a}|-k}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big)$;*
2. *if $H^i\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) = 0$ for $i > |\mathbf{a}|$ then $G_\bullet$ is a virtual resolution for M whose differentials have no degree $\mathbf{0}$ components.*

Proof. Let $\mathcal{K}$ be the resolution of the diagonal from (3.4.3) and let $\Phi_{\mathcal{K}}$ be the corresponding

Fourier–Mukai transform. The Beilinson spectral sequence has (vertical) first page $E_1^{-s,t}$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^2 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
\mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^1 p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
p_*(q^*\widetilde{M} \otimes \mathcal{K}_0) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_1) & \longleftarrow & p_*(q^*\widetilde{M} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
k=0 & & k=1 & & k=2 & &
\end{array} \tag{3.5.1}$$

The vertical differentials of E_0 in (3.4.4) are sheaves tensored with complexes of vector spaces that are global sections of Čech complexes, so they satisfy the splitting hypotheses of [EFS03, Lem. 3.5], which implies that the total complex of E_0 is homotopy equivalent to a complex $\mathcal{L}$ with terms $\mathcal{L}_k = \bigoplus_{s-t=k} E_1^{-s,t}$. Hence

$$\mathcal{L} \sim \operatorname{Tot}(E_0) = \Phi_{\mathcal{K}}(\widetilde{M}) \sim \widetilde{M}.$$

Since the terms of E_1 are direct sums of line bundles, the complex $\mathcal{L}$ is a free monad for $\widetilde{M}$.

Observe that the only terms with twist $\mathbf{a}$ appear in $\mathcal{K}_s$ for $s = |\mathbf{a}|$ and that the Betti numbers in homological index k come from the higher direct images $E_1^{-s,t}$ on diagonals with $s - t = k$. Hence $\beta_{k,\mathbf{a}}(G_\bullet)$ is the rank of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(-\mathbf{a})$ in $E_1^{-|\mathbf{a}|,|\mathbf{a}|-k}$ which is $h^{|\mathbf{a}|-k}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}(\mathbf{a}))$.

Lastly, note that the hypothesis of part (2) implies that the terms of (3.5.1) on diagonals with $k < 0$ vanish; hence the free monad $\mathcal{L}$ is a locally free resolution. Since each map in the construction from [EFS03, Lem. 3.5] increases the index $-s$, the differentials in $G_\bullet$ have no degree $\mathbf{0}$ components. □

Remark 3.5.2. In the proof of [BES20, Prop. 1.2], Berkesch, Erman, and Smith show that if M is sufficiently twisted so that all higher direct images of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$ vanish, then the E_1 page will be concentrated in one row, which results in a linear virtual resolution. Similarly in [EES15, Prop. 1.7], Eisenbud, Erman, and Schreyer prove that for sufficiently positive twists, the truncation of M has a linear free resolution. However, in both cases the positivity condition is stronger than $\mathbf{0}$-regularity for M, as illustrated by the following example.

Example 3.5.3. Write $S = \Bbbk[x_0, x_1, y_0, y_1, y_2]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2$ and consider the ideal $I = (y_0 + y_1 + y_2, x_0y_0 + x_0y_1 + x_0y_2 + x_1y_0 + x_1y_1)$. Then $M = S/I$ is a bigraded, $(0,0)$-regular

S-module. The global sections of the Beilinson spectral sequence for $\widetilde{M}$ has first page

$$\begin{tikzcd}
0 & 0 \arrow[l] & \color{purple}{S(-1,-1)} \arrow[l] \arrow[dll, dotted, "x_1y_2"'] & \color{purple}{S(-1,-2)} \arrow[l, "y_0+y_1+y_2"'] \arrow[dll, dotted, "-x_1y_2"'] & 0 \arrow[l] \\
\color{teal}{S} & \color{teal}{S(0,-1)} \arrow[l, "y_0+y_1+y_2"] & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the dotted diagonal maps are lifts of maps from the second page of the spectral sequence, which agree with the maps from [EFS03, Lem. 3.5].

In Chapter 6 we state and prove Theorem E by illustrating the restrictions on the virtual resolution above that follow from the regularity of $\widetilde{M}$ and using them to bound the shape of the minimal free resolution of a truncation of M. In a sense, we will characterize $\mathbf{d}$-regularity by showing that this virtual resolution is isomorphic to the minimal free resolution of $M_{\geq \mathbf{d}}$. To do so, we will need the following proposition on uniqueness of virtual resolutions which consist only of certain twists.

Theorem 3.5.4. *Suppose $F_\bullet$ and $G_\bullet$ are minimal virtual resolutions of an S-module M. If every term is a direct sum of $S(-\mathbf{a})$ for $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$, then $G_\bullet$ and $F_\bullet$ are isomorphic.*

The proof uses an explicit equivalence of categories to reduce the question to the uniqueness of minimal projective resolutions over the path algebra of bound quivers. For a detailed account of the representation theory of finite-dimensional algebras, see [Bon89, Bon90, Kin97].

Consider the direct sum of line bundles $\mathcal{E} = \bigoplus_{\mathbf{a}=\mathbf{0}}^{\mathbf{n}} \mathcal{O}(-\mathbf{a})$ and its endomorphism algebra $A = \operatorname{End}(\mathcal{E})$. In this situation, A is the path algebra of a bound quiver whose vertices correspond to summands of $\mathcal{E}$ and paths correspond to homomorphisms between them. In particular, A is a finite-dimensional graded algebra whose canonical basis consists of a monomial for each path in the quiver. The projective modules over A are submodules of A containing all paths starting at a vertex, and are labeled $P_{\mathbf{a}} = e_{\mathbf{a}}A$ where $e_{\mathbf{a}}$ is an idempotent. We will use the fact that summands of $\mathcal{E}$ form a full strong exceptional collection [Huy06, §1.4], which by [Bon90, Thm. 6.2] implies that

$$\mathbf{R}\operatorname{Hom}(\mathcal{E}, -)\colon\ \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}{-}A). \tag{3.5.2}$$

is an equivalence of categories. By a direct computation present in the proof of [Bon90, Thm. 6.2], this functor sends the bundles $\mathcal{O}(-\mathbf{a})$ to projective modules $P_{\mathbf{a}}$ for $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$. Moreover, minimality is preserved, in the sense that a non-constant map $\mathcal{O}(-\mathbf{a}) \leftarrow \mathcal{O}(-\mathbf{b})$ is sent to the map $e_{\mathbf{a}}A \leftarrow e_{\mathbf{b}}A$ corresponding to the monomial for the path beginning at vertex $\mathbf{b}$ and ending at vertex $\mathbf{a}$. Composing with the reverse equivalence $- \otimes^{\mathbf{L}} \mathcal{E}$ gives the identity on these objects by the proof of [Kin97, Thm. 2.1].

Another ingredient of the proof is showing that $\operatorname{Hom}(\mathcal{E}, -)$ is exact on the class of free monads constructed in Proposition 3.5.1, and in particular we may apply the additive functor (3.5.2) term-wise on locally free resolutions in this class and yield projective resolutions. See [Wei94, §2.5] for terminology of derived categories and acyclic classes.

Lemma 3.5.5. *A chain complex $\mathcal{L}$ whose terms consist of summands of $\mathcal{E}$ is $\operatorname{Hom}(\mathcal{E}, -)$-acyclic. Furthermore,*

1. *if $\mathcal{L}$ is a free monad for $\mathcal{G}$ then $\mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{G}) = \mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{L}) = \operatorname{Hom}(\mathcal{E}, \mathcal{L})$;*
2. *if $\mathcal{L}$ is a locally free resolution then $\mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{G})$ is a projective resolution over A.*

Proof. Since the summands of $\mathcal{E}$ form a strong exceptional sequence and each $\mathcal{L}_j$ consists of summands of $\mathcal{E}$, we have $\mathbf{R}^i \operatorname{Hom}(\mathcal{E}, \mathcal{L}) = 0$ for $i \neq 0$, hence $\mathcal{L}$ is $\operatorname{Hom}(\mathcal{E}, -)$-acyclic. Thus, denoting $F = \operatorname{Hom}(\mathcal{E}, -)$, the spectral sequence $E^{-j,i} = \mathbf{R}^i F(\mathcal{L}_j) \Rightarrow \mathbf{R}F(\mathcal{L})$ degenerates on the first page to $F(\mathcal{L})$. On the other hand, if $\mathcal{L}$ is a free monad for $\mathcal{G}$ we have $\mathbf{R}F(\mathcal{G}) = \mathbf{R}F(\mathcal{L}) = F(\mathcal{L})$ as triangulated functors preserve quasi-isomorphisms. Thus we can apply the functor (3.5.2) on $\mathcal{L}$ term-wise.

For the second part, first observe that if $\mathcal{L}$ is a free monad for $\mathcal{G}$ and $H^i F(\mathcal{L}) = \mathbf{R}^i F(\mathcal{G}) = 0$ for $i \neq 0$ then $F(\mathcal{L})$ is a projective monad for $F(\mathcal{G})$. We will show that this vanishing holds when $\mathcal{L}$ is a locally free resolution by inducting on the length of $\mathcal{L}$: consider $0 \leftarrow \mathcal{G} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow 0$ with length 1, then the long exact sequence of $\operatorname{Hom}(\mathcal{E}, -)$ implies that $\mathbf{R}^i \operatorname{Hom}(\mathcal{E}, \mathcal{G}) = \mathbf{R}^{i+1} \operatorname{Hom}(\mathcal{E}, \mathcal{L}_1) = 0$ for $i \neq 0$. If $\mathcal{L}$ has length n, break it into

$$0 \leftarrow \mathcal{G} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{G}' \leftarrow 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}' \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_n \leftarrow 0.$$

Then by the inductive hypothesis $\mathbf{R}^i \operatorname{Hom}(\mathcal{E}, \mathcal{G}') = 0$ for $i \neq 0$, thus $\mathbf{R}^i \operatorname{Hom}(\mathcal{E}, \mathcal{G}) = 0$ for $i \neq 0$ using the short exact sequence. Thus, when $\mathcal{L}$ is a locally free resolution as above, $\mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{G}) = \operatorname{Hom}(\mathcal{E}, \mathcal{L})$ is a projective resolution over A. ∎

Proof of Theorem 3.5.4. Since $F_\bullet$ and $G_\bullet$ are minimal virtual resolutions of the same module, the locally free resolutions $\widetilde{F}_\bullet$ and $\widetilde{G}_\bullet$ are quasi-isomorphic, and by assumption all their terms are direct sums of $\mathcal{O}(-\mathbf{a})$ for $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$. Hence by Lemma 3.5.5 applying the functor (3.5.2) gives minimal projective resolutions $C_\bullet$ and $D_\bullet$, respectively.

Since A is graded it follows from [BK99, §2] that mod$-A$ is a "perfect" category in the sense of [Eil56, §2]. Thus by [Eil56, Prop. 7] there exists an isomorphism $\psi\colon C_\bullet \to D_\bullet$ of complexes of right A-modules.

Applying the reverse equivalence $-\otimes^{\mathbf{L}} \mathcal{E}$ gives an isomorphism of complexes of $\mathcal{O}_X$-modules $\psi \otimes^{\mathbf{L}} \mathcal{E}\colon \widetilde{F}_\bullet \to \widetilde{G}_\bullet$ as desired. Finally, applying twisted global sections Γ_* yields the isomorphism between $F_\bullet$ and $G_\bullet$. $\square$

Remark 3.5.6. Note that Lemma 3.5.5 holds in more generality for any triangulated category and any additive left exact functor F such that $\mathbf{R}^i F(\mathcal{E}) = 0$ for $i > 0$, and the equivalence of categories via the functor (3.5.2) applies to any full exceptional collection of line bundles on a toric variety. In particular, translating every line bundles in the collection by a fixed line bundle results in a similar uniqueness theorem that may apply to virtual resolutions which are only free monads for the standard collection. However, note that in general free monads are not sent to projective monads.

4 Resolving the Diagonal for Toric Varieties of Picard Rank 2

The material in this chapter originally appeared in [BS22].

4.1 Introduction

In this chapter, we aim to construct a Beilinson-type resolution of the diagonal over a smooth projective toric variety X of Picard rank 2. More specifically, with a view toward proving a new case of a conjecture of Berkesch–Erman–Smith (Conjecture 3.1.2 below), we construct such a resolution of length $\dim X$—the shortest possible length—whose terms are finite direct sums of line bundles. While the existence of a full strong exceptional collection of line bundles [CM04, BH09] implies that X admits a resolution of the diagonal via a tilting bundle construction [Kin97, Prop. 3.1], it follows from a result of Ballard–Favero [BF12, Prop. 3.33] that this resolution may have length greater than $\dim X$. Our main result is as follows:

Theorem 4.1.1. *Let X be the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_s))$ over $\mathbb{P}^r$, where $1 \leq r, s$ and $0 \leq a_1 \leq \cdots \leq a_s$. Denote by $\mathbb{F}_{a_s}$ the Hirzebruch surface of type a_s, and equip $\mathrm{Pic}(\mathbb{F}_{a_s}) \cong \mathbb{Z}^2$ with the basis described in Convention 4.3.1 below. There is a complex R of finitely generated graded free modules over the Cox ring of $X \times X$ such that:*

1. *R is exact in positive degrees.*
2. *R is linear, in the sense that there exists a basis of R with respect to which the differentials of R are matrices whose entries are $\Bbbk$-linear combinations of the variables.*
3. *We have $\operatorname{rank} R_n = \binom{r+s}{n} \dim_{\Bbbk} H^0(\mathbb{F}_{a_s}, \mathcal{O}(r,s))$. In particular, R has length $\dim X = r+s$, and the equality $\operatorname{rank} R_n = \operatorname{rank} R_{r+s-n}$ holds.*
4. *The sheafification $\mathcal{R}$ of R is a resolution of the diagonal sheaf $\mathcal{O}_\Delta$ on $X \times X$.*

We note that, by a result of Kleinschmidt in [Kle88], every smooth projective toric variety of Picard rank 2 arises as a projective bundle as in the hypothesis of Theorem 4.1.1. We construct the resolution $\mathcal{R}$ in Theorem 4.1.1 using a variant of Weyman's "geometric technique" for building free resolutions, described in [Wey03, §5]. In a bit more detail: let x_i and x_i' refer to the variables corresponding to the first and second copy of X, respectively, in the Cox ring S of $X \times X$. A first, naive, idea is that the diagonal sheaf $\mathcal{O}_\Delta$ ought to be defined by the relations $x_i - x_i'$ in S. The problem is that these relations are not homogeneous with respect to the $\mathbb{Z}^4$-grading on S. To fix this, we homogenize the relations $x_i - x_i'$ in the Cox ring of a certain toric fiber bundle E over $X \times X$ with fiber given by $\mathbb{F}_{a_s}$. Our resolution $\mathcal{R}$ is obtained by taking the Koszul complex on these homogenized relations over E, twisting it by a certain line bundle, and pushing it forward to $X \times X$. Choosing the toric fiber bundle E is delicate; not only do the degrees of the variables in the Cox ring of E need to be suitable for homogenizing the relations $x_i - x_i'$, but the terms of the Koszul complex on these homogenized relations must enjoy appropriate cohomological vanishing properties in order to conclude that $\mathcal{R}$ is a resolution of the required form. See §4.3.3 for details.

The simplest case of Theorem 4.1.1 is the Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$, where $r = s = 1$ and $a = a_1$. As detailed in Example 4.3.9, the construction above yields a resolution of the diagonal for $\mathbb{F}_a$ whose terms $\mathcal{R}_0$, $\mathcal{R}_1$, and $\mathcal{R}_2$ are sums of $a+4$, $2a+8$, and $a+4$ line bundles, respectively (cf. [Buc87, §1]).

As we explain in §4.2, the resolution $\mathcal{R}$ in Theorem 4.1.1 should be considered as a natural extension of Beilinson's resolution over projective space and similar resolutions due to Buchdahl for Hirzebruch surfaces [Buc87], Canonaco–Karp for weighted projective stacks [CK08], and Kapranov for quadrics and flag varieties [Kap88]. See [BE21, §4] for a related idea, where a resolution of the diagonal—with terms given by infinite direct sums of line bundles—is obtained for any projective toric stack.

As a consequence of Theorem 4.1.1, we prove the following:

Corollary 4.1.2. *Conjectures 3.1.2 and 1.2.1 hold for smooth projective toric varieties of Picard rank 2.*

We refer the reader to the original paper [Rou08] of Rouquier for background on his notion of dimension for triangulated categories. Since the resolution of the diagonal $\mathcal{R}$ in Theorem 4.1.1

has length $\dim X$, and each term $\mathcal{R}_i$ is a sum of box products of vector bundles on X, it is an immediate consequence of [Rou08, Prop. 7.6] that Theorem 4.1.1 implies Conjecture 1.2.1 for smooth projective toric varieties of Picard rank 2. Conjecture 1.2.1 was first proven in this case by Ballard–Favero–Katzarkov [BFK19, Cor. 5.2.6] using an entirely different approach: they first observe that the conjecture holds for a smooth projective Picard rank 2 toric variety that is weakly Fano, and then they apply descent along admissible subcategories. See the discussion beneath [BC21, Conj. 1.1] for a list of known cases of Conjecture 1.2.1.

In Chapter 5, we also apply Theorem 4.1.1 to obtain a splitting criterion for vector bundles on smooth projective toric varieties of Picard rank 2.

4.2 Warm-up: the Case of $\mathbb{P}^n$

Throughout this chapter, we work over a base field $\Bbbk$. Let $\mathcal{T}_{\mathbb{P}^n}$ denote the tangent bundle on $\mathbb{P}^n$ and $\mathcal{W}$ the vector bundle $\mathcal{O}_{\mathbb{P}^n}(1)\boxtimes\mathcal{T}_{\mathbb{P}^n}(-1)$ on $\mathbb{P}^n\times\mathbb{P}^n$. There is a canonical section $s\in H^0(\mathbb{P}^n\times\mathbb{P}^n,\mathcal{W})$ whose vanishing cuts out the diagonal in $\mathbb{P}^n\times\mathbb{P}^n$ (see [Huy06, §8.3]). The Koszul complex associated to s yields Beilinson's resolution of the diagonal

$$0\leftarrow\mathcal{O}_\Delta\leftarrow\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}\leftarrow\Lambda^1\mathcal{W}^\vee\leftarrow\cdots\leftarrow\Lambda^n\mathcal{W}^\vee\leftarrow 0.$$

In this section, we construct another resolution of the diagonal sheaf on $\mathbb{P}^n\times\mathbb{P}^n$, whose terms are direct sums of line bundles (cf. [CK08, Rem. 3.3]). We explain in Remark 4.2.3(3) a sense in which this resolution resembles Beilinson's. As discussed in the introduction, our approach is similar to Weyman's "geometric technique" [Wey03, §5]. In §4.3, we explain how the approach in this section extends to smooth projective toric varieties of Picard rank 2.

Let E denote the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1,1))$ on $\mathbb{P}^n\times\mathbb{P}^n$ and let $\pi\colon E\to\mathbb{P}^n\times\mathbb{P}^n$ be the canonical map. The projective bundle E is a toric variety with $\mathbb{Z}^3$-graded Cox ring $S_E=\Bbbk[x_0,\dots,x_n,y_0,\dots,y_n,u_0,u_1]$, where $\deg(x_i)=(1,0,0)$, $\deg(y_i)=(0,1,0)$, $\deg(u_0)=(1,-1,1)$, and $\deg(u_1)=(0,0,1)$. Set $\alpha_i=u_1x_i-u_0y_i$ for all i; the intuition here is that u_0 and u_1 are homogenizing variables for the non-homogeneous equations x_i-y_i. Let $\mathcal{K}$ denote the Koszul complex on $\alpha_0,\dots,\alpha_n$, considered as a complex of sheaves on E, and set $\mathcal{V}=\mathcal{O}(-1,0,0)^{n+1}$. Twisting $\mathcal{K}$ by

$\mathcal{O}(0,0,n)$ yields a complex of the form

$$\mathcal{O}(0,0,n) \leftarrow (\Lambda^1\mathcal{V})(0,0,n-1) \leftarrow \cdots \leftarrow \Lambda^n\mathcal{V} \leftarrow (\Lambda^{n+1}\mathcal{V})(0,0,-1).$$

Using [Har77, Ch. III, Ex. 8.4(a)] and the projection formula, $\mathcal{R} = \pi_*\mathcal{K}(0,0,n)$ has the form

$$\operatorname{Sym}^n \mathcal{Q} \leftarrow \Lambda^1\mathcal{P} \otimes \operatorname{Sym}^{n-1} \mathcal{Q} \leftarrow \cdots \leftarrow \Lambda^{n-1}\mathcal{P} \otimes \operatorname{Sym}^1 \mathcal{Q} \leftarrow \Lambda^n\mathcal{P}, \tag{4.2.1}$$

where $\mathcal{P} = \mathcal{O}(-1,0)^{n+1}$ and $\mathcal{Q} = \mathcal{O} \oplus \mathcal{O}(-1,1)$. Notice that applying π_* to the $n+1^{\text{th}}$ term $(\Lambda^{n+1}\mathcal{V})(0,0,-1)$ of $\mathcal{K}(0,0,n)$ gives 0, hence the complex (4.2.1) has length n.

Proposition 4.2.1. *The complex $\mathcal{R}$ is a resolution of the diagonal sheaf on $\mathbb{P}^n \times \mathbb{P}^n$. Moreover, the complex $\mathcal{R}$ is isomorphic to (the sheafification of) the n^{th} symmetric power of the complex*

$$S(-1,1) \oplus S \xleftarrow{\begin{pmatrix} -y_0 & -y_1 & \cdots & -y_n \\ x_0 & x_1 & \cdots & x_n \end{pmatrix}} S(-1,0)^{n+1}, \tag{4.2.2}$$

concentrated in homological degrees 0 and 1, where S denotes the Cox ring of $\mathbb{P}^n \times \mathbb{P}^n$.

Proof. One can use a slight variation of the proof of Theorem 4.1.1 below to show that $\mathcal{R}$ is a resolution of the diagonal. As for the second statement: let K denote the Koszul complex on the regular sequence $\alpha_0, \ldots, \alpha_n$, considered as a complex of S_E-modules. Let R be the complex of S-modules given by $K(0,0,n)_{(*,*,0)}$. Since K is exact in positive homological degrees, R is as well. It follows from the description of $\mathcal{R}$ in (4.2.1) that R sheafifies to $\mathcal{R}$. Let R' denote the n^{th} symmetric power of (4.2.2). We observe that R' has exactly the same terms as R. The complex R' is precisely the generalized Eagon–Northcott complex of type $\mathcal{C}^n$, as defined in [Eis95, A2.6], associated to the map (4.2.2). It therefore follows from [Eis95, Thm. A2.10(c)] that R' is exact in positive homological degrees. By the uniqueness of minimal free resolutions, we need only check that the cokernels of the first differentials of R and R' are isomorphic, and this can be verified by direct computation. □

We now compute a well-known example using this approach (cf. [Kin97, Ex. 5.2]).

Example 4.2.2. Suppose $n = 2$. The monomials in the u_i's give bases for the symmetric powers of $\mathcal{Q}$, and the exterior monomials in the α_i's give bases for the terms of $\mathcal{K}$, which correspond to the exterior powers of $\mathcal{P}$. Hence, we may index the summands of (4.2.1) by monomials in $u_0, u_1, \alpha_0, \alpha_1, \alpha_2$. With this in mind, the complex (4.2.1) has terms

$$\underbrace{\mathcal{O}(-2,2)}_{u_0^2} \oplus \underbrace{\mathcal{O}(-1,1)}_{u_0u_1} \oplus \underbrace{\mathcal{O}}_{u_1^2} \xleftarrow{\partial_1} \underbrace{\mathcal{O}(-2,1)^3}_{\alpha_0u_0,\alpha_1u_0,\alpha_2u_0} \oplus \underbrace{\mathcal{O}(-1,0)^3}_{\alpha_0u_1,\alpha_1u_1,\alpha_2u_1} \xleftarrow{\partial_2} \underbrace{\mathcal{O}(-2,0)^3}_{\alpha_0\alpha_1,\alpha_0\alpha_2,\alpha_1\alpha_2}$$

and differentials

$$\partial_1 = \begin{pmatrix} -y_0 & -y_1 & -y_2 & 0 & 0 & 0 \\ x_0 & x_1 & x_2 & -y_0 & -y_1 & -y_2 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix} \quad \text{and} \quad \partial_2 = \begin{pmatrix} y_1 & y_2 & 0 \\ -y_0 & 0 & y_2 \\ 0 & -y_0 & -y_1 \\ -x_1 & -x_2 & 0 \\ x_0 & 0 & -x_2 \\ 0 & x_0 & x_1 \end{pmatrix}.$$

Remark 4.2.3. We conclude this section with the following observations:

1. We have $\operatorname{rank} \mathcal{R}_i = \operatorname{rank} \mathcal{R}_{n-i}$, just as in Theorem 4.1.1.

2. The resolutions in Theorem 4.1.1 cannot arise as symmetric powers of complexes, in general; this follows immediately from rank considerations.

3. Let us explain a sense in which our resolution $\mathcal{R}$ is modeled on Beilinson's resolution of the diagonal. Consider the external tensor product of $\mathcal{O}(1)$ with the Euler sequence: $0 \leftarrow \mathcal{O}(1) \boxtimes \mathcal{T}(-1) \leftarrow \mathcal{O}(1,0)^{n+1} \xleftarrow{\begin{pmatrix} y_0 & \cdots & y_n \end{pmatrix}^T} \mathcal{O}(1,-1) \leftarrow 0$. Letting $\mathcal{C}$ denote the subcomplex $\mathcal{O}(1,0)^{n+1} \leftarrow \mathcal{O}(1,-1)$ concentrated in degrees 0 and 1, there is a quasi-isomorphism $\mathcal{C} \xrightarrow{\simeq} \mathcal{O}(1) \boxtimes \mathcal{T}(-1)$. The morphism $s\colon \mathcal{O} \xrightarrow{\begin{pmatrix} x_0 & \cdots & x_n \end{pmatrix}^T} \mathcal{C}$, where $\mathcal{O}$ lies in degree 0, gives a hypercohomology class in $\mathbb{H}^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{C})$, which is isomorphic to $H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}(1) \boxtimes \mathcal{T}(-1))$. By Proposition 4.2.1, the n^{th} symmetric power of the dual of s, i.e. the n^{th} Koszul complex of the dual of s [Köc01, Definition 2.3], is isomorphic to the resolution $\mathcal{R}$. In short: the resolution $\mathcal{R}$ is a Koszul complex on a section of $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$, just like Beilinson's resolution.

