Math 5421

An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

2

Math 5421
Energy Balance

What determines Earth's surface temperature?

3

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Energy Balance

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Conservation of Energy
Heat is a form of energy.
Temperature measures heat.

4

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temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

5

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Heat is a form of energy.
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short wave energy from the Sun

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Everything else is detail.

6

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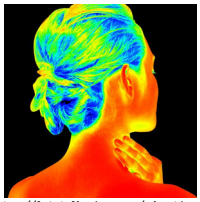
Energy Balance

Black-Body Radiation

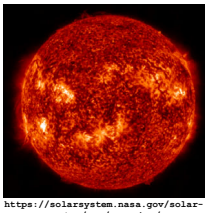
Every object emits electromagnetic radiation according to its temperature. The intensity of the radiation (power flux) is approximated by a theoretical object called a ***perfect black body***.

human



https://letstalkscience.ca/educational-resources/backgrounders/thermal-imaging

sun



https://solarsystem.nasa.gov/solar-system/sun/overview/

7

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Energy Balance

Black-Body Radiation

Stefan-Boltzmann Law

watts per square meter → power flux (W/m²)

$$F = \sigma T^4$$

kelvin → temperature (K)

0 K = −273°C = "absolute zero"

Stefan-Boltzmann constant

$\sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/m^2K^4}$

watt = joule per second = "power"

8

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Reasonable approximation:
All bodies in the solar system radiate energy according to this law.

9

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Let's try the Sun.

10

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Energy Balance

What is the surface temperature of the Sun?

11

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Energy Balance

What is the surface temperature of the Sun?

5772K ≈ 10000°F

12

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Stefan-Boltzman Law

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surface temperature of the Sun: 5772K

power flux: $5.67 \times 10^{-8} \times (5772)^4 =$

$6.29 \times 10^7\ \mathrm{W/m^2}$

13

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Stefan-Boltzman Law

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Is this a lot? **$6.29\text{x}10^7$ W/m²**

14

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Wikipedia: Prairie Island Nuclear Power Plant capacity =
522 + 519 =1041 MW ≈ 10^9 watts
$10^9/(6.29\text{x}10^7) \approx 16$ m²
Sixteen square meters on the Sun's surface produces as much power as a 1-gigawatt nuclear power plant.

`https://en.wikipedia.org/wiki/Prairie_Island_Nuclear_Power_Plant`

15

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What is the total energy output of the Sun?

16

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How many square meters on the surface of the Sun?

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What is the total energy output of the Sun?

radius of the sun = 6.96×10^8 meters

18

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Surface area of the Sun

surface temperature of the Sun: 5772K

power flux: $5.67\text{x}10^{-8} \times (5772)^4 =$

$6.29\text{x}10^7$ W/m²

total solar power output: $6.29\text{x}10^7 \times 4\pi(r_S)^2$
where r_S = radius of the sun = $6.96\text{x}10^8$ m
total solar output: **$3.83\text{x}10^{26}$ W**

19

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total solar output: **3.83×10^{26} W**

How much is that?

20

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Stefan-Boltzman Law

Global Annual Electricity Production

2022 production: about 28,000 TWh

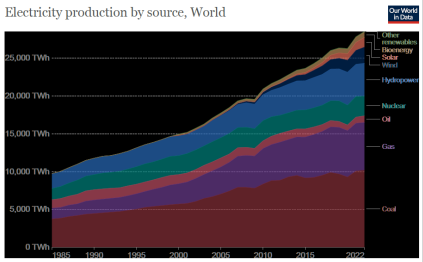
28,000 TWh = 28×10^{15} Wh

1 Wh = 3.6×10^3 J

28,000 TWh = $28 \times 3.6 \times 10^{18}$ J ≈ 10^{20} J

Solar Output = **3.83×10^{26} J/s**

seconds of solar output = $10^{20} / 3.83 \times 10^{26} \approx 0.26 \times 10^{-6} = 260 \times 10^{-9} =$ 260 nanoseconds



Source: Ember's Yearly Electricity Data; Ember's European Electricity Review; Energy Institute Statistical Review of World Energy
Note: 'Other renewables' includes waste, geothermal, wave and tidal.
OurWorldInData.org/energy • CC BY

21

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total solar output: **3.83×10^{26} W**

260 nanoseconds = time it takes for the Sun to produce the equivalent of the annual global electricity production.

22

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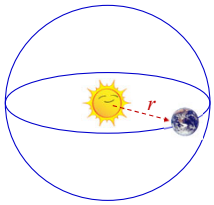
Insolation

In*coming *sol*ar Radi*ation

How much energy from the Sun is hitting the Earth?



