Math 5421

An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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Energy Balance

What determines Earth's surface temperature?

Conservation of Energy
Heat is a form of energy.
Temperature measures heat.

temperature change ~ energy in – energy out

short wave energy from the Sun (→ energy in)
long wave energy from the Earth (→ energy out)

Everything else is detail.

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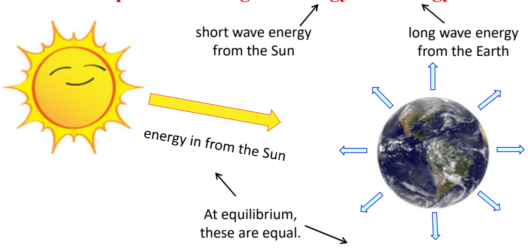
Energy Balance

What determines the Earth's surface temperature?

temperature change ~ energy in – energy out

short wave energy from the Sun
long wave energy from the Earth



At equilibrium, these are equal.

energy out from the Earth

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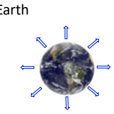
Energy Balance

What determines the Earth's surface temperature?

Simple Model
Assume that Earth is a perfectly thermally conducting black body.

energy in from the Sun
$Q = 342\ \mathrm{W/m^2}$

energy out from the Earth
$\sigma T^4\ \mathrm{W/m^2}$



$$T = (342/\sigma)^{1/4} = (342/5.67\times10^{-8})^{1/4} =$$
$$279\mathrm{K} = 6^\circ\mathrm{C} = 43^\circ\mathrm{F}$$

temperature change ~ energy in – energy out

heat capacity → $R\dfrac{dT}{dt} = Q - \sigma T^4$

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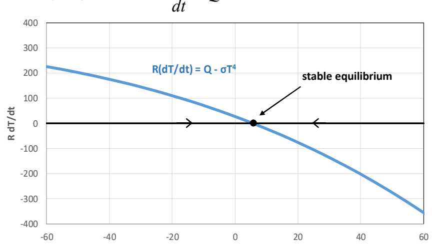
temperature change ~ energy in – energy out

heat capacity → $R\dfrac{dT}{dt} = Q - \sigma T^4$



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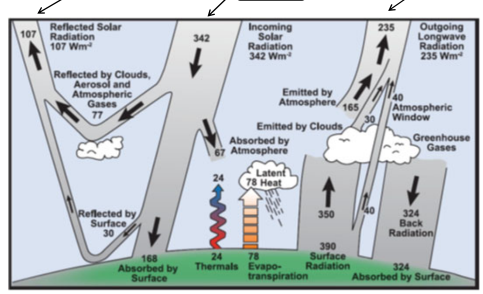
albedo — insolation — OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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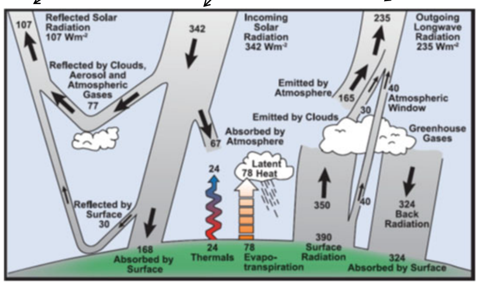
Energy Balance

albedo insolation OLR



Historical Overview of Climate Change Science, IPCC AR4, p.96
http://ipcc-wg1.ucar.edu/wg1/Report/AR4WG1_Print_CH01.pdf

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Albedo

Not all the insolation reaches the surface. Some is reflected back into space.
The proportion reflected is called the albedo, denoted α.
For Earth, $\alpha \approx 0.3$.

Simple Model
Assume that Earth is a perfectly thermally conducting black body, but only 70% of the insolation is absorbed.

$$T = (0.7 \cdot Q/\sigma)^{1/4} = (0.7 \cdot 342/5.67 \times 10^{-8})^{1/4}$$
$$= \boxed{255\text{K} = -18^\circ\text{C} = 0^\circ\text{F}}$$

Dynamics

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

stable equilibrium

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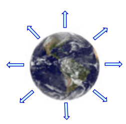
Photosphere

A photosphere is the deepest region of a luminous object, usually a star, that is transparent to photons of certain wavelengths.

https://en.wikipedia.org/wiki/Photosphere

For the Earth, the photosphere is where the long wave photons escape into space. It is high in the atmosphere where the temperature is 255 K.



