SPLITTING OF VECTOR BUNDLES VIA FOURIER–MUKAI THEORY

MAHRUD SAYRAFI

Abstract. A major program in algebraic geometry is the study of vector bundles on algebraic varieties. Intuitively, vector bundles over a space are analogous to finitely generated projective modules over a ring. In 1956, Grothendieck proved that any vector bundle E on $\mathbb{P}^1$ splits as a direct sum $E = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i)$ for some $d_i \in \mathbb{Z}$. In contrast, for $n \geq 3$ there are indecomposable vector bundles of rank $n-1$ on $\mathbb{P}^n$. Still, most of the literature in this area is concentrated on vector bundles on the projective space.

After a short survey of the landscape, this note proposes an apparatus for resolving a number of questions in combinatorial algebraic geometry towards studying derived categories of toric varieties, which generalize projective spaces, and finding a splitting criterion for vector bundles on them.

Contents

Key words and phrases. Commutative Algebra, Toric Geometry, Derived Category, Fourier–Mukai.

1. Introduction

Let S be the polynomial ring in $n+1$ variables over an algebraically closed field $\Bbbk$. A (geometric) vector bundle E of rank r on $\mathbb{P}^n = \operatorname{Proj} S$ is specified by a locally free (coherent) sheaf $\mathscr{E}$ on $\mathbb{P}^n$ associated to a finitely generated graded S-module M. This correspondence, together with the minimal free resolution of M, provides a wealth of invariants for E, such as the Castelnuovo–Mumford regularity, defined as the smallest twist d for which the cohomology $\mathrm{H}^i\big(\mathbb{P}^n, \mathscr{E}(d-i)\big)$ vanishes for every $i \geq 1$.

A prominent problem in the study of vector bundles is the construction of indecomposable bundles of low rank on algebraic varieties. Classically, this is accomplished by first considering a *monad*, which is a complex of vector bundles

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

that is exact except at B, and considering the *cohomology of the monad* $E = \ker\beta / \operatorname{im}\alpha$. Monads were used in constructing the Horrocks–Mumford bundle [HM73; Hul95], which is an indecomposable rank 2 bundle on $\mathbb{P}^4$. Existence of such a bundle on $\mathbb{P}^n, n \geq 5$ is unknown.

Such questions are often connected to commutative algebra in surprising ways. For instance, decomposability of every rank 2 vector bundle on $\mathbb{P}^n$ for $n \geq 7$ is equivalent to a conjecture of Hartshorne that every smooth subvariety of codimension 2 is a complete intersection.

Beginning in the late seventies, Bernstein–Gel'fand–Gel'fand [BGG78], Beilinson [Bei78], Buchweitz [Buc86], Gorodentsev–Rudakov [GR87], Kapranov [Kap88], Bondal [Bon90], and Orlov [Orl93] developed an apparatus for studying the derived category of bounded complexes of coherent sheaves on X, denoted $\mathcal{D}^{\mathrm{b}}(X)$, using collections of *exceptional* bundles $\{E_i\}$. Notably, Beilinson described the derived category $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ using *full strong exceptional collections* which generate the derived category, one given by twists of the structure sheaf

$$\mathcal{O}_{\mathbb{P}^n},\ \mathcal{O}_{\mathbb{P}^n}(1),\ \ldots,\ \mathcal{O}_{\mathbb{P}^n}(n) \tag{1.1}$$

and the other by the collection consisting of exterior products of the cotangent bundle

$$\mathcal{O}_{\mathbb{P}^n},\ \Omega_{\mathbb{P}^n}(1),\ \ldots,\ \Omega^n_{\mathbb{P}^n}(n). \tag{1.2}$$

The point of view of Bondal [Bon90] involves representations of a *bound quiver* (Q, R) arising from the endomorphisms of a *tilting sheaf* $T \in \mathbf{Coh}(X)$ satisfying $\operatorname{Ext}^i_X(T,T) = 0$ for $i > 0$. The direct sum of objects in a full strong exceptional collection, for instance, is a tilting bundle which generates $\mathcal{D}^{\mathrm{b}}(X)$; i.e., the smallest thick subcategory of $\mathcal{D}^{\mathrm{b}}(X)$ containing T is all of $\mathcal{D}^{\mathrm{b}}(X)$. In this case, there is an equivalence of categories between $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}})$, the category of bounded complexes of finite-dimensional right modules on the *path algebra*

$$A = \Bbbk Q/R = \operatorname{End}_X(T) = \bigoplus_{i,j=1}^{n} \operatorname{Hom}_X(E_i, E_j),$$

given by a pair of adjoint functors called the *derived Hom* and the *derived tensor product*

$$\begin{aligned} \mathbf{R}\operatorname{Hom}_X(\ T\ ,\ -\ )\colon\ \mathcal{D}^{\mathrm{b}}(X) &\longrightarrow \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) \\ -\ \otimes^{\mathbf{L}}_A T\ :\ \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) &\longrightarrow \mathcal{D}^{\mathrm{b}}(X). \end{aligned}$$

These functors will be made explicit in Section 7 through the lens of Fourier–Mukai theory, using a *resolution of the diagonal* as the kernel of a *Fourier–Mukai transform* $\mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(X)$, where complexes on each side are written in terms of different exceptional collections.

Another perspective on $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ comes from the Bernstein–Gel'fand–Gel'fand (BGG) correspondence [BGG78], which compares complexes of S-modules with complexes of modules over the associated exterior algebra E. This correspondence defines a natural and explicit exact equivalence of triangulated categories

$$\mathbb{L}\colon\ \mathcal{D}^{\mathrm{b}}(E) \rightleftharpoons \mathcal{D}^{\mathrm{b}}(S)\ :\mathbb{R}$$

where $\mathcal{D}^{\mathrm{b}}(E)$ denotes the category of bounded complexes of graded, finitely generated right modules over E. Remarkably, this machinery provides a method for calculating the cohomology of all twists of a coherent sheaf $\mathscr{F}$ via a resolution whose i-th term is isomorphic to

$$\bigoplus_{j\in\mathbb{Z}} \mathrm{H}^{i+j}\big(\mathbb{P}^n, \mathscr{F}(-j)\big) \otimes \Omega^j(j). \tag{1.3}$$

The information about all twists of $\mathscr{F}$ can be read from a doubly infinite complex over the exterior algebra known as the *Tate resolution* [DE02; EFS03]. This technique gives an efficient algorithm for evaluating the following criterion for splitting of vector bundles.

Theorem 1.1 (Horrocks' splitting criterion)**.** *Let E be a vector bundle on $\mathbb{P}^n$ corresponding to a locally free sheaf $\mathscr{E}$. If the cohomology modules $\mathrm{H}^i\big(\mathbb{P}^n, \mathscr{E}(d)\big)$ for all twists are equal to the cohomology of positive sums of line bundles, then E splits as a direct sum of line bundles.*

The overarching goal of this note is to view the results above through the lens of toric varieties and generalize the techniques for studying vector bundles using the language of derived categories. A first question is captured in a conjecture of A. King in [Kin97].

Conjecture 1.2 (King's Conjecture)**.** *Let X be a smooth complete toric variety. Then X has a tilting bundle whose summands are line bundles.*

Eisenbud, Erman, and Schreyer generalized the theory of Tate resolutions to products of projective spaces and, under an additional hypothesis, extended Horrocks' splitting criterion to that setting [EES15]. Since products of projective spaces can be interpreted as a sequence of projectivizations of line bundles, as in Example 2.8, a natural question is whether there is a splitting criterion for vector bundles over any such toric varieties. Promising evidence in this direction is given by Costa and Miró-Roig's proof that King's conjecture 1.2 holds true for this class of toric varieties [CM04]. This result motivates the following project.

Problem 1.3. Extend Horrocks' splitting criterion to sequences of projectivizations on $\mathbb{P}^n$.

In a forthcoming paper, Brown, Eisenbud, Erman, and Schreyer extend the BGG correspondence to simplicial projective toric varieties by taking advantage of a Fourier–Mukai transform, which opens the door to generalizations of this problem to Mori Dream Spaces.

1.1. **The Commutative Algebra Perspective.** Another direction in exploiting the theory of exceptional collections is investigating when classical theorems in commutative algebra, such as the *Hilbert Syzygy Theorem* or the *Auslander–Buchsbaum formula*, can be generalized to simplicial toric variety X. For instance, consider a $\mathrm{Pic}(X)$-graded module M on the Cox ring S of X. While a minimal free resolution $F_\bullet$ for M can be explicitly computed using Gröbner methods, it does not provide as faithful a reflection of geometry: when X has Picard rank higher than one, $F_\bullet$ may be longer than the dimension of X. To bridge this gap, Berkesch, Erman, and Smith introduced *virtual resolutions* [BES20].

Definition 1.4. A $\mathrm{Pic}(X)$-graded complex of free S-modules $F_\bullet$ is a *virtual resolution* of M if the locally free complex $\widetilde{F_\bullet}$ of vector bundles on X is a resolution of the sheaf $\widetilde{M}$.

This definition better reflects the geometry when X is a smooth projective toric variety, and motivates the following project in extending the theory of virtual resolutions.

Problem 1.5. Are there sufficient conditions on an exceptional collection for a variety X which ensure that a *virtual* Hilbert Syzygy Theorem holds for any coherent sheaf on X?

