Math 5421
An Introduction to Mathematical Climate Models

Spring 2025
1:25 – 3:20 Tuesdays and Thursdays
Blegen Hall 155

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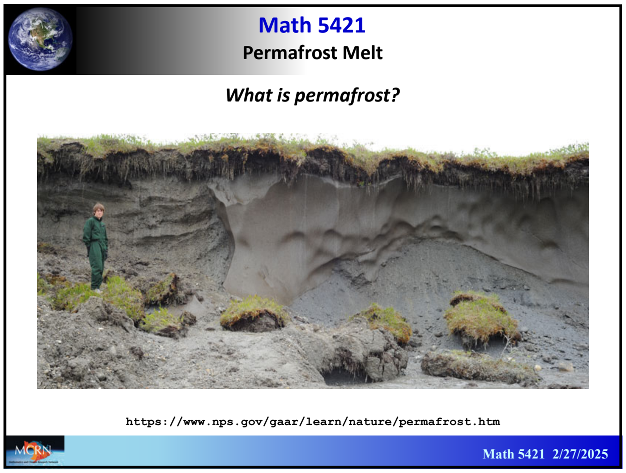
course website
`https://www-users.cse.umn.edu/~mcgehee/Course/Math5421/`



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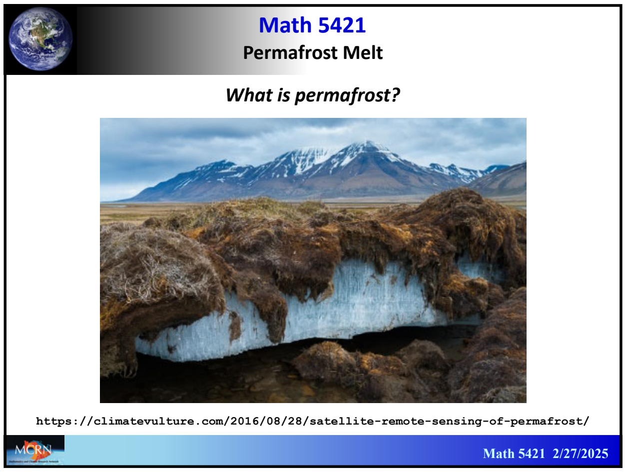
Permafrost Melt

What is permafrost?

`https://www.nps.gov/gaar/learn/nature/permafrost.htm`



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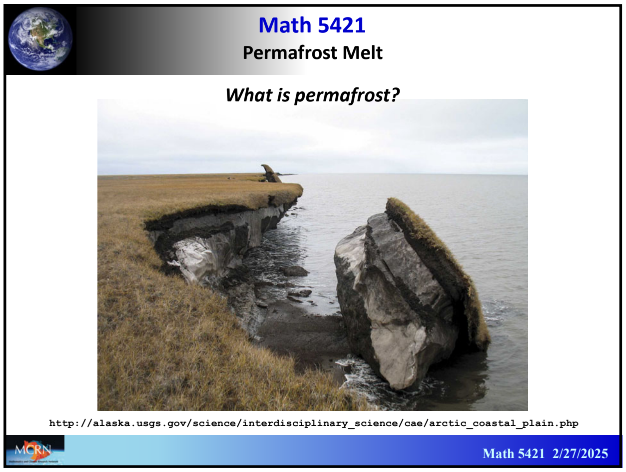
`https://climatevulture.com/2016/08/28/satellite-remote-sensing-of-permafrost/`



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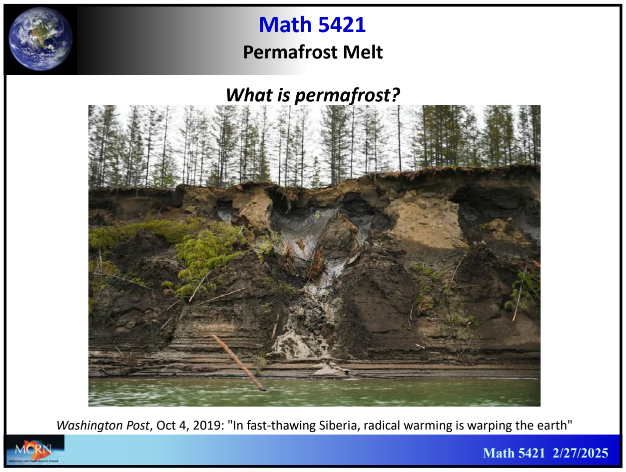
What is permafrost?

