Math 5421

An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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Arguments from Climate Skeptics

Earth's climate has changed many times in the past. Why do we think humans are responsible now?

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Humans haven't been around very long in geologic time. They couldn't have been the cause of climate changes in the distant past.

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Humans didn't cause the climate changes on Mars.

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Earth's climate has changed many times in the past.

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Earth's climate has changed many times in the past.

Paleoclimate

Paleoclimatology is the scientific study of climates predating the invention of meteorological instruments, when no direct measurement data were available. As instrumental records only span a tiny part of Earth's history, the reconstruction of ancient climate is important to understand natural variation and the evolution of the current climate.

`https://en.wikipedia.org/wiki/Paleoclimatology`

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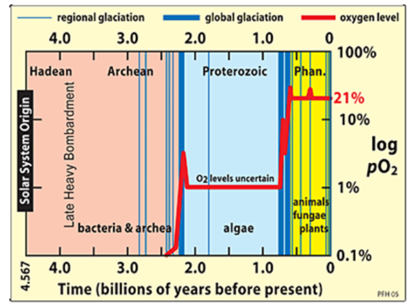
Earth's climate has changed many times in the past.



`http://www.snowballearth.org/when.html`

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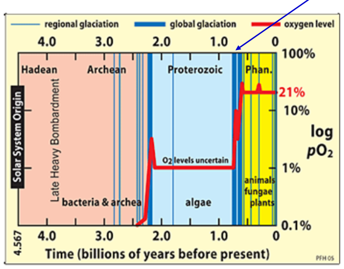

Humans didn't cause the Earth to become a snowball. Complex life did not exist then.

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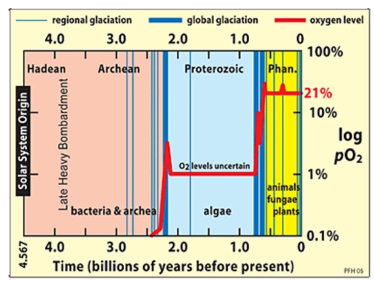

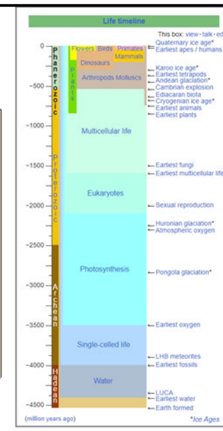

`https://en.wikipedia.org/wiki/Phanerozoic`

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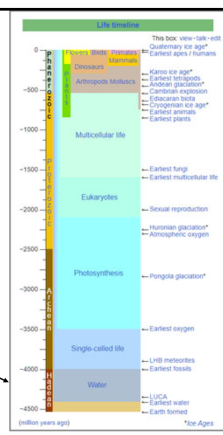
Four Geologic Eons

Phanerozoic

Proterozoic

Archean

Hadean



`https://en.wikipedia.org/wiki/Phanerozoic`

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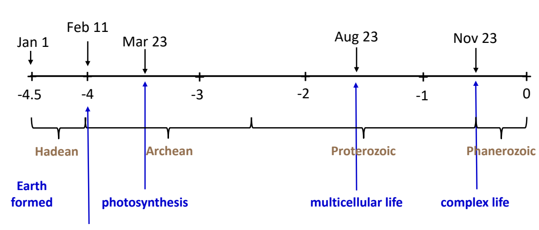
Earth Age Compressed to One Year



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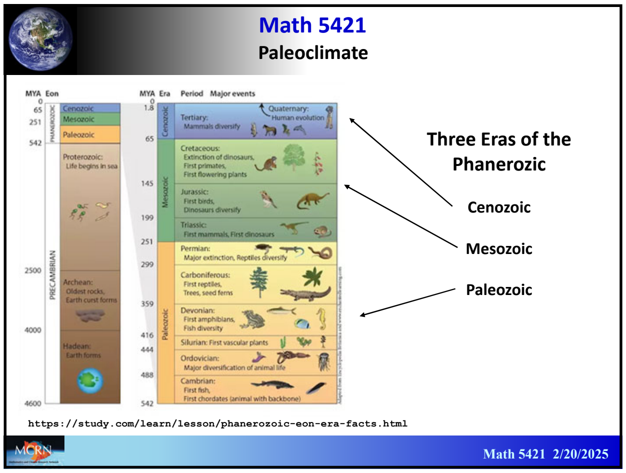