$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

T = photosphere temperature.

What about the surface temperature?

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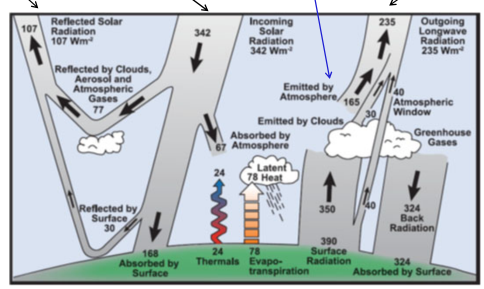
albedo insolation *photosphere* OLR



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OLR as a Function of Surface Temperature
(Outgoing Longwave Radiation)

$$\text{OLR} \approx A + BT$$

A and B are determined from satellite observations.
T is surface temperature (*in Celsius*).

$A = 202\ \text{W/m}^2$
$B = 1.90\ \text{W/m}^2\text{K}$

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

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Energy Balance

OLR as a Function of Surface Temperature

$$\text{OLR} \approx A + BT$$

Important:
$A+BT$ is **not** a linear approximation to the Stefan-Boltzmann equation.

different

Dynamics

Kelvin

$$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$$

photosphere temperature

becomes

Celsius

$$R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$$

global mean surface temperature

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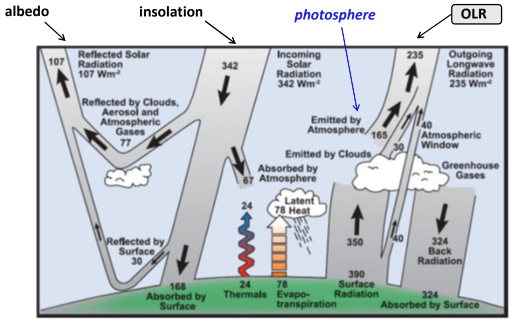

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Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

Equilibrium Temperature: $Q(1-\alpha) - A - BT^* = 0$

$$T^* = \frac{Q(1-\alpha)-A}{B}$$

Is it stable?

$$R\frac{dT}{dt} = \left(Q(1-\alpha)-A\right) - BT$$

Stable, since $B > 0$.

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Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

What's missing?

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Energy Balance

Homogeneous Earth

$$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$$

What's missing?

Earth is not homogeneous. For example, it is warmer at the equator and colder at the poles. The temperature should depend on latitude.

Make T depend on $y = \sin(\text{latitude})$

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation distribution

$s(y) = \text{distribution across latitudes}$ $\left(\int_0^1 s(y)dy = 1\right)$

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Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

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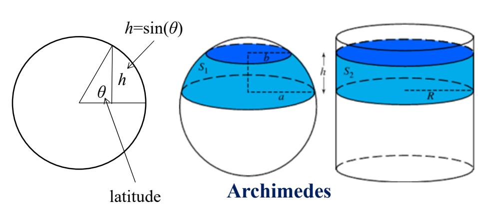
Energy Balance

Why y ?

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Why do we use y = sine(latitude) instead of just latitude?

Because y is directly proportional to surface area.



http://mathworld.wolfram.com/ArchimedesHat-BoxTheorem.html

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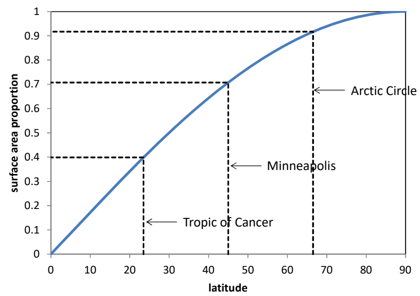
Energy Balance
Why y ?



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Energy Balance
Latitiude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$s(y) =$ distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

One can show that **β = obliquity = 23.4°**

$$s(y) = \frac{2}{\pi^2}\int_0^{2\pi}\sqrt{1-\left(\sqrt{1-y^2}\sin\beta\cos\theta - y\cos\beta\right)^2}\,d\theta$$

McGehee & Lehman, *SIAM J. Applied Dynamical Systems* **11** (2) (2012), 684–707.