`http://alaska.usgs.gov/science/interdisciplinary_science/cae/arctic_coastal_plain.php`



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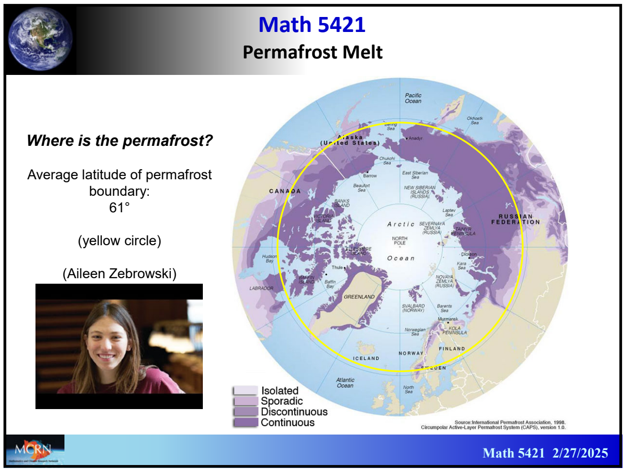
What is permafrost?

Washington Post, Oct 4, 2019: "In fast-thawing Siberia, radical warming is warping the earth"



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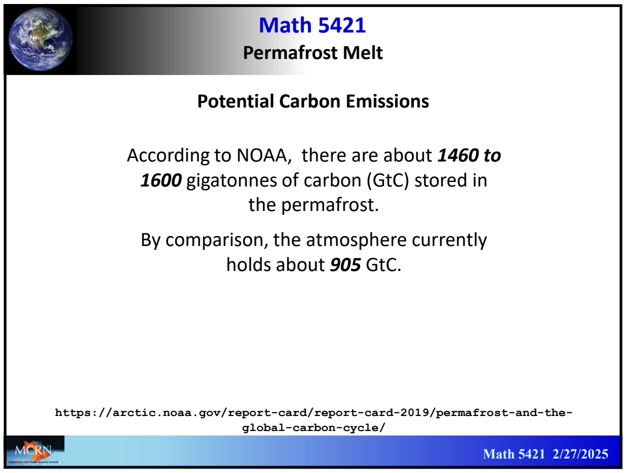
Where is the permafrost?

Average latitude of permafrost boundary:
61°

(yellow circle)

(Aileen Zebrowski)

Source: International Permafrost Association, 1998. Circumpolar Active-Layer Permafrost System (CAPS), version 1.0.



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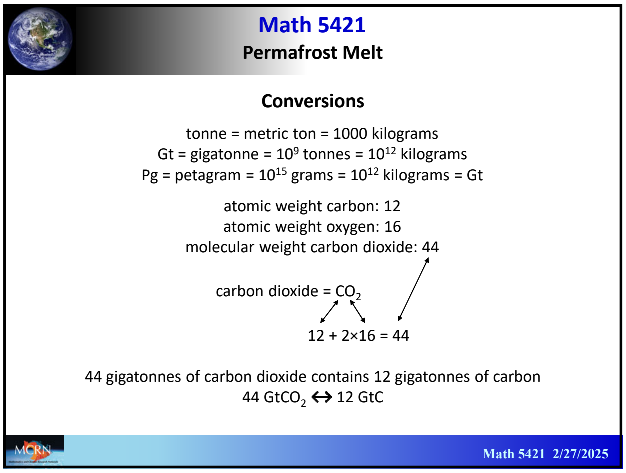
Potential Carbon Emissions

According to NOAA, there are about ***1460 to 1600*** gigatonnes of carbon (GtC) stored in the permafrost.