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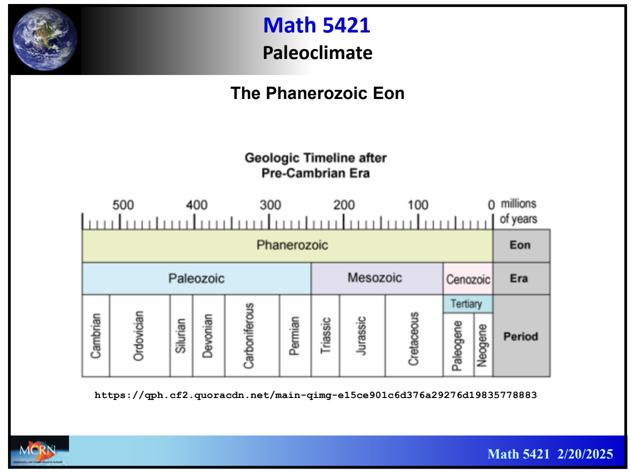

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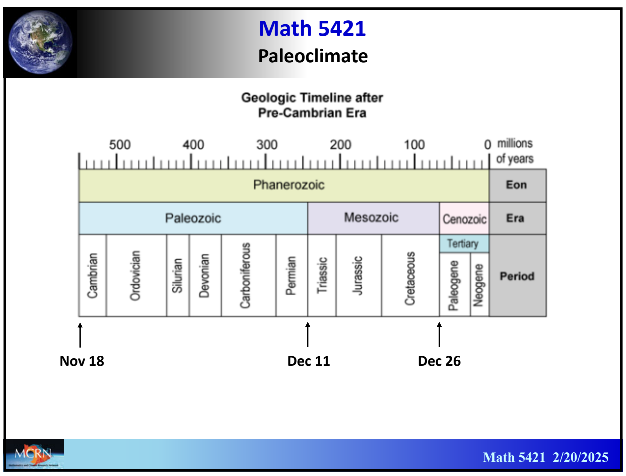

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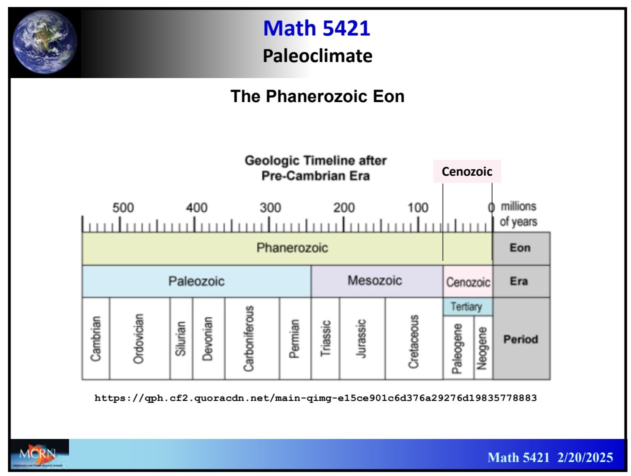

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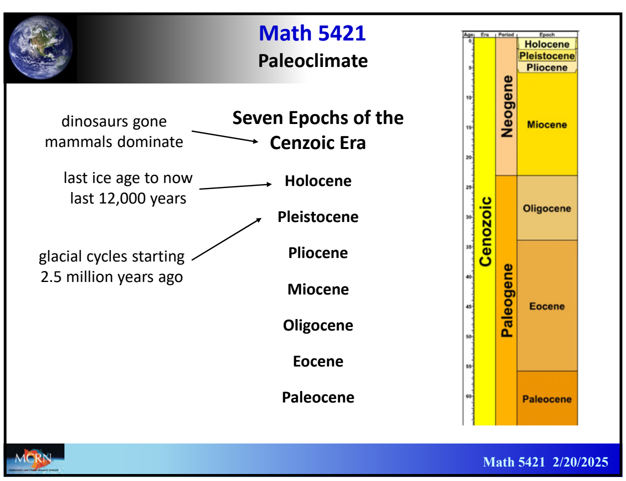