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Chylek & Coakley, *J. Atmos. Sci.* **32** (1975), 675–679.

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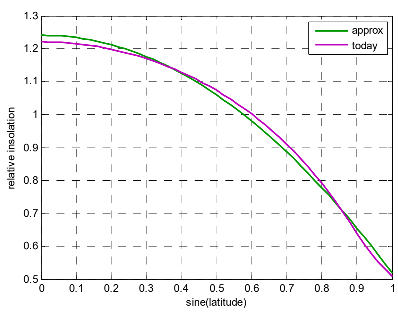
Insolation Distribution

green = quadratic approximation (Chylek & Coakley)

fuchsia = formula using obliquity of 23.5°



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Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$s(y) =$ distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Discussion Question: Is $\int_0^1 s(y)dy = 1$?

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Energy Balance
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

$s(y) =$ distribution across latitudes $\left(\int_0^1 s(y)dy = 1\right)$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 - 0.241\left(3y^2 - 1\right)$$

Discussion Question: Is $\int_0^1 s(y)dy = 1$?

Answer:

$$\int_0^1\left(3y^2-1\right)dy = y^3 - y\Big|_0^1 = 0$$

$$\int_0^1 s(y)dy = \int_0^1 1 - 0.241\left(3y^2-1\right)dy = 1 - 0.241\int_0^1\left(3y^2-1\right)dy = 1$$

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Energy Balance
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Note that an equilibrium temperature T is a function of y = sine(latitude), denoted $T^*(y)$

$$Qs(y)(1-\alpha) - (A + BT^*(y)) = 0$$

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A}{B}}$$

Recall homogeneous Earth: $R\frac{dT}{dt} = Q(1-\alpha) - (A + BT)$

equilibrium: $T^* = \frac{Q(1-\alpha) - A}{B}$

Discussion Question: Is T* the average of T*(y)?

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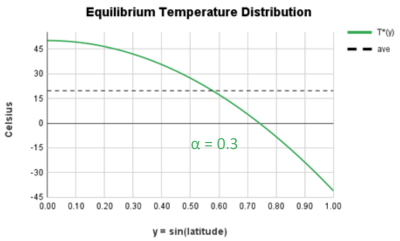
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Equilibrium Solution $T^*(y) = \dfrac{Qs(y)(1-\alpha) - A}{B}$

Note that the average temperature is about 20°C, but the equator is about 50°C and the pole is about -40°C, a 90°C difference.



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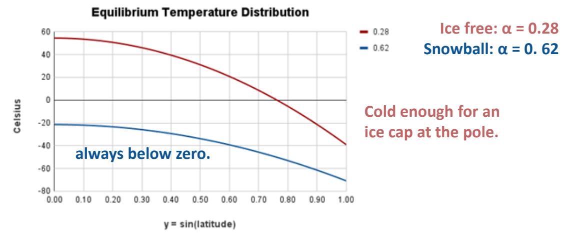
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Equilibrium Solution $T^*(y) = \dfrac{Qs(y)(1-\alpha) - A}{B}$



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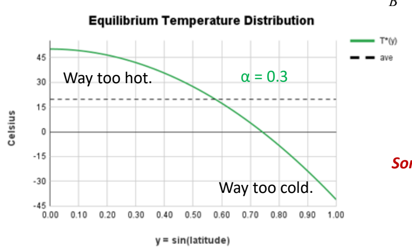
Latitude Dependence

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$$

Equilibrium Solution $T^*(y) = \dfrac{Qs(y)(1-\alpha) - A}{B}$



Something is wrong.

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latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

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Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

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Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

One simple statement of the law is that heat always moves from hotter objects to colder objects ...

https://en.wikipedia.org/wiki/Second_law_of_thermodynamics

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Energy Balance

latitude

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right)$$

insolation albedo OLR

What's Missing?

The Second Law of Thermodynamics

It's hotter at the equator than at the poles, so heat moves from the lower latitudes to the higher latitudes.

