Math 5421

An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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course website
https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/

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Paleoclimate

Earth's climate has changed many times in the past.

Paleoclimate

Paleoclimatology is the scientific study of climates predating the invention of meteorological instruments, when no direct measurement data were available. As instrumental records only span a tiny part of Earth's history, the reconstruction of ancient climate is important to understand natural variation and the evolution of the current climate.

https://en.wikipedia.org/wiki/Paleoclimatology

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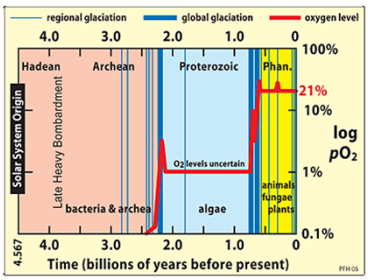

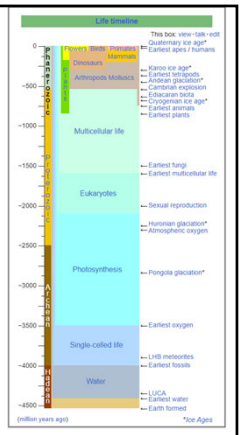

https://en.wikipedia.org/wiki/Phanerozoic

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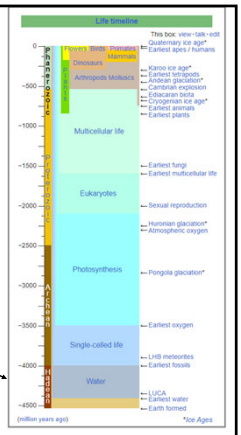
Four Geologic Eons

Phanerozoic

Proterozoic

Archean

Hadean



https://en.wikipedia.org/wiki/Phanerozoic

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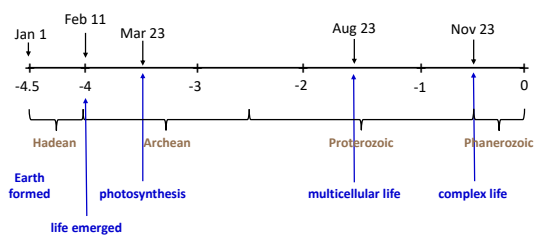
Earth Age Compressed to One Year



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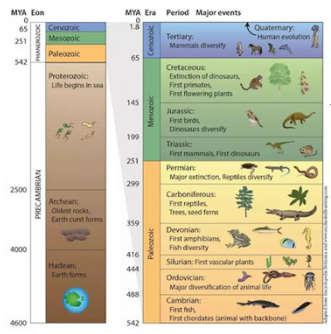

Three Eras of the Phanerozic

Cenozoic

Mesozoic

Paleozoic

https://study.com/learn/lesson/phanerozoic-eon-era-facts.html

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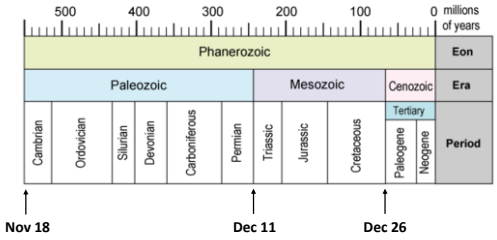
Geologic Timeline after Pre-Cambrian Era



Nov 18 — Dec 11 — Dec 26

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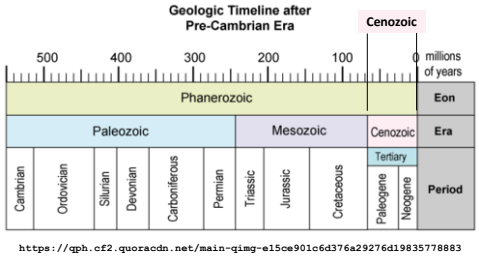
The Phanerozoic Eon



https://qph.cf2.quoracdn.net/main-qimg-e15ce901c6d376a29276d19835778883

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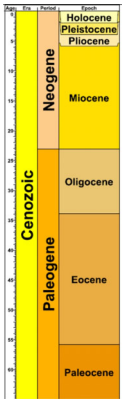
dinosaurs gone
mammals dominate