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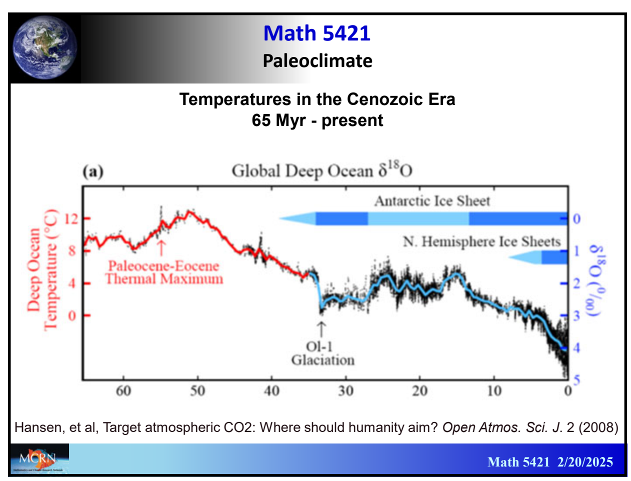

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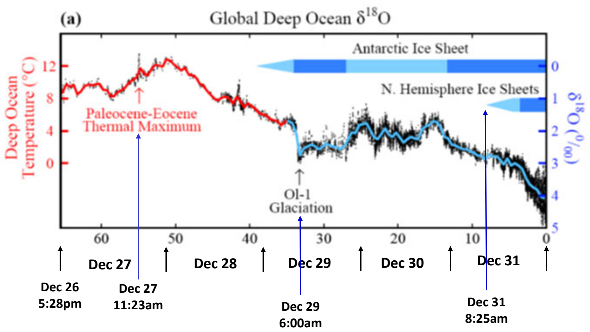

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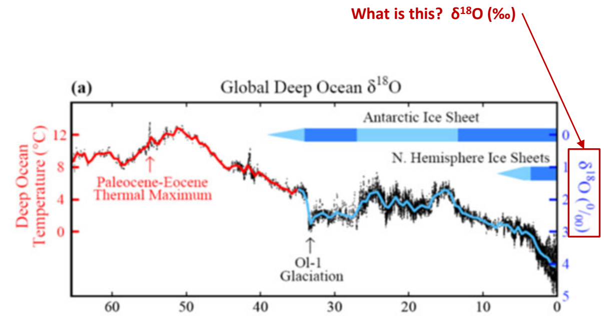

Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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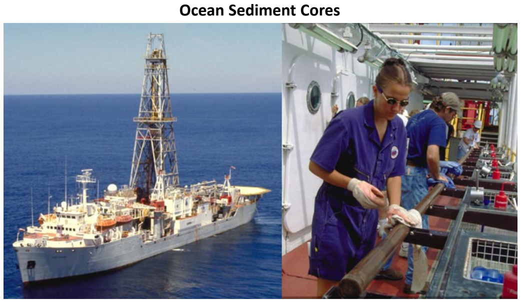
Ocean Sediment Cores



http://eo.ucar.edu/staff/rrussell/climate/paleoclimate/sediment_proxy_records.html

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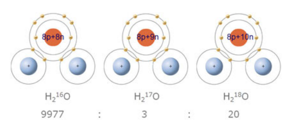
^{18}O as a Climate Proxy



The isotope ^{16}O preferentially evaporates from the ocean and is sequestered in glaciers, leaving the heavier isotope ^{18}O more highly concentrated in the ocean. Thus oceanic concentration of the isotope ^{18}O is higher during glacial periods.

Foraminifera absorb more ^{18}O into their skeletons when the water temperature is lower and when more ^{18}O is in the water.

Thus higher concentrations of ^{18}O in foraminifera fossils indicate lower ocean temperatures and higher glacier volume.