How?

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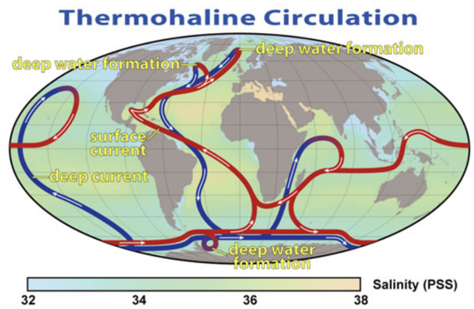
What's Missing?



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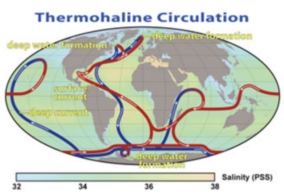
What's Missing?



Example

The Gulf Stream carries warm salty surface water from the Gulf of Mexico to the North Atlantic, where it cools, becomes denser, and sinks, flowing south in the deep ocean.

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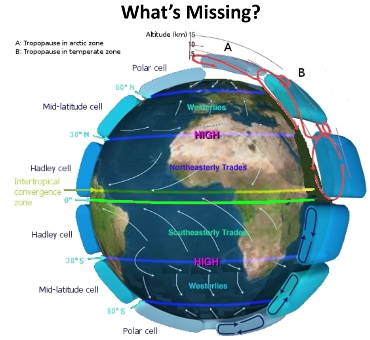
What's Missing?



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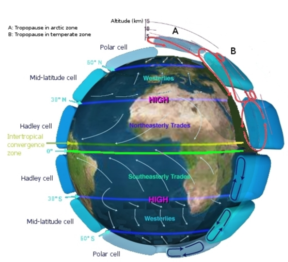
What's Missing?

Warm air from the surface along the equator rises and flows toward the poles in a series of cells moving heat from the equator to the poles.



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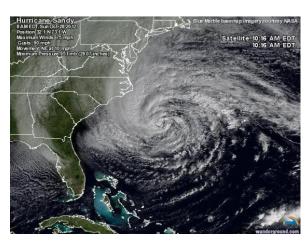
What's Missing?



Atlantic hurricanes move heat from the equatorial Atlantic up the coast of North America.

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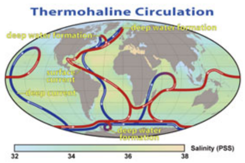
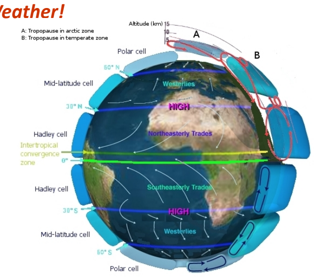
What's Missing?

Weather!





The second law of thermodynamics

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Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + \boxed{C\left(\bar{T}(t) - T(y,t)\right)}$$

2nd law

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t)dy$

$\left(\bar{T}(t) - T(y,t)\right)$ interpretation

Each point on Earth's surface is trying to assume the global mean temperature. If the temperature at a point is below the global mean, then it heats up. If the temperature at that point is above the mean, then it cools off.

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Budyko's Equation

M. I. Budyko, "The effect of solar radiation variations on the climate of the Earth," **Tellus XXI**, 611-619 , 1969.

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

$\bar{T} = \int_0^1 T(y)dy$

surface temperature: T; sin(latitude): y; heat capacity: R; insolation: $Qs(y)$; albedo: $(1-\alpha)$; OLR: $(A+BT)$; heat transport: $C(\bar{T}-T)$

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Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + C\left(\bar{T}(t) - T(y,t)\right)$$

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t)dy$

Important Point

The heat transport term only redistributes the energy around the planet. It does not affect the global mean temperature.

$$R\int_0^1 \frac{\partial T}{\partial t}(y,t)dy = Q\int_0^1 s(y)dy(1-\alpha) - \left(A + B\int_0^1 T(y,t)dy\right) + C\left(\bar{T}(t) - \int_0^1 T(y,t)dy\right)$$

$$R\frac{\partial}{\partial t}\int_0^1 T(y,t)dy = Q(1-\alpha) - \left(A + B\bar{T}(t)\right) + C \times 0$$

