Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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2

Math 5421
Energy Balance

What determines Earth's surface temperature?

Conservation of Energy
Heat is a form of energy.
Temperature measures heat.

temperature change ~ energy in – energy out

energy in: short wave energy from the Sun
energy out: long wave energy from the Earth

Everything else is detail.

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3

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Stefan-Boltzmann

Black-Body Radiation

Stefan-Boltzmann Law

$$F = \sigma T^4$$

F: power flux (W/m²) — watts per square meter
T: temperature (K) — kelvin; 0 K = −273°C = "absolute zero"
σ: Stefan-Boltzmann constant, $\sigma \approx 5.67\times10^{-8}\ \mathrm{W/m^2K^4}$

watt = joule per second = "power"

Reasonable approximation:
All bodies in the solar system radiate energy according to this law.

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4

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Stefan Boltzmann

$$F = \sigma T^4$$

F: power flux (W/m²)
T: temperature (K)
σ: Stefan-Boltzmann constant, $\sigma \approx 5.67\times10^{-8}\ \mathrm{W/m^2K^4}$

surface temperature of the Sun: 5772K
power flux: $5.67\times10^{-8} \times (5772)^4$ =
6.29×10^7 W/m²

total solar power output: $6.29\times10^7 \times 4\pi(r_S)^2$,
where r_S = radius of the sun = 6.96×10^8 m
total solar output: **3.83×10^{26} W**

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5

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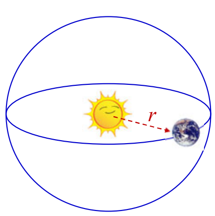
Insolation

In**coming **sol**ar Radi**ation

How much energy from the Sun is hitting the Earth?



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6

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Insolation

In**coming **sol**ar Radi**ation

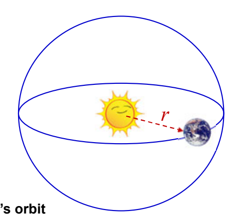
How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun:

$$F = \frac{6.29\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.29\times10^7\left(\frac{r_S}{r}\right)^2\ \mathrm{W/m^2}$$

$r_S = 7\times10^8$ m
$r = 1.5\times10^{11}$ m

$\boxed{F = 1370\ \mathrm{W/m^2}}$ — **solar flux at Earth's orbit**



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7

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Insolation

Incoming solar Radiation

How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun:

$$F = \frac{6.29\times10^7\, 4\pi r_S^2}{4\pi r^2} = 6.29\times10^7 \left(\frac{r_S}{r}\right)^2 \text{ W/m}^2$$

r_S = 7x10⁸ m
r = 1.5x10¹¹ m

$\boxed{F = 1370 \text{ W/m}^2}$

solar flux at Earth's orbit

Earth presents a disk to the Sun

area of disk

power intercepted by the Earth: $F \times \pi r_E^2$ W

Earth's surface area: $4\pi r_E^2$ m²

area of surface

Average surface flux: $\frac{F\times \pi r_E^2}{4\pi r_E^2} = \frac{F}{4} = \boxed{342 \text{ W/m}^2}$

note the 4

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8

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Insolation

Incoming solar Radiation

How much energy from the Sun is hitting the Earth?

Solar flux at a distance r from the sun: $F = 1368 \text{ W/m}^2$

Power intercepted by Earth: $F \times \pi r_E^2$ W

Earth surface area

r_E = radius of Earth $= 6.37\times10^6$ m

Power intercepted by Earth:

$F\times \pi r_E^2 = 1368 \times \pi \times 6.37^2 \times 10^{12}$ = **1.74×10^{17} W**

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9

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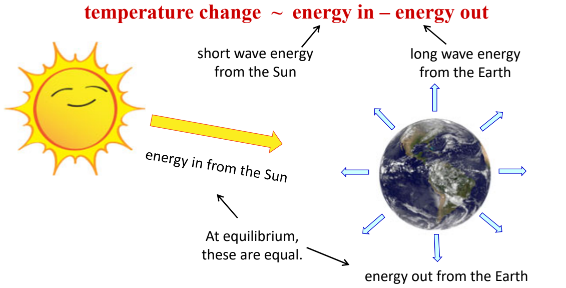
Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out



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10

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Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

energy in from the Sun — energy out from the Earth

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun: 342 W/m²
energy out from the Earth: σT^4 W/m²

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}$$

Earth's global mean temperature: 59°F *Not bad!*

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11

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Energy Balance

What determines the Earth's surface temperature?