Berkesch, Erman, and Smith gave a positive answer to this problem when X is a product of projective spaces. Their proof uses a Fourier–Mukai transform that relates the exceptional collections (1.1) and (1.2) to produce a short virtual resolution whose i-th term is given by

$$\bigoplus_{|\mathbf{a}|=i} \mathrm{H}^{|\mathbf{a}|-i}\left(X, \widetilde{M} \otimes \Omega^{\mathbf{a}}(\mathbf{a})\right) \otimes \mathcal{O}(-\mathbf{a}). \tag{1.4}$$

Comparing with (1.3) illustrates the point that Fourier–Mukai transforms are the underlying machinery applicable in a range of problems depending on the choice of the kernel for the transformation. An ongoing project on studying the spectral sequence that computes the Fourier–Mukai transform with kernel $\mathcal{K}$ as in [BES20] has lead to a generalization of a result of Eisenbud and Goto [EG84] to products of projective spaces.

Theorem 1.6 (Heller–S.). *Let S be the Cox ring of a product of projective spaces with irrelevant ideal B. A finitely-generated B-saturated S-module M is d-regular if and only if the Fourier–Mukai transform $\pi_{1*}(\pi_2^*\widetilde{M} \otimes \mathcal{K})$ yields a quasi-linear free resolution of $M_{\geq d}$.*

This theorem gives an efficient algorithm for computing the multigraded Castelnuovo–Mumford regularity in the sense of Maclagan and Smith [MS04], eliminating the need for computing higher cohomology as in [ABLS20].

1.2. **The K-Theory Perspective.** Let $K_\circ(X)$ denote the *Grothendieck group* of X, defined as the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. Similarly, $K^\circ(X)$ is defined using vector bundles (i.e., locally free coherent sheaves) on X [Har77, Ch.II Ex. 6.10]. Note that the map $K^\circ(X) \to K_\circ(X)$ is an isomorphism whenever X is a smooth quasi-projective scheme.

When X is a toric variety with an action from the torus $\mathbb{T}$, the $\mathbb{T}$-equivariant K-theory of X is denoted by $K_\circ^T(X)$ and $K_T^\circ(X)$ for the Grothendieck group of $\mathbb{T}$-equivariant coherent sheaves and toric vector bundles on X. Note that the tensor product gives a ring structure on $K_T^\circ(X)$, and that $K_\circ^T(X)$ has the structure of a module on this ring.

The relationship of these groups is an interesting area of research. For instance, the restriction maps from $\mathbb{T}$-equivariant K-theory to the ordinary, non-equivariant K-theory for vector bundles and coherent sheaves is captured in a commutative diagram.

$$\begin{array}{ccc} K_T^\circ(X) & \longrightarrow & K^\circ(X) \\ \downarrow & & \downarrow \\ K_\circ^T(X) & \longrightarrow & K_\circ(X) \end{array}$$

Merkurjev showed that the restriction of coherent sheaves is always surjective [Mer97; Mer05], while Anderson, Gonzales, and Payne recently gave projective toric threefolds X for which the restriction map from the K-theory of $\mathbb{T}$-equivariant vector bundles vector on X to the ordinary K-theory of vector bundles on X is not surjective [AGP20, §7].

K-theory can also be viewed through the lens of exceptional collections. In particular, the existence of a full strong exceptional collection for $\mathcal{D}^{\mathrm{b}}(X)$ is equivalent to finite generation of $K_\circ(X)$. A natural question is whether $K_T^\circ(X)$ is finitely generated by a collection of $\mathbb{T}$-equivariant bundles. This motivates the following modification of King's conjecture.

Conjecture 1.7. *Let X be a simplicial toric variety. Then X has a $\mathbb{T}$-equivariant tilting bundle whose summands are line bundles.*

The restriction to simplicial toric varieties is motivated by Cox's theorem on the equivalence of quasi-coherent sheaves on X and graded modules over the corresponding Cox ring S.

1.3. **The Combinatorial Perspective.** A closely related research direction is inspired by A. Klyachko's classification of toric vector bundles using filtrations of a vector space defined in terms of the rays in the polyhedral fan Δ of a toric variety X [Kly90].

Theorem 1.8 (Klyachko's filtrations)**.** *The category of toric vector bundles on a toric variety is naturally equivalent to the category of finite-dimensional $\Bbbk$-vector spaces V together with a decreasing filtration $V^\rho(i)$ for each ray $\rho \in \Delta(1)$ satisfying a certain compatibility condition.*

Klyachko's filtrations have been used in [DJS18] to specify a toric vector bundle E as a collection of convex polytopes $\mathcal{P}(E)$ and study the positivity of E. These results light a path towards finding combinatorial conditions for a toric vector bundle to be a tilting bundle, providing evidence for Conjecture 1.7.

Problem 1.9. Are there sufficient conditions on Klyachko's description of a toric vector bundle which ensure that it is a $\mathbb{T}$-equivariant tilting bundle?

A natural first step in this direction is determining a vanishing criterion for $\mathrm{Ext}^i_X(E, E)$.

1.4. **Outline of the paper.** Sections 2 through 4 summarize necessary definitions and notation for working with toric varieties, representations of quivers, and derived categories of toric varieties. Sections 5 and 6 present results on exceptional collections and tilting bundles. Section 7 introduces the machinery of Fourier–Mukai transforms as relevant to this note, including Beilinson's resolution of the diagonal, which is then used in Section 8 to compute the Beilinson spectral sequence. Finally, Section 9 concludes the discussion.

2. Simplicial Toric Varieties

Toric varieties are a prime crucible for interesting examples in algebraic geometry, owing both to their intrinsic combinatorial structure and to the structure of torus equivariant vector bundles. In this section we briefly recall the terminology for normal toric varieties and toric vector bundles. At the end of the section a number of key running examples of toric varieties are given. More in depth exposition may be found in [CLS11] and [Ful93].

Definition 2.1. A *toric variety* of dimension n over $\mathbb{C}$ is a variety X that contains an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$ as a dense open subset, together with an action of $\mathbb{T}$ on X that extends the action of $\mathbb{T}$ on itself.

The structure of an *affine toric variety* can be characterized by a *strongly convex rational cone*. Concretely, let N be a lattice in $\mathbb{Z}^n$ and σ a cone in N, then the dual cone $\sigma^\vee$ in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the set of vectors in $M \otimes_{\mathbb{Z}} \mathbb{R}$ with non-negative inner product on σ. This gives a commutative semi-group $S_\sigma = \sigma^\vee \cap M = \{\eta \in M : \eta(\nu) \geq 0 \text{ for all } \nu \in \sigma\}$, and an open affine toric variety $U_\sigma = \mathrm{Spec}\ \mathbb{C}[S_\sigma]$ corresponding to its group algebra. The gluing data for a general toric variety comes from a *fan* of cones.

Definition 2.2. A *fan* Δ of strongly convex polyhedral cones in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a set of rational strongly convex polyhedral cones $\sigma \in N_{\mathbb{R}}$ such that:

(1) each face of a cone in Δ is also a cone in Δ;
(2) the intersection of two cones in Δ is a face of each of the cones.

With this information in hand, a normal toric variety X is fully characterized as follows: given a fan Δ, $X(\Delta)$ is assembled by gluing the affine toric subvarieties U_σ for each $\sigma \in \Delta$. If every cone $\sigma \in \Delta$ is generated by a subset of a $\mathbb{Z}$-basis of N then X is smooth, and it is simplicial if every cone is generated by a subset of an $\mathbb{R}$-basis of $N_{\mathbb{R}}$.

Let $\Delta(i)$ denote the set of i-dimensional cones in Δ. A 1-dimensional cone $\rho \in \Delta(1)$ is referred to as a *ray* and corresponds to an irreducible $\mathbb{T}$-invariant *Weil divisor* D_ρ on X. Hence we identify $\mathrm{Div}(X) = \mathbb{Z}^{\Delta(1)}$ and define $\mathrm{CDiv}(X)$ to be the subgroup of $\mathbb{T}$-invariant *Cartier divisors* on X. To any Cartier divisor D on a normal variety X we can associate an *invertible sheaf* $\mathscr{L} = \mathcal{O}_X(D)$ which is a locally free sheaf of sections of a *line bundle* $L \to X$. Isomorphism classes of such line bundles on X define the *Picard group* $\mathrm{Pic}(X)$.

The relationship between these groups is captured in a commutative diagram of $\mathbb{Z}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\mathrm{div}} & \mathrm{CDiv}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & A_{n-1}(X) & \longrightarrow & 0, \end{array} \tag{2.1}$$

where $\mathrm{div}(m) = \sum_\rho \langle m, n_\rho \rangle D_\rho$ with $n_\rho \in N$ the generator of $\rho \subset N_{\mathbb{R}}$ computes the orders of poles and zeros along the divisors D_ρ. See [Cox95] or [Har77, Ch.II §6] for further context.

Definition 2.3. For a toric variety X, the *homogeneous coordinate ring* is the polynomial ring $S = \mathrm{Cox}(X) = \mathbb{C}[x_\rho : \rho \in \Delta(1)]$ where each divisor $D = \sum_\rho a_\rho D_\rho$ is represented by a monomial $x^D = \prod_\rho x_\rho^{a_\rho}$ with $\deg(x^D) = [D] \in A_{n-1}(X)$.