C disappears

Back to homogeneous Earth →
$$R\frac{d}{dt}\bar{T}(t) = Q(1-\alpha) - \left(A + B\bar{T}(t)\right)$$

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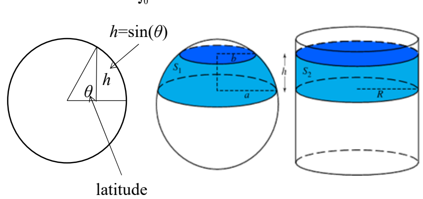
Budyko's Equation

$$R\frac{\partial T}{\partial t}(y,t) = Qs(y)(1-\alpha) - \left(A + BT(y,t)\right) + C\left(\bar{T}(t) - T(y,t)\right)$$

global mean temperature $\bar{T}(t) = \int_0^1 T(y,t)dy$

Another Important Point

Why is $\int_0^1 T(y)dy$ the average of T?



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Summary

	Model	Equilibrium
Perfect Black Body	$R\frac{dT}{dt} = Q - \sigma T^4$	$T = (Q/\sigma)^{1/4}$
Plus Albedo	$R\frac{dT}{dt} = Q(1-\alpha) - \sigma T^4$	$T = ((1-\alpha)Q/\sigma)^{1/4}$
Switch to Surface Temperature	$R\frac{dT}{dt} = Q(1-\alpha) - (A+BT)$	$T = ((1-\alpha)Q - A)/B$
Dependence on Latitude	$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha) - (A + BT(y,t))$	$T(y) = ((1-\alpha)Qs(y) - A)/B$
Add Heat Transport	$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + \boxed{C(\bar{T} - T)}$	

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Symmetry Assumption

We assume that Earth is symmetric between the northern and southern hemispheres.

Is that true?

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Energy Balance

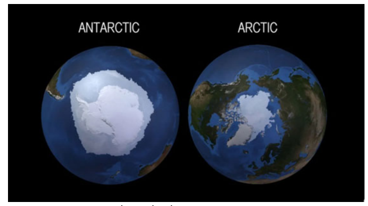
Symmetry Assumption

We assume that Earth is symmetry between the northern and southern hemispheres.

Is that true?

No, but it is good enough for now.

The south pole is in the middle of a continent surrounded by an ocean and covered in an ice sheet two miles thick.



The north pole is in the middle of an ocean surrounded by continents and covered (usually) with sea ice.

https://yaleclimateconnections.org/2021/04/researchers-examine-how-world-apart-ice-sheets-influence-each-other/

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Symmetry Assumption

We assume that Earth is symmetry between the northern and southern hemispheres.

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

We consider only northern latitudes and reflect through the equator for the southern hemispheres.

$$0 \le y = \text{sine(latitude)} \le 1$$

Recall that s is a distribution:

$$\int_0^1 s(y)dy = 1$$

We use the Chylek & Coakley quadratic approximation:

$$s(y) \approx 1 - 0.241(3y^2 - 1)$$

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Energy Balance

Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

As before, the equilibrium solution is the temperature as a function of latitude.

$$T = T^*(y)$$

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0,$$

where $\bar{T}^*$ is the global mean temperature at equilibrium, *i.e.*,

$$\bar{T}^* = \int_0^1 T^*(y)dy. \qquad \text{constant (doesn't depend on } y\text{)}$$

How can we compute the constant?

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Budyko's Equilibrium

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha) - (A+BT) + C(\bar{T} - T)$$

As before, the equilibrium solution is the temperature as a function of latitude.

$$T = T^*(y)$$

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0,$$

where $\bar{T}^*$ is the global mean temperature at equilibrium, *i.e.*,

$$\bar{T}^* = \int_0^1 T^*(y)dy. \qquad \text{constant (doesn't depend on } y\text{)}$$

Solve for $T^*(y)$

***Assume* we can somehow compute this constant.**

$$Qs(y)(1-\alpha) - A + C\bar{T}^* - \left(BT^*(y)\right) + C\left(-T^*(y)\right) = 0,$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}}$$

Now we can compute the constant!