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun: $Q = 342$ W/m²
energy out from the Earth: σT^4 W/m²

$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\text{K} = 6^\circ\text{C} = 43^\circ\text{F}$$

Note that there is a differential equation lurking here.

temperature change ~ energy in – energy out

$$R\frac{dT}{dt} = Q - \sigma T^4$$

heat capacity

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12

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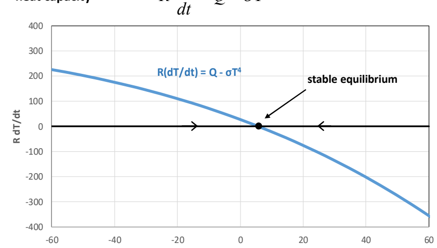
temperature change ~ energy in – energy out

heat capacity → $R\frac{dT}{dt} = Q - \sigma T^4$



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13

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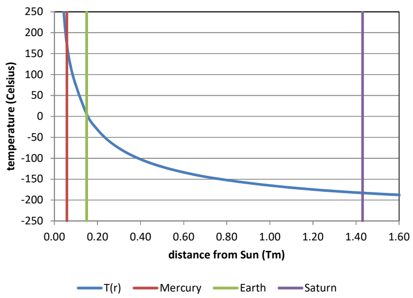
Energy Balance

Goldilocks Zone



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14

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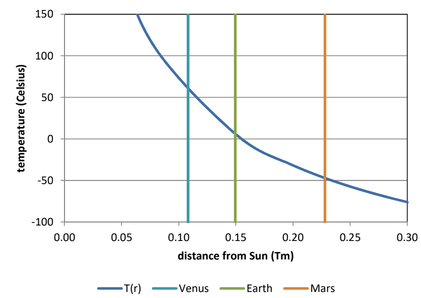
Goldilocks Zone



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15

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Energy Balance

What determines Earth's surface temperature?

Conservation of Energy

Heat is a form of energy.

Temperature measures heat.

temperature change ~ energy in – energy out

short wave energy from the Sun

long wave energy from the Earth

Does all this energy get absorbed by Earth?

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16

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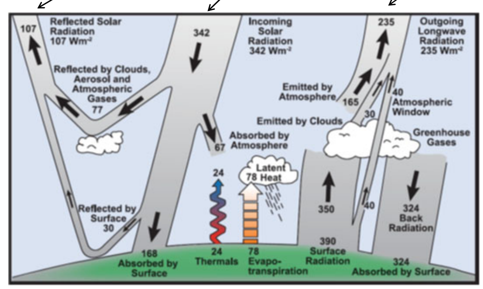
Energy Balance

albedo

insolation

OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96

http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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17

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albedo

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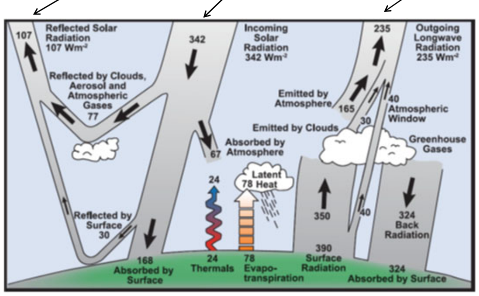
OLR



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18

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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.

The proportion reflected is called the ***albedo***, denoted α.

What is Earth's albedo?

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19

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Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

What is Earth's albedo?

According to Google's AI:

Earth's albedo refers to the proportion of sunlight that is reflected back into space by the Earth's surface, with an average value around 0.3, meaning roughly 30% of incoming solar radiation is reflected back, while the remaining 70% is absorbed by the planet; it's a key factor in regulating Earth's temperature and climate balance.

Since that is the approximation usually used, let's go with Google:

$\alpha = 0.3$

20

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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

21

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Energy Balance

Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α .

For Earth, $\alpha \approx 0.3$.

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342/5.67 \times 10^{-8}\right)^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

22

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Energy Balance

Albedo

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342/5.67 \times 10^{-8}\right)^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

??? Really cold!