4.3 Smooth Projective Toric Varieties of Picard Rank 2

In this section, we extend the construction in §4.2 and prove the main theorem. Let X denote the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_s))$ over $\mathbb{P}^r$, where $a_1 \leq \cdots \leq a_s$. As discussed in [CLS11, §7.3], the fan $\Sigma_X \subseteq \mathbb{Z}^{r+s}$ of X has $r+s+2$ ray generators given by the rows of the $(r+s+2) \times (r+s)$ matrix

$$P = \begin{pmatrix} -1 & -1 & \cdots & -1 & a_1 & a_2 & \cdots & a_s \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \vdots \\ \rho_r \\ \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_s \end{pmatrix} \tag{4.3.1}$$

and maximal cones generated by collections of rays of the form

$$\{\rho_0, \ldots, \widehat{\rho_i}, \ldots, \rho_r, \sigma_0, \ldots, \widehat{\sigma_j}, \ldots, \sigma_s\}.$$

Convention 4.3.1. Throughout this chapter, we equip $\operatorname{Pic} X \cong \operatorname{coker}(P) \cong \mathbb{Z}^2$ with the basis given by the divisors corresponding to ρ_0 and σ_0. With this choice of basis, we may view the Cox ring of X as the $\mathbb{Z}^2$-graded ring $\Bbbk[x_0, \ldots, x_r, y_0, \ldots, y_s]$ whose variables have degrees given by the columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_s \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

A main reason we use this convention is that it is also used by the function `kleinschmidt` in `Macaulay2`, which produces any smooth projective toric variety of Picard rank 2 as an object of type `NormalToricVariety`.

4.3.1 Vanishing of Sheaf Cohomology

We will need a calculation of the cohomology of a line bundle on X:

Proposition 4.3.2. *Let $\mathcal{E}$ be the vector bundle $\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_s)$ on $\mathbb{P}^r$, where $a_1 \leq \cdots \leq a_s$, so that $X = \mathbb{P}(\mathcal{E})$. Write $m = \sum_{i=1}^s a_i$, and consider a line bundle $\mathcal{O}(k,\ell)$ on X. For each $0 \leq j \leq r+s$, we have:*

$$H^j(X, \mathcal{O}(k,\ell)) \cong \begin{cases} H^j(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k) \otimes \mathrm{Sym}^\ell(\mathcal{E})), & \ell \geq 0; \\ H^{j-s}(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k-m) \otimes \mathrm{Sym}^{-\ell-s-1}(\mathcal{E})^\vee), & \ell \leq -s-1; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Let $\pi \colon X \to \mathbb{P}^r$ denote the projective bundle map. It follows from a well-known calculation (see e.g. [TT90, 4.5(e)]) and the projection formula that

$$\mathbf{R}^i\pi_*(\mathcal{O}(k,\ell)) = \begin{cases} \mathcal{O}_{\mathbb{P}^r}(k) \otimes \mathrm{Sym}^\ell(\mathcal{E}), & i = 0; \\ \mathcal{O}_{\mathbb{P}^r}(k-m) \otimes \mathrm{Sym}^{-\ell-s-1}(\mathcal{E})^\vee, & i = s; \\ 0, & 0 < i < s. \end{cases}$$

The conclusion follows from the observation that the second page of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}^r, \mathbf{R}^q\pi_*(\mathcal{O}(k,\ell))) \Rightarrow H^{p+q}(X, \mathcal{O}(k,\ell))$$

collapses to row $q = 0$ when $\ell \geq 0$ and to row $q = s$ when $\ell \leq -s-1$. □

The following result is an immediate consequence of Proposition 4.3.2. It will play a key role in the proof of Theorem 4.1.1.

Corollary 4.3.3. *Let X be the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_s))$ over $\mathbb{P}^r$ as above, where $a_1 \leq \cdots \leq a_s$. Write $m = \sum_{i=1}^s a_i$, and consider a line bundle $\mathcal{O}(k,\ell)$ on X.*

1. *We have:*

   a) *$H^i(X, \mathcal{O}(k,\ell)) = 0$ if $i \notin \{0, r, s, r+s\}$.*

b) $H^0(X, \mathcal{O}(k, \ell)) = 0$ *if and only if* $\ell < 0$ *or* $k + a_s\ell < 0$.

c) If $r \neq s$ *then*

i. $H^r(X, \mathcal{O}(k, \ell)) = 0$ *if and only if* $-r - 1 < k$ *or* $\ell < 0$*, and*

ii. $H^s(X, \mathcal{O}(k, \ell)) = 0$ *if and only if* $-s - 1 < \ell$ *or* $k < m$.

d) If $r = s$ *then* $H^r(X, \mathcal{O}(k, \ell)) = 0$ *if and only if both of the following hold:*

i. $-r - 1 < k$ *or* $\ell < 0$*, and*

ii. $-s - 1 < \ell$ *or* $k < m$.

e) Lastly, $H^{r+s}(X, \mathcal{O}(k, \ell)) = 0$ *if and only if either of the following hold:*

i. $-r - 1 - a_s(\ell + s + 1) + m < k$*, or*

ii. $-s - 1 < \ell$;

2. *In particular, the line bundle* $\mathcal{O}(k, \ell)$ *is acyclic (*$H^i(X, \mathcal{O}(k, \ell)) = 0$ *for* $i > 0$*) if and only if one of the following holds:*

   (a) $-s - 1 < \ell < 0$,

   (b) $-r - 1 < k$ *and* $0 \leq \ell$,

   (c) $-r - 1 - a_s(\ell + s + 1) + m < k < m$ *and* $\ell \leq -s - 1$.

Remark 4.3.4. Conditions (1b) and (1e) are Serre dual to one another. Ditto for the two conditions in (1c), as well as the conditions (i) and (ii) in (1d). These calculations are surely well-known; see, for instance, [LM11, Prop. 3.9] for a criterion for acyclicity of line bundles on toric varieties. We refer the reader to [BE21, Ex. 3.14] for a depiction of the regions of $\mathbb{Z}^2$ where each $H^i(X, \mathcal{O}(k, \ell))$ vanishes for the Hirzebruch surface $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$.

4.3.2 Toric Fiber Bundles

Let E and Y be smooth projective toric varieties of dimensions d_E and d_Y associated to fans Σ_E and Σ_Y. Let $\bar{\pi}\colon \mathbb{Z}^{d_E} \to \mathbb{Z}^{d_Y}$ be a $\mathbb{Z}$-linear surjection that is compatible with the fans Σ_E and Σ_Y, in the sense of [CLS11, Def. 3.3.1], so that it induces a morphism $\pi\colon E \to Y$. We denote by F the toric variety associated to the fan $\Sigma_F = \{\sigma \in \Sigma_E \colon \sigma \subseteq \ker(\bar{\pi})_{\mathbb{R}}\}$, and write $d_F = \dim F$. Let us assume that the fan Σ_E is *split* by the fans Σ_Y and Σ_F, in the sense of [CLS11, Def. 3.3.18]. In this case, the map $\pi\colon E \to Y$ is a fibration with fiber F; see [CLS11, Thm. 3.3.19].

Writing the Cox rings of Y and F as $S_Y = \Bbbk[x_1, \dots, x_{n_1}]$ and $S_F = \Bbbk[u_1, \dots, u_{n_2}]$, the Cox ring of E has the form $S_E = \Bbbk[x_1, \dots, x_{n_1}, u_1, \dots, u_{n_2}]$. We have presentations $P_Y\colon \mathbb{Z}^{d_Y} \to \mathbb{Z}^{n_1}$ and $P_F\colon \mathbb{Z}^{d_F} \to \mathbb{Z}^{n_2}$ of $\operatorname{Pic} Y$ and $\operatorname{Pic} F$ whose rows are given by the ray generators of Σ_Y and Σ_F, respectively. The analogous presentation of $\operatorname{Pic} E$ is of the form

$$\begin{pmatrix} P_Y & Q \\ 0 & P_F \end{pmatrix}$$

for some $n_1 \times d_F$ matrix Q. One may use this presentation to equip S_E with a $\mathbb{Z}^e \oplus \mathbb{Z}^f$-grading such that $\deg_{S_E}(x_i) = (\deg_{S_Y}(x_i), 0)$, and $\deg_{S_E}(u_i) = (t_i, \deg_{S_F}(u_i))$ for some $t_i \in \mathbb{Z}^e$.

Lemma 4.3.5 (cf. [Har77] Ch. III, Ex. 8.4(a))**.** *Let $\mathcal{L} = \mathcal{O}_E(b_1, \dots, b_e, c_1, \dots, c_f)$, and let $\mathcal{B}$ be a $\Bbbk$-basis of $H^0(F, \mathcal{O}_F(c_1, \dots, c_f))$ given by monomials in S_F. Given $m \in \mathcal{B}$, denote its degree in S_E by $(d_1^m, \dots, d_e^m, c_1, \dots, c_f)$. We have $\pi_*(\mathcal{L}) \cong \bigoplus_{m\in\mathcal{B}} \mathcal{O}_Y(b_1 - d_1^m, \dots, b_e - d_e^m)$. Moreover, if $H^i(F, \mathcal{O}_F(c_1, \dots, c_f)) = 0$, then $\mathbf{R}^i\pi_*(\mathcal{L}) = 0$.*

Proof. Let $g\colon \bigoplus_{m\in\mathcal{B}} \mathcal{O}_Y(b_1 - d_1^m, \dots, b_e - d_e^m) \to \pi_*(\mathcal{L})$ be the morphism given on the component corresponding to $m \in \mathcal{B}$ by multiplication by m. Let U be an affine open subset of Y over which the fiber bundle E is trivializable; abusing notation slightly, we denote by π the map $\pi^{-1}(U) \to U$ induced by π. To prove the first statement, it suffices to show that the restriction $g_U : \bigoplus_{m\in\mathcal{B}_i} \mathcal{O}_U \to \pi_*(\mathcal{L}|_U)$ of g to U is an isomorphism. Without loss of generality, we may assume that $\pi^{-1}(U) = U \times F$ and that $\pi\colon \pi^{-1}(U) \to U$ is the projection onto U. Letting $\gamma : \pi^{-1}(U) \to F$ denote the projection, we have that $\mathcal{L}|_U = \gamma^*(\mathcal{O}_F(c_1, \dots, c_f))$. Finally, we observe that g_U coincides with the base change isomorphism $\bigoplus_{m\in\mathcal{B}} \mathcal{O}_U = \mathcal{O}_U \otimes_\Bbbk H^0(F, \mathcal{O}_F(c_1, \dots, c_f)) \xrightarrow{\cong} \pi_*(\gamma^*(\mathcal{O}_F(c_1, \dots, c_f)) = \pi_*(\mathcal{L}|_U)$. As for the last statement: it suffices to observe that, by base change, $\mathbf{R}^i\pi_*(\mathcal{L}|_U) \cong \mathcal{O}_U \otimes_\Bbbk H^i(F, \mathcal{O}_F(c_1, \dots, c_f)) = 0$. $\square$

4.3.3 Constructing the Resolution of the Diagonal

Let X be as defined at the beginning of this section. We will construct our resolution of the diagonal for X as the pushforward of a certain Koszul complex on a fibration E over $X \times X$ whose fiber is the Hirzebruch surface $\mathbb{F}_{a_s}$. We begin by constructing the fiber bundle $\pi \colon E \to X \times X$. The ray generators of E are given by the rows of the $(2r+2s+8) \times (2r+2s+2)$ matrix

$$\left(\begin{array}{cc|cc} P & 0 & v & -w \\ 0 & P & -v & w \\ \hline 0 & 0 & -1 & a_s \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right), \tag{4.3.2}$$

where P is as in (4.3.1), and v (resp. w) is the $(r+s+2) \times 1$ matrix with unique nonzero entry given by a 1 in the first (resp. $(r+2)^{\text{th}}$) position. Notice that the rows in the top-left quadrant of this matrix are the ray generators of $X \times X$, and the rows in the bottom-right quadrant are the ray generators of $\mathbb{F}_{a_s}$.

Let $\bar{\pi} \colon \mathbb{Z}^{2r+2s+2} \to \mathbb{Z}^{2r+2s}$ denote the projection onto the first $2r+2s$ coordinates. We define the cones of E to be those of the form $\gamma + \gamma'$, where γ is a cone corresponding to a cone of $\mathbb{F}_{a_s}$ and is spanned by a subset of the bottom 4 rows of (4.3.2), and γ' is a cone spanned by a collection of the top $2r+2s+4$ rows of (4.3.2) such that $\bar{\pi}_{\mathbb{R}}(\gamma')$ is a cone of $X \times X$. By [CLS11, Thm. 3.3.19], the map $\bar{\pi}$ induces a fibration $\pi \colon E \to X$ with fiber $\mathbb{F}_{a_s}$.

In order to describe the Cox ring of E, first recall the matrix A from Convention 4.3.1 whose columns are the degrees of the variables of the Cox ring of X, and consider the matrices

$$B = \begin{pmatrix} 1 & -a_s & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & -a_s & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Notice that the columns of C are the degrees of the variables in the Cox ring of $\mathbb{F}_{a_s}$. We choose a basis of $\operatorname{Pic} E \cong \mathbb{Z}^6$ so that the degrees of the variables in the Cox ring

$$S_E = \Bbbk[x_0, \ldots x_r, y_0, \ldots, y_s, x'_0, \ldots, x'_r, y'_0, \ldots, y'_s, u_0, \ldots, u_3]$$

of E are given by the columns of the Gale dual of (4.3.2), which is the $6 \times (2r + 2s + 8)$ matrix

$$\begin{pmatrix} A & 0 & B \\ 0 & A & -B \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_s & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -a_s & 0 & 0 \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_s & -1 & a_s & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & -1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -a_s & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Let K be the Koszul complex corresponding to the regular sequence $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_s$ given by the homogeneous binomials

$$\begin{aligned} \alpha_i &= u_2 x_i - u_0 x'_i && \text{for } 0 \le i \le r \text{ and} \\ \beta_i &= u_3 y_i - u_0^{a_s - a_i} u_1 u_2^{a_i} y'_i && \text{for } 0 \le i \le s \ \ (a_0 := 0) \end{aligned}$$

in the Cox ring S_E. Observe that $\deg(\alpha_i) = (1, 0, 0, 0, 1, 0)$ and $\deg(\beta_i) = (-a_i, 1, 0, 0, 0, 1)$. Here, we are using that the columns of B span the effective cone of X to homogenize the relations $x_i - x'_i$ and $y_i - y'_i$. Denote by $\mathcal{K}$ the complex of sheaves on E corresponding to K. The following proposition shows that $\mathcal{K}$ twisted by $\mathcal{O}_E(0, 0, 0, 0, r, s)$ is π_*-acyclic.

Proposition 4.3.6. *The higher direct images $\mathbf{R}^i\pi_*(\mathcal{K}(0, 0, 0, 0, r, s))$ vanish for $i > 0$.*

Proof. It suffices to show that $\mathbf{R}^i\pi_*(\mathcal{K}_j(0, 0, 0, 0, r, s)) = 0$ for $i > 0$ and all j. Each term of $\mathcal{K}(0, 0, 0, 0, r, s)$ is a direct sum of line bundles of the form $\mathcal{O}_E(a, b, 0, 0, k, \ell))$ for some $a, b \in \mathbb{Z}$, $-1 \le k \le r$, and $-1 \le \ell \le s$. By Lemma 4.3.5, we need only show that $H^i(\mathbb{F}_{a_s}, \mathcal{O}(k, \ell)) = 0$ for $i > 0$ and such k and ℓ, which follows from Corollary 4.3.3(2)(a-b). □

Let S denote the Cox ring of $X \times X$ and R the complex of graded S-modules given by the subcomplex $K(0, 0, 0, 0, r, s)_{(*,*,*,*,0,0)}$ of the Koszul complex K twisted by $S_E(0, 0, 0, 0, r, s)$. We will show that R satisfies the requirements of Theorem 4.1.1. Observe that, by Lemma 4.3.5, one can alternatively construct R by applying the twisted global sections functor:

$$R = \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X \times X)} H^0(X \times X, \mathcal{L} \otimes \pi_* \mathcal{K}(0,0,0,0,r,s)).$$

In particular, writing $\mathcal{R}$ for the complex of sheaves on $X \times X$ corresponding to R, we have $\mathcal{R} \cong \pi_*\mathcal{K}(0,0,0,0,r,s)$. Note that Proposition 4.3.6 implies that $\pi_*\mathcal{K}(0,0,0,0,r,s)$ is quasi-isomorphic to $\mathbf{R}\pi_*(\mathcal{K}(0,0,0,0,r,s))$.

Before discussing some examples, we must establish a bit of notation:

Notation 4.3.7. Let $S_F = \Bbbk[u_0, u_1, u_2, u_3]$ denote the Cox ring of the Hirzebruch surface $\mathbb{F}_{a_s}$, equipped with the $\mathbb{Z}^2$-grading so that the degrees of the variables correspond to the columns of the matrix C above. Given $i, j \in \mathbb{Z}$, let $M_{i,j}$ denote the set of monomials in S_F of degree (i,j). For $m \in M_{i,j}$, let $(d_1^m, d_2^m, d_3^m, d_4^m) \in \mathbb{Z}^4$ denote the first four coordinates of the degree of m as an element of the $\mathbb{Z}^6$-graded ring S_E; notice that $d_3^m = -d_1^m$, and $d_4^m = -d_2^m$.

Example 4.3.8. Let us compute the first differential in R. Using the notation above, we have

$$R_0 = \bigoplus_{m \in M_{r,s}} S(-d_1^m, -d_2^m, d_1^m, d_2^m) \cdot m, \quad \text{and} \quad R_1 = R_1^\alpha \oplus R_1^\beta, \quad \text{where}$$

$$R_1^\alpha = \bigoplus_{i=0}^{r} \bigoplus_{m \in M_{r-1,s}} S(-d_1^m - 1, -d_2^m, d_1^m, d_2^m) \cdot \alpha_i m,$$

$$R_1^\beta = \bigoplus_{i=0}^{s} \bigoplus_{m \in M_{r,s-1}} S(-d_1^m + a_i, -d_2^m - 1, d_1^m, d_2^m) \cdot \beta_i m.$$

Here, the decorations "$\cdot\, m$" in our description of R_0 are just for bookkeeping, and similarly for the "$\cdot\alpha_i m$" and "$\cdot\beta_i m$" in R_1. Viewing the differential $\partial_1 \colon R_1 \to R_0$ as a matrix with respect to the above basis, the column corresponding to $\alpha_i m$ has exactly two nonzero entries: an entry of x_i corresponding to the monomial $u_2 m \in M_{r,s}$ and an entry of $-x_i'$ corresponding to $u_0 m \in M_{r,s}$. Similarly, the column corresponding to $\beta_i m$ has exactly two nonzero entries: an entry of y_i corresponding to $u_3 m$

and an entry of $-y_i'$ corresponding to $u_0^{a_s-a_i}u_1u_2^{a_i}m$. That is, the matrix ∂_1 has the following form:

$$
\begin{array}{cc}
\begin{pmatrix}
 & 0 & & 0 & \\
 & -x_i' & & 0 & \\
 & 0 & & 0 & \\
 & 0 & & -y_i' & \\
\cdots & 0 & \cdots & 0 & \cdots \\
 & x_i & & 0 & \\
 & 0 & & 0 & \\
 & 0 & & y_i & \\
 & 0 & & 0 &
\end{pmatrix}
&
\begin{matrix}
\vdots \\ u_0 m \\ \vdots \\ u_0^{a_s-a_i}u_1u_2^{a_i}m \\ \vdots \\ u_2 m \\ \vdots \\ u_3 m \\ \vdots
\end{matrix} \\
\begin{matrix} \alpha_i m & \cdots & \beta_i m \end{matrix} &
\end{array}
.
$$

Example 4.3.9. Suppose X is the Hirzebruch surface of type a, i.e. the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$ over $\mathbb{P}^1$. We have $r = s = 1$ and $a_1 = a$. The Koszul complex K on $\alpha_0, \alpha_1, \beta_0, \beta_1$, twisted by $(0,0,0,0,1,1)$, looks like:

$$
\begin{aligned}
\underbrace{S_E(0,0,0,0,1,1)}_{1} \leftarrow\ & \underbrace{S_E(-1,0,0,0,0,1)^2}_{\alpha_0,\ \alpha_1} \oplus \underbrace{S_E(0,-1,0,0,1,0)}_{\beta_0} \oplus \underbrace{S_E(a,-1,0,0,1,0)}_{\beta_1} \\
\leftarrow\ & \underbrace{S_E(-2,0,0,0,-1,1)}_{\alpha_0\alpha_1} \oplus \underbrace{S_E(-1,-1,0,0,0,0)^2}_{\alpha_0\beta_0,\ \alpha_1\beta_0} \oplus \underbrace{S_E(a-1,-1,0,0,0,0)^2}_{\alpha_0\beta_1,\ \alpha_1\beta_1} \\
& \oplus \underbrace{S_E(a,-2,0,0,1,-1)}_{\beta_0\beta_1} \\
\leftarrow\ & \underbrace{S_E(a-2,-1,0,0,-1,0)}_{\alpha_0\alpha_1\beta_1} \oplus \underbrace{S_E(-2,-1,0,0,-1,0)}_{\alpha_0\alpha_1\beta_0} \oplus \underbrace{S_E(a-1,-2,0,0,0,-1)^2}_{\alpha_0\beta_0\beta_1,\ \alpha_1\beta_0\beta_1} \\
\leftarrow\ & \underbrace{S_E(a-2,-2,0,0,-1,-1)}_{\alpha_0\alpha_1\beta_0\beta_1}.
\end{aligned}
$$

Letting $M_{i,j}$ be as in Notation 4.3.7 (with $a_s = a$), we have:

$$
\begin{aligned}
M_{0,0} &= \{1\}, \qquad M_{1,0} = \{u_0, u_2\}, \qquad M_{0,1} = \{u_3\} \cup \{u_0^k u_1 u_2^\ell \colon k + \ell = a\}, \\
M_{-1,1} &= \{u_0^k u_1 u_2^\ell \colon k + \ell = a - 1\}, \\
M_{1,1} &= \{u_0u_3, u_2u_3\} \cup \{u_0^k u_1 u_2^\ell \colon k + \ell = a + 1\}, \\
M_{i,j} &= \emptyset \quad \text{for } (i,j) \in \{(1,-1), (-1,0), (0,-1), (-1,-1)\}.
\end{aligned}
$$

It follows that the complex R has terms as follows:

$$
\begin{aligned}
R_0 = {} & \underbrace{S(-1,-1,1,1)}_{u_0^{a+1}u_1} \oplus \underbrace{S(0,-1,0,1)}_{u_0^a u_1 u_2} \oplus \cdots \oplus \underbrace{S(a,-1,-a,1)}_{u_1 u_2^{a+1}} \oplus \underbrace{S(-1,0,1,0)}_{u_0 u_3} \oplus \underbrace{S(0,0,0,0)}_{u_2 u_3}, \\
R_1 = {} & \underbrace{S(-1,-1,0,1)^2}_{\alpha_0 u_0^a u_1,\ \alpha_1 u_0^a u_1} \oplus \underbrace{S(0,-1,-1,1)^2}_{\alpha_0 u_0^{a-1} u_1 u_2,\ \alpha_1 u_0^{a-1} u_1 u_2} \oplus \cdots \oplus \underbrace{S(a-1,-1,-a,1)^2}_{\alpha_0 u_1 u_2^a,\ \alpha_1 u_1 u_2^a} \oplus \underbrace{S(-1,0,0,0)^2}_{\alpha_0 u_3,\ \alpha_1 u_3} \\
& \oplus \underbrace{S(-1,-1,1,0)}_{\beta_0 u_0} \oplus \underbrace{S(a-1,-1,1,0)}_{\beta_1 u_0} \oplus \underbrace{S(0,-1,0,0)}_{\beta_0 u_2} \oplus \underbrace{S(a,-1,0,0)}_{\beta_1 u_2}, \\
R_2 = {} & \underbrace{S(-1,-1,-1,1)}_{\alpha_0\alpha_1 u_0^{a-1} u_1} \oplus \underbrace{S(0,-1,-2,1)}_{\alpha_0\alpha_1 u_0^{a-2} u_1 u_2} \oplus \cdots \oplus \underbrace{S(a-2,-1,-a,1)}_{\alpha_0\alpha_1 u_1 u_2^{a-1}} \oplus \underbrace{S(-1,-1,0,0)^2}_{\alpha_0\beta_0,\ \alpha_1\beta_0} \oplus \underbrace{S(a-1,-1,0,0)^2}_{\alpha_0\beta_1,\ \alpha_1\beta_1}.
\end{aligned}
$$

The differentials $\partial_1 \colon R_0 \leftarrow R_1$ and $\partial_2 \colon R_1 \leftarrow R_2$ are given, respectively, by the matrices

$$
\partial_1 = \begin{pmatrix}
-x_0' & -x_1' & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -y_0' & 0 & 0 & 0 \\
x_0 & x_1 & -x_0' & -x_1' & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_0' & 0 \\
0 & 0 & x_0 & x_1 & -x_0' & -x_1' & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_0 & x_1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -x_0' & -x_1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & x_0 & x_1 & -x_0' & -x_1' & 0 & 0 & 0 & -y_1' & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x_0 & x_1 & 0 & 0 & 0 & 0 & 0 & -y_1' \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -x_0' & -x_1' & y_0 & y_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & 0 & y_0 & y_1
\end{pmatrix},
$$

$$
\partial_2 = \begin{pmatrix}
x_1' & 0 & \cdots & 0 & y_0' & 0 & 0 & 0 \\
-x_0' & 0 & \cdots & 0 & 0 & y_0' & 0 & 0 \\
-x_1 & x_1' & \cdots & 0 & 0 & 0 & 0 & 0 \\
x_0 & -x_0' & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & -x_1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & x_0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & x_1' & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & -x_0' & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & -x_1 & 0 & 0 & y_1' & 0 \\
0 & 0 & \cdots & x_0 & 0 & 0 & 0 & y_1' \\
0 & 0 & \cdots & 0 & -y_0 & 0 & -y_1 & 0 \\
0 & 0 & \cdots & 0 & 0 & -y_0 & 0 & -y_1 \\
0 & 0 & \cdots & 0 & -x_0' & -x_1' & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & -x_0' & -x_1' \\
0 & 0 & \cdots & 0 & x_0 & x_1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & x_0 & x_1
\end{pmatrix}.
$$

As predicted by Theorem 4.1.1 parts (2) and (3), the differentials in R are linear; and the ranks of R_0, R_1, and R_2 are $a+4$, $2a+8$, and $a+4$, respectively.