Seven Epochs of the Cenzoic Era

last ice age to now
last 12,000 years

glacial cycles starting
2.5 million years ago

- Holocene
- Pleistocene
- Pliocene
- Miocene
- Oligocene
- Eocene
- Paleocene



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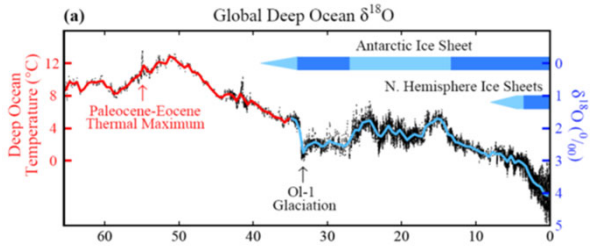
Temperatures in the Cenozoic Era
65 Myr - present



Hansen, et al, Target atmospheric CO2: Where should humanity aim? *Open Atmos. Sci. J.* 2 (2008)

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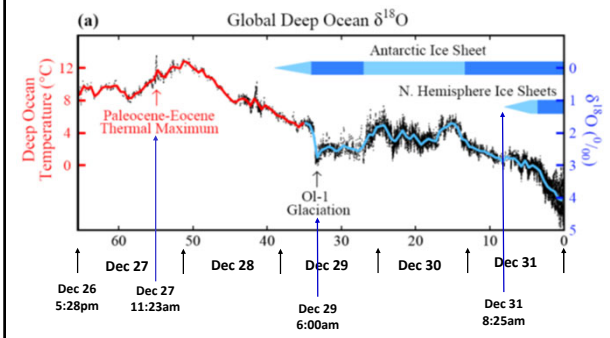

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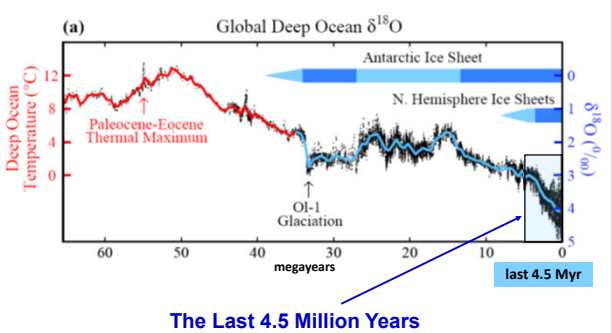

The Last 4.5 Million Years

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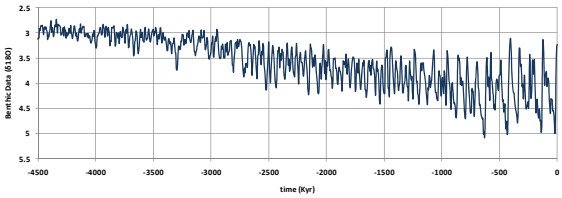
^{18}O in Foraminifera Fossils During the Past 4.5 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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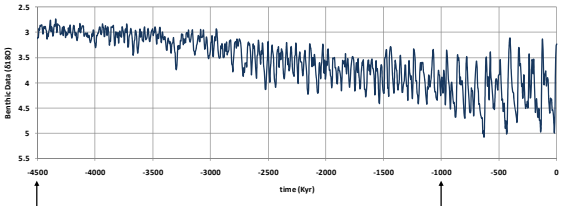
^{18}O in Foraminifera Fossils During the Past 4.5 Myr



Dec 31
3:15pm

Dec 31
10:03pm

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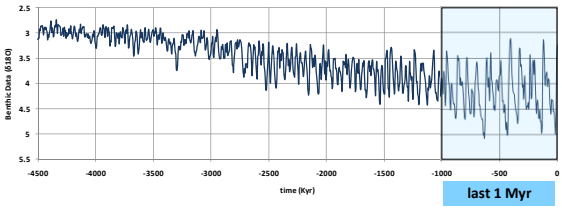
^{18}O in Foraminifera Fossils During the Past 4.5 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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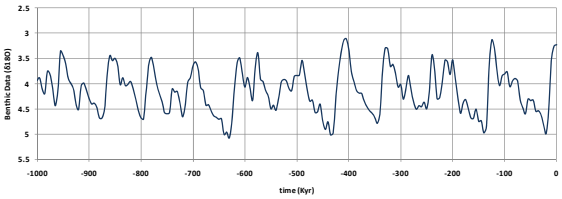
^{18}O in Foraminifera Fossils During the Past 1.0 Myr



Lisiecki, L. E., and M. E. Raymo (2005), A Pliocene-Pleistocene stack of 57 globally distributed benthic d18O records, *Paleoceanography*,20, PA1003, doi:10.1029/2004PA001071.