23

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Insolation

In*coming *sol*ar Radi*ation

How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun:

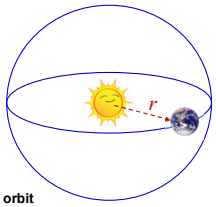
$$F = \frac{6.29 \times 10^7\, 4\pi r_S^2}{4\pi r^2} = 6.29 \times 10^7 \left(\frac{r_S}{r}\right)^2 \mathrm{W/m^2}$$

$r_S = 7 \times 10^8$ m

$r = 1.5 \times 10^{11}$ m

$\boxed{F = 1370\ \mathrm{W/m^2}}$

solar flux at Earth's orbit



https://solarsystem.nasa.gov/planets/earth/overview/

24

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$r_S = 7 \times 10^8$ m

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$\boxed{F = 1370\ \mathrm{W/m^2}}$

solar flux at Earth's orbit

Earth presents a disk to the Sun

area of disk

power intercepted by the Earth: $F \times (\pi r_E^2)$ W

Earth's surface area: $4\pi r_E^2$ m²

area of surface

Average surface flux: $\frac{F \times \pi r_E^2}{4\pi r_E^2} = \frac{F}{4} = \boxed{342\ \mathrm{W/m^2}}$

note the 4

25

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Insolation

Incoming solar Radiation

How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun:

$$F = \frac{6.29\times10^{7}\,4\pi r_S^2}{4\pi r^2} = 6.29\times10^{7}\left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

r_S = 7x10^8 m
r = 1.5x10^{11} m

$$\boxed{F = 1370 \text{ W/m}^2}$$

solar flux at Earth's orbit

Note:
The IPCC uses F = 1368 and F/4 = 342, which are the values we will use.

Average surface flux: $\frac{F\times\pi r_E^2}{4\pi r_E^2} = \frac{F}{4} = \boxed{342 \text{ W/m}^2}$

26

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Insolation

Incoming solar Radiation

How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun: $F = 1368 \text{ W/m}^2$

Power intercepted by Earth: $F\times\pi r_E^2$ W

Earth surface area

r_E = radius of Earth $= 6.37\times10^6$ m

https://en.wikipedia.org/wiki/Earth_radius

Power intercepted by Earth:

$F\times\pi r_E^2 = 1368\times\pi\times6.37^2\times10^{12} = \mathbf{1.74\times10^{17}}$ W

How much is that?

27

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Insolation

Incoming solar Radiation

How much energy from the Sun is hitting the Earth?

Solar power intercepted by Earth: 1.74×10^{17} W

Biologically Stored Energy
total coal reserves: 10^{15} kg
energy content: 3×10^7 J/kg
total energy in coal reserves: 3×10^{22} J
How long: $3\times10^{22}/1.74\times10^{17}$
$\approx 1.7\times10^5$ seconds
$\approx$ 47 hours
$\approx$ ***2 days of insolation***

28

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Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

short wave energy from the Sun
long wave energy from the Earth
energy in from the Sun
energy out from the Earth
At equilibrium, these are equal.

29

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Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

energy in from the Sun
energy out from the Earth

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun
342 W/m^2

energy out from the Earth
σT^4 W/m^2

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\text{K} = 6^\circ\text{C} = (43^\circ\text{F})$$

Earth's global mean temperature: (57°F) ***Not bad!***

30

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Energy Balance

What determines the Earth's surface temperature?

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun
$Q = 342$ W/m^2

energy out from the Earth
σT^4 W/m^2

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}$$

Note that there is a differential equation lurking here.

temperature change ~ energy in – energy out

heat capacity

$$R\frac{dT}{dt} = Q - \sigma T^4$$

31

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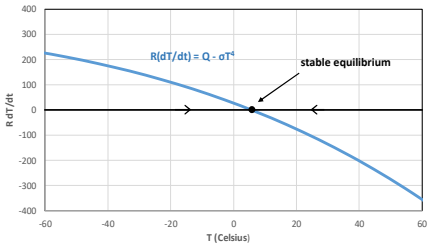
Energy Balance

temperature change ~ energy in – energy out

heat capacity → $R\frac{dT}{dt} = Q - \sigma T^4$



32

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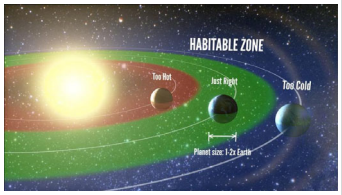
Goldilocks Zone

Goldilocks Zone

Earthling compatible life on exoplanets would be unlikely to occur unless there was the possibility of liquid water on the surface. This would mean the planet would not be too close to the star (too hot) or too far from the star (too cold). Instead, with an irresistible analogy with *Goldilocks and the Three Bears*, the temperature would be just right.



33

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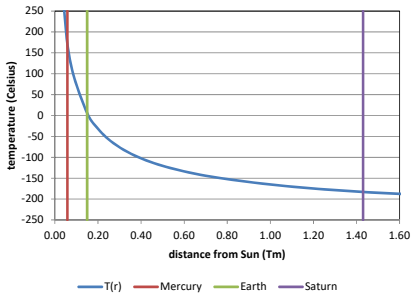

34

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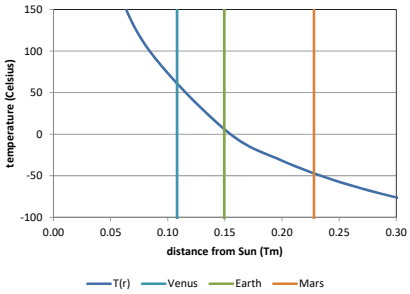

35

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Energy Balance

https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/assignments/A04.html

36