4.3.4 The Fourier–Mukai Transform

Let π_1 and π_2 denote the projections of $X \times X$ onto X, and let $\Phi_{\mathcal{R}}$ denote the following Fourier–Mukai transform:

$$\Phi_{\mathcal{R}} \colon \mathrm{D^b}(X) \xrightarrow{\pi_1^*} \mathrm{D^b}(X \times X) \xrightarrow{\cdot \otimes \mathcal{R}} \mathrm{D^b}(X \times X) \xrightarrow{\mathbf{R}\pi_{2*}} \mathrm{D^b}(X).$$

We will prove that $\mathcal{R}$ is a resolution of the diagonal by showing that $\Phi_{\mathcal{R}}$ is isomorphic to the identity functor, and we will do so by directly exhibiting a natural isomorphism $\Phi_\nu : \Phi_{\mathcal{R}} \to \Phi_{\mathcal{O}_\Delta}$. In fact, we show this by proving that Φ_ν induces a quasi-isomorphism on a full exceptional collection. To perform this calculation, we will need an explicit model for the functor $\Phi_{\mathcal{R}}$, which we present in this section. We refer the reader to [Huy06, §8.3] for further background.

Let $\mathrm{coh}(X)$ denote the category of coherent sheaves on X, and suppose $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{coh}(X)$, where $\mathcal{F}_1$ is locally free. By the projection formula and base change, we have canonical isomorphisms

$$\mathbf{R}\pi_{2*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \mathbf{R}\pi_{2*}\pi_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} \mathcal{F}_2 \cong \mathbf{R}\Gamma(X, \mathcal{F}_1) \otimes_{\Bbbk} \mathcal{F}_2$$

in $\mathrm{D^b}(X)$. Given $\mathcal{F} \in \mathrm{coh}(X)$, we can use this to explicitly compute $\Phi_{\mathcal{R}}(\mathcal{F})$ as follows. Given $\mathcal{G} \in \mathrm{coh}(X)$, let $\check{C}_{\mathcal{G}}$ denote the Čech complex of $\mathcal{G}$ associated to the affine open cover of X arising from the maximal cones in its fan. Consider the following bicomplex, where the horizontal maps are induced by the differentials in $\mathcal{R}$, the vertical maps are induced by the Čech differentials, N is the length of $\mathcal{R}$, and "$\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_i$" is shorthand for "$\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a summand of $\mathcal{R}_i$":

$$0 \longleftarrow \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_0} \check{C}_{\mathcal{F} \otimes \mathcal{L}_1} \otimes \mathcal{L}_2 \longleftarrow \cdots \longleftarrow \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_N} \check{C}_{\mathcal{F} \otimes \mathcal{L}_1} \otimes \mathcal{L}_2 \longleftarrow 0. \tag{4.3.3}$$

Since the differentials of $\check{C}_{\mathcal{G}}$ have entries in $\Bbbk$, the columns of (4.3.3) split. Thus, we may apply [EFS03, Lem. 3.5] to conclude that the totalization of (4.3.3) is homotopy equivalent to a complex

$\mathbf{B}(\mathcal{F})$ concentrated in degrees $k = -N, \ldots, N$ with terms

$$\mathbf{B}(\mathcal{F})_k = \bigoplus_{i-j=k} \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_i} H^j(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{L}_2 \cong \bigoplus_{i-j=k} \mathbf{R}^j \pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{R}_i). \tag{4.3.4}$$

The terms of $\mathbf{B}(\mathcal{F})$ arise from the totalization of the vertical homology of (4.3.3).

Over projective space, the analogue of this Fourier–Mukai transform involving Beilinson's resolution of the diagonal is called the Beilinson monad (see e.g. [EFS03]), hence the notation $\mathbf{B}(-)$. Note that "the" complex $\mathbf{B}(\mathcal{F})$ is only well-defined up to homotopy equivalence, since the differential depends on a choice of splitting of the columns in the bicomplex (4.3.3). More precisely, for each term $Y_{i,j}$ of (4.3.3), choose a decomposition $Y_{i,j} = B_{i,j} \oplus H_{i,j} \oplus L_{i,j}$ such that $B_{i,j} \oplus H_{i,j} = Z_{i,j}^{\text{vert}}$, where $Z_{i,j}^{\text{vert}}$ denotes the vertical cycles in $Y_{i,j}$. Notice that there is a canonical isomorphism $H_{i,j} \cong \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_i} H^{-j}(\mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{L}_2$. Let $\sigma_H \colon Y_{\bullet,\bullet} \to H_{\bullet,\bullet}$ and $\sigma_B \colon Y_{\bullet,\bullet} \to B_{\bullet,\bullet}$ denote the projections, let $g \colon L_{\bullet,\bullet} \xrightarrow{\cong} B_{\bullet,\bullet-1}$ denote the isomorphism induced by the vertical differential, and let $\pi = g^{-1}\sigma_B$. By [EFS03, Lem. 3.5], the differential on $\mathbf{B}(\mathcal{F})$ is given by

$$\partial_{\mathbf{B}(\mathcal{F})} = \sum_{i \geq 0} \sigma_H (d_{\text{hor}} \pi)^i d_{\text{hor}},$$

where d_{hor} is the horizontal differential in the bicomplex (4.3.3).

Remark 4.3.10. The $i = 0$ term in the formula for $\partial_{\mathbf{B}(\mathcal{F})}$ is simply the map induced by the differential on $\mathcal{R}$; it is independent of the choices of splittings of the columns of (4.3.3). Since this is the only part of the differential on $\mathbf{B}(\mathcal{F})$ that we will need to explicitly compute, we will ignore the ambiguity of $\mathbf{B}(\mathcal{F})$ up to homotopy equivalence from now on.

4.3.5 Proof of Theorem 4.1.1

Proof. To prove parts (1) and (2), first recall that R is the direct sum of the degree $(d_1, d_2, d_3, d_4, 0, 0)$ components of $K(0,0,0,0,r,s)$ for all $d_1, \ldots, d_4 \in \mathbb{Z}$. Thus, since K is exact in positive homological degrees, R is as well; moreover, the differentials of R are linear[1]. We now check that R has property

[1] Free complexes that are linear in the sense of Theorem 4.1.1(2) are called *strongly linear* in [BE22].

(3). For all $k, \ell \in \mathbb{Z}$, we have

$$\dim_{\Bbbk} H^0(\mathbb{F}_{a_s}, \mathcal{O}(k,\ell)) = \begin{cases} (k+1)(\ell+1) + \binom{\ell+1}{2} a_s, & \ell \geq 0; \\ 0, & \ell < 0. \end{cases} \tag{4.3.5}$$

We now compute:

$$\begin{aligned}
\operatorname{rank} \mathcal{R}_n &= \sum_{i=0}^{n} \binom{r+1}{i}\binom{s+1}{n-i} \dim_{\Bbbk} H^0(\mathbb{F}_{a_s}, \mathcal{O}(r-i, s-(n-i))) \\
&= \sum_{i=0}^{r} \binom{r+1}{i}\binom{s+1}{n-i}\left((r-i+1)(s-(n-i)+1) + \binom{s-(n-i)+1}{2} a_s\right) \\
&+ \binom{s+1}{n-(r+1)}\binom{s-(n-(r+1))+1}{2} a_s \\
&= \sum_{i=0}^{r} \binom{r}{i}\binom{s}{n-i}(r+1)(s+1) + \sum_{i=0}^{r} \binom{r+1}{i}\binom{s-1}{n-i}\binom{s+1}{2} a_s \\
&+ \binom{s-1}{n-(r+1)}\binom{s+1}{2} a_s \\
&= \sum_{i=0}^{r} \binom{r}{i}\binom{s}{n-i}(r+1)(s+1) + \sum_{i=0}^{r+1} \binom{r+1}{i}\binom{s-1}{n-i}\binom{s+1}{2} a_s \\
&= \binom{r+s}{n} \dim_{\Bbbk} H^0(\mathbb{F}_{a_s}, \mathcal{O}(r,s)).
\end{aligned}$$

The first equality follows from the definition of $\mathcal{R}$, the second from (4.3.5), the third from some straightforward manipulations, the fourth by combining the second and third terms, and the last by Vandermonde's identity and the equality $\dim_{\Bbbk} H^0(\mathbb{F}_{a_s}, \mathcal{O}(r,s)) = (r+1)(s+1) + \binom{s+1}{2} a_s$. This proves (3).

Finally, we check property (4): namely, that the cokernel of the differential $\partial_1 \colon \mathcal{R}_1 \to \mathcal{R}_0$ is $\mathcal{O}_\Delta$. Just as in the proof of [CK08, Prop. 3.2], we will prove that $\mathcal{R}$ is a resolution of $\mathcal{O}_\Delta$ by showing there is a chain map $\mathcal{R} \to \mathcal{O}_\Delta$ that induces a natural isomorphism on certain Fourier–Mukai transforms. In detail: given any $i, j \in \mathbb{Z}$, there is a natural map $\mathcal{O}(i,j,-i,-j) \to \mathcal{O}_\Delta$ given by multiplication. These maps determine a natural map $\nu_0 \colon \mathcal{R}_0 \to \mathcal{O}_\Delta$, and it is clear from the description of ∂_1 in Example 4.3.8 that ν_0 determines a chain map $\nu \colon \mathcal{R} \to \mathcal{O}_\Delta$. Recall that $\Phi_{\mathcal{R}}$ denotes the Fourier–Mukai transform associated to $\mathcal{R}$. To show that ν is a quasi-isomorphism, we need only prove that the induced natural transformation $\Phi_\nu \colon \Phi_{\mathcal{R}} \to \Phi_{\mathcal{O}_\Delta}$ on Fourier–Mukai transforms is a natural isomorphism; indeed, this immediately implies that $\Phi_{\mathrm{cone}(\nu)}$ is isomorphic to the 0 functor, and so $\mathrm{cone}(\nu) = 0$ by [CK08, Lem. 2.1].

The category $\mathrm{D}^{\mathrm{b}}(X)$ is generated by the line bundles $\mathcal{O}(b,c)$ with $0 \le b \le r$ and $0 \le c \le s$; in fact, these bundles form a full exceptional collection in $\mathrm{D}^{\mathrm{b}}(X)$ [Orl93, Cor. 2.7]. Since $\Phi_{\mathcal{O}_\Delta}$ is the identity functor, we need only show that the map $\Phi_{\mathcal{R}}(\mathcal{O}(b,c)) \to \mathcal{O}(b,c)$ induced by Φ_ν is an isomorphism in $\mathrm{D}^{\mathrm{b}}(X)$.

Say $\mathcal{O}(d_1,d_2,d_3,d_4)$ is a summand of $\mathcal{R}$. We first show that the line bundle $\mathcal{O}(d_1+b, d_2+c)$ on X is acyclic, i.e. $H^i(X, \mathcal{O}(d_1+b,d_2+c)) = 0$ for $i > 0$. Say the summand $\mathcal{O}(d_1,d_2,d_3,d_4)$ of $\mathcal{R}$ corresponds to the monomial $\alpha_{i_1}\cdots\alpha_{i_k}\beta_{j_1}\cdots\beta_{j_\ell}m$, where $k \le r+1$, $\ell \le s+1$, and $m \in M_{r-k,s-\ell}$. It follows that $d_1 = -k - t_1$ and $d_2 = -\ell - t_2$ for some $t_1 \le r-k$ and $t_2 \le s - \ell$. In particular, we have $d_1 + b \ge d_1 \ge -r$, and $d_2 + c \ge d_2 \ge -s$. Thus, $\mathcal{O}(d_1+b,d_2+c)$ satisfies either (a) or (b) in Corollary 4.3.3(2), and so $\mathcal{O}(d_1+b,d_2+c)$ is acyclic.

Recall from §4.3.4 that, given any sheaf $\mathcal{F}$ on X, $\Phi_{\mathcal{R}}(\mathcal{F})$ may be modeled explicitly as the complex $\mathbf{B}(\mathcal{F})$. The previous paragraph implies that the terms in $\mathbf{B}(\mathcal{O}(b,c))$ involving higher cohomology vanish; that is, the nonzero terms of $\mathbf{B}(\mathcal{O}(b,c))$ are of the form $H^0(\mathcal{L}_1(b,c)) \otimes \mathcal{L}_2$, where $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a summand of $\mathcal{R}$. In particular, $\mathbf{B}(\mathcal{O}(b,c))$ is concentrated in nonnegative degrees, the map $\mathbf{B}_0(\mathcal{O}(b,c)) \to \mathcal{O}(b,c)$ induced by ν is the natural multiplication map, and the differential on $\mathbf{B}(\mathcal{O}(b,c))$ is induced by the differential on $\mathcal{R}$. It follows that $\mathbf{B}(\mathcal{O}(b,c))$ is exact in positive degrees, since $\mathcal{R}$ has this property. We now show, by direct computation, that the induced map $H_0(\mathbf{B}(\mathcal{O}(b,c))) \to \mathcal{O}(b,c)$ is an isomorphism.

It follows from our explicit descriptions of the terms R_0 and R_1 in Example 4.3.8 that

$$\mathbf{B}(\mathcal{O}(b,c))_0 = \bigoplus_{m \in M_{r,s}} H^0(X, \mathcal{O}(b - d_1^m, c - d_2^m)) \otimes \mathcal{O}(d_1^m, d_2^m)\cdot m, \quad \text{and}$$

$$\mathbf{B}(\mathcal{O}(b,c))_1 = \mathbf{B}(\mathcal{O}(b,c))_1^\alpha \oplus \mathbf{B}(\mathcal{O}(b,c))_1^\beta, \quad \text{where}$$

$$\mathbf{B}(\mathcal{O}(b,c))_1^\alpha = \bigoplus_{i=0}^{r} \bigoplus_{m \in M_{r-1,s}} H^0(X, \mathcal{O}(b - d_1^m - 1, c - d_2^m)) \otimes \mathcal{O}(d_1^m, d_2^m)\cdot \alpha_i m,$$

$$\mathbf{B}(\mathcal{O}(b,c))_1^\beta = \bigoplus_{i=0}^{s} \bigoplus_{m \in M_{r,s-1}} H^0(X, \mathcal{O}(b - d_1^m + a_i, c - d_2^m - 1)) \otimes \mathcal{O}(d_1^m, d_2^m)\cdot \beta_i m.$$

We represent the first differential on $\mathbf{B}(\mathcal{O}(b,c))$ as a matrix with respect to the above decomposition, along with the monomial bases of each cohomology group. The column of this matrix corresponding to $\alpha_i m$ and a monomial z in the Cox ring $S = k[x_0, \dots, x_r, y_0, \dots, y_s]$ of X of degree $(b - d_1^m - 1, c - d_2^m)$ has exactly two nonzero entries:

- an entry of 1 for $u_2 m \in M_{r,s}$ and $x_i z \in H^0(X, \mathcal{O}(b - d_1^{u_2 m}, c - d_2^{u_2 m}))$;
- an entry of $-x_i'$ for $u_0 m \in M_{r,s}$ and $z \in H^0(X, \mathcal{O}(b - d_1^{u_0 m}, c - d_2^{u_0 m}))$.

Similarly, the column corresponding to $\beta_i m$ and a monomial $w \in S$ of degree $(b - d_1^m + a_i, c - d_2^m - 1)$ has exactly two nonzero entries:

- an entry of 1 for $u_3 m$ and $y_i w \in H^0(X, \mathcal{O}(b - d_1^{u_3 m}, c - d_2^{u_3 m}))$;
- an entry of $-y_i'$ for $u_0^{a_s - a_i} u_1 u_2^{a_i} m$ and $w \in H^0(X, \mathcal{O}(b - d_1^{u_0^{a_s-a_i} u_1 u_2^{a_i} m}, c - d_2^{u_0^{a_s-a_i} u_1 u_2^{a_i} m}))$.

That is, the first differential on $\mathbf{B}(\mathcal{O}(b,c))$ has the following form:

$$
\begin{array}{cc}
\begin{pmatrix}
 & 0 & & 0 & \\
 & -x_i' & & 0 & \\
 & 0 & & 0 & \\
 & 0 & & -y_i' & \\
\cdots & 0 & \cdots & 0 & \cdots \\
 & 1 & & 0 & \\
 & 0 & & 0 & \\
 & 0 & & 1 & \\
 & 0 & & 0 &
\end{pmatrix} &
\begin{matrix} \vdots \\ z \otimes u_0 m \\ \vdots \\ w \otimes u_0^{a_s - a_i} u_1 u_2^{a_i} m \\ \vdots \\ x_i z \otimes u_2 m \\ \vdots \\ y_i w \otimes u_3 m \\ \vdots \end{matrix} \\
\begin{matrix} z \otimes \alpha_i m & \cdots & w \otimes \beta_i m \end{matrix} &
\end{array}.
$$

Now observe: every column of this matrix contains exactly one "1", and there is exactly one row that does not contain a "1": namely, the row corresponding to the summand $H^0(X, \mathcal{O}) \otimes \mathcal{O}(b,c) \cdot u_0^{b+ca_s} u_1^c u_2^{r-b} u_3^{s-c}$. It follows immediately that the cokernel of this matrix is isomorphic to the summand $H^0(X, \mathcal{O}) \otimes \mathcal{O}(b,c)$, and the multiplication map induced by ν from this summand to $\mathcal{O}(b,c)$ is clearly an isomorphism. $\square$

Remark 4.3.11*.* Our construction of the resolution $\mathcal{R}$ realizes it as a subcomplex of the (infinite rank) resolution of the diagonal obtained in [BE21, Thm. 4.1] and therefore yields a positive answer to [BE21, Conj. 7.2] for smooth projective toric varieties of Picard rank 2.

Corollary 4.3.12. *Given a coherent sheaf $\mathcal{F}$ on X, we have $\mathbf{B}(\mathcal{F}) \cong \mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(X)$.*

Corollary 4.3.13. *Consider the ideal $I = (\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_s) \subseteq S_E$, and let $\mathcal{D}$ denote the sheaf $\widetilde{S_E/I}$ on E. We have an isomorphism $\pi_*\mathcal{D}(0,0,0,0,r,s) \cong \mathcal{O}_\Delta$ of sheaves on $X \times X$.*

Proof. Recall that $\mathcal{K}$ is the sheafification of the Koszul complex on the generators of I, which form a regular sequence. Therefore $\mathcal{K}$ is a locally free resolution of $\mathcal{D}$, and using Proposition 4.3.6 and Theorem 4.1.1(4) we have $\pi_*\mathcal{D}(0,0,0,0,r,s) \cong \pi_*\mathcal{K}(0,0,0,0,r,s) \cong \mathcal{R} \cong \mathcal{O}_\Delta$. □

We will now prove Conjecture 3.1.2 for X as in Theorem 4.1.1.

Proof of Corollary 4.1.2. Our proof is nearly the same as that of [BES20, Prop. 1.2]. Given a finitely generated graded module M over the Cox ring of X, let $\mathcal{F}$ be the associated sheaf on X. Applying the Fujita Vanishing Theorem, choose $i, j \gg 0$ such that, for all summands $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ of the resolution of the diagonal $\mathcal{R}$ from Theorem 4.1.1, we have $H^q(X, \mathcal{F}(i,j) \otimes \mathcal{L}_1) = 0$ for $q > 0$. The complex $\mathbf{B}(\mathcal{F}(i,j))$ is a resolution of $\mathcal{F}(i,j)$ of length at most $\dim(X)$ consisting of finite sums of line bundles, and twisting back by $(-i,-j)$ gives a resolution of $\mathcal{F}$. Now applying the functor $\mathcal{G} \mapsto \bigoplus_{(k,\ell)\in\mathbb{Z}^2} H^0(X, \mathcal{G}(k,\ell))$ to the complex $\mathbf{B}(\mathcal{F}(i,j))(-i,-j)$ gives a virtual resolution of M. □

5 Splitting of Vector Bundles on Smooth Projective Toric Varieties

The material in this chapter will appear in a forthcoming preprint [Say24].

5.1 Introduction

The study of algebraic vector bundles, especially as a source for interesting higher dimensional varieties, is a deep and classical aspect of algebraic geometry [Har79]. Moreover, the equivalence of the categories of algebraic and holomorphic vector bundles on a complex algebraic variety connects this study to problems in mathematical physics.

A central problem is existence of indecomposable vector bundles of low rank on $\mathbb{P}^n$ [Har74]. A well-known result of Horrocks [Hor64] states that a vector bundle on $\mathbb{P}^n$ splits as a sum of line bundles if and only if it has no intermediate cohomology. This splitting criterion has been extended to many different spaces: products of projective spaces [CM05, EES15], Grassmannians and quadrics [Ott89], rank 2 vector bundles on Hirzebruch surfaces [Buc87, AM11, FM11, Yas15], Segre–Veronese varieties [Sch22], among others.

We prove an analogous splitting criterion for vector bundles on smooth projective toric varieties, under an additional hypothesis similar to Eisenbud–Erman–Schreyer's criterion for products of projective spaces [EES15, Thm. 7.2] and the Picard rank 2 case in [BS22, Thm. 1.5].

Theorem 5.1.1. *Suppose $\mathcal{E}$ is a vector bundle on a smooth projective toric variety X and $\mathcal{E}' = \oplus_{i=1}^n \mathcal{O}(D_i)^{r_i}$ is a sum of line bundles on X such that $D_{i+1} - D_i$ is ample for $0 < i < n$. If $H^p(X, \mathcal{E} \otimes \mathcal{L}) = H^p(X, \mathcal{E}' \otimes \mathcal{L})$ for all $p \geq 0$ and $\mathcal{L} \in \operatorname{Pic} X$, then $\mathcal{E} \cong \mathcal{E}'$.*

The first ingredient is the recent construction of short resolutions of the diagonal for smooth normal toric stacks due to Hanlon–Hicks–Lazarev and Brown–Erman [HHL23, BE23b], which consist of line bundles from the Thomsen collection [Tho00]. The proof of Theorem 5.1.1 uses a Beilinson-type spectral sequence which computes the corresponding Fourier–Mukai transform. Similar ideas have been used to great success in [CM05, FM11, AM11, EES15, BS22].

In our case, a significant obstacle is introduced by the difference between the nef and effective cones for arbitrary toric varieties. In all previous incarnations of the criterion, either the nef and effective cones are identical or Picard rank is low (one or two). Without either of these restrictions, the analysis of the cohomology of line bundles requires new ideas.

We begin in §5.2 with a recipe for proving Horrocks-type splitting criteria for any smooth projective variety, which illustrates the proof. Then in §5.3 we prove Theorem 5.1.1.

5.2 A General Recipe for Splitting Criteria

Let X be a smooth projective variety with a resolution of the diagonal $\mathcal{K}$ and $\mathcal{E}$ a coherent sheaf on X. Similar to the case in §4.3.4, we use a Fourier–Mukai functor with kernel $\mathcal{K}$ to construct a monad which is quasi-isomorphic to $\mathcal{E}$ and whose terms are prescribed by the terms of $\mathcal{K}$ with ranks given by sheaf cohomology of twists of $\mathcal{E}$. The recipe for splitting criteria described in this section is a consequence of appropriate vanishing of the terms of this spectral sequence.

The diagonal embedding $X \to X \times X$ defines a closed subscheme $\Delta \subset X \times X$. Let π_1 and π_2 denote the projections of $X \times X$ onto X and for the rest of this paper suppose $\mathcal{K}$ is a locally free resolution for $\mathcal{O}_\Delta$, the structure sheaf of Δ, with terms given as a direct sums of sheaves of the form $\mathcal{G} \boxtimes \mathcal{L} \coloneqq \pi_1^*\mathcal{G} \otimes \pi_2^*\mathcal{L}$, where $\mathcal{G}$ a locally free sheaf and $\mathcal{L} = \mathcal{O}(E)$ a line bundle corresponding to a divisor E on X.