By comparison, the atmosphere currently holds about ***905*** GtC.

`https://arctic.noaa.gov/report-card/report-card-2019/permafrost-and-the-global-carbon-cycle/`



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Conversions

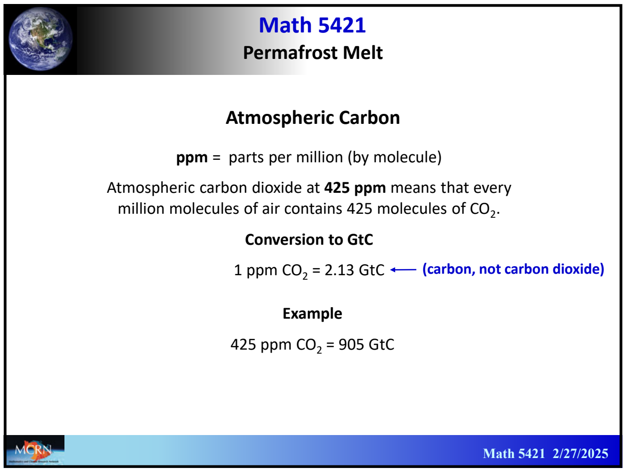
tonne = metric ton = 1000 kilograms
Gt = gigatonne = 10^9 tonnes = 10^{12} kilograms
Pg = petagram = 10^{15} grams = 10^{12} kilograms = Gt

atomic weight carbon: 12
atomic weight oxygen: 16
molecular weight carbon dioxide: 44

carbon dioxide = CO_2

12 + 2×16 = 44

44 gigatonnes of carbon dioxide contains 12 gigatonnes of carbon
44 $GtCO_2$ ↔ 12 GtC



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Atmospheric Carbon

ppm = parts per million (by molecule)

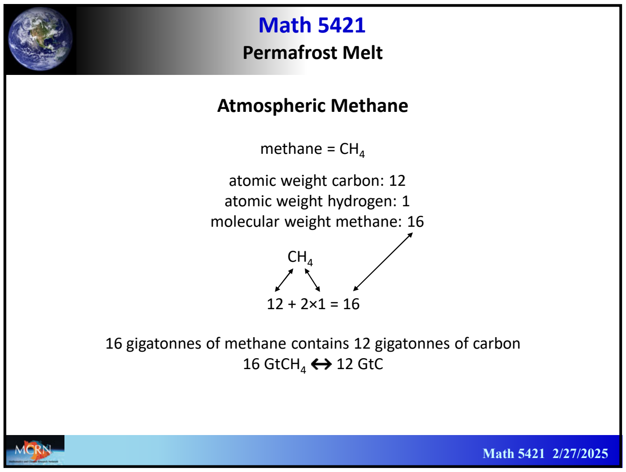
Atmospheric carbon dioxide at **425 ppm** means that every million molecules of air contains 425 molecules of CO_2.