The data of the fan determine an exceptional set $Z(\Delta) \subset \mathbb{C}^{\Delta(1)}$ which is the zero set of the *irrelevant ideal* $B(\Delta) = \langle \hat{x}_\sigma : \sigma \in \Delta \rangle$ generated by the square-free monomials $\hat{x}_\sigma = \prod_{\rho \not\subset \sigma} x_\rho$.

2.6 $\mathbb{P}^1$	2.7 $\mathbb{P}^2$	2.8 $\mathbb{P}^1 \times \mathbb{P}^1$	2.9 $\mathbb{F}_a$

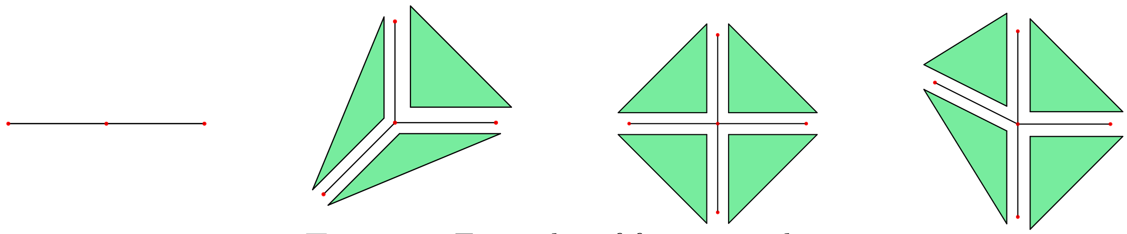

Table 1. Examples of fans on a plane

Theorem 2.4 (Thm. 3.2, [Cox95])**.** *Every quasi-coherent sheaf on a simplicial toric variety X corresponds to a finitely generated* $\mathrm{Pic}(X)$*-graded S-module.*

This theorem is central to the algebraic geometry dictionary of a simplicial toric variety: nonempty closed subvarieties of X correspond to a B-saturated radical homogeneous ideals in S and every quasi-coherent sheaf on X arises as the sheafification of a graded S-module M.

Toric varieties can be equivalently defined via a quotient construction.

Theorem 2.5. *Let $X = X(\Delta)$ be a toric variety and write $G = \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}(X), \mathbb{C}^*\right)$. Then:*
(1) X is the universal categorical quotient $\left(\mathbb{C}^{\Delta(1)} - Z(\Delta)\right)//G$.
(2) X is a geometric quotient $\left(\mathbb{C}^{\Delta(1)} - Z(\Delta)\right)/G$ if and only if Δ is simplicial.

Note that while the Cox ring S and the group G depend only on the rays $\Delta(1)$, the exceptional set $Z(\Delta)$ depends on the entire fan. In particular, when X is simplicial, $Z(\Delta)$ can be described in terms of *primitive collections*, which are subsets $\mathcal{P} \subset \Delta(1)$ with the property that any proper subset of $\mathcal{P}$ generates a cone in Δ but $\mathcal{P}$ does not. Taking the union over coordinate subspaces $\mathbb{A}(\mathcal{P})$ determined by $\{x_\rho : \rho \in \mathcal{P}\}$, we can write $Z(\Delta) = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P})$. Primitive collections were introduced by V.V. Batyrev in [Bat91] to classify d-dimensional smooth projective toric varieties. For such X, he conjectured that the number of primitive collections is bounded by a constant depending only on the Picard number of X.

2.1. **Key examples.**

Example 2.6. The Projective Line $\mathbb{P}^1$

Let V be an 2-dimensional $\Bbbk$-vector space with dual space $W = V^*$. Classically, the projective space $\mathbb{P}^1 = \mathbb{P}V$ is defined to be the set of 1-dimensional subspaces in V.

As a projective variety, $\mathbb{P}^1$ is the simplest case of the Proj construction: let $S = \mathrm{Sym}\, W$ be the symmetric algebra on W and $E = \bigwedge V$ the exterior algebra on V; the set $\mathbb{P}^1 = \mathbb{P}(W) = \mathrm{Proj}\, S$ is the set of all homogeneous prime ideals that are strictly contained in the irrelevant ideal $\mathfrak{m} = \langle x_0, x_1 \rangle$. Note that S is a polynomial ring with generators corresponding to a set of coordinates on W, while E is a skew symmetric algebra with generators corresponding to the dual basis. In particular, observe that $\mathrm{H}^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\right) = W$ and $\mathrm{H}^n\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}\right) = V$. Crucially, since $\mathrm{Hom}_{\Bbbk}(E, \Bbbk) \simeq E$, the exterior algebra is Gorenstein and has finite dimension over the field. Section 8.2 explains the BGG correspondence between $\mathcal{D}^{\mathrm{b}}(S)$ and $\mathcal{D}^{\mathrm{b}}_{\mathrm{Sing}}(E)$.

As a toric variety, $\mathbb{P}^1$ is the simplest non-affine example. let Δ be a fan with cones $\mathbb{R}_{\geq 0}$, $\{0\}$, and $\mathbb{R}_{\leq 0}$, corresponding to the affine toric varieties $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{C}$, respectively, which are glued to form $\mathbb{P}^1$ as

$$\mathbb{C}\left[\frac{x_0}{x_1}\right] \hookrightarrow \mathbb{C}\left[\frac{x_0}{x_1}, \frac{x_1}{x_0}\right] \hookleftarrow \mathbb{C}\left[\frac{x_1}{x_0}\right].$$

In this case the total coordinate ring $\mathrm{Cox}(\mathbb{P}^1)$ coincides with S and is graded by $\mathbb{Z}$, hence $\mathbb{P}^1$ has Picard group $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}^1$ and Picard rank 1. Generalizing this definition yields $\mathbb{P}^n$ as the set of lines in a vector space of dimension $n + 1$.

Example 2.7. The Projective Plane $\mathbb{P}^2$

A second way of constructing a toric variety is through the quotient construction: let $\mathbb{C}^*$ act on $\mathbb{C}^3\setminus\{0\}$ by scalar multiplication; since the $\mathbb{C}^*$-orbits are closed, $\mathbb{P}^2 = (\mathbb{C}^3\setminus\{0\})/\mathbb{C}^*$ is a geometric quotient. This construction extends to arbitrary n as $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. See [Cox95, Thm. 2.1] or [CLS11, Ch.5] for the quotient construction of toric varieties.

Example 2.8. Product of Projective Spaces $\mathbb{P}^1 \times \mathbb{P}^1$

The Proj construction of a projective variety can be extended to the relative case: let $\mathscr{S} = \mathcal{O}_{\mathbb{P}^1}[y_0, y_1]$ be a sheaf of graded algebras over $\mathbb{P}^1$ with $\deg y_i = 1$; the projectivization $\mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{Proj}\,\mathscr{S}$ is a rational ruled surface which also carries the structure of a toric variety induced by the equivariant morphism $\pi_1\colon \mathrm{Proj}\,\mathscr{S} \to \mathbb{P}^1$. In particular, the pullback $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ yields the invertible sheaf $\mathcal{O}(1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Seen through this lens, the definition of $\mathbb{P}^1$ as the Proj of a graded ring is a relative Proj of the sheaf $\mathcal{O}_{\mathrm{Spec}\,\mathbb{C}}[x_0, x_1]$. Repeating this construction yields products of projective spaces with higher Picard rank, such as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 =: \mathbb{P}^{(1,1,1)}$, while increasing the number of generators in $\mathscr{S}$ increases the dimension of each projective space factor.

Recall that a vector bundle on variety X is a locally free sheaf on X. An important class of varieties, called *projective bundles*, arise as projectivizations of vector bundles. A *toric vector bundle* on X is a vector bundle E together with a $\mathbb{T}$-action compatible with the action on X.

Example 2.9. The Hirzebruch Surface $\mathbb{F}_a$

Another construction is the projective bundle $\mathbb{P}(\mathscr{E})$ associated with a locally free sheaf $\mathscr{E}$: let $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}$ be a locally free coherent sheaf on $\mathbb{P}^1$, and consider the sheaf of graded algebras $\mathscr{S} = \bigoplus_{m\geq 0} \mathrm{Sym}^m \mathscr{E}^\vee$. The projective bundle $\mathbb{F}_a = \mathbb{P}(\mathscr{E}) = \mathrm{Proj}\,\mathscr{S}$ is the Hirzebruch surface of type a. In particular, there is a projection $\pi\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^1$ which induces a natural surjection $\pi^*\mathscr{E} \to \mathcal{O}_{\mathbb{P}^1}(1)$. See [Har77, Ch.II §7] for further context on this construction.

Remark 2.10. Every line bundle on a toric variety X is also a toric line bundle. However, in general not all vector bundles over X are toric vector bundles.

Remark 2.11. While projectivization of a toric vector bundle on a toric variety yields a variety with a $\mathbb{T}$-action, it is not a toric variety unless the vector bundle splits as a sum of line bundles [OM78, §7].

Theorem 2.12 ([Bat91]; Theorem 4.3 and Corollary 4.4)**.** *The following are equivalent for a smooth projective toric variety X with fan Σ:*

- *there exists a sequence $X = X_r, \ldots, X_0 = \mathbb{P}^n$ of toric varieties such that for each $0 < i \leq r$, X_i is the projectivization of a decomposable vector bundle on X_{i-1};*
- *any two primitive collections of Σ are disjoint (i.e., Σ is a* splitting fan*).*

This criterion, in conjunction with Lemma 5.5, provides a large class of toric varieties for which full strong exceptional collections are known [CM04].