The Fourier–Mukai transform with kernel $\mathcal{K}$ is the composition of functors:

$$\Phi_{\mathcal{K}} \colon \mathcal{D}^{\mathrm{b}}(X) \xrightarrow{\pi_1^*} \mathcal{D}^{\mathrm{b}}(X \times X) \xrightarrow{\cdot \otimes \mathcal{K}} \mathcal{D}^{\mathrm{b}}(X \times X) \xrightarrow{\mathbf{R}\pi_{2*}} \mathcal{D}^{\mathrm{b}}(X).$$

In particular, $\Phi_{\mathcal{K}}$ is the identity functor on the derived categories, meaning that $\Phi_{\mathcal{K}}(\mathcal{E})$ will be quasi-isomorphic to $\mathcal{E}$. We compute the last functor, derived pushforward, by resolving the first term of each box product with a Čech complex to obtain a spectral sequence

$$E_1^{-s,t} = \mathbf{R}^t \pi_{2*}(\pi_1^* \mathcal{E} \otimes \mathcal{K}_s) = \bigoplus_i \mathcal{G}_i \otimes H^t(X, \mathcal{L}_i) \Rightarrow \mathbf{R}^{s-t}\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}) \cong \begin{cases} \mathcal{E} & i = j \\ 0 & i \neq j, \end{cases}$$

where the direct sum ranges over summands $\mathcal{G}_i \boxtimes \mathcal{L}_i$ of $\mathcal{K}_s$ (c.f. §4.3.4 and Chapter 6, §3.3).

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathbf{R}^2\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^2\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_1) & \longleftarrow & \mathbf{R}^2\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
\mathbf{R}^1\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_0) & \longleftarrow & \mathbf{R}^1\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_1) & \longleftarrow & \color{red}{\mathbf{R}^1\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_2)} & \longleftarrow & \cdots \\
\color{teal}{\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_0)} & \longleftarrow & \color{red}{\pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_1)} & \longleftarrow & \pi_{2*}(\pi_1^*\mathcal{E} \otimes \mathcal{K}_2) & \longleftarrow & \cdots \\
k=0 & & k=1 & & k=2 & & s
\end{array} \tag{5.2.1}$$

Definition 5.2.1. For a convex cone $\mathcal{A} \subset \operatorname{Pic} X$, we say $\mathcal{K}$ is *cohomologically supported in* $\mathcal{A}$ if for any summand $\mathcal{G} \boxtimes \mathcal{O}(E)$ of $\mathcal{K}_q$ we have $H^p(X, \mathcal{O}(E - D)) = 0$ for all $p < q$ and $D \in \mathcal{A}$.

For instance, Beilinson's resolution of the diagonal for $\mathbb{P}^n$, its variant for products of projective spaces, and the resolutions constructed in Chapter 4 are all cohomologically supported in $\operatorname{Nef} X \subset \operatorname{Pic} X$. The main difference between these examples is that only the Picard group for $\mathbb{P}^n$ has a total ordering.

Lemma 5.2.2. *Let $\mathcal{E}$ be a coherent sheaf on a smooth projective variety X with a resolution of the diagonal $\mathcal{K}$ such that $\Phi_{\mathcal{K}}(\mathcal{O}_X) = \mathcal{O}_X$. Consider the spectral sequence $E_1^{-s,t} \Rightarrow \mathcal{E}$ above.*

1. *If $E_1^{-s-1,s} = 0$ for all s (i.e. the red terms vanish) then $E_1^{0,0}$ is a direct summand of $\mathcal{E}$.*
2. *If $\mathcal{K}$ is supported in $\mathcal{A}$ and $\mathcal{E} = \bigoplus \mathcal{O}(D_i)^{r_i}$ with $-D_i \in \mathcal{A}$ then $E_1^{-s-1,s} = 0$ for all s.*

Proof. The proof of the first part is identical to [BS22, Lem. 4.1] and [EES15, Lem. 7.3]. Using [EFS03, Lem. 3.5], there exists a complex with terms the same as the totalization $\operatorname{Tot}(E_1)$ (along the dotted diagonals in (5.2.1)) which is quasi-isomorphic to $\mathcal{E}$. The vanishing of the first term of the totalization (colored in red) implies that all differentials with source or target $E_r^{0,0}$ are zero, therefore $E_1^{0,0}$ is a summand of $E_\infty^{0,0} = \mathcal{E}$.

The second part immediately follows from Definition 5.2.1, as $\Phi_{\mathcal{K}}$ commutes with direct sums and the totalization $\mathrm{Tot}(E_1)$ corresponding to $\Phi_{\mathcal{K}}(\mathcal{O}(-D))$ for a divisor $D \in \mathcal{A}$ is supported in non-positive homological degrees. □

Remark 5.2.3. The additional hypotheses present in the splitting criteria proved in [EES15] and [BS22] are meant to sidestep the problem of missing total ordering in higher Picard rank by proving a criterion for a smaller cone $\mathcal{A} = \mathrm{Nef}\, X$. Nevertheless, to date we do not know whether such hypotheses are necessary. In contrast, for certain varieties (e.g. Hirzebruch surfaces), it is possible to compute a resolution of the diagonal supported in $\mathrm{Eff}\, X$, which yields a strictly stronger splitting criterion.

Proposition 5.2.4. *Suppose X is a smooth projective variety X with a locally free resolution of the diagonal $\mathcal{K}$ such that $\mathcal{K}$ is cohomologically supported in $\mathcal{A}$ and $\Phi_{\mathcal{K}}(\mathcal{O}_X) = \mathcal{O}_X$. Let $\mathcal{E}$ be a vector bundle and $\mathcal{E}' = \oplus_{i=1}^n \mathcal{O}(D_i)^{r_i}$ a sum of line bundles on X such that $D_{i+1} - D_i \in \mathcal{A}$ for $0 < i < n$. If $H^p(X, \mathcal{E} \otimes \mathcal{L}) = H^p(X, \mathcal{E}' \otimes \mathcal{L})$ for all $p \geq 0$ and $\mathcal{L} \in \mathrm{Pic}\, X$, then $\mathcal{E} \cong \mathcal{E}'$.*

Proof. Following a similar road map as [BS22, Thm. 1.5] and [EES15, Thm. 7.2], twist $\mathcal{E}$ and $\mathcal{E}'$ by the highest line bundle $\mathcal{O}(-D_n)$ so that without loss of generality we can assume $\mathcal{E}' = \oplus_{i=1}^{n-1} \mathcal{O}(D_i)^{r_i} \oplus \mathcal{O}^{r_n}$. Let $E_1(\mathcal{E})$ denote the spectral sequence corresponding to $\Phi_{\mathcal{K}}(\mathcal{E})$.

By hypothesis, $E_1(\mathcal{E})$ and $E_1(\mathcal{E}')$ have the same terms, so $E_1^{0,0}(\mathcal{E}) = E_1^{0,0}(\mathcal{E}') = \mathcal{O}_X^{r_n}$. Since $-D_i \in \mathcal{A}$ for all i, by Lemma 5.2.2(b) we have $E_1^{-s-1,s}(\mathcal{E}) = E_1^{-s-1,s}(\mathcal{E}') = 0$ for all s. Using Lemma 5.2.2(a), the term $E_1^{0,0}(\mathcal{E}) = \mathcal{O}_X^{r_n}$ is a summand of $\mathcal{E}$. Induction on the complement of $\mathcal{O}_X^{r_n}$ in $\mathcal{E}$ and $\mathcal{E}'$ finishes the proof. □

Remark 5.2.5. The existence of a resolution of the diagonal of appropriate shape is a strong assumption. For instance, sufficiently complicated varieties, for example K3 surfaces, may not admit such a resolution consisting of sheaves of the form $\mathcal{F} \boxtimes \mathcal{G}$ (See [Huy06, pp. 180]).

Remark 5.2.6. It is straightforward to prove that given resolutions of the diagonal $\mathcal{K}$ and $\mathcal{K}'$ supported in $\mathcal{A}$ and $\mathcal{A}'$ for X and X', respectively, $\mathcal{K} \boxtimes \mathcal{K}'$ is a resolution of the diagonal for $X \times X'$ supported in $\mathcal{A} \times \mathcal{A}'$. In particular, the splitting criterion [EES15, Thm. 7.2] for products of projective spaces can be recovered from Beilinson's resolution of the diagonal and Horrocks' splitting criterion for $\mathbb{P}^n$.

5.3 A Splitting Criterion for Toric Varieties

In [HHL23], Hanlon, Hicks, and Lazarev construct resolutions of toric subvarieties by line bundles on a smooth toric variety X. The case which is of interest here is the diagonal subvariety, where the resolution of the structure sheaf of the diagonal $\mathcal{K}$ consists of line bundles from the Thomsen collection on $X \times X$ (c.f. [BE23b]).

We will need the following technical lemma on properties of the terms of $\mathcal{K}$.

Lemma 5.3.1. *Suppose $\mathcal{O}(E') \boxtimes \mathcal{O}(E)$ is a summand of $\mathcal{K}_q$ constructed as in* [HHL23].

(I). *The divisor $-E$ is an effective Cartier divisor on X; that is:*

$$E = -\sum d_\rho D_\rho \textit{ for } d_\rho \in \mathbb{Z}_{\geq 0} \textit{ and } \rho \in \Sigma(1).$$

(II). *The bundle $\mathcal{O}(E)$ is a summand of a high toric Frobenius pushforward of $\mathcal{O}_X$; that is, there is a Cartier $\mathbb{Q}$-divisor $\widetilde{E}$ linearly equivalent to E such that:*

$$\widetilde{E} = -\sum c_\rho D_\rho \textit{ for } c_\rho \in [0,1) \textit{ and } \rho \in \Sigma(1).$$

(III). *The dimension of the polytope P_{-E} is at least q; that is:*

$$\textit{If } \mathcal{O}(E') \boxtimes \mathcal{O}(E) \textit{ is a summand of } \mathcal{K}_q \textit{ then } q \leq \dim P_{-E}.$$

Proof. The Thomsen collection for $X \times X$ consists of products of bundles from the Thomsen collection for X [HHL23, Rem. 1.3], hence the first two properties follow from $\mathcal{O}(E)$ being in the Thomsen collection for X

The third point is more subtle, as it implies that not all line bundles from the Thomsen collection for $X \times X$ appear in $\mathcal{K}$, and only few may appear in a given term. The diagonal embedding is induced by an inclusion of lattices $\phi\colon N_X \to N_{X\times X}$. The dual map on the character lattices $\phi^*\colon M_{X\times X} \to M_X$ induces a short exact sequence of real tori:

$$0 \to L_\mathbb{R} \to M_{X\times X,\mathbb{R}}/M_{X\times X} \to M_{X,\mathbb{R}}/M_X \to 0.$$

In the case of the diagonal, the dimension of $L_{\mathbb{R}} \cong M_{X,\mathbb{R}}/M_X$ equals the dimension of X, and it inherits a stratification labeled by divisors on $X \times X$. It follows from the construction in [HHL23, eq. (17)] that the line bundle summands of $\mathcal{K}_q$ correspond to q-dimensional strata. In particular, the strata on $L_{\mathbb{R}}$ are the same as the strata on $M_{X,\mathbb{R}}$ when resolving a point on X, only in that case the labels are divisors on X. Specifically, if a q-dimensional strata S_σ has label $\mathcal{O}(E') \boxtimes \mathcal{O}(E)$ in $\mathcal{K}$, then S_σ has label $\mathcal{O}(E)$ in the resolution of a point on X.

Given a line bundle $\mathcal{O}(E)$ in the Thomsen collection, let S_{-E} denote the union of strata with that label. It follows from [FH22, Lem. 5.6] that $S_{-E} = (P_{-E} \setminus \bigcup_{\rho\in\Sigma(1)} P_{-E-D_\rho})/M_X$. Since $-E$ is effective P_{-E} is nonempty and since the section polytopes are closed S_{-E} is open, hence $\dim S_{-E} = \dim P_{-E}$. Putting this all together: any q-dimensional strata S_σ which corresponds to a line bundle $\mathcal{O}(E)$ in $\mathcal{K}_q$ must satisfy $q = \dim S_\sigma \leq \dim S_{-E} = \dim P_{-E}$. □

Remark 5.3.2. The stratifications considered in [HHL23, §3.4] and [FH22, §5] are both versions of the stratification studied by Bondal in [Bon06], but they have subtle differences: the union of the strata with the same label in [HHL23] is the unique strata with that label in [FH22], which is contractible by [FH22, Lem. 5.6].

In order to use Lemma 5.2.2, we need the following analysis of the support of $\mathcal{K}$.

Proposition 5.3.3. *The resolution of the diagonal $\mathcal{K}$ is cohomologically supported in* $\mathrm{Ample}(X)$.

Proof. Suppose $\mathcal{O}(E') \boxtimes \mathcal{O}(E)$ is a summand of $\mathcal{K}_q$. We show that:

$$H^p(X, \mathcal{O}(E-D)) = 0 \text{ for } p < q \text{ and any ample divisor } D$$

First, using notation from (I) and (II), since $\lceil d_\rho - (1-\epsilon)c_\rho \rceil = d_\rho$ for $0 \leq \epsilon \leq 1$, we have:

$$\begin{aligned} -\lceil D + (1-\epsilon)\widetilde{E} - E \rceil &= -\lceil D + \sum (d_\rho - (1-\epsilon)c_\rho) D_\rho \rceil \\ &= -\lceil D + \sum d_\rho D_\rho \rceil = -\lceil D - E \rceil = E - D. \end{aligned}$$

Second, by linear equivalence in (II) we have $\widetilde{E} - E \sim 0$, hence:

$$\begin{aligned} D + (1-\epsilon)\widetilde{E} - E &= D + (\widetilde{E} - E) - \epsilon\widetilde{E} \\ &\sim D - \epsilon\widetilde{E}, \end{aligned}$$

which, for sufficiently small ϵ, is ample, and hence nef, because by hypothesis D is ample. Third, since both D and $-\epsilon\widetilde{E}$ are effective, we have:

$$\begin{aligned} \dim P_{D+(1-\epsilon)\widetilde{E}-E} &= \dim P_{D-\epsilon\widetilde{E}} \\ &\geq \dim P_{-\epsilon\widetilde{E}} = \dim P_{-\widetilde{E}} = \dim P_{-E}. \end{aligned}$$

Hence by Batyrev–Borisov vanishing (see [CLS11, Thm. 9.3.5(b)]), we have:

$$H^p(X, \mathcal{O}(E - D)) = H^p(X, \mathcal{O}(-\lceil D + (1-\epsilon)\widetilde{E} - E\rceil)) = 0 \text{ for all } p < \dim P_{-E}.$$

Therefore by (III) we have the weaker vanishing $H^p(X, \mathcal{O}(E - D)) = 0$ for $p < q$. □

The proof of the main theorem is a direct application of the recipe from Section 5.2.

Proof of Theorem 5.1.1. Since $\mathcal{K}$ is cohomologically supported in Ample(X) by Proposition 5.3.3 and the consecutive differences $D_{i+1} - D_i$ are ample by hypothesis, the proof follows immediately using the recipe in Proposition 5.2.4. □

6 Characterizing Multigraded Regularity on Products of Projective Spaces

The material in this chapter originally appeared in [BCHS21].

6.1 Introduction

Castelnuovo–Mumford regularity of coherent sheaves on a projective variety is a measure of complexity given in terms of vanishing of sheaf cohomology [BM93]. Its geometric significance has been studied extensively for projective spaces [Mum66], abelian varieties [PP03], Grassmannians [Chi00], and smooth projective toric varieties [MS04]. In many of these situations Castelnuovo–Mumford regularity has deep connections to minimal free resolutions and syzygies of graded modules [Mum70, PP04].

Consider the projective space case. Let S be the polynomial ring on $n+1$ variables over an algebraically closed field $\Bbbk$ and $\mathfrak{m}$ its maximal homogeneous ideal. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n = \operatorname{Proj} S$ is d-regular for $d \in \mathbb{Z}$ if

1. $H^i(\mathbb{P}^n, \mathscr{F}(b)) = 0$ for all $i > 0$ and all $b \geq d - i$.

The Castelnuovo–Mumford regularity of $\mathscr{F}$ is then the minimum d such that $\mathscr{F}$ is d-regular. In [EG84], Eisenbud and Goto considered the analogous condition on the local cohomology of a finitely generated graded S-module M, proving the equivalence of the following:

2. $H^i_{\mathfrak{m}}(M)_b = 0$ for all $i \geq 0$ and all $b > d - i$;

3. the truncation $M_{\geq d}$ has a linear free resolution;

4. $\operatorname{Tor}_i(M, \Bbbk)_b = 0$ for all $i \geq 0$ and all $b > d + i$.

In particular, conditions (1) through (4) are equivalent when $M = \bigoplus_p H^0(\mathbb{P}^n, \mathscr{F}(p))$ is the graded S-module corresponding to $\mathscr{F}$, so that $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$ (c.f. [Eis05, Prop. 4.16]).

In [MS04], Maclagan and Smith introduced the notion of multigraded Castelnuovo–Mumford regularity for finitely generated $\operatorname{Pic}(X)$-graded modules over the Cox ring of a smooth projective toric variety X. In essence their definition is a generalization of conditions (1) and (2). In this setting the multigraded regularity of a module is a subset of $\operatorname{Pic} X$ rather than a single integer. When $X = \mathbb{P}^n$ the minimum element of this region is the classical regularity.

In the multigraded case, translating the geometric definition of Maclagan and Smith into algebraic conditions like (3) and (4) above has been an open problem. In this direction, Maclagan–Smith and later Berkesch–Erman–Smith demonstrated connections between multigraded regularity and the existence of virtual resolutions with certain twists in [MS04, Thm. 7.8] and [BES20, Thm. 2.9]. In a more general setting, Botbol–Chardin sharpened the relationshop between local cohomology and multigraded Betti numbers [BC17, Thm. 4.14]. More recently, Brown and Erman explored different notions of linearity for weighted projective spaces [BE23a] in relation to Green's N_p-conditions and Benson's weighted regularity [Ben04].

In this chapter, taken from [BCHS21], we focus on the case when X is a product of projective spaces and establish a tight relationship between multigraded regularity, truncations, Betti numbers, and virtual resolutions. Our main results strengthen and clarify previous work in a number of directions: First, we extend the equivalence of (2) and (3) by modifying the notion of a linear resolution. Second, we prove a uniqueness theorem for virtual resolutions considered in [BES20, Thm. 2.9] and use it to show that they are precisely the minimal free resolutions of truncated modules. Finally, as a consequence we provide an effective method for determining whether a specific element $\mathbf{d} \in \operatorname{Pic} X$ lies in $\operatorname{reg}(M)$ without a cohomology computation.

The obvious way one might hope to generalize Eisenbud and Goto's result to products of projective spaces is false: the truncation $M_{\geq \mathbf{d}}$ of a $\mathbf{d}$-regular multigraded module M can have nonlinear maps in its minimal free resolution (see Example 6.3.2). We show that under a mild saturation hypothesis, multigraded Castelnuovo–Mumford regularity is determined by a different linearity condition, which we call *quasilinearity* (see Definition 6.3.3).

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and B the corresponding irrelevant ideal. The following complex contains all allowed twists for a quasilinear resolution generated in degree zero on a product of 2 projective spaces:

$$
0 \longleftarrow S \longleftarrow \begin{matrix} S(-1,\ \ 0) \\ \oplus \\ S(\ \ 0,-1) \end{matrix} \ \oplus\ S(-1,-1) \longleftarrow \begin{matrix} S(-2,\ \ 0) \\ \oplus \\ S(-1,-1) \\ \oplus \\ S(\ \ 0,-2) \end{matrix} \ \oplus\ \begin{matrix} S(-2,-1) \\ \oplus \\ S(-1,-2) \end{matrix} \longleftarrow \cdots .
$$

Within each term, the summands in the left column (green) are linear syzygies while those in the right column (pink) are nonlinear syzygies. In general, for twists $-\mathbf{b}$ appearing in the i-th step of a quasilinear resolution, the sum of the positive components of $\mathbf{b}-\mathbf{d}-\mathbf{1}$ is at most $i-1$, where $\mathbf{d}$ is the degree of all generators. This condition is inspired by the criterion from [BES20, Thm. 2.9], which suggest a close relationship between multigraded regularity and properties of the irrelevant ideal.

Our main theorem characterizes multigraded regularity of modules on products of projective spaces in terms of the Betti numbers of their truncations.

Theorem G. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$.*

The proof of Theorem G is based in part on a Čech–Koszul spectral sequence that relates the Betti numbers of $M_{\geq \mathbf{d}}$ to the terms of the Beilinson spectral sequence which computes the Fourier–Mukai transform of $\widetilde{M}(\mathbf{d})$. Precisely, if M is $\mathbf{d}$-regular and $H^0_B(M) = 0$ then

$$
\dim_{\Bbbk} \operatorname{Tor}^S_j(M_{\geq \mathbf{d}}, \Bbbk)_{\mathbf{a}} = h^{|\mathbf{a}|-j}\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{d}) \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\big) \quad \text{for } |\mathbf{a}| \geq j \geq 0, \tag{6.1.1}
$$

where the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ are cotangent sheaves on $\mathbb{P}^{\mathbf{n}}$. The regularity of M implies certain cohomological vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, which, using (6.1.1), implies quasilinearity of the resolution of $M_{\geq \mathbf{d}}$. Conversely, a computation of $H^i_B(S)$ in Section 6.3.2 shows that the cokernel of a quasilinear resolution generated in degree $\mathbf{d}$ is $\mathbf{d}$-regular. Thus we give a practically computable criterion for regularity in degree $\mathbf{d}$.

Our proof of Theorem G extends the same argument used in [BES20, Thm. 2.9], showing that the Fourier–Mukai transform of a $\mathbf{0}$-regular module M has the same graded Betti numbers as

$M_{\geq \mathbf{0}}$. Since free resolutions of $M_{\geq \mathbf{d}}$ are virtual resolutions of M, this naturally suggests that the virtual resolutions exhibited in [BES20, Thm. 2.9] are precisely the minimal free resolutions of the truncations of M. We will use prove this to be true using Theorem 3.5.4.

Finally, note that since a linear resolution is necessarily quasilinear, having a linear truncation at $\mathbf{d}$ is strictly stronger than being $\mathbf{d}$-regular. That is to say, when $H^0_B(M) = 0$:

$$\begin{array}{c} M_{\geq \mathbf{d}} \text{ has a linear resolution} \\ \text{generated in degree } \mathbf{d} \end{array} \implies \begin{array}{c} M_{\geq \mathbf{d}} \text{ has a quasilinear resolution} \\ \text{generated in degree } \mathbf{d} \end{array} \iff M \text{ is } \mathbf{d}\text{-regular.}$$

Despite not fully characterizing multigraded regularity, having a linear resolution after truncation remains a useful condition. Understanding the geometric implications of these vanishing conditions is an interesting open question.

6.2 Notation and Background

Throughout we denote the natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$. When referring to vectors in $\mathbb{Z}^r$ we use a bold font. Given a vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ we denote the sum $v_1 + \cdots + v_r$ by $|\mathbf{v}|$. For $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^r$ we write $\mathbf{v} \leq \mathbf{w}$ when $v_i \leq w_i$ for all i, and use $\max\{\mathbf{v}, \mathbf{w}\}$ to denote the vector whose i-th component is $\max\{v_i, w_i\}$. We reserve $\mathbf{e}_1, \ldots, \mathbf{e}_r$ for the standard basis of $\mathbb{Z}^r$ and for brevity we write $\mathbf{1}$ for $(1, 1, \ldots, 1) \in \mathbb{Z}^r$ and $\mathbf{0}$ for $(0, 0, \ldots, 0) \in \mathbb{Z}^r$.

Fix a Picard rank $r \in \mathbb{N}$ and dimension vector $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. We denote by $\mathbb{P}^{\mathbf{n}}$ the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of r projective spaces over a field $\Bbbk$. Given $\mathbf{b} \in \mathbb{Z}^r$ we let

$$\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{b}) \coloneqq \pi_1^* \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \otimes \cdots \otimes \pi_r^* \mathcal{O}_{\mathbb{P}^{n_r}}(b_r)$$

where π_i is the projection of $\mathbb{P}^{\mathbf{n}}$ to $\mathbb{P}^{n_i}$. This gives an isomorphism $\operatorname{Pic} \mathbb{P}^{\mathbf{n}} \cong \mathbb{Z}^r$, which we use implicitly throughout.

Let S be the $\mathbb{Z}^r$-graded Cox ring of $\mathbb{P}^{\mathbf{n}}$, which is isomorphic to the polynomial ring $\Bbbk[x_{i,j} \mid 1 \leq i \leq r,\ 0 \leq j \leq n_i]$ with $\deg(x_{i,j}) = \mathbf{e}_i$. Further, let $B = \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle \subset S$ be the irrelevant ideal. For a description of the Cox ring and the relationship between coherent $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$-modules and $\mathbb{Z}^r$-graded S-modules, see [Cox95, CLS11]. In particular, the twisted global sections functor Γ_* given by $\mathscr{F} \mapsto \bigoplus_{\mathbf{p} \in \mathbb{Z}^r} H^0(\mathbb{P}^{\mathbf{n}}, \mathscr{F}(\mathbf{p}))$ takes coherent sheaves on $\mathbb{P}^{\mathbf{n}}$ to S-modules. Given a $\mathbb{Z}^r$-graded

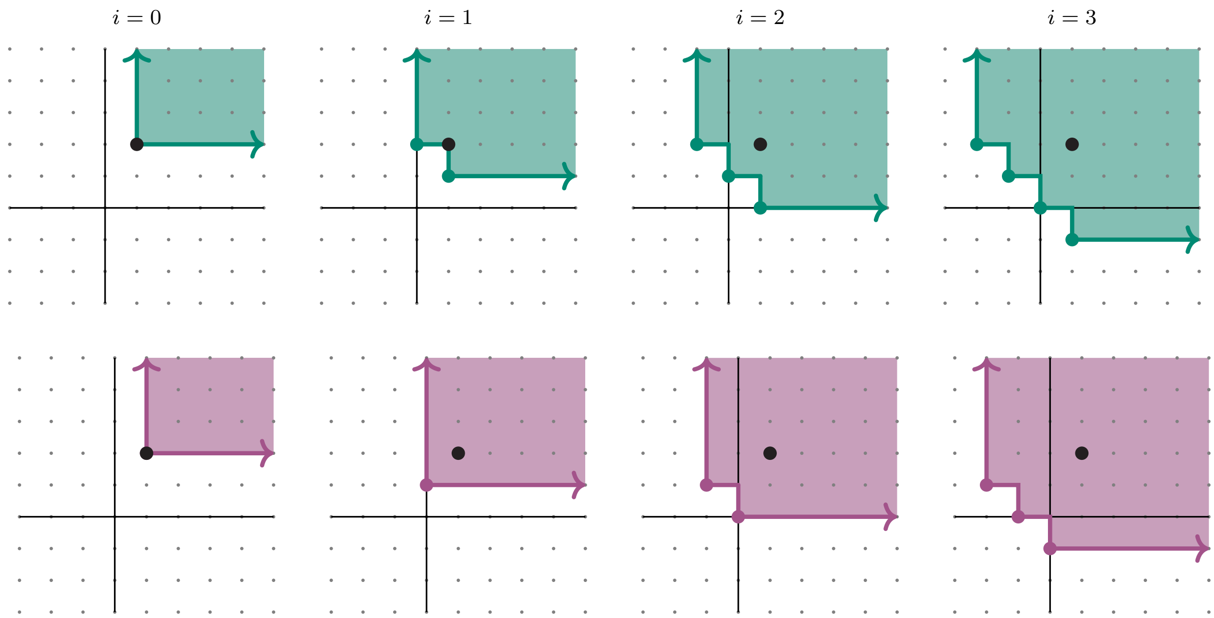


Figure 6.1: The top row shows the regions $L_i(1,2)$ in green, and the bottom row $Q_i(1,2)$ in pink for $i = 0, 1, 2, 3$, from left to right, as defined in Section 6.2.1.