Conversion to GtC

1 ppm CO_2 = 2.13 GtC ⟵ **(carbon, not carbon dioxide)**

Example

425 ppm CO_2 = 905 GtC



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Atmospheric Methane

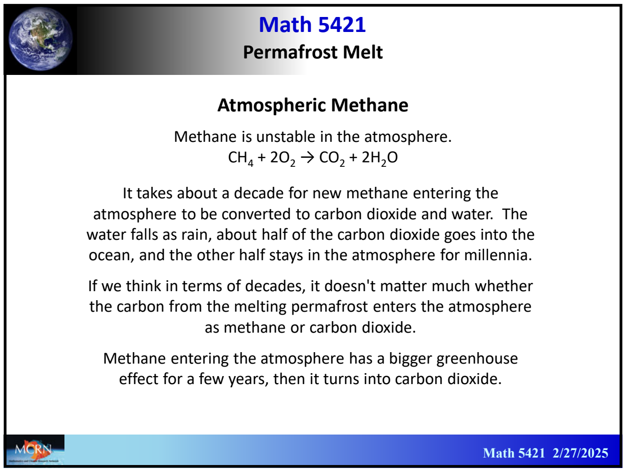
methane = CH_4

atomic weight carbon: 12
atomic weight hydrogen: 1
molecular weight methane: 16

CH_4

12 + 2×1 = 16

16 gigatonnes of methane contains 12 gigatonnes of carbon
16 $GtCH_4$ ↔ 12 GtC



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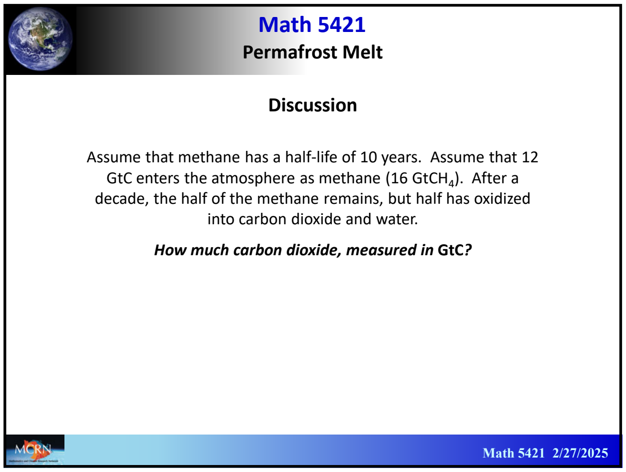
Atmospheric Methane

Methane is unstable in the atmosphere.
$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O$

It takes about a decade for new methane entering the atmosphere to be converted to carbon dioxide and water. The water falls as rain, about half of the carbon dioxide goes into the ocean, and the other half stays in the atmosphere for millennia.

If we think in terms of decades, it doesn't matter much whether the carbon from the melting permafrost enters the atmosphere as methane or carbon dioxide.

Methane entering the atmosphere has a bigger greenhouse effect for a few years, then it turns into carbon dioxide.



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Discussion

Assume that methane has a half-life of 10 years. Assume that 12 GtC enters the atmosphere as methane (16 $GtCH_4$). After a decade, the half of the methane remains, but half has oxidized into carbon dioxide and water.

How much carbon dioxide, measured in* GtC*?

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Assume that methane has a half-life of 10 years. Assume that 12 GtC enters the atmosphere as methane (16 GtCH$_4$). After a decade, the half of the methane remains, but half has oxidized into carbon dioxide and water.

How much carbon dioxide, measured in* GtC*?
6 GtC

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6 GtC

How much carbon dioxide, measured in* GtCO$_2$*?

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How much carbon dioxide, measured in* GtC*?
6 GtC

How much carbon dioxide, measured in* GtCO$_2$*?
22 GtC0$_2$

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Assume that methane has a half-life of 10 years. Assume that 12 GtC enters the atmosphere as methane (16 GtCH$_4$). After a decade, the half of the methane remains, but half has oxidized into carbon dioxide and water.

How much carbon dioxide, measured in* GtC*?
6 GtC

How much carbon dioxide, measured in* GtCO$_2$*?
22 GtC0$_2$

How much carbon dioxide, measured in* ppm*?

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Discussion

Assume that methane has a half-life of 10 years. Assume that 12 GtC enters the atmosphere as methane (16 GtCH$_4$). After a decade, the half of the methane remains, but half has oxidized into carbon dioxide and water.