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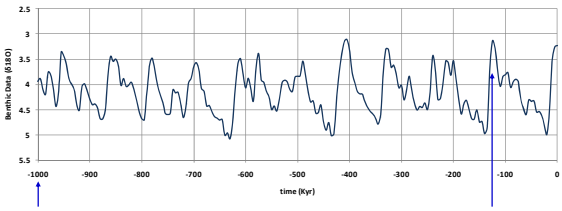
^{18}O in Foraminifera Fossils During the Past 1.0 Myr



Dec 31
10:03pm

Dec 31
11:45pm

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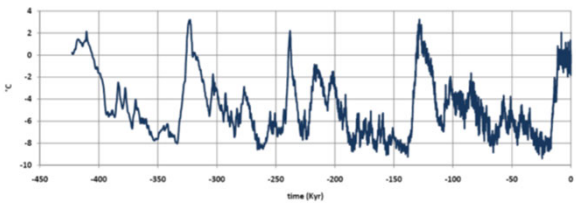
Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data



J.R. Petit, *et al* (1999) Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature* 399, 429-436.

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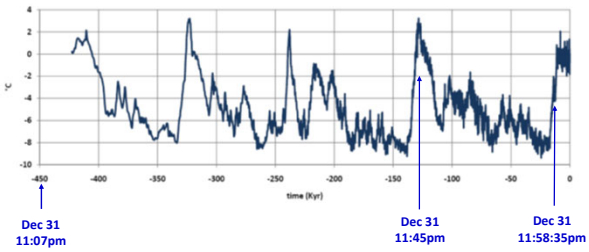
Recent (last 400 Kyr) Temperature Cycles

Vostok Ice Core Data



Dec 31 11:07pm

Dec 31 11:45pm

Dec 31 11:58:35pm

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Glacial Cycles

What Causes Glacial Cycles?

Widely Accepted Hypothesis

The glacial cycles are driven by the variations in the Earth's orbit (Milankovitch Cycles), causing a variation in incoming solar radiation (insolation).

This hypothesis is widely accepted, but also widely regarded as insufficient to explain the observations.

The additional hypothesis is that there are feedback mechanisms and/or triggering mechanisms that amplify the Milankovitch cycles. What these feedbacks are and how they work are not fully understood.

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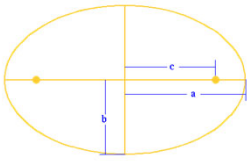
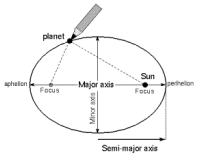
Earth's Orbit

Kepler's First Law: The orbit of every planet is an ellipse with the Sun at one of the two foci.



Eccentricity = c/a

Johannes Kepler (1571-1630)



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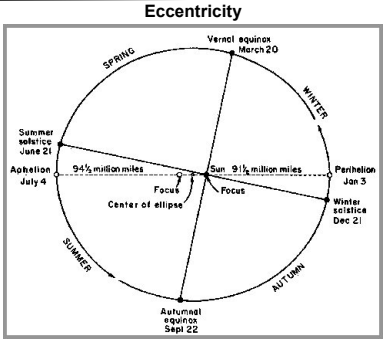
Eccentricity



John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery,* Harvard Univ. Press, 1979.

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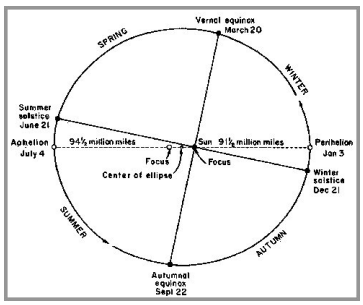
Eccentricity

Perihelion: 91.5×10^6 mi
Aphelion: 94.5×10^6 mi

Semimajor axis: 93×10^6 mi

Eccentricity: 1.5/93 = 0.016



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Glacial Cycles

Eccentricity

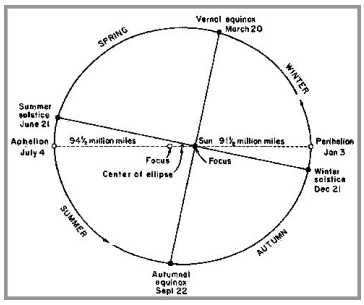
Perihelion: 91.5
Aphelion: 94.5

Change in radius:
3/93 = 3.2%

Change in insolation:
6.4%

Six percent less insolation in the southern winter than the northern winter.