S-module M, let $\beta_i(M) \coloneqq \{\mathbf{b} \in \mathbb{Z}^r \mid \operatorname{Tor}_i^S(M, \Bbbk)_{\mathbf{b}} \neq 0\}$ denote the set of multidegrees of i-th syzygies of M.

6.2.1 Multigraded Regularity

In order to streamline our definitions of regions inside the Picard group of $\mathbb{P}^{\mathbf{n}}$, we introduce the following subsets of $\mathbb{Z}^r$: for $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{N}$ let

$$L_i(\mathbf{d}) \coloneqq \bigcup_{|\lambda|=i} (\mathbf{d} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r + \mathbb{N}^r) \quad \text{for } \lambda_1, \ldots, \lambda_r \in \mathbb{N}$$

$$Q_i(\mathbf{d}) \coloneqq L_{i-1}(\mathbf{d} - \mathbf{1}) \quad \text{for } i > 0 \quad \text{and} \quad Q_0(\mathbf{d}) = \mathbf{d} + \mathbb{N}^r.$$

Note that for fixed $\mathbf{d} \in \mathbb{Z}^r$ we have $L_i(\mathbf{d}) \subseteq Q_i(\mathbf{d})$ for all i.

Example 6.2.1. When $r = 2$ the regions $L_i(\mathbf{d})$ and $Q_i(\mathbf{d})$ can be visualized as in Figure 6.1. For $i > 1$ they are shaped like staircases with $i + 1$ and i "corners," respectively; in other words $L_i(\mathbf{d})$ contains $i + 1$ minimal elements and $Q_i(\mathbf{d})$ contains i.

Remark 6.2.2. An alternate description of $L_i(\mathbf{d})$ will also be useful: it is the set of $\mathbf{b} \in \mathbb{Z}^r$ so that the sum of the positive components of $\mathbf{d} - \mathbf{b}$ is at most i. (This ensures that we can distribute the λ_j so that $\mathbf{b} + \sum_j \lambda_j \mathbf{e}_j \geq \mathbf{d}$.)

With this notation in hand we can recall the definition of multigraded regularity.

Definition 6.2.3. [MS04, Def. 1.1] Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. We say M is $\mathbf{d}$-*regular* for $\mathbf{d} \in \mathbb{Z}^r$ if the following hold:

1. $H^0_B(M)_{\mathbf{p}} = 0$ for all $\mathbf{p} \in \bigcup_{1 \leq j \leq r}(\mathbf{d} + \mathbf{e}_j + \mathbb{N}^r)$,
2. $H^i_B(M)_{\mathbf{p}} = 0$ for all $i > 0$ and $\mathbf{p} \in L_{i-1}(\mathbf{d})$.

The *multigraded Castelnuovo–Mumford regularity* of M is then the set

$$\operatorname{reg}(M) \coloneqq \left\{ \mathbf{d} \in \mathbb{Z}^r \;\middle|\; M \text{ is } \mathbf{d}\text{-regular} \right\} \subset \operatorname{Pic} \mathbb{P}^{\mathbf{n}} \cong \mathbb{Z}^r.$$

It follows directly from the definition that if M is $\mathbf{d}$-regular, then M is $\mathbf{d}'$-regular for all $\mathbf{d}' \geq \mathbf{d}$. For other properties of multigraded regularity, such as $\mathbf{0}$-regularity of S, see [MS04].

Remark 6.2.4. Several alternate notions of Castelnuovo–Mumford regularity for the multigraded setting exist in the literature. The initial extension was introduced by Hoffman and Wang for a product of two projective spaces [HW04]. Following Maclagan and Smith's definition, Botbol and Chardin gave a more general definition working over an arbitrary base ring [BC17]. Recently, in their work on Tate resolutions on toric varieties, Brown and Erman introduced a modified notion of multigraded regularity for a weighted projective space, which they then extended to other toric varieties [BE21, §6.1].

6.2.2 Truncations and Local Cohomology

In this section we collect facts about truncations and local cohomology that will be used repeatedly. As in the case of a single projective space, the truncation of a graded module on a product of projective spaces at multidegree $\mathbf{d}$ contains all elements of degree at least $\mathbf{d}$.

Definition 6.2.5. For $\mathbf{d} \in \mathbb{Z}^r$ and M a $\mathbb{Z}^r$-graded S-module, the *truncation* of M at $\mathbf{d}$ is the $\mathbb{Z}^r$-graded S-submodule $M_{\geq \mathbf{d}} \coloneqq \bigoplus_{\mathbf{d}' \geq \mathbf{d}} M_{\mathbf{d}'}$.

Immediate from the definition is the following lemma.

Lemma 6.2.6. *The truncation map $M \mapsto M_{\geq \mathbf{d}}$ is an exact functor of $\mathbb{Z}^r$-graded S-modules.*

Remark 6.2.7. Since truncation is exact, if $F_\bullet$ is a graded free resolution of a module M then the term by term truncation $(F_\bullet)_{\geq \mathbf{d}}$ is a resolution of $M_{\geq \mathbf{d}}$. However, in general the truncation of a free module is not free, so $(F_\bullet)_{\geq \mathbf{d}}$ is generally not a free resolution of $M_{\geq \mathbf{d}}$.

We denote by $H^p_B(M)$ the p-th local cohomology of M supported at the irrelevant ideal B. For $p > 0$ and $\mathbf{a} \in \mathbb{Z}^r$ there exist natural isomorphisms

$$H^p\big(\mathbb{P}^{\mathbf{n}}, \widetilde{M}(\mathbf{b})\big) \cong H^{p+1}_B(M)_{\mathbf{b}},$$

and for $p = 0$ there is a $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow H^0_B(M) \longrightarrow M \longrightarrow \Gamma_*(\widetilde{M}) \longrightarrow H^1_B(M) \longrightarrow 0. \tag{6.2.1}$$

An important tool for computing local cohomology is the local Čech complex

$$\check{C}^\bullet(B, M) \colon 0 \longrightarrow M \longrightarrow \bigoplus M[g_i^{-1}] \longrightarrow \bigoplus M[g_i^{-1}, g_j^{-1}] \longrightarrow \cdots$$

where the g_i range over the generators of B. We index the local Čech complex so that the summands of $\check{C}^p(B, M)$ are localizations of M at p distinct generators of B. Then we have

$$H^p_B(M) \cong H^p(\check{C}^\bullet(B, M)).$$

See [ILL+07] and [CLS11, §9] for more details.

Note that inverting a generator of B inverts a variable from each factor of $\mathbb{P}^{\mathbf{n}}$, so the distinguished open sets corresponding to the generators of B form an affine cover $\mathfrak{U}_B$ of $\mathbb{P}^{\mathbf{n}}$. Denote by $\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F})$ the Čech complex of a sheaf $\mathscr{F}$ with respect to $\mathfrak{U}_B$:

$$\check{C}^\bullet(\mathfrak{U}_B, \mathscr{F}) \colon 0 \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i} \longrightarrow \bigoplus \mathscr{F}|_{\mathfrak{U}_i \cap \mathfrak{U}_j} \longrightarrow \cdots .$$

Lemma 6.2.8. *Given a complex of graded S-modules $L \to M \to N$ such that $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact, the complex $\check{C}^p(B, L) \to \check{C}^p(B, M) \to \check{C}^p(B, N)$ is exact for each $p \geq 0$.*

Proof. Fix p. Then $\check{C}^p(B, L) \to \check{C}^p(B, M) \to \check{C}^p(B, N)$ splits as a direct sum of complexes

$$L[g_1^{-1}, \ldots, g_p^{-1}] \to M[g_1^{-1}, \ldots, g_p^{-1}] \to N[g_1^{-1}, \ldots, g_p^{-1}]$$

each of which can be obtained by applying $\Gamma(U, -)$ to $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$, where U is the complement of $V(g_1, \ldots, g_p)$. Since U is affine they are exact. □

Since $M/M_{\geq \mathbf{d}}$ is annihilated by a power of B, a module M and its truncation define the same sheaf on $\mathbb{P}^{\mathbf{n}}$. In particular $H^p_B(M) = H^p_B(M_{\geq \mathbf{d}})$ for $p \geq 2$. The long exact sequence of local cohomology applied to $0 \to M_{\geq \mathbf{d}} \to M \to M/M_{\geq \mathbf{d}} \to 0$ gives

$$0 \longrightarrow H^0_B(M_{\geq \mathbf{d}}) \longrightarrow H^0_B(M) \longrightarrow M/M_{\geq \mathbf{d}} \longrightarrow H^1_B(M_{\geq \mathbf{d}}) \longrightarrow H^1_B(M) \longrightarrow 0.$$

Hence $H^0_B(M) = 0$ implies $H^0_B(M_{\geq \mathbf{d}}) = 0$. Since $M/M_{\geq \mathbf{d}}$ is zero in degrees larger than $\mathbf{d}$ we also have $H^1_B(M_{\geq \mathbf{d}})_{\geq \mathbf{d}} = H^1_B(M)_{\geq \mathbf{d}}$. An immediate consequence is the following lemma, which we will use repeatedly to reduce to the case when $\mathbf{d} = \mathbf{0}$.

Lemma 6.2.9. *A $\mathbb{Z}^r$-graded S-module M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ is $\mathbf{d}$-regular.*

6.2.3 Koszul Complexes and Cotangent Sheaves

For each factor $\mathbb{P}^{n_i}$ of $\mathbb{P}^{\mathbf{n}}$, the Koszul complex on the variables of $S_i = \operatorname{Cox} \mathbb{P}^{n_i}$ is a resolution of $\Bbbk$:

$$K^i_\bullet \colon 0 \leftarrow S_i \leftarrow S_i^{n_i+1}(-1) \leftarrow \bigwedge^2 \left[S_i^{n_i+1}(-1)\right] \leftarrow \cdots \leftarrow \bigwedge^{n_i+1} \left[S_i^{n_i+1}(-1)\right] \leftarrow 0. \tag{6.2.2}$$

The Koszul complex $K_\bullet$ on the variables of S is the tensor product of the complexes $\pi_i^* K^i_\bullet$.

For $1 \leq a \leq n$ let $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ be the kernel of $\bigwedge^{a-1} \left[S_i^{n_i+1}(-1)\right] \leftarrow \bigwedge^a \left[S_i^{n_i+1}(-1)\right]$ and let $\Omega^a_{\mathbb{P}^{n_i}}$ denote its sheafification. The minimal free resolution of $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ then consists of the terms of $K^i_\bullet$ with homological index greater than a. Write $\hat{\Omega}^0_{\mathbb{P}^{n_i}}$ for the kernel of $\Bbbk \leftarrow S_i$ (so that $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$) and take $\hat{\Omega}^a_{\mathbb{P}^{n_i}}$ to be

0 otherwise. For $\mathbf{a} \in \mathbb{Z}^r$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$ define

$$\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} \coloneqq \pi_1^* \Omega^{a_1}_{\mathbb{P}^{n_1}} \otimes \cdots \otimes \pi_r^* \Omega^{a_r}_{\mathbb{P}^{n_r}}$$

and write $\hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ for the analogous tensor product of the modules $\hat{\Omega}^{a}_{\mathbb{P}^{n_i}}$.

Given a free complex $F_\bullet$ and a multidegree $\mathbf{a} \in \mathbb{Z}^r$, denote by $F_\bullet^{\leq \mathbf{a}}$ the subcomplex of $F_\bullet$ consisting of free summands generated in degrees at most $\mathbf{a}$.

Lemma 6.2.10. *Fix $\mathbf{a} \in \mathbb{Z}^r$ and let $K_\bullet$ be the Koszul complex on the variables of S. The subcomplex $K_\bullet^{\leq \mathbf{a}}$ is equal to $K_\bullet$ in degrees $\leq \mathbf{a}$, and its sheafification is exact except at homological index $|\mathbf{a}|$, where it has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.*

Proof. The first statement follows from the fact that the terms appearing in $K_\bullet$ but not $K_\bullet^{\leq \mathbf{a}}$ have no elements in degrees $\leq \mathbf{a}$.

Note that $K_\bullet^{\leq \mathbf{a}}$ is a tensor product of pullbacks of subcomplexes of the $K_\bullet^i$ in (6.2.2):

$$K_\bullet^{\leq \mathbf{a}} = \pi_i^*(K_\bullet^1)^{\leq a_1} \otimes \cdots \otimes \pi_r^*(K_\bullet^r)^{\leq a_r}.$$

After sheafification, each complex $\pi_i^*(K_\bullet^i)^{\leq \mathbf{a}}$ is exact away from its kernel $\pi_i^* \Omega^{a_i}_{\mathbb{P}^{n_i}}$, which appears at homological index a_i. Thus $\widetilde{K}_\bullet^{\leq \mathbf{a}}$ has homology $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$, appearing in index $|\mathbf{a}|$. $\square$

6.3 A Criterion for Multigraded Regularity

To investigate the relationship between multigraded regularity and resolutions of truncations we first need to establish a definition of linearity for a multigraded resolution. We would like the differentials to be given by matrices with entries of total degree at most 1. However, we will examine only the twists in the resolution, requiring that they lie in the L regions from Section 6.2.1. In particular, we will identify a complex with a map of degree > 1 as nonlinear even if that map is zero.

Definition 6.3.1. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *linear* if F_0 is generated in a single multidegree $\mathbf{d}$ and the twists appearing in F_j lie in $L_j(-\mathbf{d})$.

We require F_0 to be generated in a single degree so that the truncation of a module with a linear resolution also has a linear resolution (see Proposition 6.3.5). Otherwise, for instance, the minimal resolution of M in the following example would be considered linear, yet the resolution of its truncation $M_{\geq(1,0)}$ would not.

Example 6.3.2. Write $S = \Bbbk[x_0, x_1, y_0, y_1]$ for the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ and let M be the module with resolution $S(0,-1)^2 \oplus S(-1,0)^2 \leftarrow S(-1,-1)^4 \leftarrow 0$ given by the presentation matrix

$$\begin{bmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & x_0 \\ -y_0 & 0 & -y_0 & 0 \\ 0 & -y_1 & 0 & -y_1 \end{bmatrix}.$$

A *Macaulay2* computation shows that M is $(1,0)$-regular. However, the minimal graded free resolution of the truncation $M_{\geq(1,0)}$ is

$$0 \longleftarrow S(-1,0)^2 \longleftarrow S(-2,-1)^2 \longleftarrow 0$$

which is not linear because $(-2,-1) \notin L_1(-1,0)$.

This example shows that a module can be $\mathbf{d}$-regular yet have a nonlinear resolution for $M_{\geq\mathbf{d}}$. Thus in order to characterize regularity in terms of truncations we need to weaken the definition of linear. We will use the larger Q regions from Section 6.2.1 in order to allow some maps of higher degree.

Definition 6.3.3. Let $F_\bullet$ be a $\mathbb{Z}^r$-graded free resolution. We say $F_\bullet$ is *quasilinear* if F_0 is generated in a single multidegree $\mathbf{d}$ and for each j the twists appearing in F_j lie in $Q_j(-\mathbf{d})$.

Example 6.3.4. Unlike on a single projective space, the resolution of S/B for the irrelevant ideal B on a product of projective spaces is not linear. However it is quasilinear. On $\mathbb{P}^1 \times \mathbb{P}^2$, for instance, S/B has resolution

$$0 \longleftarrow S \longleftarrow S(-1,-1)^6 \longleftarrow \begin{matrix} S(-1,-2)^6 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^3 \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0,$$

which has generators in degree $(0,0)$ and relations in degree $(1,1)$. Thus the resolution is not linear, since $(-1,-1) \notin L_1(0,0)$. However $(-1,-1) \in Q_1(0,0)$ is compatible with quasilinearity.

This condition is inspired by [BES20, Thm. 2.9], which characterized regularity in terms of the existence of virtual resolutions with Betti numbers similar to those of S/B—see Corollary 6.3.13 and Section 6.3.2 for a more complete discussion. Note that both linear and quasilinear reduce to the standard definition of linear on a single projective space.

Proposition 6.3.5. *Let M be a $\mathbb{Z}^r$-graded S-module. If $M_{\geq \mathbf{d}}$ has a linear (respectively quasilinear) resolution and $\mathbf{d}' \geq \mathbf{d}$ then $M_{\geq \mathbf{d}'}$ has a linear (respectively quasilinear) resolution.*

A linear resolution for $M_{\geq \mathbf{d}}$ implies that M is $\mathbf{d}$-regular when $H^0_B(M) = 0$. To obtain a converse that generalizes Eisenbud–Goto's result one should instead check that the resolution is quasilinear. This gives a criterion for regularity that does not require computing cohomology.

Theorem 6.3.6. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. Then M is $\mathbf{d}$-regular if and only if $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ such that F_0 is generated in degree $\mathbf{d}$.*

Example 6.3.7. A smooth hyperelliptic curve of genus 4 can be embedded into $\mathbb{P}^1 \times \mathbb{P}^2$ as a curve of degree $(2,8)$. An example of such a curve is given explicitly in [BES20, Ex. 1.4] as the B-saturation I of the ideal

$$\langle x_0^2 y_0^2 + x_1^2 y_1^2 + x_0 x_1 y_2^2, x_0^3 y_2 + x_1^3 (y_0 + y_1) \rangle.$$

Using Theorem 6.3.9 it is relatively easy to check that S/I is not $(2,1)$-regular: the minimal, graded, free resolution of $(S/I)_{\geq (2,1)}$ is

$$0 \longleftarrow S(-2,-1)^9 \longleftarrow \begin{matrix} S(-3,-1)^7 \\ \oplus \\ S(-2,-2)^{10} \\ \oplus \\ S(-2,-3)^2 \end{matrix} \longleftarrow \begin{matrix} S(-3,-2)^6 \\ \oplus \\ S(-2,-3)^3 \\ \oplus \\ S(-3,-3)^3 \end{matrix} \longleftarrow S(-3,-3)^2 \longleftarrow 0$$

which is not quasilinear because $(-2,-3) \notin Q_1(-2,-1)$.

We prove one direction of Theorem 6.3.6 in Section 6.3.1 (Theorem 6.3.9) and the other in Section 6.3.2 (Theorem 6.3.15).

6.3.1 Regularity Implies Quasilinearity

In Proposition 3.5.1 we constructed a virtual resolution with Betti numbers determined by the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By resolving the $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$ in terms of line bundles and tensoring with $\widetilde{M}$, we can relate the cohomological vanishing in the definition of multigraded regularity to the shape of this virtual resolution. The following lemma implies that when M is $\mathbf{d}$-regular the virtual resolution is quasilinear, i.e., the coefficients of twists outside of $Q_i(-\mathbf{d})$ are zero. The lemma is a variant of [BES20, Lem. 2.13] (see Section 6.3.2).

Lemma 6.3.8. *If a $\mathbb{Z}^r$-graded S-module M is $\mathbf{0}$-regular then $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) = 0$ for all $-\mathbf{a} \notin Q_i(\mathbf{0})$ and all $i > 0$.*

Proof. Fix i and $\mathbf{a} \in \mathbb{Z}^r$ with $-\mathbf{a} \notin Q_i(\mathbf{0})$, and suppose that $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \neq 0$. We will show that M is not $\mathbf{0}$-regular. We must have $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$, else $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}) = 0$. Let ℓ be the number of nonzero coordinates in $\mathbf{a}$.

A tensor product of locally free resolutions for the factors $\pi_i^*(\Omega^{a_i}_{\mathbb{P}^{n_i}})$ gives a locally free resolution for $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. Since $\Omega^0_{\mathbb{P}^{n_i}} = \mathcal{O}_{\mathbb{P}^{n_i}}$ we can use $r - \ell$ copies of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ and ℓ linear resolutions, each generated in total degree 1, to obtain such a resolution $\mathcal{F}_\bullet$ (see Section 6.2.3). Thus the twists in $\mathcal{F}_j$ have nonpositive coordinates and total degree $-j - \ell$, so they are in $L_{j+\ell}(\mathbf{0})$.

Since $\mathcal{F}$ is locally free the cokernel of $\widetilde{M} \otimes \mathcal{F}$ is isomorphic to $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$. By a standard spectral sequence argument, explained in the proof of Theorem 6.3.15, the nonvanishing of $H^{|\mathbf{a}|-i}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$ implies the existence of some j such that $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$.

If $i = 0$ then

$$|\mathbf{a}| - i + j \geq \ell - i + j = j + \ell.$$

If $i > 0$ then $\mathbf{a} - \mathbf{1}$ has ℓ nonnegative coordinates that sum to $|\mathbf{a}| - \ell$. Thus $|\mathbf{a}| - \ell > i - 1$, since $-\mathbf{a} \notin Q_i(\mathbf{0}) = L_{i-1}(-\mathbf{1})$ (see Remark 6.2.2). This also gives

$$|\mathbf{a}| - i + j \geq (\ell + i) - i + j = j + \ell.$$

so in either case $L_{j+\ell}(\mathbf{0}) \subseteq L_{|\mathbf{a}|-i+j}(\mathbf{0})$. Therefore $H^{|\mathbf{a}|-i+j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \mathcal{F}_j) \neq 0$ for F_j with twists in $L_{j+\ell}(\mathbf{0})$ implies that M is not $\mathbf{0}$-regular. $\square$

See [CM07, Thm. 5.5] for a similar result relating Hoffman and Wang's definition of regularity [HW04] to a different cohomology vanishing for $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Motivated by the quasilinearity of the virtual resolution in Proposition 3.5.1, we will prove that the $\mathbf{d}$-regularity of M implies that the minimal free resolution of $M_{\geq \mathbf{d}}$ is quasilinear. Let K be the Koszul complex from Section 6.2.3 and $\check{C}^p(B, \cdot)$ the Čech complex as in Section 6.2.2. We will use the spectral sequence of a double complex with rows from subcomplexes of K and columns given by Čech complexes in order to relate the Betti numbers of $M_{\geq \mathbf{d}}$ to the sheaf cohomology of $\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$.

Theorem 6.3.9. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M)_{\mathbf{d}} = 0$. If M is $\mathbf{d}$-regular then $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$.*

Proof. Without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 6.2.9).

By Proposition 3.5.1 there exists a free monad $G_\bullet$ of M with j-th Betti number given by $h^{|\mathbf{a}|-j}(\widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a}))$. Since M is $\mathbf{0}$-regular the vanishing of these cohomology groups results in a quasilinear virtual resolution by Lemma 6.3.8 and (2) from Proposition 3.5.1. Let $F_\bullet$ be the minimal free resolution of M. We will show that the Betti numbers of $F_\bullet$ are equal to those of $G_\bullet$, so that $F_\bullet$ is also quasilinear and $F_0 = G_0$ is generated in degree $\mathbf{d}$. (In fact this is enough to show that $F_\bullet$ and $G_\bullet$ are isomorphic, as we will do in Corollary 6.3.13.)