How much carbon dioxide, measured in* GtC*?
6 GtC

How much carbon dioxide, measured in* GtCO$_2$*?
22 GtC0$_2$

How much carbon dioxide, measured in* ppm*?
6/2.13 = 2.8 ppm

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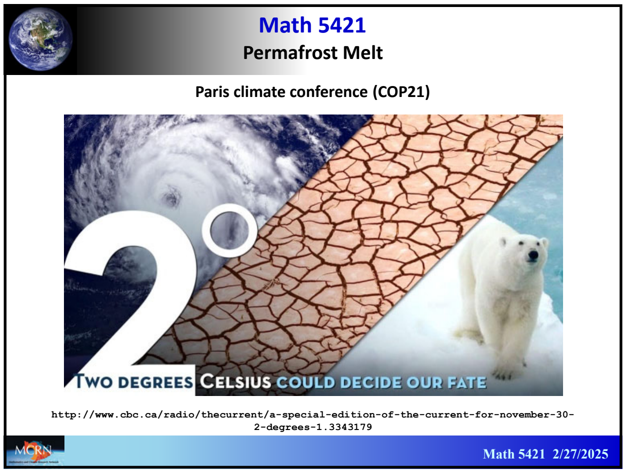
Paris climate conference (COP21)

`http://www.npr.org/sections/thetwo-way/2015/12/12/459502597/2-degrees-100-billion-the-world-climate-agreement-by-the-numbers`



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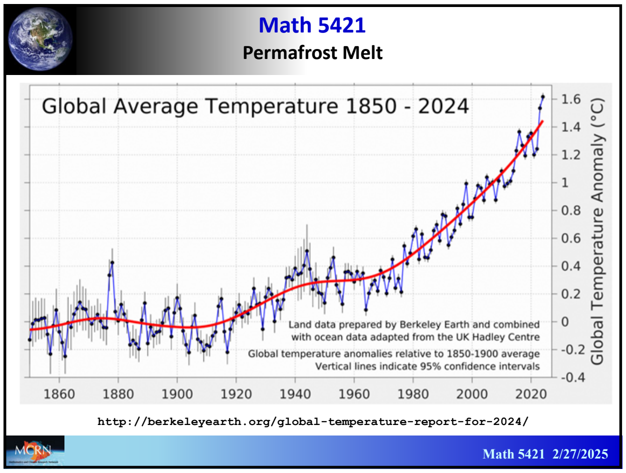
Paris climate conference (COP21)

`http://www.cbc.ca/radio/thecurrent/a-special-edition-of-the-current-for-november-30-2-degrees-1.3343179`



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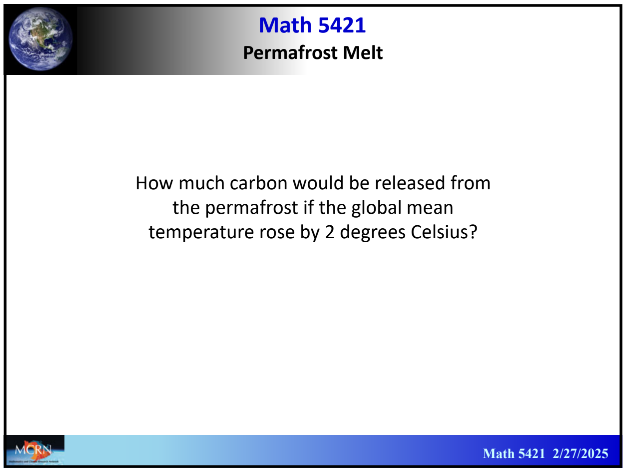
`http://berkeleyearth.org/global-temperature-report-for-2024/`



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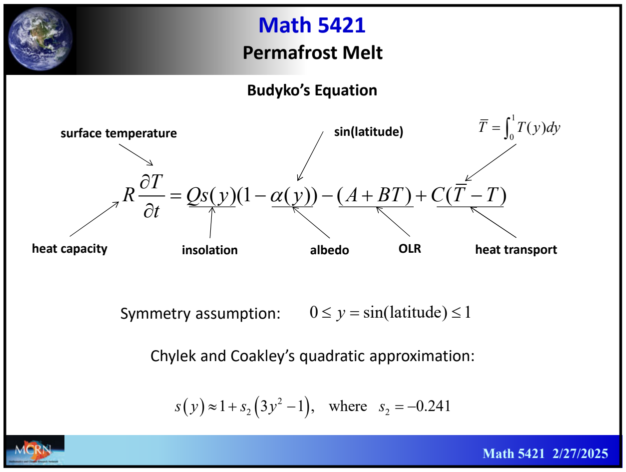
How much carbon would be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?