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$$T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)dy$$

$$(B+C)\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} = Q(1-\alpha)\underbrace{\int_0^1 s(y)dy}_{1} - A\underbrace{\int_0^1 dy}_{1} + C\bar{T}^*\underbrace{\int_0^1 dy}_{1}$$

$$(B+\cancel{C})\bar{T}^* = Q(1-\alpha) - A + \cancel{C\bar{T}^*}$$

$$\bar{T}^* = \frac{Q(1-\alpha) - A}{B}$$

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Energy Balance

Budyko's Equilibrium

$$T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}$$

$$(B+C)T^*(y) = Qs(y)(1-\alpha) - A + C\bar{T}^*$$

Integrate:

$$\int_0^1 (B+C)T^*(y)dy = \int_0^1 \left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)dy$$

$$(B+C)\underbrace{\int_0^1 T^*(y)dy}_{\bar{T}^*} = Q(1-\alpha)\underbrace{\int_0^1 s(y)dy}_{1} - A\underbrace{\int_0^1 dy}_{1} + C\bar{T}^*\underbrace{\int_0^1 dy}_{1}$$

$$(B+C)\bar{T}^* = Q(1-\alpha) - A + C\bar{T}^*$$

$$\boxed{\bar{T}^* = \frac{Q(1-\alpha)-A}{B}}$$

We computed the constant!

$$\boxed{T^*(y) = \frac{Qs(y)(1-\alpha) - A + C\bar{T}^*}{B+C}, \quad \text{where} \quad \bar{T}^* = \frac{Q(1-\alpha)-A}{B}}$$

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Energy Balance

Budyko's Equilibrium

$$Qs(y)(1-\alpha) - \left(A + BT^*(y)\right) + C\left(\bar{T}^* - T^*(y)\right) = 0$$

Equilibrium temperature profile: $T^*(y) = \frac{1}{B+C}\left(Qs(y)(1-\alpha) - A + C\bar{T}^*\right)$

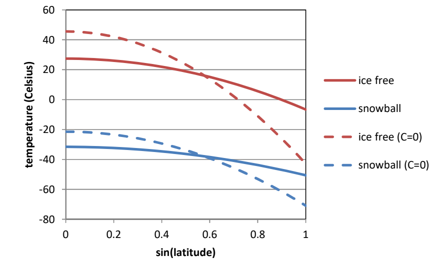
Tung*

$C = 3.04$

$\alpha = 0.32$: ice free

$\alpha = 0.62$: snowball

* K.K. Tung, *Topics in Mathematical Modeling*, Princeton U. Press, 2007



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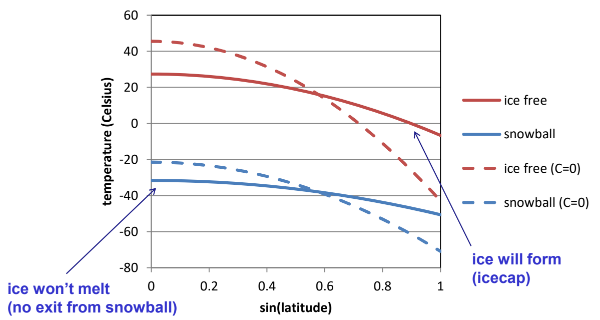
Budyko's Equilibrium



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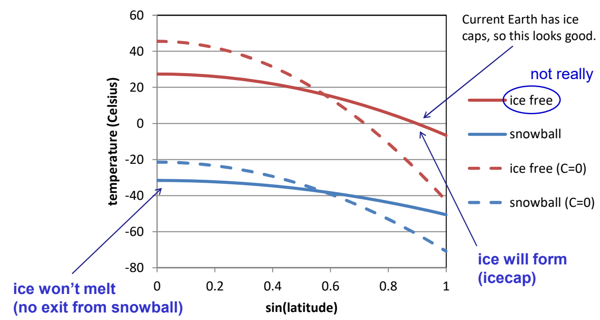
Budyko's Equilibrium



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Next Time

The current Earth has ice caps.

How can we model them?

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