Fix a degree $\mathbf{a} \in \mathbb{Z}^r$. Construct a double complex $E^{\bullet,\bullet}$ by taking the Čech complex of each term in $M \otimes K^{\leq \mathbf{a}}_\bullet$ and including the Čech complex of $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ as an additional column. Index $E^{\bullet,\bullet}$ so that

$$E^{s,t} = \begin{cases} \check{C}^t\left(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|+1-s}\right) & \text{if } s > 0, \\ \check{C}^t\left(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\right) & \text{if } s = 0. \end{cases}$$

We will compare the vertical and horizontal spectral sequences of $E^{\bullet,\bullet}$ in degree $\mathbf{a}$. By Lemma 6.2.10 and the fact that $K^{\leq \mathbf{a}}_\bullet$ is locally free, the sheafification of the 0-th row $E^{\bullet,0}$ is exact. Thus by Lemma 6.2.8 the rows of $E^{\bullet,\bullet}$ are exact for $t \neq 0$.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots \\
\check{C}^2(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) \arrow[u] \arrow[r] & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|}) \arrow[u] \arrow[r] & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \check{C}^2(B, M \otimes K^{\leq \mathbf{a}}_{0}) \arrow[u] \\
\check{C}^1(B, M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}) \arrow[u] \arrow[r] & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|}) \arrow[u] \arrow[r] & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \check{C}^1(B, M \otimes K^{\leq \mathbf{a}}_{0}) \arrow[u] \\
M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} \arrow[u] \arrow[r] & M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|} \arrow[u] \arrow[r] & M \otimes K^{\leq \mathbf{a}}_{|\mathbf{a}|-1} \arrow[u] \arrow[r] & \cdots \arrow[r] & M \otimes K^{\leq \mathbf{a}}_{0} \arrow[u]
\end{tikzcd}
$$

Since the elements of M have degrees $\geq \mathbf{0}$, the elements of degree $\mathbf{a}$ in $M \otimes K_\bullet$ come from elements of degree $\leq \mathbf{a}$ in $K_\bullet$. Thus by Lemma 6.2.10 the homology of $M \otimes K^{\leq \mathbf{a}}_\bullet$ in degree $\mathbf{a}$ is the same as that of $M \otimes K_\bullet$. Hence the cohomology of the 0-th row $E^{\bullet,0}$ in degree $\mathbf{a}$ computes the degree $\mathbf{a}$ Betti numbers of F_j for $0 \leq j \leq |\mathbf{a}|$, i.e., for $s > 0$,

$$H^s(E^{\bullet,0})_{\mathbf{a}} = \operatorname{Tor}_{|\mathbf{a}|+1-s}(M, \Bbbk)_{\mathbf{a}}. \tag{6.3.1}$$

The vertical cohomology of $E^{\bullet,\bullet}$ gives the local cohomology of the terms of $M \otimes K^{\leq \mathbf{a}}_\bullet$ along with $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Consider the degree $\mathbf{a}$ part of this double complex. The cohomology coming from $M \otimes K^{\leq \mathbf{a}}_\bullet$ has summands of the form $H^i_B(M(-\mathbf{b}))_{\mathbf{a}} = H^i_B(M)_{\mathbf{a}-\mathbf{b}}$ where $\mathbf{b} \leq \mathbf{a}$. These vanish because M is $\mathbf{0}$-regular, except possibly $H^0_B(M)_{\mathbf{0}}$ which vanishes by hypothesis, so the only nonzero terms come from $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$.

Since $K^{\leq \mathbf{a}}_\bullet$ is a resolution of $\Bbbk$ in degrees $\leq \mathbf{a}$, there are no elements of degree $\mathbf{a}$ in $M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$. Hence, using (6.2.1),

$$H^1_B\Big(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}\Big)_{\mathbf{a}} = H^0\Big(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\Big).$$

Therefore the cohomology of the 0-th column $E^{0,\bullet}$ in degree $\mathbf{a}$ is

$$H^t(E^{0,\bullet})_{\mathbf{a}} = H^t_B(M \otimes \hat{\Omega}^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}})_{\mathbf{a}} = H^{t-1}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{6.3.2}$$

for $t > 0$, i.e., the Betti numbers of $G_\bullet$ indexed differently.

Since both spectral sequences of the double complex $E^{\bullet,\bullet}$ converge after the first page, their total complexes agree in degree $\mathbf{a}$, so by equating the dimensions of (6.3.1) and (6.3.2) in total degree $|\mathbf{a}| + 1 - j$ we get

$$\dim_{\Bbbk} \operatorname{Tor}_j(M, \Bbbk)_{\mathbf{a}} = \dim_{\Bbbk} H^{|\mathbf{a}|-j}(\mathbb{P}^{\mathbf{n}}, \widetilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})) \tag{6.3.3}$$

for $|\mathbf{a}| \geq j \geq 0$. When $j > |\mathbf{a}|$, neither $F_\bullet$ nor $G_\bullet$ has a nonzero Betti number for degree reasons, and when $\mathbf{a}$ has $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}} = 0$ the argument above still holds. Hence the Betti numbers of $G_\bullet$ and $F_\bullet$ are equal in degree $\mathbf{a}$. □

To check that a module M is $\mathbf{d}$-regular directly from Definition 6.2.3, condition (2) requires one to show that $H^i_B(M)_{\mathbf{p}}$ vanishes for all $i > 0$ and all $\mathbf{p} \in \bigcup_{|\lambda|=i}(\mathbf{d} - \lambda_1\mathbf{e}_1 - \cdots - \lambda_r\mathbf{e}_r + \mathbb{N}^r)$ with $\lambda \in \mathbb{N}^r$. The proof of Theorem 6.3.9, when combined with Theorem 6.3.6 and Lemma 6.3.8, shows that on a product of projective spaces the full strength of this condition is unnecessary. In particular, one only needs to consider λ_j with $\lambda_j \leq n_j + 1$.

Proposition 6.3.10. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If*

1. $H^0_B(M)_{\mathbf{p}} = 0$ *for all* $\mathbf{p} \geq \mathbf{d}$
2. $H^i_B(M)_{\mathbf{p}} = 0$ *for all* $i > 0$ *and all* $\mathbf{p} \in \bigcup_{|\lambda|=i}(\mathbf{d} - \sum_1^r \lambda_j\mathbf{e}_j + \mathbb{N}^r)$ *where* $0 \leq \lambda_j \leq n_j + 1$

then M is $\mathbf{d}$-regular.

Proof. The only difference between (2) above and condition (2) in Definition 6.2.3 is the restriction to $\lambda_j \leq n_j + 1$. By the proof of Theorem 6.3.9, if $H^0_B(M)_{\mathbf{b}} = 0$ and M satisfies the hypotheses of Proposition 3.5.1 and Lemma 6.3.8 then M has a quasilinear resolution generated in degree $\mathbf{d}$ and is thus $\mathbf{d}$-regular by Theorem 6.3.6. In the proof of Lemma 6.3.8 it is sufficient for the cohomology of $M(\mathbf{d})$ to vanish in degrees appearing in the resolution of some $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})$, which excludes those with coordinates not $\leq \mathbf{n} + \mathbf{1}$. □

Example 6.3.11. On $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, to show that a module M is $\mathbf{0}$-regular using Definition 6.2.3 one must check that $H^3_B(M)_{\mathbf{p}} = 0$ for $\mathbf{p}$ in the region with minimal elements

$$(-3, 0, 0), (-2, -1, 0), (-2, 0, -1), \ldots, (0, -3, 0), \ldots, (0, 0, -3).$$

However, Proposition 6.3.10 implies that a smaller region is sufficient. For instance, we need not check that $H^3_B(M)_{\mathbf{p}} = 0$ for $\mathbf{p}$ equal to each of $(-3,0,0)$, $(0,-3,0)$, and $(0,0,-3)$.

Remark *6.3.12.* One may also deduce Proposition 6.3.10 from the proofs in [BES20] without the hypothesis that $H^0_B(M)_{\mathbf{d}} = 0$.

The proof of Theorem 6.3.9 also implies that when M is $\mathbf{d}$-regular the resolution of $M_{\geq \mathbf{d}}$ is isomorphic to the virtual resolution constructed in Proposition 3.5.1. In other words, the minimal free resolution of $M_{\geq \mathbf{d}}$ is a splitting of the Beilinson spectral sequence for $M(\mathbf{d})$, giving a concrete construction of the abstractly defined virtual resolutions used in [BES20, Thm. 2.9] to witness the regularity of $M(\mathbf{d})$.

Corollary 6.3.13. *The complexes $F_\bullet$ and $G_\bullet$ in the proof of Theorem 6.3.9 are isomorphic.*

Proof. From Proposition 3.5.1 and the fact that $\Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}$ is nonzero only for $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$ it follows that $G_\bullet$ is a minimal virtual resolution consisting of twists $S(-\mathbf{a})$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{n}$. Therefore the isomorphism follows from Theorem 3.5.4. $\square$

6.3.2 Quasilinearity Implies Regularity

We will now prove the reverse implication of Theorem 6.3.6, namely that a quasilinear resolution generated in degree $\mathbf{d}$ for $M_{\geq \mathbf{d}}$ implies that M is $\mathbf{d}$-regular. We use a hypercohomology spectral sequence argument, which relates the local cohomology of M to the local cohomology of the terms in a resolution for $M_{\geq \mathbf{d}}$.

The following lemma will show that entire diagonals in our spectral sequence vanish when the resolution is quasilinear. Thus the local cohomology modules $H^i_B(M)$ to which the diagonals converge also vanish in the same degrees.

Lemma 6.3.14. *If $i, j \in \mathbb{N}$ then $H^{i+j+1}_B(S)_{\mathbf{a}+\mathbf{b}} = 0$ for all $\mathbf{a} \in L_i(\mathbf{0})$ and all $\mathbf{b} \in Q_j(\mathbf{0})$.*

Proof. Note that $L_i(\mathbf{0}) + Q_j(\mathbf{0}) = L_i(\mathbf{0}) + L_{j-1}(-\mathbf{1}) = L_{i+j-1}(-\mathbf{1})$ as sets. We also have $H^0_B(S) = H^1_B(S) = 0$, so it suffices to show that $H^{k+1}_B(S)_{\mathbf{c}} = H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c})) = 0$ for $k \geq 1$ and $\mathbf{c} \in L_{k-1}(-\mathbf{1})$.

The cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ is given by the Künneth formula. Fix a nonempty set of indices $J \subseteq$

$\{1, \ldots, r\}$ and consider the term

$$\left[\bigotimes_{j \in J} H^{n_j}(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right] \otimes \left[\bigotimes_{j \notin J} H^0(\mathbb{P}^{n_j}, \mathcal{O}_{\mathbb{P}^{n_j}}(d_j))\right],$$

which contributes to $H^k(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{c}))$ for $k = \sum_{j \in J} n_j$. It will be nonzero if and only if $d_j \leq -n_j - 1$ for $j \in J$ and $d_j \geq 0$ for $j \notin J$. If $\mathbf{c} \in L_{k-1}(-\mathbf{1})$ then

$$\mathbf{c} \geq -\mathbf{1} - \lambda_1 \mathbf{e}_1 - \cdots - \lambda_r \mathbf{e}_r$$

for some λ_i with $\sum \lambda_i = k - 1 = -1 + \sum_{j \in J} n_j$. It is not possible for the right side to have components $\leq -n_j - 1$ for all $j \in J$. Since all cohomology of $\mathcal{O}_{\mathbb{P}^{\mathbf{n}}}$ arises in this way, the lemma follows. $\square$

In [BES20, Thm. 2.9] Berkesch, Erman, and Smith show for M with $H^0_B(M) = H^1_B(M) = 0$ that M is $\mathbf{d}$-regular if and only if M has a virtual resolution $F_\bullet$ so that the degrees of the generators of $F(\mathbf{d})_\bullet$ are at most those appearing in the minimal free resolution of S/B. This Betti number condition is stronger than quasilinearity, but the additional strength is not used in their proof, so the existence of such a virtual resolution is equivalent to the existence of a quasilinear one.

Since a resolution of $M_{\geq \mathbf{d}}$ is a type of virtual resolution, the reverse implication of Theorem 6.3.6 mostly reduces to this result. We present a modified proof for completeness. In particular, we do not need to require $H^1_B(M) = 0$ because we have more information about the cokernel of our resolution.

From this perspective Theorem 6.3.6 says that the regularity of M is determined not only by the Betti numbers of its virtual resolutions, but by the Betti numbers of only those virtual resolutions that are actually minimal free resolutions of truncations of M. Thus we provide an explicit method for checking whether M is $\mathbf{d}$-regular.

Theorem 6.3.15. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $H^0_B(M) = 0$. If $M_{\geq \mathbf{d}}$ has a quasilinear resolution $F_\bullet$ with F_0 generated in degree $\mathbf{d}$, then M is $\mathbf{d}$-regular.*

Proof. Without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$ and $M = M_{\geq \mathbf{0}}$ (see Lemma 6.2.9).

Let $F_\bullet$ be a quasilinear resolution of M, so that the twists of F_j are in $Q_j(\mathbf{0})$. Then the spectral sequence of the double complex $E^{\bullet,\bullet}$ with terms

$$E^{s,t} = \check{C}^t(B, F_{-s})$$

converges to the cohomology $H^i_B(M)$ of M in total degree i. The first page of the vertical spectral sequence has terms $H^t_B(F_{-s})$, so $H^{i+j}_B(F_j)_{\mathbf{a}} = 0$ for all j (i.e., for all $(s,t) = (-j, i+j)$) implies $H^i_B(M)_{\mathbf{a}} = 0$.

Therefore it suffices to show that $H^{i+j}_B(S(\mathbf{b}))_{\mathbf{a}} = 0$ for $i \geq 1$ and all $\mathbf{a} \in L_{i-1}(\mathbf{0})$ and $\mathbf{b} \in Q_j(\mathbf{0})$, as is done in Lemma 6.3.14. $\square$

7 Bounds on Multigraded Regularity on Toric Varieties

The material in this chapter originally appeared in [BCHS22].

7.1 Introduction

Building on the work of Swanson in [Swa97], Cutkosky–Herzog–Trung in [CHT99] and Kodiyalam in [Kod00] described the surprisingly predictable asymptotic behavior of Castelnuovo–Mumford regularity for powers of ideals on a projective space $\mathbb{P}^r$: given an ideal $I \subset \mathbb{K}[x_0, \ldots, x_r]$, there exist $d, e \in \mathbb{Z}$ such that for $n \gg 0$ the regularity of I^n satisfies

$$\operatorname{reg}(I^n) = dn + e.$$

Due to the importance of regularity as a measure of complexity for syzygies and its geometric interpretation in terms of the cohomology of coherent sheaves [BEL91, CEL01], this phenomenon has received substantial attention [GGP95, Cha97, SS97, Röm01, TW05, BCH13], focused mostly on projective spaces. See [Cha13] for a survey.

Motivated by toric geometry, we turn our focus toward ideals in the multigraded total coordinate ring S of a smooth projective toric variety X, for which a generalized notion of regularity was introduced by Maclagan and Smith [MS04]. In this setting the regularity of a $\operatorname{Pic}(X)$-graded module is a subset of $\operatorname{Pic} X$ that is closed under the addition of nef divisors. A natural question is thus whether there is an analogous description for the asymptotic shape of $\operatorname{reg}(I^n) \subset \operatorname{Pic} X$.

In Theorem 7.4.1 we bound multigraded regularity by establishing regions "inside" and "outside" of $\operatorname{reg}(I^n)$ which translate linearly by a fixed vector as n increases (see the figure in Example 7.4.2). The inner bound depends on the Betti numbers of the Rees ring $S[It]$, while the outer bound depends only on the degrees of the generators of I.

Theorem 7.4.1. *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on I, such that for each integer $n > 0$ and each pair of degrees $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ satisfying $\mathbf{q}_1 \geq \deg f_i \geq \mathbf{q}_2$ for all generators f_i of I, we have*

$$n\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^n) \subseteq n\mathbf{q}_2 + \operatorname{Nef} X.$$

It is worth emphasizing that our result holds over smooth projective toric varieties with arbitrary Picard rank. Indeed, toric varieties of higher Picard rank introduce a wrinkle that is not present in existing asymptotic results on Castelnuovo–Mumford regularity: in general there are infinitely many possible regularity regions compatible with two given bounds. (In contrast, when $\operatorname{Pic} X = \mathbb{Z}$, inner and outer bounds correspond to upper and lower bounds, respectively, with only finitely many integers between each pair.) Nevertheless, since multigraded regularity is invariant under positive translation by $\operatorname{Nef} X$, an outer bound in the shape of the nef cone cannot contain an infinite expanding chain of regularity regions.

Surprisingly, we will see in Example 7.3.2 that even on a Hirzebruch surface X the regularity of a finitely generated module may not be contained in the union of finitely many translates of $\operatorname{Nef} X$. In the case of powers of ideals, however, the absence of torsion over S implies that the regularity has finitely many minimal elements. More generally, in Theorem 7.3.11 we construct a nef-shaped outer bound determined by the degrees of generators of a torsion-free module (see the figure in Example 7.3.13). We use the idea that if the truncation $M_{\geq \mathbf{d}}$ is not generated in a single degree $\mathbf{d}$ then M is not $\mathbf{d}$-regular (see Theorem 7.3.3 for a simpler case).

Theorem 7.3.11. *Let M be a finitely generated graded torsion-free S-module with $\widetilde{M} \neq 0$. Then $\operatorname{reg} M$ is contained in a translate of $\operatorname{Nef} X$. In particular, $\operatorname{reg} M$ has finitely many minimal elements.*

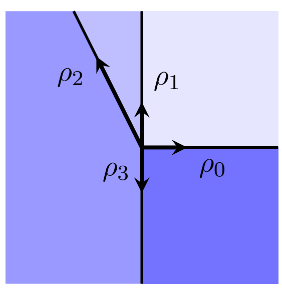
It remains an interesting problem to characterize modules with torsion whose regularity is contained in a translate of $\operatorname{Nef} X$. Note that the regularity of a finitely generated module is always contained in a translate of $\operatorname{Eff} X$ (see Proposition 7.3.7). In fact, the existence of a module whose regularity contains infinitely many minimal elements is a consequence of the difference between the effective and nef cones of X. This possibility highlights a theme from [BCHS21, BKLY21] that algebraic properties which coincide over projective spaces can diverge in higher Picard rank.

7.2 Notation and Definitions

Throughout we work over a base field $\mathbb{K}$ and denote by $\mathbb{N}$ the set of non-negative integers. Let X be a smooth projective toric variety determined by a fan. The total coordinate ring of X is a $\operatorname{Pic}(X)$-graded polynomial ring S over $\mathbb{K}$ with an irrelevant ideal $B \subset S$. Write $\operatorname{Eff} X$ for the monoid in $\operatorname{Pic} X$ generated by the degrees of the variables in S.

Fix minimal generators $\mathbb{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$ for the monoid $\operatorname{Nef} X$ of classes in $\operatorname{Pic} X$ represented by numerically effective divisors. For $\lambda \in \mathbb{Z}^r$, write $\lambda \cdot \mathbb{C}$ to represent the linear combination $\lambda_1 \mathbf{c}_1 + \cdots + \lambda_r \mathbf{c}_r \in \operatorname{Pic} X$, and similarly for other tuples in $\operatorname{Pic} X$. Write $|\lambda|$ for the sum $\lambda_1 + \cdots + \lambda_r$. We use a partial order on $\operatorname{Pic} X$ induced by $\operatorname{Nef} X$: given $\mathbf{a}, \mathbf{b} \in \operatorname{Pic} X$, we write $\mathbf{a} \leq \mathbf{b}$ when $\mathbf{b} - \mathbf{a} \in \operatorname{Nef} X$.

Example 7.2.1. The Hirzebruch surface $\mathcal{H}_t = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(t))$ is a smooth projective toric variety whose associated fan, shown left in Figure 7.1, has rays $(1,0)$, $(0,1)$, $(-1,t)$, and $(0,-1)$. For each ray there is a corresponding prime torus-invariant divisor. In particular, the total coordinate ring of $\mathcal{H}_t$ is the polynomial ring $S = \mathbb{K}[x_0, x_1, x_2, x_3]$ and its irrelevant ideal is $B = \langle x_0, x_2 \rangle \cap \langle x_1, x_3 \rangle$.



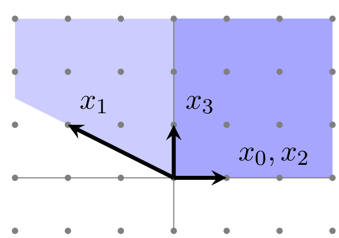


Figure 7.1: Left: fan of $\mathcal{H}_2$. Right: the cones $\operatorname{Nef} \mathcal{H}_2$ (dark blue) and $\operatorname{Eff} \mathcal{H}_2$ (blue).

Choosing a basis for $\operatorname{Pic} \mathcal{H}_t \cong \mathbb{Z}^2$, the grading on S can be given as $\deg x_0 = \deg x_2 = (1,0)$, $\deg x_1 = (-t, 1)$, and $\deg x_3 = (0,1)$. The effective and nef cones are illustrated on the right.

For a $\operatorname{Pic}(X)$-graded S-module M and $\mathbf{d} \in \operatorname{Pic} X$, denote by $M_{\geq \mathbf{d}}$ the submodule of M generated by all elements of degrees $\mathbf{d}'$ satisfying $\mathbf{d}' \geq \mathbf{d}$ (c.f. [MS04, Def. 5.1]). Denote by $\widetilde{M}$ the quasi-coherent sheaf on X associated to M, as in [Cox95, §3].

We now recall the notion of multigraded Castelnuovo–Mumford regularity for an arbitrary toric variety introduced by Maclagan and Smith.

Definition 7.2.2 (c.f. [MS04, Def. 1.1])**.** Let M be a graded S-module. For $\mathbf{d} \in \operatorname{Pic} X$, we say M is $\mathbf{d}$-*regular* if the following hold:

1. $H^i_B(M)_\mathbf{b} = 0$ for all $i > 0$ and all $\mathbf{b} \in \bigcup_{|\lambda|=i-1}(\mathbf{d} - \lambda \cdot \mathbb{C} + \operatorname{Nef} X)$ where $\lambda \in \mathbb{N}^r$.

2. $H^0_B(M)_\mathbf{b} = 0$ for all $\mathbf{b} \in \bigcup_j(\mathbf{d} + \mathbf{c}_j + \operatorname{Nef} X)$.

We write $\operatorname{reg} M$ for the set of $\mathbf{d}$ such that M is $\mathbf{d}$-regular.

7.3 Finite Generation of Multigraded Regularity

We begin by constructing an outer bound for the regularity of I^n—a subset of $\operatorname{Pic} X$ that contains $\operatorname{reg}(I^n)$. In [Kod00], Kodiyalam constructs this from a bound on the degrees of the generators of I^n. However, more nuanced behavior can occur in the multigraded setting. The following example shows that the degree of a minimal generator of an ideal does not bound its regularity on an arbitrary toric variety.

Example 7.3.1. Let $I = \langle x_0x_3, x_0x_2, x_1x_2\rangle$ be an ideal in the total coordinate ring of the Hirzebruch surface $\mathcal{H}_t$, with notation as in Example 7.2.1. A local cohomology computation verifies that I is $(1,1)$-regular. However x_0x_2 is a minimal generator with $\deg(x_0x_2) = (2,0) \not\leq (1,1)$.

The existence of a similar example with $H^0_B(M) \neq 0$ was noted by Macalagan and Smith, who asked whether B-torsion was necessary in [MS04, §5]. Example 7.3.1 shows that it is not.

Perhaps more unexpectedly, it is also possible for the regularity of a finitely generated module to have infinitely many minimal elements with respect to $\operatorname{Nef} X$, as is the case in the following simple example pointed out by Daniel Erman.

Example 7.3.2. Let $M = S/\langle x_2, x_3\rangle$ be the coordinate ring of a single point on $\mathcal{H}_t$ (see Example 7.2.1). Since $\langle x_2, x_3\rangle$ is saturated we have $H^0_B(M) = 0$. Further, since the support of $\widetilde{M}$ has dimension 0 we must have $H^i_B(M) = 0$ for $i \geq 2$. Thus $\operatorname{reg} M$ is determined entirely by $H^1_B(M)$, which vanishes exactly where the Hilbert function of M agrees with its Hilbert polynomial.

The Hilbert function of M is equal to 1 inside $\operatorname{Eff} \mathcal{H}_t$ and 0 outside of it. Hence $\operatorname{reg} M = \operatorname{Eff} \mathcal{H}_t$. When $t > 0$ this cone does not contain finitely many minimal elements with respect to $\operatorname{Nef} X$, as illustrated in Figure 7.2.

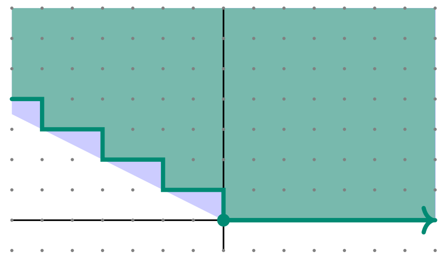


Figure 7.2: The multigraded regularity of M (green) is an infinite staircase contained in a translate of the effective cone of $\mathcal{H}_2$ (blue).

The regularity of the module in Example 7.3.2 is contained in a translate of $\operatorname{Eff} X$, which does give an outer bound. We will see in Proposition 7.3.7 that this is true for all M. At the same time many modules, for instance $S/\langle x_0, x_1\rangle$, do have regularity regions contained in translates of $\operatorname{Nef} X$. Thus an outer bound in the shape of $\operatorname{Eff} X$ would not be tight in general. In particular, we will see in Corollary 7.3.12 that an outer bound in the shape of $\operatorname{Nef} X$ exists for an ideal $I \subseteq S$ and thus $\operatorname{reg} I$ has finitely many minimal elements. We begin with the case $I = S$.

7.3.1 Regularity of the Coordinate Ring

In this section we show that the pathology seen in Example 7.3.2—a regularity region contained in no translate of $\operatorname{Nef} X$—does not occur for the total coordinate ring of a smooth projective toric variety. In particular we show that $\operatorname{reg} S \subseteq \operatorname{Nef} X$.

In [MS04, Prob. 6.12], Maclagan and Smith asked for a combinatorial characterization of toric varieties X such that $\operatorname{Nef} X \subseteq \operatorname{reg} S$. Theorem 7.3.3 below shows that when X is smooth and projective, $\operatorname{Nef} X \subseteq \operatorname{reg} S$ is in fact equivalent to the a priori stronger condition that $\operatorname{reg} S = \operatorname{Nef} X$. It still remains an interesting question to characterize such toric varieties. For instance, the only Hirzebruch surface with this property is $\mathcal{H}_1$.

Theorem 7.3.3. *Using the notation from Section 7.2, we have* $\operatorname{reg} S \subseteq \operatorname{Nef} X$*. In particular,* $\operatorname{reg} S$ *contains finitely many minimal elements.*

Proof. Take $\mathbf{d} \in \operatorname{reg} S$. By [MS04, Thm. 5.4] the truncation $S_{\geq \mathbf{d}}$ is generated by the monomials of $S_{\mathbf{d}}$, so there is a surjection $S_{\mathbf{d}} \otimes_{\mathbb{K}} S \to S_{\geq \mathbf{d}}(\mathbf{d})$ which sheafifies to a surjection $S_{\mathbf{d}} \otimes \mathcal{O} \to \mathcal{O}(\mathbf{d})$. Hence $\mathcal{O}(\mathbf{d})$ is generated by global sections, so by [CLS11, Thm. 6.3.11] $\mathbf{d}$ is nef.