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Budyko's Equation

$$\overset{\text{heat capacity}}{R}\,\overset{\text{surface temperature}}{\frac{\partial T}{\partial t}} = \underbrace{Qs(y)}_{\text{insolation}}(1-\underbrace{\alpha(y)}_{\text{albedo}}) - \underbrace{(A+BT)}_{\text{OLR}} + \underbrace{C(\bar{T}-T)}_{\text{heat transport}}$$

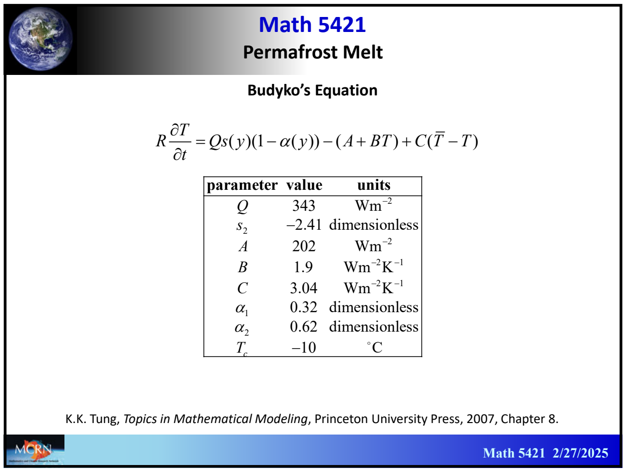
sin(latitude)

$$\bar{T} = \int_0^1 T(y)\,dy$$

Symmetry assumption: $0 \le y = \sin(\text{latitude}) \le 1$

Chylek and Coakley's quadratic approximation:

$$s(y) \approx 1 + s_2\left(3y^2 - 1\right), \quad \text{where} \quad s_2 = -0.241$$



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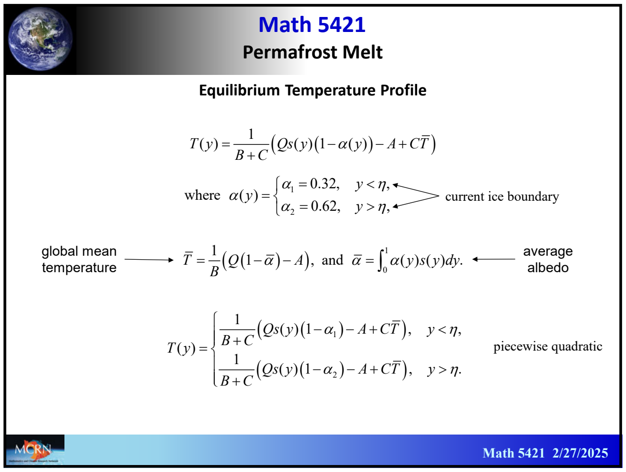
Budyko's Equation

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) + C(\bar{T}-T)$$

parameter	value	units
Q	343	Wm^{-2}
s_2	–2.41	dimensionless
A	202	Wm^{-2}
B	1.9	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$
C	3.04	$\mathrm{Wm}^{-2}\mathrm{K}^{-1}$
α_1	0.32	dimensionless
α_2	0.62	dimensionless
T_c	–10	$^\circ\mathrm{C}$

K.K. Tung, *Topics in Mathematical Modeling*, Princeton University Press, 2007, Chapter 8.