6.4% of 342 W/m^2 = 22 W/m^2



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Global Annual Average Insolation

Solar output: $K \approx 4 \times 10^{26}$ Watts

Solar intensity at distance r from the sun:

$$Q(t) = \frac{K}{4\pi r(t)^2} \quad \mathrm{Wm}^{-2}$$

Cross section of Earth: πr_E^2 m^2

Global solar input: $\frac{Kr_E^2}{4r(t)^2}$ W

Total annual solar input (P = one year (in seconds)):

$$\int_0^P \frac{Kr_E^2}{4r(t)^2}\,dt = \frac{Kr_E^2}{4}\int_0^P \frac{dt}{r(t)^2} \quad \text{Joules}$$

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Global Annual Average Insolation

Specific angular momentum (angular momentum per unit mass):

$$\Omega = r^2\dot{\theta} \quad \mathrm{m}^2\mathrm{s}^{-1}$$

Total annual solar input:

$$\frac{Kr_E^2}{4}\int_0^P \frac{dt}{r(t)^2} = \frac{Kr_E^2}{4}\int_0^P \frac{\dot{\theta}dt}{\Omega} = \frac{Kr_E^2}{4\Omega}\int_0^{2\pi} d\theta = \frac{\pi Kr_E^2}{2\Omega} \quad \text{Joules}$$

Mean annual solar input: $\frac{\pi Kr_E^2}{2P\Omega}$ Watts

Mean annual solar intensity on the Earth's surface:

$$\frac{\pi Kr_E^2}{2P\Omega}\cdot\frac{1}{4\pi r_E^2} = \frac{K}{8P\Omega} \quad \mathrm{Wm}^{-2}$$

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Global Annual Average Insolation

Kepler's Third Law: $P \sim a^{3/2}$ a = semimajor axis

Derived from Kepler: $\Omega^2 \sim a(1-e^2)$ e = eccentricity

Mean annual solar intensity:

$$\frac{K}{8P\Omega} = \frac{\hat{K}}{a^{3/2}a^{1/2}\sqrt{1-e^2}} = \frac{\hat{K}a^{-2}}{\sqrt{1-e^2}} \quad \mathrm{Wm}^{-2}$$

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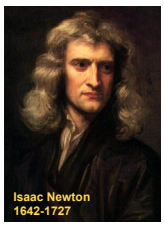
Planetary Motion

$$m_i\frac{d^2x_i}{dt^2} = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_im_j(x_j - x_i)}{|x_j - x_i|^3}$$



Jacques Laskar (1955-)

The orbits of all the planets can be computed (both forward and backward in time) for billions of years.

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Earth's Semi-major Axis

Laskar:

Global Annual Average Insolation: $\frac{\hat{K}a^{-2}}{\sqrt{1-e^2}}$

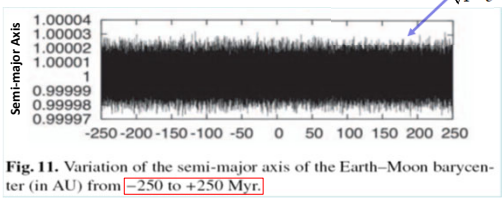

Fig. 11. Variation of the semi-major axis of the Earth–Moon barycenter (in AU) from −250 to +250 Myr.

Semi major axis does not change much:
0.005% corresponding to .01% change in global average insolation

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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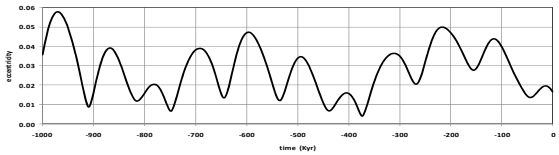
Earth's Eccentricity



Note periods of about 100 kyr and 400 kyr.

The effect due to eccentricity is more significant:

As e varies between 0 and 0.06, $(1\text{-}e^2)^{-1/2}$ varies between 1 and 1.0018, or about 0.2%. (Twenty times the effect due to a.)