3. Representations of Bound Quivers

A *quiver* Q is a directed graph consisting of a finite set Q_0 of vertices and Q_1 of arrows. Notably, the category of representations of a quiver of a field $\Bbbk$ is equivalent to the category of finite dimensional left modules over a $\Bbbk$-algebra. This equivalence is significant to the study of bounded derived categories of certain toric varieties, hence this section reviews the basic terminology for quivers and their representations. See Section 3.1 for examples of quivers corresponding to the running examples. See [Cra07, §1] for further details.

To each arrow $x \xrightarrow{\alpha} y$, the maps $t, h \colon Q_1 \to Q_0$ correspond a tail $t(\alpha) = x$ and head $h(\alpha) = y$ such that a sequence of arrows $p = \alpha_l \cdots \alpha_1$ form a *path* of length l whenever $h(\alpha_i) = t(\alpha_{i+1})$ for $1 \leq i < l$. By convention, for each vertex x there is a trivial path $x \xrightarrow{e_x} x$.

Definition 3.1. A *representation W of a quiver Q over a field* $\Bbbk$ consists of

- a $\Bbbk$-vector space W_x for each vertex $x \in Q_0$;
- a $\Bbbk$-linear map $w_\alpha \colon W_{t\alpha} \to W_{h\alpha}$ for each arrow $\alpha \in Q_1$.

A representation W is finite dimensional if each W_i has finite dimension over $\Bbbk$. More generally, a representation may be defined over any ring, but here working over a field suffices.

A key object in this note is the *path algebra* associated to a quiver.

Definition 3.2. The *path algebra* $\Bbbk Q$ *of a quiver* Q is the graded $\Bbbk$-algebra where

- $(\Bbbk Q)_l$ is a $\Bbbk$-vector space with basis the set of paths of length l, and
- multiplication is defined by concatenation of paths, when possible, or zero otherwise.

The path algebra is an associative algebra with identity $\sum_{x \in Q_0} e_x$ and it is finite dimensional when it is acyclic. Moreover, the subring $(\Bbbk Q)_0$ generated by the trivial paths e_x is a semi-simple ring with e_x as orthogonal idempotents; that is, $e_x e_y = e_x$ when $x = y$ and zero otherwise.

Our interest in quiver representations stems from the construction of tilting algebras as the quotient of a path algebra with vertices corresponding to the exceptional objects and arrows corresponding to homomorphisms between them.

Definition 3.3. A *bound quiver* (Q, R) is a quiver Q together with a finite set of relations R, given as $\Bbbk$-linear combinations of paths of length at least 2 with the same head and tail. Note that R can be identified with an ideal in $\Bbbk Q$.

A representation of (Q, R) is a representation of Q where for each $p - p' \in R$ the homomorphisms associated to p and p' coincide; that is, $\mathrm{Hom}(W_{tp}, W_{hp}) = \mathrm{Hom}(W_{tp'}, W_{hp'})$. Note that for each quiver representation W we may associate a $\Bbbk Q/R$-module $\bigoplus_{x \in Q_0} W_x$. Conversely, for any left $\Bbbk Q/R$-module M we have a quiver representation given by the $\Bbbk$-vector spaces $W_x = e_x M$ for $x \in Q_0$ and maps $w_\alpha \colon W_{t\alpha} \to W_{h\alpha}$ given by $m \mapsto \alpha(m)$ for $\alpha \in Q_1$.

Proposition 3.4. *The category* $\mathrm{Rep}_\Bbbk(Q, R)$ *of representations of bound quivers is equivalent to the category of finitely generated left* $\Bbbk Q/R$*-modules.*

Observe that if $(\Bbbk Q)^{\mathrm{op}}$ is the opposite algebra with product $a \cdot b = ba$, then $(\Bbbk Q)^{\mathrm{op}} \simeq \Bbbk Q^{\mathrm{op}}$ where Q^{op} is the opposite quiver with arrows reversed. In particular, the proposition above gives an equivalence of $\mathrm{mod}(A^{\mathrm{op}})$ and $\mathrm{Rep}_\Bbbk(Q^{\mathrm{op}}, R)$.

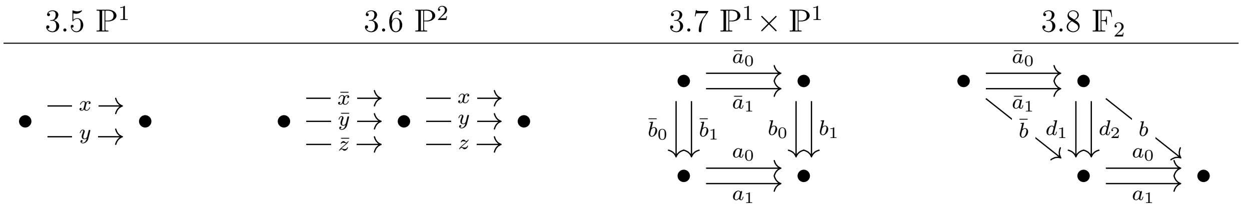

TABLE 2. Examples of Beilinson or Bondal quivers of toric varieties

3.1. **Key examples.**

Example 3.5. Kronecker quiver

This is the unique acyclic quiver with two vertices and two nontrivial arrows. A representation W of this quiver consists of a pair of vector spaces (W_0, W_1) and maps $w, w' : W_0 \to W_1$.

Beilinson's exceptional collection for $\mathbb{P}^1$ proved the equivalence $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1)\rangle$. The importance of the Kronecker quiver, also known as the Beilinson quiver or Bondal quiver for $\mathbb{P}^1$, is due to the isomorphism $\Bbbk Q \simeq \mathrm{End}_{\Bbbk}(\mathcal{O} \oplus \mathcal{O}(1))$; that is, Q encodes the data of endomorphisms of the tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$. Writing $\mathbb{P}^1 = \mathrm{Proj}\, \Bbbk[x, y]$, the arrows of Q correspond to the maps $\mathcal{O}(1) \xrightarrow{\cdot x} \mathcal{O}$ and $\mathcal{O}(1) \xrightarrow{\cdot y} \mathcal{O}$.

Recall that a vector bundle E on $\mathbb{P}^1$ splits as a direct sum $E = \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i)$ for $d_i \in \mathbb{Z}$, hence E has a trivial locally free resolution consisting only of twists $\mathcal{O}_{\mathbb{P}^1}(d_i)$. However, the existence of Beilinson's exceptional collection $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1)\rangle$ implies that there is also a locally free resolution consisting only of those two twists, though *a priori* this resolution may be longer. This observation points to the fact that the fixing an exceptional collection amounts to limiting permissible representations of objects in the derived category.

Example 3.6. Beilinson quiver for $\mathbb{P}^2$

This quiver is the first example of a Beilinson quiver for which the tilting algebra of interest is isomorphic to a quotient of the path algebra. Therefore we introduce an ideal of relations

$$R = \langle \bar{x}y - \bar{y}x, \bar{x}z - \bar{z}x, \bar{y}z - \bar{z}y\rangle.$$

We can then write $\Bbbk Q/R \cong \mathrm{End}_{\Bbbk}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$ as before.

Example 3.7. Quiver of sections for $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the collection of line bundles $\{\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\}$, with $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ the pullbacks of $\mathcal{O}_{\mathbb{P}^1}(1)$ along the first and second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

The *quiver of sections* of this collection is the quiver with vertices corresponding to the bundles L_i and an arrow $i \to j$ for each indecomposable $\mathbb{T}$-invariant section

$$s \in \mathrm{H}^0(\mathbb{P}^1 \times \mathbb{P}^1, L_j \otimes L_i^{-1}).$$

A $\mathbb{T}$-invariant section is indecomposable if the divisor $div(s)$ does not split as a sum $div(s') + div(s'')$ for nonzero sections s', s'' of $L_j \otimes L_k^{-1}$ and $L_k \otimes L_i^{-1}$.

Example 3.8. Quiver of sections for $\mathbb{F}_2$

The path algebra of this quiver, modulo the ideal of relations, represents the homomorphisms among the bundles $\{\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_4), \mathcal{O}(D_1 + D_4)\}$, where D_i are the $\mathbb{T}$-invariant irreducible divisors of X. For more complicated examples, such as this one, the ideal of relations is easier to write as having generators coming from the composition rule; i.e., the relations that arise are of the form $p - p' \in R$.

4. The Derived Category of Coherent Sheaves

Let X be a smooth projective variety, and $\mathbf{Coh}(X)$ the Abelian category of coherent sheaves on X. Every morphism $f : X \to Y$ between such varieties induces two functors:

- the inverse image functor $f^* : \mathbf{Coh}(Y) \to \mathbf{Coh}(X)$ (pullback), and
- the direct image functor $f_* : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ (pushforward).

However, these two functors are not exact, in the sense that exact sequences are not preserved. To preserve functoriality, Cartan and Eilenberg introduced the notion of derived functors.

The derived category of coherent sheaves on X, denoted $\mathcal{D}(X)$, contains geometric information about X. In some cases one can even recover X from $\mathcal{D}(X)$, but there are also examples of different varieties, for instance non-isomorphic K3 surfaces, with equivalent derived categories. This section provides an introduction to the derived category theory used in the rest of this note. See [Huy06] or [Orl03] for further context and topics.