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Equilibrium Temperature Profile

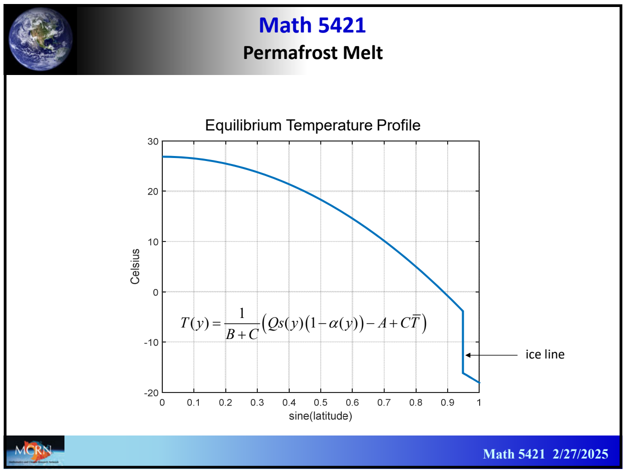
$$T(y) = \frac{1}{B+C}\left(Qs(y)\left(1-\alpha(y)\right) - A + C\bar{T}\right)$$

$$\text{where} \quad \alpha(y) = \begin{cases} \alpha_1 = 0.32, & y < \eta, \\ \alpha_2 = 0.62, & y > \eta, \end{cases} \quad \longleftarrow \text{current ice boundary}$$

global mean temperature ⟶ $\bar{T} = \frac{1}{B}\left(Q\left(1-\bar{\alpha}\right) - A\right)$, and $\bar{\alpha} = \int_0^1 \alpha(y)s(y)\,dy.$ ⟵ average albedo

$$T(y) = \begin{cases} \frac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right) - A + C\bar{T}\right), & y < \eta, \\ \frac{1}{B+C}\left(Qs(y)\left(1-\alpha_2\right) - A + C\bar{T}\right), & y > \eta. \end{cases}$$

piecewise quadratic



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Equilibrium Temperature Profile

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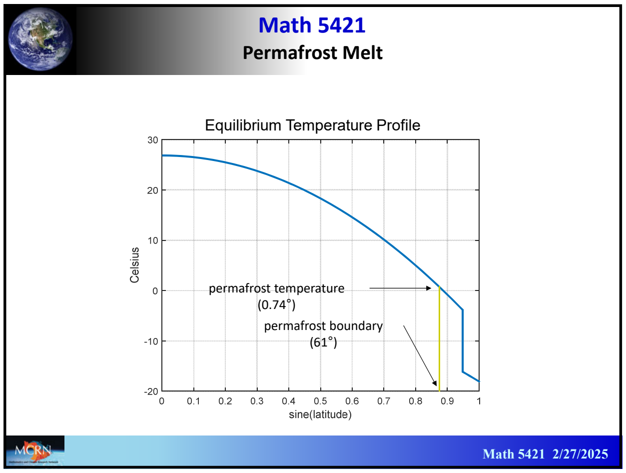

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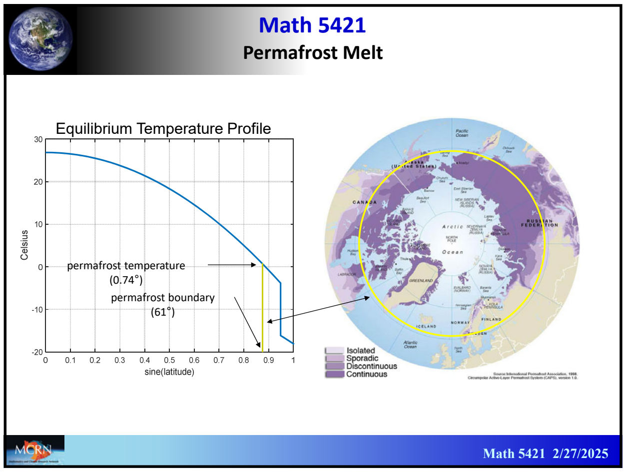
Equilibrium Temperature Profile

permafrost temperature (0.74°)

permafrost boundary (61°)