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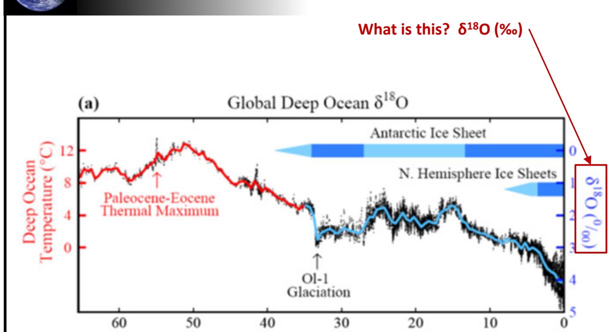

Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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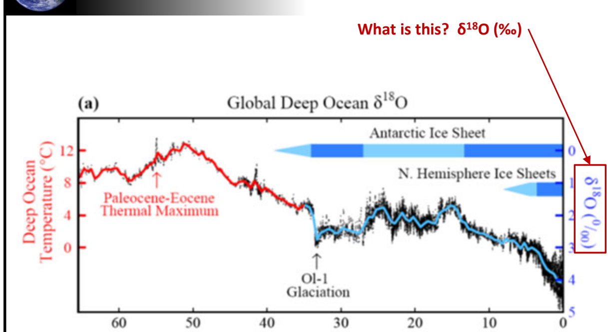

Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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What is this? $\delta^{18}O$ (‰)

‰ : "per mil," "per thousand"
1000‰ = 100% = 1
10‰ = 1% = 0.01
1‰ = 0.1% = 0.001

^{18}O: Oxygen 18: 8 protons 8 electrons 10 neutrons
^{17}O: Oxygen 17: 8 protons 8 electrons 9 neutrons
^{16}O: Oxygen 16: 8 protons 8 electrons 8 neutrons
Most of the oxygen atoms on Earth are ^{16}O.
About 1 in 500 atoms is ^{18}O. About 1 in 2500 is ^{17}O.
There are other oxygen isotopes, but they are unstable.

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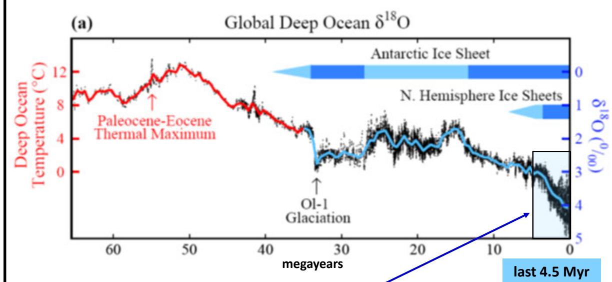

The Last 4.5 Million Years

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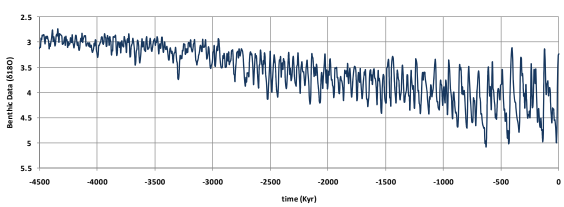
^{18}O in Foraminifera Fossils During the Past 4.5 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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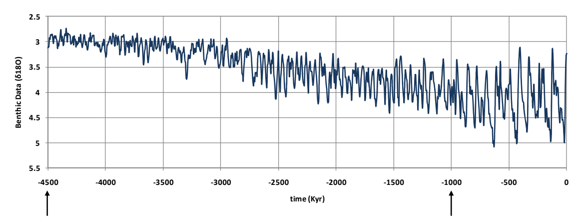