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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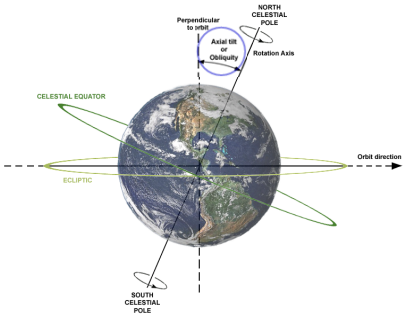
Earth's Obliquity



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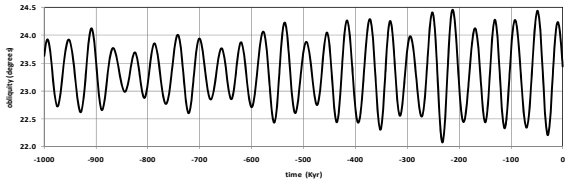
Earth's Obliquity



Note period of about 41 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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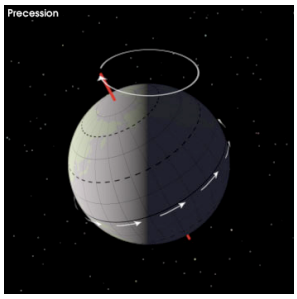
Precession



`http://earthobservatory.nasa.gov/Library/Giants/Milankovitch/milankovitch_2.html`

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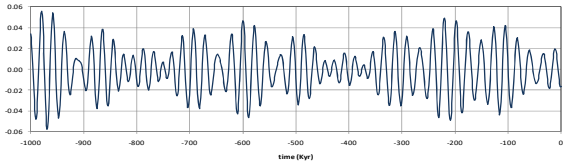
Earth's Precession Index



index = $e \sin\rho$, where e = eccentricity and ρ = precession angle (measured from spring equinox)

Note period of about 23 Kyr.

J. Laskar, et al (2004) A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics* **428**, 261–285.

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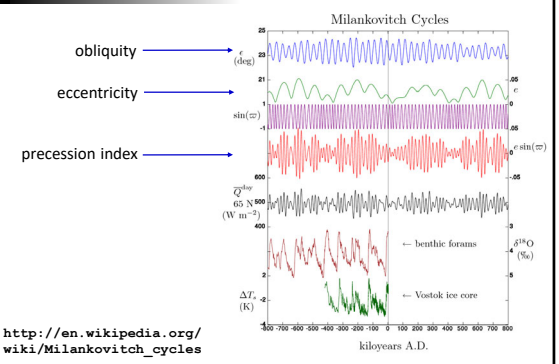

`http://en.wikipedia.org/wiki/Milankovitch_cycles`

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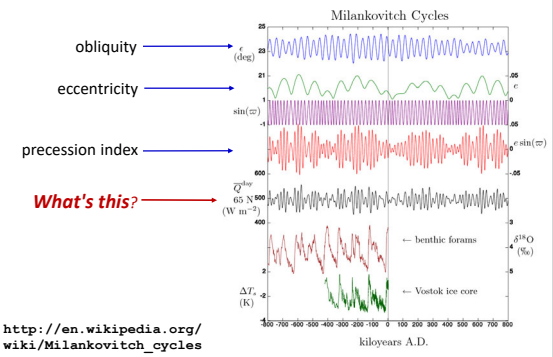

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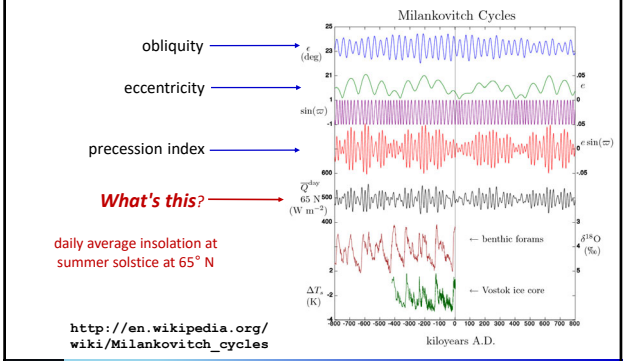

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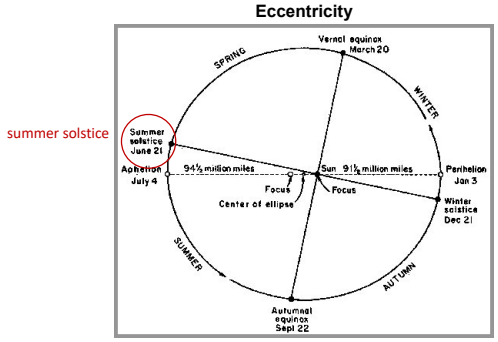
Eccentricity



John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery,* Harvard Univ. Press, 1979.