Definition 4.1. We denote by $\mathbf{K}^b(X)$ the *homotopy category* on $\mathbf{Coh}(X)$, where the objects are bounded chain complexes of coherent sheaves on X modulo the relation of homotopy and chain maps as morphisms. The *bounded derived category* on $\mathbf{Coh}(X)$, denoted $\mathcal{D}^{\mathrm{b}}(X)$, has the same objects as $\mathbf{K}^b(X)$, but each quasi-isomorphism is endowed with an inverse morphism. Explicitly, morphisms in the derived category can be expressed as roofs $A \leftarrow A' \to B$ where $A' \to A$ is a quasi-isomorphism. Note that caution is needed in order to check whether a morphism $A \to B$ in $\mathcal{D}^{\mathrm{b}}(X)$ can be lifted back to $\mathbf{K}^b(X)$.

Unlike Abelian categories, short exact sequences do not exist in derived categories, and kernels and cokernels of morphisms are not defined. However, derived categories are endowed with the structure of a *triangulated category*, formalized by Verdier.

Definition 4.2. A category $\mathcal{D}$ is a *triangulated category* if for any morphism $f : A \to B$ in $\mathcal{D}$ there exists a *distinguished triangle* $A \to B \to C \to A[1]$, where $C = \operatorname{Cone} f$ is the cone of the morphism f, satisfying certain axioms.

A triangulated subcategory is a full subcategory $\mathcal{D}$ that is closed under the shift functor and taking the mapping cone of morphisms. In other words, if two objects of some triangle belong to a triangulated subcategory, then so does the third object. We say that $\mathcal{D}$ is *thick* (or epaisse) if it is further closed under isomorphisms and direct summands of objects. The thick *envelope* of an object E in $\mathcal{D}$ is the smallest thick triangulated subcategory of $\mathcal{D}$ containing E. When the thick envelope is equal to $\mathcal{D}$ we say that E generates $\mathcal{D}$.

Remark 4.3. The derived category $\mathcal{D}^{\mathrm{b}}(X)$ is a triangulated category, with the shift $E^\bullet \mapsto E^\bullet[1]$ given by $E^\bullet[1]^i = E^{i+1}$ and $d^i_{E[1]} = -d^{i+1}_E$, and the cone of morphism $f : E^\bullet \to F^\bullet$ given by $\operatorname{Cone} f^i = E^{i+1} \oplus F^i$.

There is a fully faithful functor $\mathbf{Coh}(X) \hookrightarrow \mathcal{D}^{\mathrm{b}}(X)$ that guarantees $\operatorname{Ext}^i(E^\bullet, F^\bullet) \simeq \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(X)}(E^\bullet, F^\bullet[i])$, hence we will identify the two going forward.

Derived categories of coherent sheaves appear in many other areas of algebraic geometry as well. For instance, the Homological Mirror Symmetry Conjecture states that there is an equivalence of categories between the derived category of coherent sheaves on a Calabi–Yau variety and the derived Fukai category of its mirror. While this is beyond the scope of this note, it is worth mentioning that a large number of Calabi–Yau manifolds are realized as subspaces of toric varieties, in particular weighted projective spaces.

In the next two sections the structure of $\mathcal{D}^{\mathrm{b}}(X)$ is determined using two related approaches: exceptional collections and tilting bundles.

5. Exceptional Collections

Studying the structure of the derived category is an important step towards studying the underlying scheme. Recall that an object E in a triangulated category $\mathcal{D}$ generates the category $\mathcal{D}$ if any full triangulated subcategory containing it is equivalent to $\mathcal{D}$. In this section, this idea is generalized to that of a *full strong exceptional collection* for $\mathcal{D}^{\mathrm{b}}(X)$, the existence of which implies that $\mathcal{D}^{\mathrm{b}}(X)$ is freely and finitely generated. Refer to [Kuz14] for further details on semiorthogonal decompositions.

Consider a full triangulated subcategory $\mathcal{B}$ in a triangulated category $\mathcal{D}$. The *right (resp. left) orthogonal* to $\mathcal{B}$ is the full subcategory $\mathcal{B}^{\perp} \subset \mathcal{D}$ (resp. ${}^{\perp}\mathcal{B}$) consisting of the objects C such that $\operatorname{Hom}(B, C) = 0$ (resp. $\operatorname{Hom}(C, B) = 0$) for all $B \in \mathcal{B}$. Both right and left orthogonal subcategories are also triangulated.

A sequence of triangulated subcategories $(\mathcal{B}_0, \ldots, \mathcal{B}_n)$ in a triangulated category $\mathcal{D}$ is a *semiorthogonal sequence* if $\mathcal{B}_j \subset \mathcal{B}_i^{\perp}$ for all $0 \leq j < i \leq n$.

Definition 5.1. A *semiorthogonal decomposition* $\mathcal{D} = \langle \mathcal{B}_0, \ldots, \mathcal{B}_n \rangle$ is a semiorthogonal sequence that generates $\mathcal{D}$ as a triangulated category.

The first examples of semiorthogonal decompositions arise from full exceptional collections.

Definition 5.2. Let $\mathcal{D}$ be a $\Bbbk$-linear triangulated category.

- An object $E \in \mathcal{D}$ is *exceptional* if
$$\operatorname{Hom}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0. \end{cases}$$
  Equivalently, using the notation of the derived functors, the condition states $\operatorname{Hom}(E, E) = \mathbb{C}$ and $R^i \operatorname{Hom}(E, E) = \operatorname{Ext}^i(E, E) = 0$ when $i \neq 0$.
- An *exceptional collection* is an ordered sequence $E_1, \ldots, E_n$ of exceptional objects such that $\operatorname{Hom}(E_i, E_j[l]) = 0$ for all $i > j$ and all l.
  Equivalently, $R^{\bullet} \operatorname{Hom}(E_i, E_j) = \operatorname{Ext}^{\bullet}(E_i, E_j) = 0$ when $i > j$.
- An exceptional collection is *strong* if, in addition, $\operatorname{Hom}(E_i, E_j[l]) = 0$ for all $i < j$.
  Equivalently, $R^{\bullet} \operatorname{Hom}(E_i, E_j) = \operatorname{Ext}^{\bullet}(E_i, E_j) = 0$ when $i < j$.
- An exceptional collection is *full* if $\mathcal{D}$ is generated by $\{E_i\}$; that is, any full triangulated subcategory containing all objects E_i is equivalent to $\mathcal{D}$ via inclusion of E_i.

There is mutative structure available on such sets, including an action by the braid group, which we will not need for the purposes of this note.

Example 5.3 ([Bei78])**.** The projective space $\mathbb{P}^n$ has a full strong exceptional collection consisting of twists of the structure sheaf,
$$\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) = \big\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n) \big\rangle,$$
as well as one consisting of exterior products of the cotangent bundle $\Omega^1(1)$,
$$\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) = \big\langle \mathcal{O}, \Omega^1(1), \Omega^2(2), \ldots, \Omega^n(n) \big\rangle.$$

Observe that while the first exceptional collection is comprised of symmetric products of $\mathcal{O}(1)$, any collection of $n+1$ consecutive twists of the structure sheaf is a full strong exceptional collection for $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$.

Remark 5.4. The existence of a full strong exceptional collection of size $n+1$ consisting of coherent sheaves on a smooth projective variety X implies that the Grothendieck group $K_\circ(X)$ is isomorphic to $\mathbb{Z}^{n+1}$.

5.1. **Exceptional Collections for Projective Bundles.** Let E be a vector bundle of rank r corresponding to a locally free coherent sheaf $\mathscr{E}$ on a smooth projective variety X. Then there exists a projective bundle $\mathbb{P}(\mathscr{E})$ with invertible sheaf $\mathcal{O}_{\mathscr{E}}(1)$. The projection map $p\colon \mathbb{P}(\mathscr{E}) \to X$ induces a canonical surjection $p^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1)$ [Har77, Ch.II Prop. 7.11].

Lemma 5.5 ([Orl93], Corollary 2.7)**.** *Let $p\colon \mathbb{P}(\mathscr{E}) \to X$ be the projectivization of a vector bundle E of rank r over a smooth projective variety X equipped with a full exceptional sequence $\mathcal{D}^{\mathrm{b}}(X) = \langle E_0, \dots, E_n\rangle$, then the derived category $\mathcal{D}^{\mathrm{b}}(E) := \mathcal{D}^{\mathrm{b}}(\mathbb{P}(\mathscr{E}))$ also has a full exceptional sequence*

$$\begin{aligned}\mathcal{D}^{\mathrm{b}}(E) = \langle p^*E_0 \otimes \mathcal{O}_{\mathscr{E}}(-r+1), \dots, p^*E_n \otimes \mathcal{O}_{\mathscr{E}}(-r+1), \dots,\\ p^*E_0 \otimes \mathcal{O}_{\mathscr{E}}(-r+2), \dots, p^*E_n \otimes \mathcal{O}_{\mathscr{E}}(-r+2), \dots, p^*E_0, \dots, p^*E_n\rangle.\end{aligned}$$

Proof sketch. Let $d = \dim X$ and consider the fiber square over X:

$$\begin{array}{ccccc} & & \mathbb{P}(\mathscr{E}) \times_X \mathbb{P}(\mathscr{E}) & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathbb{P}(\mathscr{E}) & & & & \mathbb{P}(\mathscr{E}) \\ & \searrow_{p} & & \swarrow_{p} & \\ & & X & & \end{array}$$

Recall that $\mathscr{F} \boxtimes \mathscr{G} = p_1^*\mathscr{F} \otimes p_2^*\mathscr{G}$ is the *external tensor product* with respect to the projections p_1 and p_2. The proof involves the following steps.