Dec 31
3:15pm

Dec 31
10:03pm

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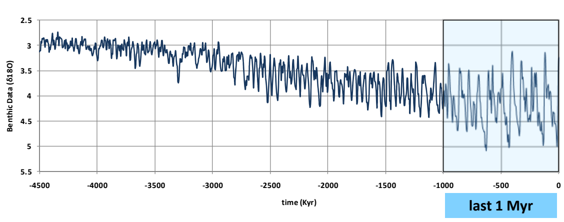
^{18}O in Foraminifera Fossils During the Past 4.5 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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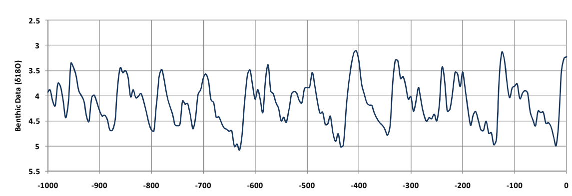
^{18}O in Foraminifera Fossils During the Past 1.0 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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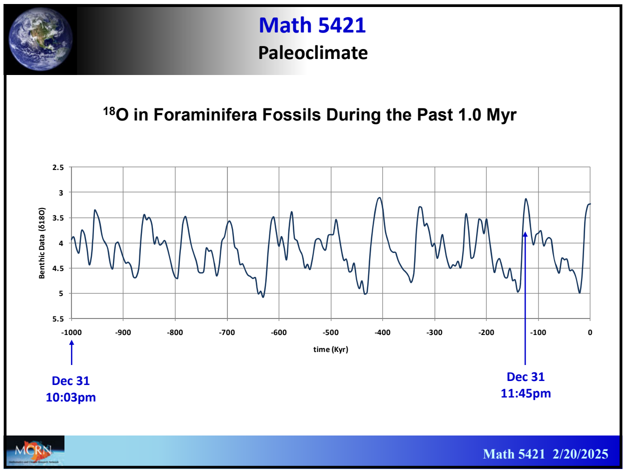

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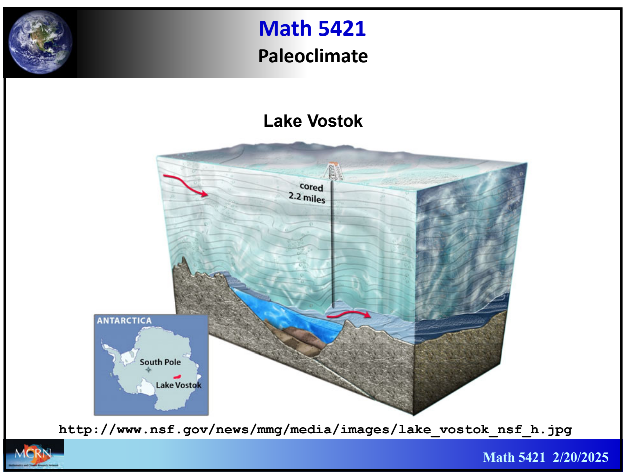

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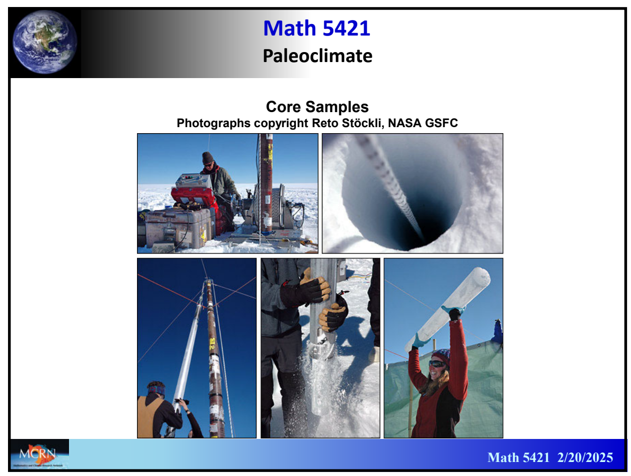

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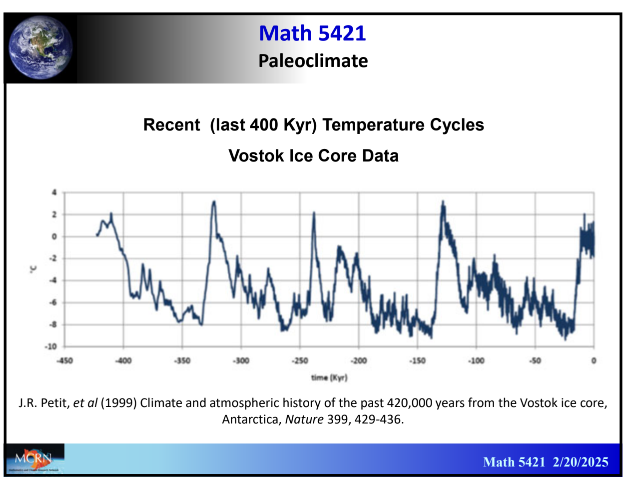