An application of Dickson's lemma (e.g. [CLO15, §2.4 Thm. 5]) shows that $\operatorname{reg} S$ has finitely many minimal elements, finishing the proof.

Lemma 7.3.4. *A subset $V \subseteq \operatorname{Nef} X$ contains finitely many minimal elements with respect to $\leq$ on $\operatorname{Pic} X$.*

Elements of V can be written as linear combinations $\lambda \cdot \mathbb{C}$ of the monoid generators of $\operatorname{Nef} X$. The minimal elements of V must have coefficients $\lambda \in \mathbb{N}^r$ that are minimal in the component-wise partial order on $\mathbb{N}^r$. By Dickson's lemma only finitely many possible coefficients exist. □

Example 7.3.5. The multigraded regularity of the coordinate ring of the Hirzebruch surface $\mathcal{H}_2$ is contained in the nef cone of $\mathcal{H}_2$, as illustrated in Figure 7.3.

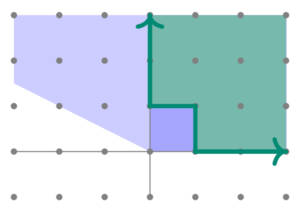


Figure 7.3: The regularity of S (dark green) is contained in $\operatorname{Nef} \mathcal{H}_2$ (dark blue).

Though we do not directly use Theorem 7.3.3 in the next section, we do rely on the idea of the proof. For an arbitrary module M, if $\mathbf{d} \in \operatorname{reg} M$ then the truncation $M_{\geq \mathbf{d}}$ is generated in a single degree $\mathbf{d}$, meaning that $\widetilde{M}(\mathbf{d})$ is globally generated. This no longer immediately implies that $\mathbf{d}$ is nef, but Lemma 7.3.6 below connects the difference between $\mathbf{d}$ and the degrees of the generators of M to monomials in truncations of S itself.

We also use the chamber complex of the rays of $\operatorname{Eff} X$, which is described in [MS04, §2]. By definition, this chamber complex is the coarsest fan with support $\operatorname{Eff} X$ which refines all triangulations of the degrees of the variables of S. It partitions $\operatorname{Eff} X$ into cones that govern many geometric properties of $\operatorname{Spec} S$, including its GIT quotients, birational geometry, and Hilbert polynomials (c.f. [CLS11, Ch. 14-15], [HKP06, §5]).

For our purposes we need only the existence of a strongly convex rational polyhedral fan that covers $\operatorname{Eff} X$ and contains $\operatorname{Nef} X$ as a cone. We will refer to the maximal cones as chambers and the codimension one cones as walls. In particular, $\operatorname{Nef} X$ is a chamber.

Lemma 7.3.6. *Let* Γ *be a chamber of* $\operatorname{Eff} X$ *other than* $\operatorname{Nef} X$*, and let* $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \operatorname{Pic} X$*. If* $\mathbf{a}_i \in \Gamma \setminus \operatorname{Nef} X$ *for all* i*, then there exist monomials* $m_i \in S_{\geq \mathbf{a}_i}$ *such that* $\prod_i m_i$ *is not generated by the monomials of* $S_{\sum \mathbf{a}_i}$*.*

Proof. Since Γ and $\operatorname{Nef} X$ intersect at most in a wall of Γ and no $\mathbf{a}_i$ lies in $\Gamma \cap \operatorname{Nef} X$, their sum $\mathbf{b} = \sum \mathbf{a}_i$ must also be in $\Gamma \setminus \operatorname{Nef} X$. Consider the multiplication maps

$$\begin{tikzcd} S_{\mathbf{b}} \otimes_{\mathbb{K}} S \arrow[r, "\varphi"] & S(\mathbf{b}) \\ & \bigotimes_{\mathbb{K}} S_{\geq \mathbf{a}_i}(\mathbf{a}_i). \arrow[u, "\psi"'] \end{tikzcd}$$

Suppose the proposition is false. Then the image of ψ must be contained in the image of φ, else we could choose $(m_i) \in \bigotimes_{\mathbb{K}} S_{\geq \mathbf{a}_i}(\mathbf{a}_i)$ with image not generated by the monomials of $S_{\mathbf{b}}$. Note that each $S_{\geq \mathbf{a}_i}(\mathbf{a}_i)$ sheafifies to $\mathcal{O}(\mathbf{a}_i)$, so sheafifying the entire diagram gives

$$\begin{tikzcd} S_{\mathbf{b}} \otimes \mathcal{O} \arrow[r, "\varphi"] & \mathcal{O}(\mathbf{b}) \\ & \mathcal{O}(\mathbf{b}). \arrow[u, "\psi"'] \end{tikzcd}$$

In particular, the image of ψ is still contained in the image of φ. Since ψ sheafifies to an isomorphism, φ sheafifies to a surjection. This implies $\mathbf{b} \in \operatorname{Nef} X$, which is a contradiction. $\square$

7.3.2 Regularity of Torsion-Free Modules

The goal of this section is to prove that the multigraded regularity of an ideal $I \subseteq S$ has only finitely many minimal elements. We will prove this more generally for finitely generated torsion-free S-modules.

Proposition 7.3.7 shows that the regularity of an arbitrary finitely generated module is contained in some translate of $\operatorname{Eff} X$. Under the stronger assumption that M is torsion-free, Proposition 7.3.8 shows that we can also eliminate degrees that are in a translate of $\operatorname{Eff} X$ but not $\operatorname{Nef} X$.

Proposition 7.3.7. *Let* M *be a finitely generated graded* S*-module with* $\widetilde{M} \neq 0$*. Suppose the degrees of all minimal generators of* M *are contained in* $\operatorname{Eff} X$*. Then* $\operatorname{reg} M \subseteq \operatorname{Eff} X$*.*

Proof. Take $\mathbf{d} \in \operatorname{reg} M$ and suppose for contradiction that $\mathbf{d} \notin \operatorname{Eff} X$. The degree $\mathbf{d}$ part $M_{\mathbf{d}}$ generates $M_{\geq \mathbf{d}}$ by [MS04, Thm. 5.4]. By hypothesis all elements of M have degrees inside $\operatorname{Eff} X$, so $M_{\mathbf{d}} = 0$ and thus $M_{\geq \mathbf{d}} = 0$. The modules M and $M_{\geq \mathbf{d}}$ define the same sheaf by [MS04, Lem. 6.8], so $M_{\geq \mathbf{d}} = 0$ contradicts $\widetilde{M} \neq 0$. □

Proposition 7.3.8. *Let M be a finitely generated graded torsion-free S-module with $\widetilde{M} \neq 0$. Suppose Γ is a chamber of $\operatorname{Eff} X \setminus \operatorname{Nef} X$. If $\mathbf{d} - \deg f_i \in \Gamma \setminus \operatorname{Nef} X$ for all generators f_i of M, then M is not $\mathbf{d}$-regular.*

Proof. Assume on the contrary that M is $\mathbf{d}$-regular. Let $\mathbf{a}_i = \mathbf{d} - \deg f_i$ for each i. By choice of $\mathbf{d}$ we have $\mathbf{a}_i \in \Gamma \setminus \operatorname{Nef} X$. Hence by Lemma 7.3.6 there exist monomials $m_i \in S_{\geq \mathbf{a}_i}$ such that $\prod_i m_i$ is not generated by the monomials of $S_{\sum \mathbf{a}_i}$. Consider the elements $m_i f_i \in M_{\geq \mathbf{d}}$.

Since M is $\mathbf{d}$-regular, the degree $\mathbf{d}$ part $M_{\mathbf{d}}$ generates $M_{\geq \mathbf{d}}$ by [MS04, Thm. 5.4]. Let $g_1, \ldots, g_s$ with $\deg g_j = \mathbf{d}$ be generators for $M_{\geq \mathbf{d}}$. Thus we must have relations

$$m_i f_i = \sum_j b_{i,j} g_j = \sum_j b_{i,j} \left(\sum_k a_{j,k} f_k \right) = \sum_k c_{i,k} f_k$$

for some $b_{i,j}, a_{j,k}, c_{i,k} \in S$ with $\deg b_{i,j} = \deg m_i - \mathbf{a}_i$ and $\deg a_{j,k} = \mathbf{a}_k$. These relations form a partial presentation matrix

$$A = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} - \begin{bmatrix} c_{1,1} & c_{2,1} & \cdots & c_{n,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1,n} & c_{2,n} & \cdots & c_{n,n} \end{bmatrix}. \tag{7.3.1}$$

for M. In particular, $\det(A) \in \operatorname{Fitt}_0 M \subseteq \operatorname{ann} M$ by [Eis95, Prop. 20.7], so $\det(A) M = 0$.

Since there are no zerodivisors on a torsion-free S-module, we must have $\det(A) = 0$, but this is impossible: note that $\det(A)$ contains the monomial $m = \prod_i m_i$ and that $\det(A) \in m + I$ for $I = \prod_k \langle c_{1,k}, c_{2,k}, \ldots, c_{n,k} \rangle$, then observe that $I \subseteq \prod_k \langle a_{1,k}, a_{2,k}, \ldots, a_{n,k} \rangle \subseteq S \otimes_{\mathbb{K}} S_{\sum \mathbf{a}_k}$ since $\deg a_{j,k} = \mathbf{a}_k$. Hence $\det(A) = 0$ implies $m \in I \subseteq S \otimes_{\mathbb{K}} S_{\sum \mathbf{a}_k}$ and contradicts our choice of m_i. □

Remark 7.3.9. Example 7.3.2 shows that Theorem 7.3.11 is not true without the torsion-free hypothesis. In practice, however, we only need that the element $\det A$ from (7.3.1) is a nonzerodivisor on M for some choice of m_i as in Lemma 7.3.6. Given a specific toric variety, this may be possible to verify directly in some cases where M is not torsion-free.

We will use the following technical lemma about the walls of $\operatorname{Nef} X$ to find a vector satisfying the hypotheses of Proposition 7.3.8.

Lemma 7.3.10. *Given $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \operatorname{Nef} X$ and $\mathbf{d} \in \operatorname{Eff} X \setminus \operatorname{Nef} X$, there exists a chamber Γ sharing a wall W with $\operatorname{Nef} X$ and $\mathbf{w}$ in the relative interior of W such that $\mathbf{d} + \mathbf{w} \in \Gamma$ and $\mathbf{d} + \mathbf{w} \in \mathbf{a}_i + \Gamma$ for all i.*

Proof. Consider the cone P defined by all rays of $\operatorname{Nef} X$ in addition to a primitive element along $\mathbf{d}$. Since $\operatorname{Nef} X \subsetneq P$, at least one wall W of $\operatorname{Nef} X$ must be in the interior of $P \subseteq \operatorname{Eff} X$. Let Γ be the chamber across W from $\operatorname{Nef} X$. Since $\mathbf{d} \notin \operatorname{Nef} X$, for each $\mathbf{w} \in W$ we have $\mathbf{d} + \mathbf{w} \notin \operatorname{Nef} X$.

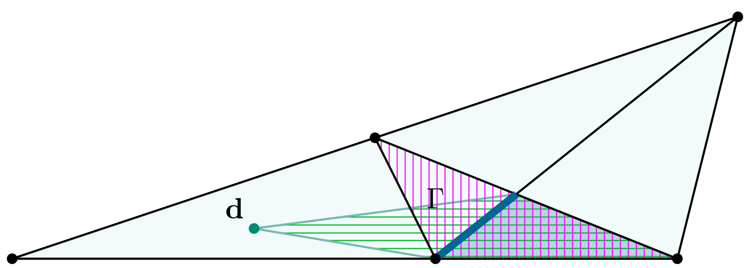


Figure 7.4: A section of a hypothetical chamber complex with P (green, horizontal) and Q (red, vertical) inside $\operatorname{Eff} X$. The chamber $\operatorname{Nef} X$ and its wall W are in blue.

Now consider the cone Q defined by all supporting hyperplanes of $\operatorname{Nef} X$ and Γ except the hyperplane containing W. Since W is in the intersection of the open half-spaces defining Q, it lies in the interior of Q. Therefore we can find $\mathbf{w}$ in the relative interior of $W \subset Q$ so that $\mathbf{d} + \mathbf{w} \in \mathbf{a}_i + Q \subseteq \mathbf{a}_i + (\Gamma \cup \operatorname{Nef} X)$ for all i. By hypothesis $\mathbf{a}_i + \operatorname{Nef} X \subseteq \operatorname{Nef} X$ so $\mathbf{d} + \mathbf{w} \notin \mathbf{a}_i + \operatorname{Nef} X$. Hence $\mathbf{d} + \mathbf{w} \in \mathbf{a}_i + \Gamma$ for all i. □

Theorem 7.3.11. *Let M be a finitely generated graded torsion-free S-module with $\widetilde{M} \neq 0$. Suppose the degrees of all minimal generators of M are contained in $\operatorname{Nef} X$. Then $\operatorname{reg} M \subseteq \operatorname{Nef} X$. In particular, $\operatorname{reg} M$ has finitely many minimal elements.*

Proof. Suppose there exists $\mathbf{d} \in \operatorname{reg} M \setminus \operatorname{Nef} X$. Since M satisfies the hypothesis of Proposition 7.3.7, we can assume that $\mathbf{d} \in \operatorname{Eff} X$. Using Lemma 7.3.10, we can find $\mathbf{w}$ in the relative interior of a wall separating Nef X and an adjacent chamber Γ such that $\mathbf{d}+\mathbf{w} \in \Gamma$ and $\mathbf{d}+\mathbf{w} \in \deg f_i + \Gamma$ for all i. It follows from Proposition 7.3.8 that $\mathbf{d}+\mathbf{w} \notin \operatorname{reg} M$, which is a contradiction because $\mathbf{w} \in \operatorname{Nef} X$ and $\operatorname{reg} M$ is invariant under positive translation by $\operatorname{Nef} X$.

The conclusion that $\operatorname{reg} M$ has finitely many minimal elements follows from Lemma 7.3.4. □

Corollary 7.3.12. *Let M be a finitely-generated torsion-free S-module. If $\deg f_i \in \mathbf{b} + \operatorname{Nef} X$ for all generators f_i of M then $\operatorname{reg} M \subseteq \mathbf{b} + \operatorname{Nef} X$.*

Example 7.3.13. Consider the Hirzebruch surface $\mathcal{H}_2$, with notation from Example 7.2.1, and let M be the torsion-free module with presentation

$$S(3,-3) \oplus S(2,-2) \oplus S(1,-2) \xleftarrow{\begin{pmatrix} x_0^5 x_1 & x_1^2 x_2^6 & x_1^2 x_2^5 \end{pmatrix}^T} S(0,-4).$$

Since the degrees of the generators are contained in $(-3,2) + \operatorname{Nef} \mathcal{H}_2$, by Corollary 7.3.12 the multigraded regularity of M is contained in a translate of the nef cone, illustrated in Figure 7.5.

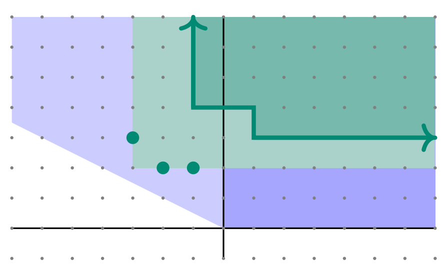


Figure 7.5: The multigraded regularity (dark green) of the module M is contained in a translate $(-3,2) + \operatorname{Nef} X$ (light green) of the nef cone of $\mathcal{H}_2$ (dark blue).

7.4 Powers of Ideals and Multigraded Regularity

Throughout this section let $I = \langle f_1, \ldots, f_s \rangle \subseteq S$ be an ideal and let $\mathbf{P}$ be the vector with coordinates $\mathbf{p}_i = \deg f_i \in \operatorname{Pic} X$. We are interested in the asymptotic behavior of the multigraded regularity of I^n as n increases. In particular, we prove the following theorem:

Theorem 7.4.1. *There exists a degree* $\mathbf{a} \in \operatorname{Pic} X$*, depending only on* I*, such that for each integer* $n > 0$ *and each pair of degrees* $\mathbf{q}_1, \mathbf{q}_2 \in \operatorname{Pic} X$ *satisfying* $\mathbf{q}_1 \geq \mathbf{p}_i \geq \mathbf{q}_2$ *for all* i*, we have*

$$n\mathbf{q}_1 + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^n) \subseteq n\mathbf{q}_2 + \operatorname{Nef} X.$$

Proof. The inner bound will follow from Proposition 7.4.8. The outer bound follows from Corollary 7.3.12 by noting that $\deg \prod_{j=1}^n f_{i_j} = \sum_{j=1}^n \mathbf{p}_{i_j} \in n\mathbf{q}_2 + \operatorname{Nef} X$ for all products of n choices of generators of I, and such products generate I^n. $\square$

Example 7.4.2. Let $I = \langle x_0x_3, x_1^2x_2^4 \rangle$ and $J = \langle x_3, x_0^3x_1 \rangle$ be two ideals in the total coordinate ring of the Hirzebruch surface $\mathcal{H}_2$, with notation as in Example 7.2.1. Figure 7.6 shows the multigraded regularity of powers of I and J along with the bounds from Theorem 7.4.1.

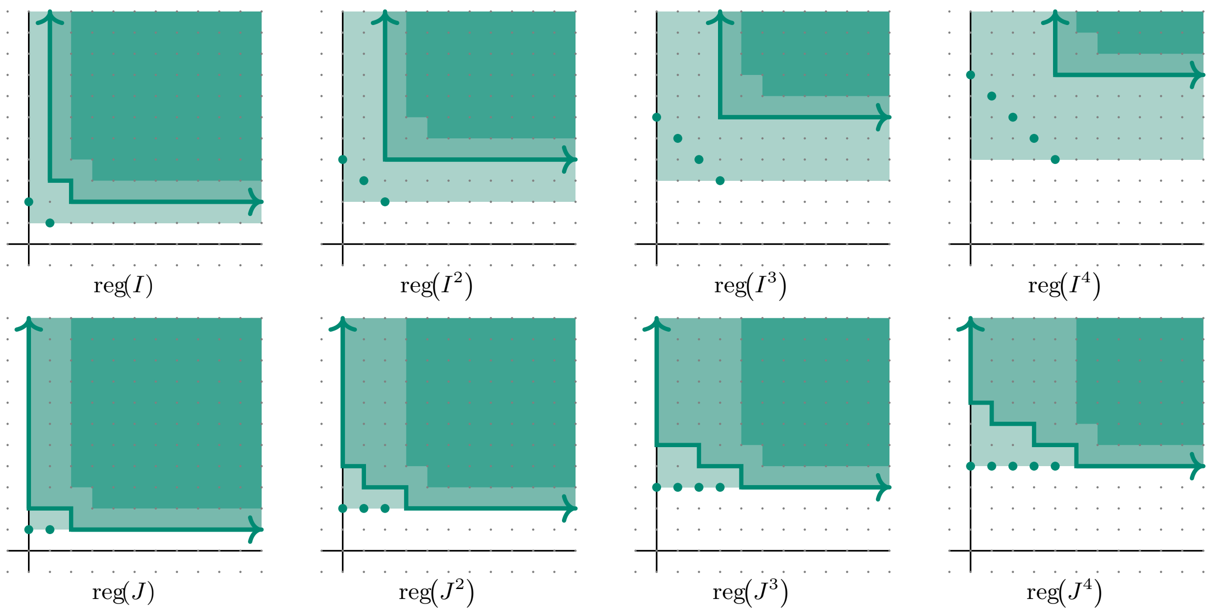


Figure 7.6: The inner (dark green) and outer (light green) bounds for powers of I and J. The circles correspond to the degrees of the generators of each power.

Remark 7.4.3. If $\mathbf{q}_2$ is not nef, then the bounds in Theorem 7.4.1 will not increase with n in the partial order on $\operatorname{Pic} X$. We can see that this behavior is necessary by taking I to be a principal ideal generated outside of $\operatorname{Nef} X$.

7.4.1 The Rees Ring

One way to find a subset of the regularity of a module is by using its multigraded Betti numbers. In order to describe $\operatorname{reg}(I^n)$, we would thus like a uniform description of the Betti numbers of I^n for all n. For this purpose, consider the multigraded Rees ring of I:

$$S[It] := \bigoplus_{n\geq 0} I^n t^n \subseteq S[t],$$

which is a $\operatorname{Pic}(X)\times\mathbb{Z}$-graded noetherian ring with $\deg ft^k = (\deg f, k)$ for $f \in S$. Let $R = S[T_1,\dots,T_s]$ be the $\operatorname{Pic}(X)\times\mathbb{Z}$-graded ring with $\deg(T_i) = (\deg f_i, 1) = (\mathbf{p}_i, 1)$. Notice that there is a surjective map of graded S-algebras:

$$\begin{aligned} R &\longrightarrow\!\!\!\!\!\rightarrow S[It] \\ T_i &\longmapsto f_i\, t \end{aligned}$$

Since R is a finitely generated standard graded algebra over S, taking a single degree of a finitely generated R-module in the auxiliary $\mathbb{Z}$ grading yields a finitely generated S-module.

Definition 7.4.4. For a $\operatorname{Pic}(X)\times\mathbb{Z}$-graded R-module M, define $M^{(n)}$ to be the $\operatorname{Pic}(X)$-graded S-module

$$M^{(n)} := \bigoplus_{\mathbf{a}\in\operatorname{Pic}X} M_{(\mathbf{a},n)}.$$

Following [Kod00], we record three important properties of this operation.

Lemma 7.4.5. *Consider the functor $-^{(n)}\colon M\mapsto M^{(n)}$ from the category of $\operatorname{Pic}(X)\times\mathbb{Z}$-graded R-modules to the category of $\operatorname{Pic}(X)$-graded S-modules.*

(i) $-^{(n)}$ is an exact functor.

(ii) $S[It]^{(n)} \cong I^n$.

(iii) $R(-\mathbf{a},-b)^{(n)} \cong R^{(n-b)}(-\mathbf{a}) \cong \bigoplus_{|\nu|=n-b} S(-\nu\cdot\mathbf{P}-\mathbf{a})$ where $\nu\in\mathbb{N}^s$.

Since $S[It]$ is a finitely generated module over the polynomial ring R, it has a finite free resolution. Applying $-^{(n)}$ gives a resolution by (i), which has cokernel I^n by (ii) and whose terms are finitely generated free S-modules by (iii). Thus we can constrain the Betti numbers of I^n in terms of those of $S[It]$.

7.4.2 Regularity of Powers of Ideals

Given a description of the Betti numbers of I^n in terms of n, we obtain an inner bound on $\operatorname{reg}(I^n)$ using the following lemma.

Lemma 7.4.6. *If $F_\bullet$ is a finite free resolution for M with $F_j = \bigoplus_i S(-\mathbf{a}_{i,j})$ and $H^0_B(M) = 0$ then*

$$\bigcap_{i,j} \bigcup_{|\lambda|=j} (\mathbf{a}_{i,j} - \lambda \cdot \mathbb{C} + \operatorname{reg} S) \subseteq \operatorname{reg} M \tag{7.4.1}$$

where $\mathbb{C} = (\mathbf{c}_1, \dots, \mathbf{c}_r)$ is the sequence of nef generators for X and the union is over $\lambda \in \mathbb{N}^r$.

Remark 7.4.7. This result amounts to switching the union and intersection in the statement of [MS04, Cor. 7.3] for modules with $H^0_B(M) = 0$, which increases the size of the subset by allowing a different choice of λ for each i, j.

Proof. Fix $\mathbf{d}$ in the left hand side of (7.4.1) and consider the hypercohomology spectral sequence for $F_\bullet$ (see [BCHS21, Thm. 4.14] for a description of this spectral sequence). We must show that M is $\mathbf{d}$-regular, meaning that $H^k_B(M)_{\mathbf{d}-\mu\cdot\mathbb{C}} = 0$ for all k and all μ with $|\mu| = k-1$. Since $F_\bullet$ is a resolution for M, a diagonal of our spectral sequence converges to $H^k_B(M)$. Thus it is sufficient to prove that this entire diagonal vanishes in degree $\mathbf{d} - \mu \cdot \mathbb{C}$, i.e. that

$$H^{k+j}_B(F_j)_{\mathbf{d}-\mu\cdot\mathbb{C}} = \bigoplus_i H^{k+j}_B(S(-\mathbf{a}_{i,j}))_{\mathbf{d}-\mu\cdot\mathbb{C}} = 0 \tag{7.4.2}$$

for all j. This is satisfied for $k = 0$ by hypothesis. Now fix $k > 0$, μ, j, and i. By choice of $\mathbf{d}$ we have $\mathbf{d} \in \mathbf{a}_{i,j} - \lambda \cdot \mathbb{C} + \operatorname{reg} S$ for some λ with $|\lambda| = j$, so that $\mathbf{d} - \mathbf{a}_{i,j} + \lambda \cdot \mathbb{C} \in \operatorname{reg} S$. Call this degree $\mathbf{d}'$, and let $\mathbf{c}' = (\lambda + \mu) \cdot \mathbb{C}$, where $|\lambda + \mu| = k + j - 1$. Then by the definition of the regularity of S we have $H^{k+j}_B(S)_{\mathbf{d}'-\mathbf{c}'} = 0$ where

$$\mathbf{d}' - \mathbf{c}' = \mathbf{d} - \mathbf{a}_{i,j} + \lambda \cdot \mathbb{C} - (\lambda + \mu) \cdot \mathbb{C} = \mathbf{d} - \mu \cdot \mathbb{C}.$$

Hence each summand in (7.4.2) is zero for $k > 0$, as desired. □

Proposition 7.4.8. *There exists a degree $\mathbf{a} \in \operatorname{Pic} X$, depending only on the Rees ring of I, such that for each integer $n > 0$ and degree $\mathbf{q} \in \operatorname{Pic} X$ satisfying $\mathbf{q} \geq \deg f_i$ for all homogeneous generators f_i*

of I, we have

$$n\mathbf{q} + \mathbf{a} + \operatorname{reg} S \subseteq \operatorname{reg}(I^n).$$

Proof. Let $F_\bullet$ be a minimal $\operatorname{Pic}(X) \times \mathbb{Z}$-graded free resolution of $S[It]$ as an R-module, and write $F_j = \bigoplus_i R(-\mathbf{a}_{i,j}, -b_{i,j})$ for $\mathbf{a}_{i,j} \in \operatorname{Pic} X$ and $b_{i,j} \in \mathbb{Z}$. By Lemma 7.4.5, applying the $-^{(n)}$ functor to $F_\bullet$ yields a (potentially non-minimal) resolution of $S[It]^{(n)} \cong I^n$ consisting of free S-modules

$$F_j^{(n)} \cong \bigoplus_i R(-\mathbf{a}_{i,j}, -b_{i,j})^{(n)} \cong \bigoplus_i \left[\bigoplus_{|\nu|=n-b_{i,j}} S(-\nu \cdot \mathbf{P} - \mathbf{a}_{i,j}) \right],$$

where $\mathbf{P} = (\deg f_1, \ldots, \deg f_s)$ is the sequence of degrees of the homogeneous generators f_i of I. From this Lemma 7.4.6 gives the following bound on the regularity of I^n:

$$\bigcap_{\substack{i,j \\ |\nu|=n-b_{i,j}}} \bigcup_{|\lambda|=j} [\nu \cdot \mathbf{P} + \mathbf{a}_{i,j} - \lambda \cdot \mathbb{C} + \operatorname{reg} S] \subseteq \operatorname{reg}(I^n). \tag{7.4.3}$$

Note that $b_{0,0} = 0$, as $S[It]$ is a quotient of R, and thus $b_{i,j} \geq 0$ for all i, j, as R is positively graded in the $\mathbb{Z}$ coordinate.