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Glacial Cycles

Daily Average Insolation at Summer Solstice at 65° N

Insolation at a point on the Earth's surface

$$I(\beta,\rho,r,\theta,\varphi,\gamma)=\frac{K}{4\pi r^2}\left[-\cos\varphi\left(\cos\beta\cos(\theta-\rho)\cos\gamma+\sin(\theta-\rho)\sin\gamma\right)-\sin\varphi\sin\beta\cos(\theta-\rho)\right]^+$$

(φ,γ) = (latitude, longitude)
(r,θ) = position of Earth in orbital plane
β = obliquity angle
ρ = precession angle

Daily average insolation at latitude φ at summer solstice

$$\overline{I}(e,\beta,\rho',\varphi)=Q\frac{(1-e\sin\rho')^2}{(1-e^2)^2}\frac{1}{2\pi}\int_0^{2\pi}\left[\cos\varphi\cos\beta\cos\gamma+\sin\varphi\sin\beta\right]^+ d\gamma$$

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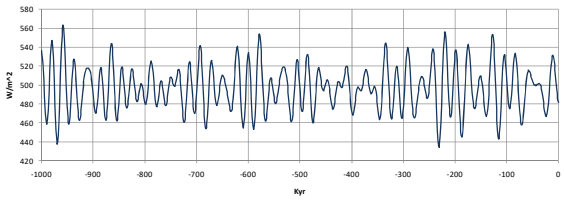
Glacial Cycles

Daily Average Insolation at Summer Solstice at 65° N



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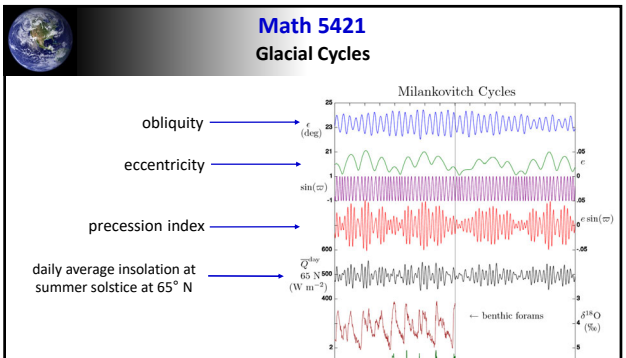

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Glacial Cycles

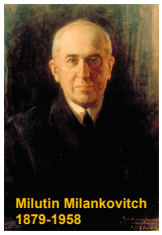
Who was Milankovitch?

Milutin Milankovitch was a Serbian mathematician and professor at the University of Belgrade.

In 1920 he published his seminal work on the relation between insolation and the Earth's orbital parameters.

In 1941 he published a book explaining his entire theory.

His work was not fully accepted until 1976.



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Glacial Cycles

What happened in 1976?


James D. Hays

Hays, Imbrie, and Shackleton, "Variations in the Earth's Orbit: Pacemaker of the Ice Ages," *Science* **194**, 10 December 1976.


John Imbrie

"It is concluded that changes in the earth's orbital geometry are the fundamental cause of the succession of Quaternary ice ages."


Nicholas Shackleton

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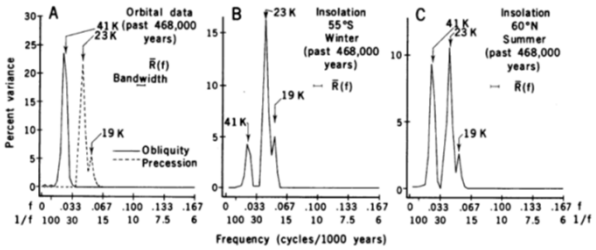
Glacial Cycles

Solar Forcing (Hays, et al)



Hays, *et al*, *Science* **194** (1976), p. 1125

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Hays, et al, Summary

1) Three indices of global climate have been monitored in the record of the past 450,000 years in Southern Hemisphere ocean-floor sediments.