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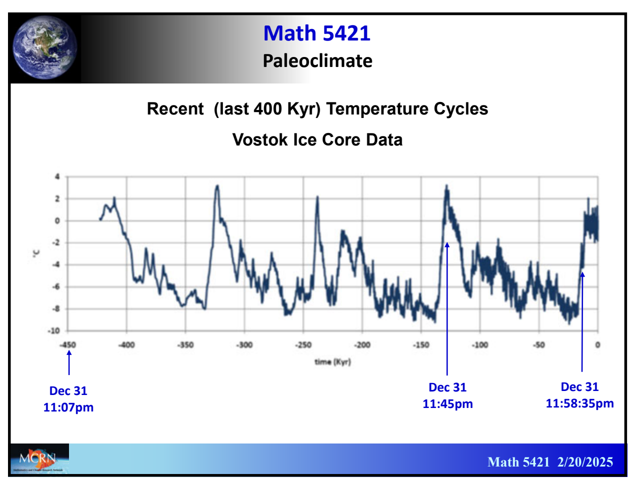

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event	years before present	Earth age = 1 year
end of last ice age	12,000 ybp	- 84 seconds
First Egyptian pyramid	4,800 ybp	-34 seconds
First Chinese dynasty	4,000 ybp	-28 seconds
0 AD	2,024 ybp	-14 seconds
Norman Conquest	956 ybp	-7 seconds
Declaration of Indepedence	248 ybp	-1740 milliseconds
Industrialization (280ppm)	163 ybp	-1140 milliseconds
Atomic Age	79 ybp	-550 milliseconds
Mauna Loa (315ppm)	66 ybp	-460 milliseconds
350 ppm	37 ybp	-260 milliseconds
400 ppm	8 ybp	-56 milliseconds

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What Causes Glacial Cycles?

Widely Accepted Hypothesis

The glacial cycles are driven by the variations in the Earth's orbit (Milankovitch Cycles), causing a variation in incoming solar radiation (insolation).

This hypothesis is widely accepted, but also widely regarded as insufficient to explain the observations.

The additional hypothesis is that there are feedback mechanisms and/or triggering mechanisms that amplify the Milankovitch cycles. What these feedbacks are and how they work are not fully understood.

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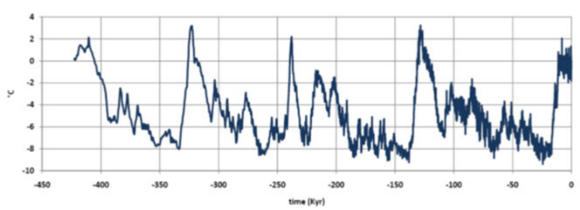
Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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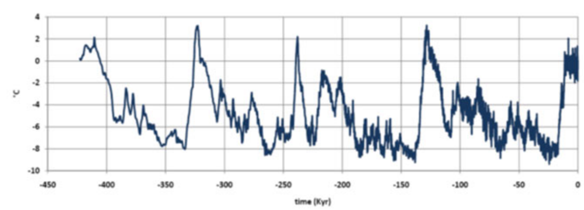

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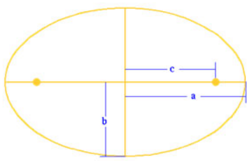
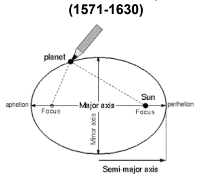
Earth's Orbit

Kepler's First Law: The orbit of every planet is an ellipse with the Sun at one of the two foci.