Take $\mathbf{a} \in \operatorname{Pic} X$ so that $\mathbf{a} \geq \mathbf{a}_{i,j}$ for all i, j. There are only finitely many $\mathbf{a}_{i,j}$ because $S[It]$ is a finitely generated R-module and R is noetherian. We may now simplify the left hand side of (7.4.3) by noting three things: (i) for all $|\lambda| = j$ and all j we have $\operatorname{reg} S \subseteq -\lambda \cdot \mathbb{C} + \operatorname{reg} S$, (ii) if $|\nu| = n - b_{i,j}$ then $(n - b_{i,j})\mathbf{q} \in \nu \cdot \mathbf{P} + \operatorname{reg} S$, and (iii) for all i and all j we have $n\mathbf{q} + \mathbf{a} \in (n - b_{i,j})\mathbf{q} + \mathbf{a}_{i,j} + \operatorname{reg} S$. Combining these facts gives that

$$\begin{aligned}
\operatorname{reg}(I^n) &\supseteq \bigcap_{\substack{i,j \\ |\nu|=n-b_{i,j}}} \bigcup_{|\lambda|=j} [\nu \cdot \mathbf{P} + \mathbf{a}_{i,j} - \lambda \cdot \mathbb{C} + \operatorname{reg} S] \\
&\supseteq \bigcap_{\substack{i,j \\ |\nu|=n-b_{i,j}}} [\nu \cdot \mathbf{P} + \mathbf{a}_{i,j} + \operatorname{reg} S] \\
&\supseteq \bigcap_{i,j} [(n - b_{i,j})\mathbf{q} + \mathbf{a}_{i,j} + \operatorname{reg} S] \\
&\supseteq n\mathbf{q} + \mathbf{a} + \operatorname{reg} S.
\end{aligned}$$

□

A similar problem is to characterize the asymptotic behavior of regularity for symbolic powers of I. Note that the symbolic Rees ring of I is not necessarily noetherian (see [GS21], for instance), so our argument for the existence of the degree $\mathbf{a}$ in the proof of Proposition 7.4.8 does not work in this case. More generally, if $\mathcal{I} = \{I_n\}$ is a filtration of ideals, then one may ask for sufficient conditions so that $\operatorname{reg}(I_n)$ is uniformly bounded.

8 Computing Direct Sum Decompositions

The material in this chapter will appear in a forthcoming joint work with Devlin Mallory.

8.1 Introduction

The problems of finding isomorphism classes of indecomposable modules with certain properties, or determining the indecomposable summands of a module, are ubiquitous in commutative algebra, group theory, representation theory, and other fields. Within commutative algebra, for instance, the classification of Cohen–Macaulay local rings R for which there are only finitely many indecomposable maximal Cohen–Macaulay R-modules (the *finite CM-type* property), or determining whether iterated Frobenius pushforwards of a Noetherian ring in positive characteristic have finitely many isomorphism classes of indecomposable summands (the *finite F-representation type* property) are two well-established research problems. For both these problems, and many others, making and testing conjectures depends on finding summands of modules and verifying their indecomposability.

Currently there are no efficient algorithms available for checking indecomposability or finding summands of modules over commutative rings. In contrast, variants of the "Meat-Axe" algorithm for determining irreducibility of finite-dimensional modules over a group algebra have wide ranging applications in computational group theory [Par84, HR94, Hol98] and are available through symbolic algebra software such as Magma and GAP [MAGMA, GAP].

The purpose of this paper is to describe and prove correctness of a practical algorithm for computing indecomposable summands of finitely generated modules over a finitely generated k-algebra, for k a field of positive characteristic. In particular, our algorithm works over multigraded rings, which enables the computation of indecomposable summands of coherent sheaves on subvarieties of toric varieties (in particular, for varieties embedded in projective space). After describing the algorithm and proving its correctness, we present multiple examples in the end, including some which present previously unknown phenomena regarding the behavior of summands of Frobenius pushforwards and syzygies over Artinian rings.

An accompanying implementation in Macaulay2 is available online via the GitHub repository

https://github.com/mahrud/DirectSummands.

Remark 8.1.1. Although the algorithm described below is only proved to result (probabilistically) in a decomposition into indecomposable summands in positive characteristic, in practice it often does produce nontrivial indecomposable decompositions even in characteristic 0. Moreover, if a module over a ring of characteristic 0 is decomposable, its reductions modulo p will be as well; thus, our algorithm provides a heuristic for verifying decomposability in characteristic 0.

We also point out that while the discussion below, and our implementation in Macaulay2, concerns the case where R is a *commutative* k-algebra, many of the techniques extend beyond this case; we plan to extend the results and algorithms to noncommutative rings such as Weyl algebras in future.

8.2 The Main Algorithm

Throughout, R will be an $\mathbb{Z}^r$-graded ring with $R_0 = k$ a field of positive characteristic and homogeneous maximal ideal m. Note that if M is a finitely generated R-module, and $A \in \operatorname{End}_R(M)$, then A acts on the k-vector space M/mM.

We begin with the observation that if $A \in \operatorname{End}_R(M)$ is an idempotent, then M decomposes as $\operatorname{im} A \oplus \ker A$; if A is nontrivial (i.e., neither an isomorphism nor the zero morphism), then M is decomposable. The following lemma allows us to check only for idempotents modulo the maximal ideal:

Lemma 8.2.1. *Let M be an R-module, and let $A \in \operatorname{End}_R(M)$. If the induced action of A on M/mM is idempotent, then M admits a direct sum decomposition $N \oplus M/N$, where $N \cong \operatorname{im} A$.*

Proof. By assumption, we can write $A^2 = A + B$, where $B \in \operatorname{End}_R(M)$ with $B(M) \subset mM$. Note that if $n \in m^k M$, then $A^2(n) - A(n) = B(n)$ lies in $m^{k+1}M$.

Let $N = \operatorname{im} A$. Consider the composition $N \hookrightarrow M \xrightarrow{A} \operatorname{im}(A) = N$. We claim that this composition is surjective. We may complete at the maximal ideal to check this, and thus assume R and M are complete. Let $n_0 \in N$. By assumption, $n_0 = A(m_1)$ for some $m_1 \in M$. Applying A again, we get

$$A(n_0) = A^2(m_1) = A(m_1) + n_1 = n_0 + n_1,$$

or equivalently $n_0 = A(n_0) - n_1$ for some $n_1 \in mM$. In fact, since n_0 and $A(n_0)$ are both in N, we have $n_1 \in N$ also, so $n_1 \in mM \cap N$.

Thus, we can write $n_1 = A(m_2)$ for $m_2 \in M$. Now, apply A to both sides: by the assumption that A is idempotent modulo m, we have

$$A(n_1) = A^2(m_2) = A(m_2) + n_2 = n_1 + n_2,$$

Thus, $n_2 = A(n_1) - n_1$, so $n_2 \in m^2M$; clearly also $n_2 \in N$ as well. Combining the previous equations, we can write

$$n_0 = A(n_0) - n_1 = A(n_0) - A(n_1) + n_2 = A(n_0 - n_1) + n_2,$$

with $n_1 \in mM \cap N$ and $n_2 \in m^2M \cap N$.

Continuing in this fashion, for any k we can write

$$n_0 = A(n_0 - n_1 + \cdots \pm n_k) \mp n_{k+1},$$

with $n_i \in m^iM \cap N$.

By the Artin–Rees lemma, there's some positive integer k such that for $n \gg 0$ we can write

$$m^nM \cap N = m^{n-k}(m^kM \cap N) \subset m^{n-k}N.$$

That is, the terms of $n_0 - n_1 + n_2 - \ldots$ are going to 0 in the m-adic topology on N. Thus, we write

$$n_0 = A(n_0 - n_1 + n_2 - \ldots),$$

with $n_0 - n_1 + n_2 - \cdots \in N$. Thus, we have that A is surjective as a map $N \to N$.

Since a surjective endomorphism of finitely generated modules is invertible, we have that this composition is an isomorphism on N; say α. Finally, we then have that

$$0 \hookrightarrow N \hookrightarrow M$$

is split by the map of R-modules $M \xrightarrow{A} N \xrightarrow{\alpha^{-1}} N$. $\square$

Lemma 8.2.2. *Let k be a finite field of characteristic p, and let A be an endomorphism of a k-vector space such that all eigenvalues of A are contained in k. If λ is an eigenvalue of A, then some power of $A - \lambda I$ is an idempotent. Furthermore, if λ is not the only eigenvalue of A, then a power of $A - \lambda I$ that is nonzero is an idempotent.*

Proof. Since all eigenvalues of A are contained in k, we can without loss of generality put A in Jordan canonical form, with each Jordan block having the form

$$r_i\left\{\underbrace{\begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}}_{r_i}\right.$$

with each λ_i an eigenvalue of A. In this basis, $A - \lambda I$ will be block-diagonal with blocks of form

$$\begin{pmatrix} \lambda_i - \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda_i - \lambda & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i - \lambda \end{pmatrix}$$

Set $\mu_i = \lambda_i - \lambda$. Then $(A - \lambda I)^l$ is block-diagonal with blocks of the form

$$\begin{pmatrix} \mu_i^l & \binom{l}{1}\mu_i^{l-1} & \binom{l}{2}\mu_i^{l-2} & \dots & \binom{l}{r_i}\mu_i^{l-r_i} \\ 0 & \mu_i^l & \binom{l}{1}\mu_i^{l-1} & \dots & \binom{l}{r_i-1}\mu_i^{l-r_i+1} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \mu_i^l \end{pmatrix}$$

If $l > r_i$ and l divisible by p, then all non-diagonal terms will vanish, so all blocks will have the form

$$\begin{pmatrix} \mu_i^l & 0 & 0 & \dots & 0 \\ 0 & \mu_i^l & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \mu_i^l \end{pmatrix}.$$

Finally, choosing l to be divisible also by $p^e - 1$ for some $e > 0$, we have that

$$\mu_i^{l-1} = (\mu_i^{p^e-1})^{l/(p^e-1)} = 1,$$

since μ_i is contained in a finite field of characteristic p. Thus, $(A - \lambda I)^l$ is simply a diagonal matrix with 1 or 0 on the diagonal, hence idempotent. Note moreover that if some $\lambda_i \neq \lambda$, then $(A - \lambda I)^l$ is not the zero matrix. □

This leads to a probabilistic algorithm for finding the indecomposable summands of a module M:

1. Take a general element A_0 of $[\operatorname{End} M]_0$, the degree-0 part of $\operatorname{End} M$, and consider the resulting endomorphism A of the k-vector space M/mM.
2. Find the eigenvalues of A.
3. If A has at least two eigenvalues, choose one eigenvalue λ, and compute a sufficiently high power of A (with the power explicitly as in the proof of Lemma 8.2.2). This power will be a nonzero idempotent, and thus produce a splitting of M as $\operatorname{im} A_0 \oplus \operatorname{coker} A_0$ by Lemma 8.2.1.
4. Repeat steps (1)–(3) for both $\operatorname{im} A_0$ and $\operatorname{coker} A_0$.

The following observation implies that if M is indecomposable, then the above algorithm should find the decomposition of M:

Lemma 8.2.3. *If M is not indecomposable, then a general degree-0 endomorphism of M reduces to an endomorphism of M/mM with at least two distinct eigenvalues.*

Remark 8.2.4. By "general" we mean that a general linear combination of a basis for $[\operatorname{End}(M)]_0$ over the algebraic closure of k, or equivalently over a sufficiently large algebraic extension of k.

Proof. We may assume that the base field k is algebraically closed. Let $\Phi_1, \dots, \Phi_r$ be a basis for $[\mathrm{End}(M)]_0$, and $\phi_1, \dots, \phi_r$ their images modulo m, which we view as matrices with entries in k. Let $U \subset \mathbb{A}^r$ be the subset of r-tuples $(\lambda_1, \dots, \lambda_r)$ such that $\lambda_1\phi_1 + \cdots + \lambda_r\phi_r$ has at least two distinct eigenvalues, i.e., such that $\lambda_1\Phi_1 + \cdots + \lambda_r\Phi_r$ reduces to an endomorphism of M/mM with at least two distinct eigenvalues.

It suffices to show that U is a nonempty open subset of $\mathbb{A}^r$. First, we show U is nonempty: Say $M = M_1 \oplus M_2$, with M_1 a nontrivial indecomposable summand. We may choose Φ_1 to be the projection to M_1, and Φ_2 the projection to M_2. Then for any $\lambda_1, \lambda_2 \in k$, $\lambda_1\phi_1 + \lambda_2\phi_2$ has eigenvalues λ_1, λ_2; thus in particular there is *an* element of $[\mathrm{End}(M)]_0$ reducing to an endomorphism of M/mM with distinct eigenvalues, so U is nonempty.

Now, we show that U is open. This is a purely linear algebraic statement: we claim that given a matrix ϕ with at least two distinct eigenvalues, and any r matrices $\phi_1, \dots, \phi_r$, that

$$A_{\lambda_1,\dots,\lambda_r} := \phi + \lambda_1\phi_1 + \cdots + \lambda_r\phi_r$$

has at least two distinct eigenvalues for $\lambda_1, \dots, \lambda_r$ outside a Zariski-closed subset of $\mathbb{A}^r$. The eigenvalues of $A_{\lambda_1,\dots,\lambda_r}$ are the roots of $\det(A_{\lambda_1,\dots,\lambda_r} - tI)$, which is of course a polynomial in t with coefficients in $\lambda_1, \dots, \lambda_r$. $A_{\lambda_1,\dots,\lambda_r}$ fails to have at least two distinct eigenvalues exactly when this polynomial factors as a power of a linear term. This is a polynomial condition in the coefficients of powers of t in $\det(A_{\lambda_1,\dots,\lambda_r} - tI)$ and thus in the λ_i; to see this, note that if $f := t^n b_n + \cdots + tb_1 + b_0$ has an n-fold root exactly when $f, \partial f/\partial t, \dots, \partial^n f/\partial t^n$ vanish simultaneously; the resultant of these n polynomials in n equations gives polynomial conditions in the b_i for this to occur; in our setting, the b_i are themselves polynomials in the λ_i, and thus we have obtained polynomial equations defining the locus where $A_{\lambda_1,\dots,\lambda_r}$ fails to have distinct roots. □

Remark 8.2.5. Note that the above algorithm is quite sensitive to the ground field k, because it needs an eigenvalue of the endomorphism A of M/mM to be contained in k. While theoretically the issue can be avoided by working over an algebraically closed ground field $\overline{k}$, for practical use on a computer algebra system it is better to extend k to some larger finite field. However, the general linear combinations we take in Step 1 should be taken with respect to the prime subfield (otherwise, as we increase the size of the finite field k, the eigenvalues of a general linear combination will live in higher and higher field extensions). See Example 8.3.3 for a demonstration of the necessity of extending the base field.

If the above algorithm fails to produce a nontrivial idempotent, it does not certify that M is indecomposable. However, there are a few sufficient conditions to be indecomposable, which in practice often (but not always) produce such a certification. The following sufficiency condition is immediate, but can be quite useful in practice for verifying indecomposability:

Lemma 8.2.6. *Let M be a finitely generated R-module and let $[\operatorname{End} M]_0$ be the k-vector space of degree-0 endomorphisms. Suppose that either:*

1. *$[\operatorname{End} M]_0$ is 1-dimensional and thus spanned by the identity endomorphism, or*
2. *every non-identity element of $[\operatorname{End} M]_0$, viewed as a matrix, has all entries contained in m;*

then M is indecomposable.

Proof. If M decomposes non-trivially as $M_1 \oplus M_2$, then the projections onto each factor are nontrivial degree-0 endomorphisms not equal to the identity, and which does not have entries contained in m. □

This is the local analog of the following fact about indecomposability of coherent sheaves:

Corollary 8.2.7. *Let X be a projective variety over a field k, and $\mathcal{F}$ a coherent sheaf on X. If $H^0(\operatorname{End} \mathcal{F}) = k$, then $\mathcal{F}$ is indecomposable.*

8.3 Examples

While the preceding section was written in the language of modules, by the standard translation to global (multi)projective varieties, it works equally well to find indecomposable decompositions of coherent sheaves. In this section, we give examples of the kind of calculations and observations the algorithm from the previous section allows us to make.

Example 8.3.1 (Frobenius pushforward on the projective space $\mathbb{P}^n$)**.** Let $S = k[x_0, \ldots, x_n]$ be a polynomial ring with $\operatorname{char} k = p$ and $\deg x_i = 1$ and consider the Frobenius endomorphism

$$F \colon S \to S \quad \text{given by} \quad f \to f^p.$$

Hartshorne [Har70] proved that for any line bundle $L \in \operatorname{Pic} \mathbb{P}^n$, the Frobenius pushforward F_*L splits as a sum of line bundles. While the following calculations are straightforward to do by hand, they are immediately calculated via our algorithm:

$$\text{When } p = 3, n = 2:$$
$$F_*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O} \oplus \mathcal{O}(-1)^7 \oplus \mathcal{O}(-2).$$
$$\text{When } p = 2, n = 5:$$
$$F_*\mathcal{O}_{\mathbb{P}^5} = \mathcal{O} \oplus \mathcal{O}(-1)^{15} \oplus \mathcal{O}(-2)^{15} \oplus \mathcal{O}(-3),$$
$$F_*^2\mathcal{O}_{\mathbb{P}^5} = \mathcal{O} \oplus \mathcal{O}(-1)^{120} \oplus \mathcal{O}(-2)^{546} \oplus \mathcal{O}(-3)^{336} \oplus \mathcal{O}(-4)^{21}.$$

Example 8.3.2 (Frobenius pushforward on toric varieties)**.** Let X be a smooth toric variety and consider its Cox ring

$$S = \bigoplus_{[D] \in \operatorname{Pic} X} \Gamma(X, \mathcal{O}(D)).$$

Similar to the case of the projective space, Bøgvad and Thomsen [Bøg98, Tho00] showed that F_*L totally splits as a direct sum of line bundles for any line bundle $L \in \operatorname{Pic} X$.

As an example, consider the third Hirzebruch surface $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$ over a field of characteristic 3. We have, for example, that

$$F_*\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{O}_X(-1,0)^2 \oplus \mathcal{O}_X(0,-1)^2 \oplus \mathcal{O}_X(1,-1)^3 \oplus \mathcal{O}_X(2,-1),$$
$$F_*\mathcal{O}_X(1,1) = \mathcal{O}_X^3 \oplus \mathcal{O}_X(-1,0) \oplus \mathcal{O}_X(1,-1) \oplus \mathcal{O}_X(1,0)^2 \oplus \mathcal{O}_X(2,-1)^2.$$

In fact, Achinger [Ach15] showed that the total splitting of F_*L for every line bundle L characterizes smooth projective toric varieties.

Example 8.3.3 (Frobenius pushforward on elliptic curves)**.** Consider the elliptic curve

$$X = \operatorname{Proj} \mathbb{F}_7[x, y, z]/(x^3 + y^3 + z^3).$$

This is an ordinary elliptic curve, hence F-split; thus $\mathcal{O}_X$ is a summand of $F_*\mathcal{O}_X$. Over the algebraic closure of $\mathbb{F}_7$, $F_*\mathcal{O}_X$ will decompose as $\bigoplus_{p=1}^7 \mathcal{O}_X(p_i)$, where $p_1, \ldots, p_7$ are the 7-torsion points of X.

However, over $\mathbb{F}_7$, our algorithm calculates that $F_*\mathcal{O}_X$ decomposes only as

$$F_*\mathcal{O}_X = \mathcal{O}_X \oplus M_1 \oplus M_2 \oplus M_3,$$

with M_i indecomposable (over $\mathbb{F}_7$) of rank 2.

If one extends the ground field to $\mathbb{F}_{49}$, however, our algorithm calculates the full decomposition

$$F_*\mathcal{O}_X = \bigoplus_{p=1}^{7} \mathcal{O}_X(p_i).$$

This reflects the fact that the 7-torsion points p_i of X, and thus the sheaves $\mathcal{O}_X(p_i)$, are not defined over $\mathbb{F}_7$, but they are defined over $\mathbb{F}_{49}$.

Example 8.3.4 (Frobenius pushforward on Grassmannians)**.** Consider the Grassmannian $X = \mathrm{Gr}(2,4)$. We may work over the Cox ring S, which in this case coincides with the coordinate ring

$$S = \frac{k[p_{0,1}, p_{0,2}, p_{0,3}, p_{1,2}, p_{1,3}, p_{2,3}]}{p_{1,2}p_{0,3} - p_{0,2}p_{1,3} + p_{0,1}p_{2,3}}.$$

Then in characteristic $p = 3$ we have:

$$F_*\mathcal{O}_X = \mathcal{O} \oplus \mathcal{O}(-1)^{44} \oplus \mathcal{O}(-2)^{20} \oplus A^4 \oplus B^4,$$

where A and B are rank-2 indecomposable bundles (c.f. [RŠV22]).

Example 8.3.5 (Frobenius pushforward on Mori Dream Spaces)**.** Continuing with the theme of computations over the Cox ring, the natural geometric setting is to consider the class of projective varieties known as Mori dream spaces [HK00].

For instance, consider $X = \mathrm{Bl}_4\mathbb{P}^2$, the blowup of $\mathbb{P}^2$ at 4 general points. We will working over the $\mathbb{Z}^5$-graded Cox ring

$$S = k[x_1, \ldots, x_{10}]/(\text{five quadric Plücker relations})$$

with degrees

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

Then in characteristic 2 we have:

$$\begin{aligned} F_*^2\mathcal{O}_X = \mathcal{O}_X^1 &\oplus \mathcal{O}_X^2\ (-2, 1, 1, 1, 1) \oplus \mathcal{O}_X^2\ (-1, 0, 0, 0, 1) \\ &\oplus \mathcal{O}_X^2\ (-1, 0, 0, 1, 0) \oplus \mathcal{O}_X^2\ (-1, 0, 1, 0, 0) \\ &\oplus \mathcal{O}_X^2\ (-1, 1, 0, 0, 0) \oplus B \oplus G, \end{aligned}$$

where B, G are rank-3 and rank-2 indecomposable modules, as calculated in [Har15].

Example 8.3.6 (Frobenius pushforward on cubic surfaces)**.** Let X be a smooth cubic surface. Aside from a single exception in characteristic 0, X will be globally F-split, so that any $F_*^e\mathcal{O}_X$ admits $\mathcal{O}_X$ as a direct summand. The other summands of Frobenius pushforwards of $\mathcal{O}_X$ have yet to be studied, and in particular it is not known whether such rings should have the finite F-representation type property.

The use of our algorithm to compute examples in small p and e suggest the following behavior:

$$F_*\mathcal{O}_X = \mathcal{O}_X \oplus M,$$

with M indecomposable, and furthermore F_e^*M remains indecomposable for all $e \geq 0$. In other words, the indecomposable decomposition of $F_*^e\mathcal{O}_X$ is

$$F_*^e\mathcal{O}_X \cong \mathcal{O}_X \oplus M \oplus F_*M \oplus \cdots \oplus F_*^{e-1}M.$$

In particular, $\mathcal{O}_X$ will fail to have the finite F-representation type property. In fact, we believe a similar description holds true for quartic del Pezzos, which arise as an intersection of quadrics in P^4.

Example 8.3.7 (Syzygies over Artinian rings)**.** Let $R = k[x, y]/(x^3, x^2y^3, y^5)$ and consider the (infinite) minimal free resolution of the residue field, which has rank 2^n in homological index n. In forthcoming work based in part on examples calculated using our algorithm, Dao, Eisenbud, and Polini study the indecomposable summands of syzygy modules in examples such as this one, showing unexpected periodicity behavior, and in particular proving that the syzygy modules are direct sums of only three indecomposable modules: the residue field k, the maximal ideal m, and an additional module M.

For example, the fourth syzygy module decomposes (ignoring the grading) as the direct sum

$$k^3 \oplus m^2 \oplus M^3.$$

and the fifth syzygy module as

$$k^8 \oplus m^9 \oplus M^2.$$

The use of our algorithm was essential to the observation that only the one additional module M beyond the “guaranteed” summands of k and m appealrs.

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