(1) Consider the Euler sequence $0 \to \mathcal{O}_{\mathscr{E}}(-1) \to p^*\mathscr{E}^\vee \to Q \to 0$;
(2) Construct $W = \mathcal{O}_{\mathscr{E}}(1) \boxtimes Q$ as a rank r vector bundle on $\mathbb{P}(\mathscr{E}) \times_X \mathbb{P}(\mathscr{E})$;
(3) There is a canonical section $s \in \mathrm{H}^0(\mathbb{P}(\mathscr{E}) \times_X \mathbb{P}(\mathscr{E}), W)$ whose vanishing cuts out the diagonal subscheme $\Delta \subset \mathbb{P}(\mathscr{E}) \times_X \mathbb{P}(\mathscr{E})$;
(4) There is an exact Koszul resolution K for $\mathcal{O}_\Delta$
$$0 \to \wedge^{r-1} W^\vee \to \cdots \wedge^2 W^\vee \to W^\vee \to \mathcal{O}_{\mathbb{P}(\mathscr{E}) \times_X \mathbb{P}(\mathscr{E})} \to \mathcal{O}_\Delta \to 0;$$
(5) Given arbitrary $\mathscr{F} \in \mathbf{Coh}(\mathbb{P}(\mathscr{E}))$, pullback to $p_2^*\mathscr{F}$ and tensor with K;
(6) From this complex, a spectral sequence with E_1-term is constructed;
(7) Observe that the spectral sequence converges and each term belongs to a subcategory of $\mathcal{D}^{\mathrm{b}}(E)$ generated by a term in the exceptional collection above.

□

Example 5.6 ([CM04], Example 3.2.(1))**.** Consider the Hirzebruch surface $\mathbb{F}_a$ from Example 2.9, identified as a projective bundle

$$p\colon \mathbb{P}\big(\mathcal{O}(a) \oplus \mathcal{O}\big) \to \mathbb{P}^1.$$

The bundle structure endows $\mathbb{F}_a$ with a full strong exceptional collection

$$\mathcal{D}^{\mathrm{b}}(\mathbb{F}_a) = \big\langle \mathcal{O},\ \mathcal{O}(D_1),\ \mathcal{O}(D_4),\ \mathcal{O}(D_1 + D_4)\big\rangle,$$

where $D_1, \dots, D_4$ are all toric divisors of $\mathbb{F}_a$, corresponding to the rays $\rho_1, \dots, \rho_4$.

In the next two sections we will use full strong exceptional collections to construct equivalences of bounded derived categories.

6. Tilting Bundles

Bondal's work established a correspondence between the category of coherent sheaves on a projective space and the category of finitely generated representations of a bound quiver (Q, R) through an equivalence of their bounded derived categories. See [Cra07] for the details and proofs of theorems cited.

Borrowing terminology from representation theory (cf. [Bae88]), the notion of a *tilting sheaf* on a scheme X aims to generalize Beilinson's result in the following sense: a tilting sheaf is a sheaf T of $\mathcal{O}_X$-modules that induces an equivalence of triangulated categories $\mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}})$ which sends T to $A = \mathrm{End}_X(T)$, its endomorphism algebra.

Recall that the *global dimension* of an algebra A is defined to be the maximal projective dimension of its right modules.

Definition 6.1. A sheaf of $\mathcal{O}_X$-modules is a *tilting sheaf* (resp. *bundle*) if the following hold:
(i) T has no higher self-extensions, that is, $\mathrm{Ext}^i_X(T, T) = 0$ for $i > 0$,
(ii) the endomorphism algebra $A = \mathrm{Hom}_X(T, T)$ has finite global dimension, and
(iii) T generates the bounded derived category $\mathcal{D}^{\mathrm{b}}(X)$.

More specifically, Bondal showed in [Bon90] that a triangulated category generated by a strong exceptional collection is equivalent to the derived category of right modules over the algebra of homomorphisms of the collection, which we represented as a bound quiver in Section 3.

Theorem 6.2 ([Bae88; Bon90])**.** *Let T be a tilting sheaf on a smooth projective variety X, with associated tilting algebra $A = \mathrm{End}_X(T)$. Then the functors*

$$\begin{aligned} \mathrm{Hom}_X(\ T\ ,\ -\ )\colon \mathbf{Coh}(X) &\longrightarrow \mathrm{mod}(A^{\mathrm{op}}) \quad \textit{and} \\ - \otimes_A T\ \ :\ \mathrm{mod}(A^{\mathrm{op}}) &\longrightarrow \mathbf{Coh}(X) \end{aligned}$$

induce derived equivalences of triangulated categories

$$\begin{aligned} \mathbf{R}\,\mathrm{Hom}_X(\ T\ ,\ -\ )\colon\ \mathcal{D}^{\mathrm{b}}(X) &\longrightarrow \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) \quad \textit{and} \\ - \otimes^{\mathbf{L}}_A T\ \ :\ \mathcal{D}^{\mathrm{b}}(A^{\mathrm{op}}) &\longrightarrow \mathcal{D}^{\mathrm{b}}(X) \end{aligned}$$

which are quasi-inverse to each other.

Corollary 6.3. *Suppose T is a coherent sheaf on X satisfying (i) and (ii), then T satisfies (iii) if and only if for any $E \in \mathcal{D}^{\mathrm{b}}(X)$ we have $\mathbf{R}\,\mathrm{Hom}_X(T, E) \otimes^{\mathbf{L}}_A T \cong E$.*

Establishing that a sheaf is a tilting sheaf on X is essentially done in two steps: first find a strong exceptional collection on X, then show that the exceptional collection is full. With this information, the tilting sheaf can be constructed simply as a direct sum of the exceptional collection. When the exceptional collection contains only vector bundles, then T is a tilting bundle, and if it contains only line bundles, then T is a particularly useful invariant of the derived category.

Proposition 6.4 ([Cra07], Prop. 2.7)**.** *Let $T = \oplus_{i=0}^{n} E_i$ be a locally-free sheaf on X with each E_i a line bundle (in particular,* $\mathrm{Hom}_X(E_i, E_i) = 0$ *for all i). Then*

(1) if T satisfies (i) and (ii), then $(E_0, \ldots, E_n)$ forms a strong exceptional collection.

(2) if T satisfies (iii), then $(E_0, \ldots, E_n)$ is a full strong exceptional collection.

Conversely, every full strong exceptional collection defines a tilting sheaf.

The connection with Beilinson's work is apparent from the following theorem, which can be seen as a strengthening of Corollary 6.3.

Theorem 6.5 ([Kin97], Theorem 1.2)**.** *Let X be a smooth projective variety and T be a bundle satisfying conditions (i) and (ii). Then T is a tilting bundle if and only if the map $T^\vee \boxtimes_A^{\mathbf{L}} T \to \mathcal{O}_\Delta$ is an isomorphism in $\mathcal{D}^{\mathrm{b}}(X \times X)$.*

Once again, the notion of a resolution of $\mathcal{O}_\Delta$ as an object in $\mathcal{D}^{\mathrm{b}}(X \times X)$ appears to have a close relationship with the derived category of X. In the next two sections, this relationship is formalized and expanded upon.

7. Fourier–Mukai Transforms

Let X and Y be smooth projective varieties. Every morphism $f : X \to Y$ induces two exact derived functors:

- the inverse image functor $\mathbf{L}f^* : \mathcal{D}^{\mathrm{b}}(Y) \to \mathcal{D}^{\mathrm{b}}(X)$ (pullback), and
- the direct image functor $\mathbf{R}f_* : \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$ (pushforward).

Moreover, any object $E \in \mathcal{D}^{\mathrm{b}}(X)$ defines an exact tensor functor $\cdot \otimes^{\mathbf{L}} E : \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(X)$.

A Fourier–Mukai transform is a functor $\mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(Y)$ represented by an object of $\mathcal{D}^{\mathrm{b}}(X \times Y)$ and constructed as a composition of the above functors. In particular, it turns out that any equivalence between $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ arises in this way. See [Huy06] or [Orl03] for further exposition.

Consider the two projections from $X \times Y$:

$$X \xleftarrow{\;q\;} X \times Y \xrightarrow{\;p\;} Y.$$

Then p and q induce functors on the corresponding derived categories.

Definition 7.1 ([Huy06] Definition 5.1)**.** A *Fourier–Mukai transform* is a functor

$$\Phi_{\mathcal{P}} \colon \qquad \mathcal{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{D}^{\mathrm{b}}(Y),$$

$$\text{given by} \qquad \mathscr{F} \longmapsto \mathbf{R}p_*\big(\mathbf{L}q^*\mathscr{F} \otimes^{\mathbf{L}} \mathcal{P}\big)\,.$$

The object $\mathcal{P} \in \mathcal{D}^{\mathrm{b}}(X \times Y)$ is called the *Fourier–Mukai kernel* of the functor $\Phi_{\mathcal{P}}$.