Johannes Kepler (1571-1630)





Eccentricity = c/a

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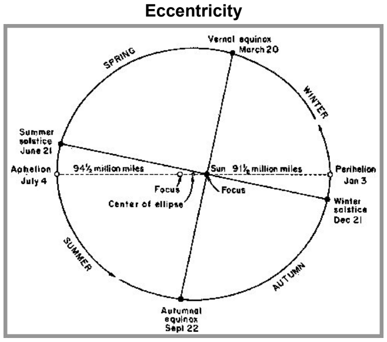
Eccentricity



John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery*, Harvard Univ. Press, 1979.

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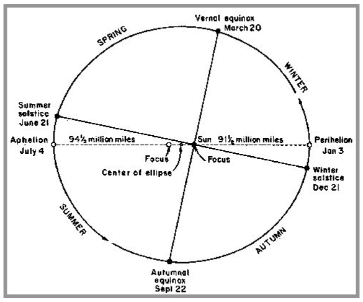
Eccentricity

Perihelion: 91.5x10^6 mi
Aphelion: 94.5x10^6 mi

Semimajor axis: 93x10^6 mi

Eccentricity: 1.5/93 = 0.016



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Eccentricity

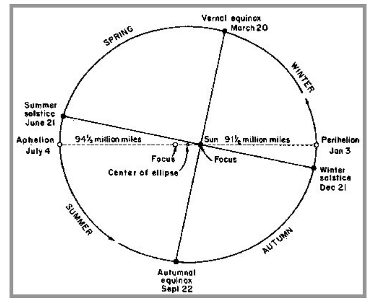
Perihelion: 91.5
Aphelion: 94.5

Change in radius:
3/93 = 3.2%

Change in insolation:
6.4%

Six percent less insolation in the southern winter than the northern winter.

6.4% of 342 W/m^2 =
22 W/m^2



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Global Annual Average Insolation

Solar output: $K \approx 4 \times 10^{26}$ Watts

Solar intensity at distance r from the sun:

$$Q(t) = \frac{K}{4\pi r(t)^2} \quad \mathrm{Wm}^{-2}$$

Cross section of Earth: $\pi r_E^2 \quad \mathrm{m}^2$

Global solar input: $\frac{K r_E^2}{4 r(t)^2} \quad \mathrm{W}$

Total annual solar input (P = one year (in seconds)):

$$\int_0^P \frac{K r_E^2}{4 r(t)^2} dt = \frac{K r_E^2}{4} \int_0^P \frac{dt}{r(t)^2} \quad \text{Joules}$$

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Global Annual Average Insolation

Specific angular momentum (angular momentum per unit mass):

$$\Omega = r^2 \dot{\theta} \quad \mathrm{m^2 s^{-1}}$$

Total annual solar input:

$$\frac{K r_E^2}{4} \int_0^P \frac{dt}{r(t)^2} = \frac{K r_E^2}{4} \int_0^P \frac{\dot{\theta} dt}{\Omega} = \frac{K r_E^2}{4\Omega} \int_0^{2\pi} d\theta = \frac{\pi K r_E^2}{2\Omega} \quad \text{Joules}$$

Mean annual solar input: $\frac{\pi K r_E^2}{2P\Omega}$ Watts

Mean annual solar intensity on the Earth's surface:

$$\frac{\pi K r_E^2}{2P\Omega} \cdot \frac{1}{4\pi r_E^2} = \frac{K}{8P\Omega} \quad \mathrm{Wm}^{-2}$$

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Global Annual Average Insolation

Kepler's Third Law: $P \sim a^{3/2}$ a = semimajor axis

Derived from Kepler: $\Omega^2 \sim a(1-e^2)$ e = eccentricity

Mean annual solar intensity:

$$\frac{K}{8P\Omega} = \frac{\hat{K}}{a^{3/2} a^{1/2} \sqrt{1-e^2}} = \frac{\hat{K} a^{-2}}{\sqrt{1-e^2}} \quad \mathrm{Wm}^{-2}$$

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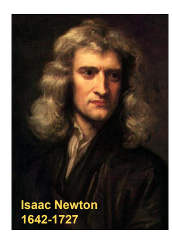
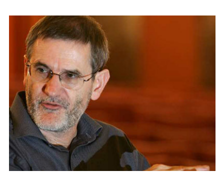
Planetary Motion

$$m_i \frac{d^2 x_i}{dt^2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j (x_j - x_i)}{|x_j - x_i|^3}$$




Jacques Laskar (1955-)

The orbits of all the planets can be computed (both forward and backward in time) for billions of years.