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

23

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Energy Balance

Albedo

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

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$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Wait a minute! Way too cold!
What just happened???

What happened to 59°F?

24

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Energy Balance

Albedo

Simple Model

Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = \left(0.7 \cdot Q/\sigma\right)^{1/4} = \left(0.7 \cdot 342/5.67 \times 10^{-8}\right)^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

Wait a minute! Way too cold!
What just happened???

255 K is actually the *photosphere* temperature.

25

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Energy Balance

Albedo

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T=(0.7\cdot Q/\sigma)^{1/4}=(0.7\cdot 342/5.67\times10^{-8})^{1/4}$$
$$=\boxed{255\text{K}=-18^\circ\text{C}=0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt}=Q(1-\alpha)-\sigma T^4$$

This model is actually very accurate, but it applies to Earth's photosphere, not Earth's surface, where we live.

26

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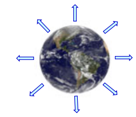
Photosphere

A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

`https://en.wikipedia.org/wiki/Photosphere`

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.



$$R\frac{dT}{dt}=Q(1-\alpha)-\sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

27

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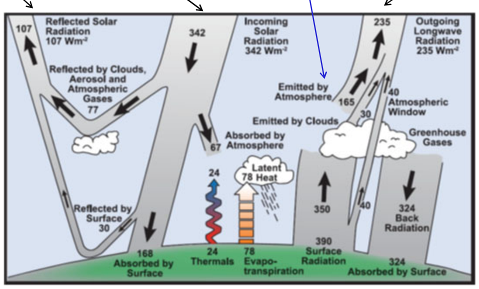
albedo — insolation — *photosphere* — OLR



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28

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OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR}\approx A+BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$$A=202\ \text{W/m}^2$$
$$B=1.90\ \text{W/m}^2\text{K}$$

Dynamics

Kelvin

$$R\frac{dT}{dt}=Q(1-\alpha)-\sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt}=Q(1-\alpha)-(A+BT)$$

global mean surface temperature

29

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OLR as a Function of Surface Temperature

$$\text{OLR}\approx A+BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

Dynamics

Kelvin

$$R\frac{dT}{dt}=Q(1-\alpha)-\sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt}=Q(1-\alpha)-(A+BT)$$

global mean surface temperature

different

30

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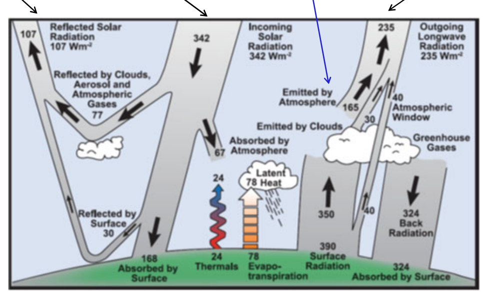
albedo — insolation — *photosphere* — OLR



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31

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Recall: Q = 342 W/m², A = 202 W/m², B = 1.9 W/m²K

$$\alpha = 0.3$$

$$T^* = 19.7^\circ\mathrm{C} = 67^\circ\mathrm{F}$$ ← ***Not bad!***

Earth's global mean temperature: 59°F

32

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Is it stable?

33

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha)-A\right) - BT$$

Stable, since $B > 0$.

34

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha)-A\right) - BT$$

Stable, since $B > 0$.

What if Earth had more ice or less ice?
That would change the albedo α.

35

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Current Earth: $\alpha = 0.30$, $T^* = 19.7^\circ\mathrm{C} = 67^\circ\mathrm{F}$

guesses → Ice-free Earth: $\alpha = \mathbf{0.28}$, $T^* = 23.3^\circ\mathrm{C} = 74^\circ\mathrm{F}$

guesses → Snowball Earth: $\alpha = \mathbf{0.62}$, $T^* = -38^\circ\mathrm{C} = -36^\circ\mathrm{F}$

Why do we have ice caps?

If Earth was ever a snowball, how did we get out?

36

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

What's missing?

37

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \big(A + BT(y,t)\big)$$

insolation distribution

$$s(y) = \text{distribution across latitudes} \quad \left(\int_0^1 s(y)dy = 1\right)$$

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38