Remark 7.2. In this situation, the pull-back $\mathbf{L}q^* \colon \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(X \times Y)$, the direct image functor $\mathbf{R}p_* \colon \mathcal{D}^{\mathrm{b}}(X \times Y) \to \mathcal{D}^{\mathrm{b}}(Y)$, and the tensor product $\cdot \otimes^{\mathbf{L}} \mathcal{P} \colon \mathcal{D}^{\mathrm{b}}(X \times Y) \to \mathcal{D}^{\mathrm{b}}(X \times Y)$ are viewed as derived functors between the derived categories. However, since q is flat, $\mathbf{L}q^*$ is the usual pull-back q^*, and if $\mathcal{P}$ is a complex of locally free sheaves, then $\cdot \otimes^{\mathbf{L}} \mathcal{P}$ is the usual tensor product. Moreover, since all three functors are exact, $\Phi_{\mathcal{P}}$ is also exact.

Remark 7.3. The analogy to the classical Fourier transform is roughly that L^2-functions are replaced by complexes of coherent sheaves.

An active area of research is whether every functor between derived categories can be represented by a Fourier–Mukai transform induced by an object on the product. Orlov gave a positive answer for equivalences of derived categories of coherent sheaves on smooth projective varieties.

We are interested in the case where $Y = X$ and $\mathcal{P} \in \mathcal{D}^{\mathrm{b}}(X \times X)$ is a resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal subscheme $\Delta \xrightarrow{\iota} X \times X$. Such $\mathcal{P}$ is referred to as the *resolution of the diagonal*. In particular, the following calculation shows that $\Phi_{\mathcal{O}_\Delta}$ is simply the identity in the derived category:

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(\mathscr{F}) &= \mathbf{R}p_*(q^*\mathscr{F} \otimes \mathcal{O}_\Delta)\\ &= \mathbf{R}p_*(q^*\mathscr{F} \otimes \iota_*\mathcal{O}_X)\\ &\cong \mathbf{R}p_*(\iota_*(\iota^*q^*\mathscr{F} \otimes \mathcal{O}_X)) \quad \text{(projection formula)}\\ &\cong \mathbf{R}(p\circ\iota)_*(q\circ\iota)^*\mathscr{F} \cong \mathscr{F}.\end{aligned}$$

That is to say, the Fourier–Mukai transform with a resolution of the diagonal as its kernel produces quasi-isomorphisms.

Two constructions for resolutions for the diagonal have been mentioned in this note so far:

- Koszul resolution of the diagonal in proof of Lemma 5.5;
- Bar resolution of the diagonal in Theorem 6.5.

From the point of view of derived categories any resolution of the diagonal induces a Fourier–Mukai transform that produces quasi-isomorphisms. Other constructions for a resolution of the diagonal have been used in the literature to accomplish different goals:

- Priddy resolution of the diagonal for Koszul algebras [Fab+20];
- Cellular resolution of the diagonal from a toric cell complex [CQ12].

Moreover, Orlov's result that for smooth projective varieties X and Y any equivalence of $\mathcal{D}^{\mathrm{b}}(X)$ and $\mathcal{D}^{\mathrm{b}}(Y)$ is a Fourier–Mukai transform implies that many results in commutative and homological algebra can be restated in terms of special resolutions of the diagonal. For instance, the content of Theorem 1.6 is that for a 0-regular graded module M on a product of projective spaces, resolution of any truncation $M_{\geq d}$ is a Fourier–Mukai transform that produces a virtual resolution.

To avoid excess abstraction, in the following sections applications of Beilinson's resolution of the diagonal, which is itself a Koszul resolution on a projective space $\mathbb{P}^n$, are explored.

8. The Beilinson Spectral Sequence

Returning to coherent sheaves on the projective space $\mathbb{P}^n$, in this section we detail the construction of Beilinson's resolution of the diagonal and the spectral sequence that computes the corresponding Fourier–Mukai transform. See [OSS80, §3.1] for a geometric exposition and [Huy06, §8.3] or [AO89, §3] for an algebraic exposition on the spectral sequence.

Let $p_1, p_2\colon : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ be the two projections, and consider the rank n vector bundle

$$W = \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes Q = p_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes p_2^*Q \cong \mathrm{Hom}_{\mathbb{P}^n\times\mathbb{P}^n}\left(p_1^*\mathcal{O}_{\mathbb{P}^n}(-1), p_2^*Q\right), \tag{8.1}$$

where $Q = \mathcal{T}_{\mathbb{P}^n}(-1)$ is a twist of the tangent bundle as in the cokernel of the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to Q \to 0.$$

There is a canonical section $s \in \mathrm{H}^0(\mathbb{P}^n \times \mathbb{P}^n, W)$ whose vanishing cuts out the diagonal subscheme $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$. This section yields an exact Koszul complex

$$\mathcal{K}\colon 0 \to \wedge^n W^\vee \to \cdots \wedge^2 W^\vee \to W^\vee \to \mathcal{O}_{\mathbb{P}(\mathscr{E})\times_X \mathbb{P}(\mathscr{E})} \to \mathcal{O}_\Delta \to 0.$$

Note that as objects in $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n \times \mathbb{P}^n)$, $\mathcal{K}$ is isomorphic to the sheaf $\mathcal{O}_\Delta$. The corresponding Fourier–Mukai transform $\Phi_{\mathcal{O}_\Delta} : \mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) \to \mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$ yields the *Beilinson spectral sequence*, an important instrument for studying coherent sheaves on $\mathbb{P}^n$.

Theorem 8.1 (Beilinson)**.** *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. There exist two spectral sequences*

$$E_1^{r,s} = \mathrm{H}^s\big(\mathbb{P}^n, \mathscr{F} \otimes \mathcal{O}(r)\big) \otimes \Omega^{-r}(-r) \quad \Rightarrow \quad E^{r+s} = \begin{cases} \mathscr{F} & \text{if } r+s=0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$E_1^{r,s} = \mathrm{H}^s\big(\mathbb{P}^n, \mathscr{F} \otimes \Omega^{-r}(-r)\big) \otimes \mathcal{O}(r) \quad \Rightarrow \quad E^{r+s} = \begin{cases} \mathscr{F} & \text{if } r+s=0, \\ 0 & \text{otherwise.} \end{cases}$$

Historically, Beilinson used the resolution of the diagonal to represent a vector bundle via the cohomology of its twists, hence the classical application of constructing vector bundles with prescribed cohomology.

Remark 8.2. The structure sheaf $\mathcal{O}_\Delta$ for more complicated varieties, such as K3 surfaces, cannot be resolved by sheaves of the form $\mathscr{F} \boxtimes \mathscr{G}$.

In the next two subsections we describe a number of applications relevant to the problems in this note, including two cases when the Beilinson spectral sequence can be extended to more complicated toric varieties.

8.1. **Virtual Resolutions.** Let X be a product of r projective spaces, so the Picard group of X is $\mathbb{Z}^r$. In [BES20, §2], Berkesch, Erman, Smith construct a short virtual resolution whose length is at most the dimension of a product of projective spaces. Given a graded module M on the Cox ring of X, the construction involves four steps:

(1) find a twist $d \in \mathbb{Z}^r$ such that $\Omega^u(u+d) \otimes \widetilde{M}$ has no higher cohomology;
(2) compute a Koszul resolution of the diagonal in $X \times X$ given by

$$\mathcal{K}\colon \mathcal{O}_X(-n) \boxtimes \Omega_X^n(n) \to \cdots \to \bigwedge_{|e_i|=k} \big(\mathcal{O}_X(-e_i) \boxtimes \Omega_X^{e_i}(e_i)\big) \to \cdots \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0$$

(3) find the image of M under the Fourier–Mukai transform with kernel $\mathcal{K}$
(4) apply the global sections functor on the resulting complex.

Step (3) relies on the convergence of a spectral sequence, which follows from a variation for the case of products of projective spaces on the theme of Beilinson's resolution of the diagonal. However, apart from being a resolution of the diagonal, the existence of a Koszul resolution for the diagonal subscheme in Step (2), whose codimension is equal to the dimension of the scheme, implies that the length of the resolution will be bounded by the dimension. In fact, any other resolution of the diagonal with the appropriate length would yield the same result here.

Remark 8.3. Short virtual resolutions constructed by the method above have the consequential property of being *linear free complexes*; i.e., all the differentials are represented by matrices of linear forms. In particular, on the projective space, this is the same as a linear free resolution of a truncation $M_{\geq d}$ for d at least the Castelnuovo-Mumford regularity [EG84]. Theorem 1.6 generalizes this characterization to products of projective spaces.

In experiments, the Bar resolution constructed via a bound quiver in [Kin97, §5] has simpler differentials, though finding such a resolution that is also minimal is more involved.

8.2. **Tate Resolutions.** Let E be the exterior algebra and $\omega = \operatorname{Hom}_{\Bbbk}(E, \Bbbk)$, which is "non-canonically" isomorphic to $E(-n-1)$. The Tate resolution for sheaves over the projective space and products of projective spaces was defined and constructed using the BGG correspondence in [EFS03] and [EES15], respectively.