2) ... climatic variance of these records is concentrated in three discrete spectral peaks at periods of 23,000, 42,000, and approximately 100,000 years. These peaks correspond to the dominant periods of the earth's solar orbit, and contain respectively about 10, 25, and 50 percent of the climatic variance.

Hays, *et al*, *Science* **194** (1976), p. 1125

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Hays, et al, Summary

3) The 42,000-year climatic component has the same period as variations in the obliquity of the earth's axis and retains a constant phase relationship with it.

4) The 23,000-year portion of the variance displays the same periods (about 23,000 and 19,000 years) as the quasiperiodic precession index.

5) The dominant, 100,000-year climatic component has an average period close to, and is in phase with, orbital eccentricity. Unlike the correlations between climate and the higher-frequency orbital variations (which can be explained on the assumption that the climate system responds linearly to orbital forcing), an explanation of the correlation between climate and eccentricity probably requires an assumption of nonlinearity.

Hays, *et al*, *Science* **194** (1976), p. 1125

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Hays, et al, Summary

6) It is concluded that changes in the earth's orbital geometry are the fundamental cause of the succession of Quaternary ice ages.

7) A model of future climate based on the observed orbital-climate relationships, ***but ignoring anthropogenic effects***, predicts that the long-term trend over the next seven thousand years is toward extensive Northern Hemisphere glaciation*.

*Quoted by George Will, Washington Post, February 5, 2009

Hays, *et al*, *Science* **194** (1976), p. 1125

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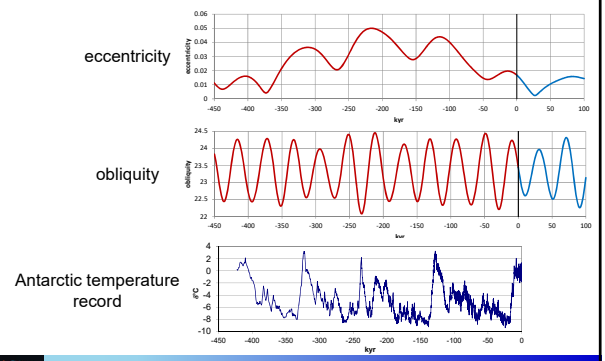
Glacial Cycles



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Glacial Cycles

Earth's climate has changed many times in the past.
Why do we think humans are responsible now?

Here's a reason.

The paleoclimate evidence points to the conclusion that the Earth should be entering a new ice age.

But we're not.

Instead, the Earth is warming and the ice sheets are melting.

The climate is not following the patterns of the last million years.

Something has changed.

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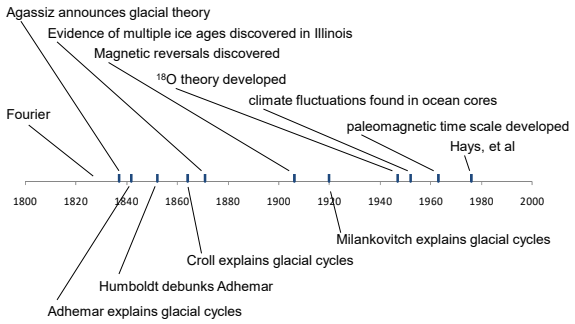
Glacial Cycles

History of Discovery



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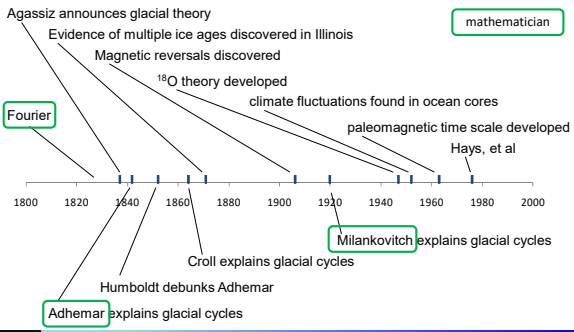
History of Discovery



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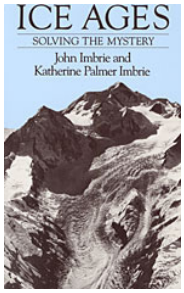
Suggested Reading



John Imbrie & Katherine Palmer Imbrie, *Ice Ages: Solving the Mystery*, Harvard University Press, 1979

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