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Earth's Semi-major Axis

Laskar: Global Annual Average Insolation: $\frac{Ka^{-2}}{\sqrt{1-e^2}}$

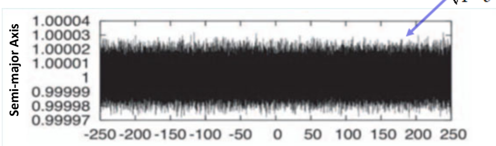

Fig. 11. Variation of the semi-major axis of the Earth–Moon barycenter (in AU) from −250 to +250 Myr.

Semi major axis does not change much:
0.005% corresponding to .01% change in global average insolation

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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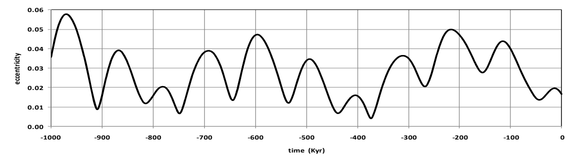
Earth's Eccentricity



Note periods of about 100 kyr and 400 kyr.

The effect due to eccentricity is more significant:
As e varies between 0 and 0.06, $(1-e^2)^{-1/2}$ varies between 1 and 1.0018, or about 0.2%. (Twenty times the effect due to a.)

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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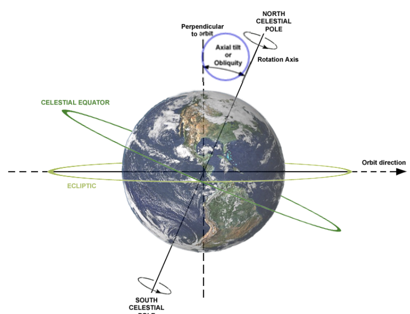
Earth's Obliquity



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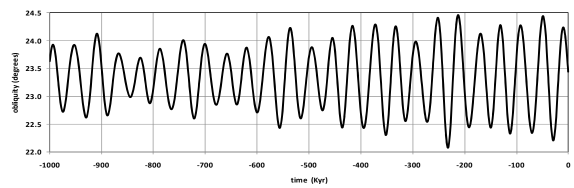
Glacial Cycles

Earth's Obliquity



Note period of about 41 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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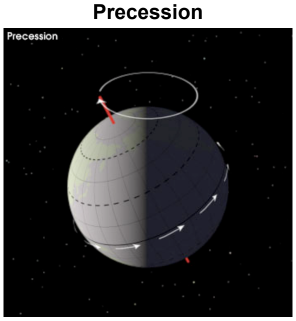
Glacial Cycles

Precession



http://earthobservatory.nasa.gov/Library/Giants/Milankovitch/milankovitch_2.html

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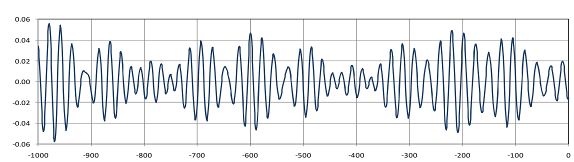
Glacial Cycles

Earth's Precession Index



index $= e \sin\rho$, where e = eccentricity and ρ = precession angle (measured from spring equinox)

Note period of about 23 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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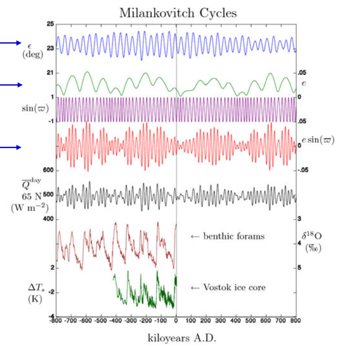
Glacial Cycles

obliquity →
eccentricity →
precession index →



http://en.wikipedia.org/wiki/Milankovitch_cycles

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