However, in a forthcoming paper by Brown, Eisenbud, Erman, and Schreyer, this construction is reinterpreted as the Fourier–Mukai transform

$$\begin{array}{ccccc} & & \mathcal{D}^{\mathrm{b}}(\mathbb{P}^n \times E) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathcal{D}^{\mathrm{b}}(\mathbb{P}^n) & & \xrightarrow{\quad\Phi_{\mathcal{K}}\quad} & & \mathcal{D}^{\mathrm{b}}(E) \end{array}$$

with the resolution of the diagonal $\mathcal{K}$ in $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n \times E)$ given by

$$\mathcal{K}\colon\ \cdots \to \mathcal{O}(-1) \otimes \omega(1) \to \mathcal{O}(1) \otimes \omega(-1) \to \cdots .$$

9. Concluding remarks

Fourier–Mukai transforms have been implicitly used to prove various results in commutative algebra. While the exceptional collection and kernel of transformation used in each case is different, contrasting different applications of Fourier–Mukai transforms can shed light on the respective results and improve understanding of the use of derived categories in commutative algebra. Moreover, better understanding of the necessary conditions for each of these results in terms of the exceptional collection and kernel used or bridging this theory with the BGG correspondence can potentially lead to further generalizations.

For the case of toric varieties, understanding how the derived category is reflected in the polyhedral geometry of the fan and representation theory arising from quivers of sections can lead to more explicit computational methods for working in the derived category. This will provide researchers with computational methods for testing conjectures and practitioners with better algorithmic tools.

Lastly, extending older techniques to toric varieties, such as using monads for constructing vector bundles with prescribed cohomology, has the potential to reinvigorate classical problems with new tools. As mentioned in the introduction, one example is the question of constructing indecomposable bundles of low rank, which motivates many of the themes explored here.

References.

[ABLS20] Ayah Almousa, Juliette Bruce, Michael C. Loper, and Mahrud Sayrafi. "The Virtual Resolutions Package for Macaulay2". *Journal of Software for Algebra and Geometry* 10.1 (2020), pp. 50–60. DOI: 10.2140/jsag.2020.10.51.

[AGP20] Dave Anderson, Richard Gonzales, and Sam Payne. *Equivariant Grothendieck-Riemann-Roch and localization in operational K-theory*. 2020. arXiv: 1907.00076 [math.AG].

[AO89] Vincenzo Ancona and Giorgio Ottaviani. "An introduction to the derived categories and the theorem of Beilinson". *Atti della Accademia Peloritana dei Pericolanti* 67 (Jan. 1989).

[Bae88] Dagmar Baer. "Tilting sheaves in representation theory of algebras". *Manuscripta Mathematica* 60.3 (Sept. 1988), pp. 323–347. DOI: 10.1007/bf01169343.

[Bat91] Victor V. Batyrev. "On the classification of smooth projective toric varieties". *Tohoku Math. J. (2)* 43.4 (1991), pp. 569–585. DOI: 10.2748/tmj/1178227429.

[Bei78] A. A. Beilinson. "Coherent sheaves on P^n and problems of linear algebra". *Functional Analysis and Its Applications* 12.3 (July 1978), pp. 214–216. DOI: 10.1007/BF01681436.

[BES20] Christine Berkesch, Daniel Erman, and Gregory G. Smith. "Virtual resolutions for a product of projective spaces". *Algebraic Geometry* (July 2020), pp. 460–481. DOI: 10.14231/ag-2020-013. URL: https://doi.org/10.14231/ag-2020-013.

[BGG78] I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand. "Algebraic vector bundles on $\mathbb{P}^n$ and problems of linear algebra". *Funktsional. Anal. i Prilozhen.* 12.3 (1978).

[Bon90] A I Bondal. "Representation of Associative Algebras and Coherent Sheaves". *Mathematics of the USSR-Izvestiya* 34.1 (Feb. 1990), pp. 23–42. DOI: 10.1070/im1990v034n01abeh000583.

[Buc86] Ragnar-Olaf Buchweitz. "Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings". In preparation for posthumous publication. 1986. URL: http://hdl.handle.net/1807/16682.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, 2011.

[CM04] L. Costa and R. M. Miró-Roig. "Tilting sheaves on toric varieties". *Mathematische Zeitschrift* 248.4 (Dec. 2004), pp. 849–865. DOI: 10.1007/s00209-004-0684-6.

[Cox95] David A. Cox. "The homogeneous coordinate ring of a toric variety". *J. Algebraic Geom.* 4.1 (1995), pp. 17–50.

[CQ12] Alastair Craw and Alexander Quintero Vélez. "Cellular resolutions of noncommutative toric algebras from superpotentials". *Advances in Mathematics* 229.3 (Feb. 2012), pp. 1516–1554. DOI: 10.1016/j.aim.2011.11.012.

[Cra07] Alastair Craw. *Explicit Methods for Derived Categories of Sheaves*. June 2007. URL: https://www.math.utah.edu/dc/tilting.pdf.

[DE02] Wolfram Decker and David Eisenbud. "Sheaf Algorithms Using the Exterior Algebra". *Computations in Algebraic Geometry with Macaulay 2*. Springer, 2002, pp. 215–249. DOI: 10.1007/978-3-662-04851-1_9.

[DJS18] S. Di Rocco, K. Jabbusch, and G. G. Smith. "Toric vector bundles and parliaments of polytopes". *Transactions of the American Mathematical Society* 370.11 (May 2018).

[EES15] D. Eisenbud, D. Erman, and F. Schreyer. "Tate Resolutions for Products of Projective Spaces". *Acta Mathematica Vietnamica* 40.1 (Mar. 2015).

[EFS03] David Eisenbud, Gunnar Floystad, and Frank-Olaf Schreyer. "Sheaf cohomology and free resolutions over exterior algebras". *Transactions of the American Mathematical Society* 355.11 (July 2003), pp. 4397–4426. DOI: 10.1090/s0002-9947-03-03291-4.

[EG84] David Eisenbud and Shiro Goto. "Linear free resolutions and minimal multiplicity". *Journal of Algebra* 88.1 (1984), pp. 89–133. DOI: 10.1016/0021-8693(84)90092-9.

[Fab+20] Eleonore Faber et al. *Canonical Resolutions over Koszul Algebras*. 2020. arXiv: 2011.12409 [math.AC].

[Ful93] William Fulton. *Introduction to Toric Varieties*. Princeton University Press, Dec. 1993. DOI: 10.1515/9781400882526.

[GR87] A. L. Gorodentsev and A. N. Rudakov. "Exceptional vector bundles on projective spaces". *Duke Math. J.* 54.1 (1987), pp. 115–130. DOI: 10.1215/S0012-7094-87-05409-3.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer, 1977.

[HM73] G. Horrocks and D. Mumford. "A rank 2 vector bundle on P4 with 15,000 symmetries". *Topology* 12.1 (1973), pp. 63–81. DOI: 10.1016/0040-9383(73)90022-0.

[Hul95] Klaus Hulek. "The Horrocks–-Mumford bundle". *Vector Bundles in Algebraic Geometry*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995, pp. 139–178. DOI: 10.1017/CBO9780511569319.007.

[Huy06] Daniel Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford Mathematical Monographs. Clarendon Press, 2006.

[Kap88] M. M. Kapranov. "On the derived categories of coherent sheaves on some homogeneous spaces". *Inventiones Mathematicae* 92.3 (Oct. 1988), pp. 479–508. DOI: 10.1007/bf01393744.

[Kin97] Alastair King. "Tilting bundles on some rational surfaces". Unpublished manuscript. 1997. URL: https://people.bath.ac.uk/masadk/papers/tilt.pdf.

[Kly90] A. A. Klyachko. "Equivariant bundle on toral varieties". *Mathematics of the USSR-Izvestiya* 35.2 (Oct. 1990).

[Kuz14] Alexander Kuznetsov. "Semiorthogonal decompositions in algebraic geometry". *Proc. of the International Congress of Mathematicians*. Vol. 2. Seoul, Korea, Aug. 2014, pp. 635–660. URL: https://arxiv.org/abs/1404.3143.

[Mer05] A. Merkurjev. "Equivariant K-Theory". *Handbook of K-Theory*. Vol. 1. Springer, 2005, pp. 925–954.

[Mer97] A. Merkurjev. "Comparison of the equivariant and the standard K-theory of algebraic varieties". *Algebra i Analiz* 9.4 (1997), pp. 175–214.

[MS04] Diane Maclagan and Gregory G. Smith. "Multigraded Castelnuovo-Mumford regularity". *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2004.571 (Jan. 2004). DOI: 10.1515/crll.2004.040.

[OM78] T. Oda and K. Miyake. *Lectures on Torus Embeddings and Applications*. Lectures on mathematics and physics. Tata Institute of Fundamental Research, 1978.

[Orl03] Dmitri Olegovich Orlov. "Derived categories of coherent sheaves and equivalences between them". *Russian Mathematical Surveys* 58.3 (June 2003), pp. 511–591. DOI: 10.1070/rm2003v058n03abeh000629.

[Orl93] Dmitri Olegovich Orlov. "Projective Bundles, Monoidal Transformations, and Derived Categories of Coherent Sheaves". *Russian Academy of Sciences. Izvestiya Mathematics* 41.1 (Feb. 1993), pp. 133–141. DOI: 10.1070/im1993v041n01abeh002182.

[OSS80] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces*. Progress in Mathematics. Boston, MA: Birkhäuser, 1980. DOI: 10.1007/978-1-4757-1460-9.

(Mahrud Sayrafi) University of Minnesota, Minneapolis, MN 55414, USA
Email address: mahrud@umn.edu