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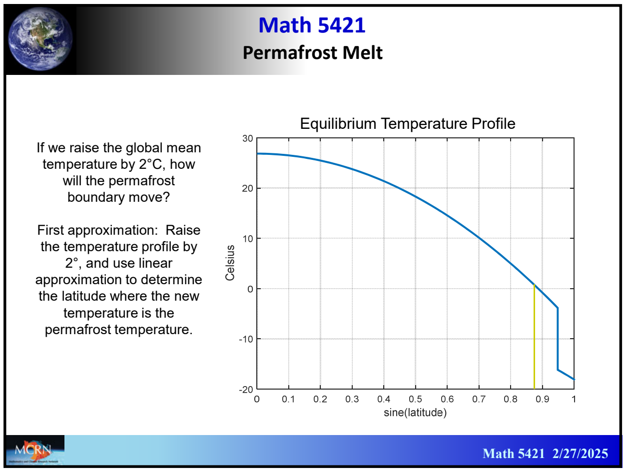
Equilibrium Temperature Profile

permafrost temperature (0.74°)

permafrost boundary (61°)



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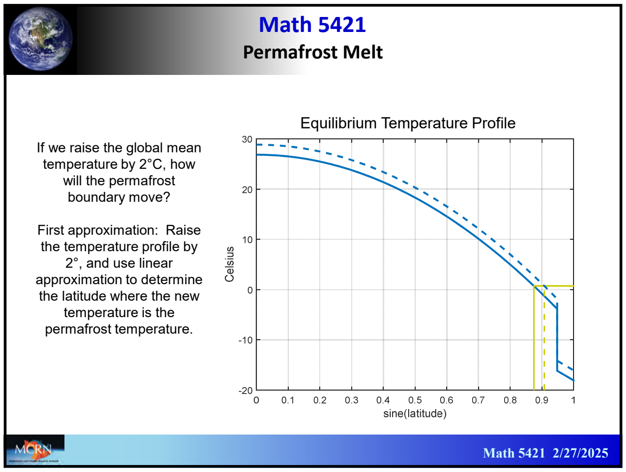
Permafrost Melt

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and use linear approximation to determine the latitude where the new temperature is the permafrost temperature.

Equilibrium Temperature Profile



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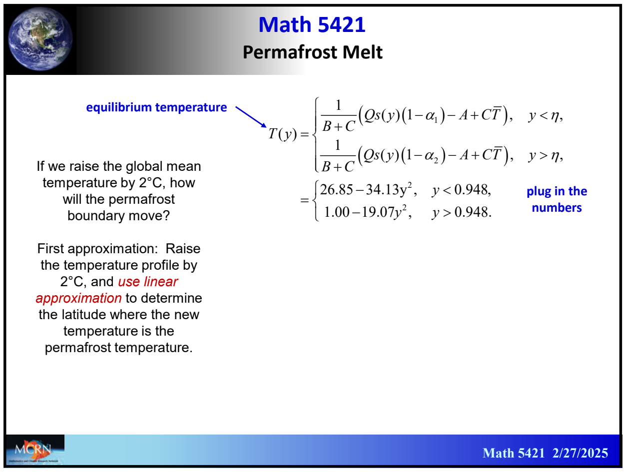

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equilibrium temperature

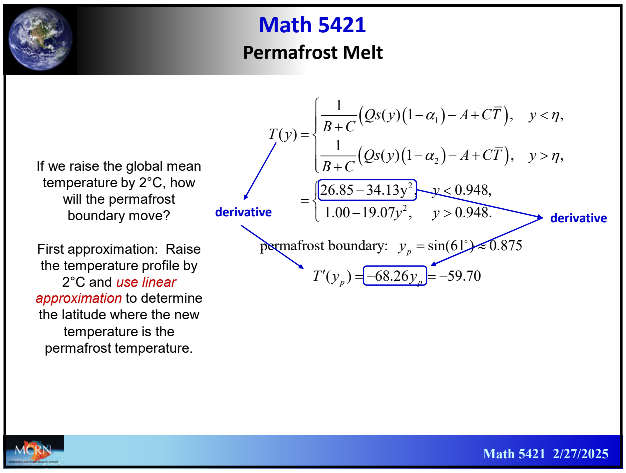
$$T(y)=\begin{cases}\dfrac{1}{B+C}\left(Qs(y)\left(1-\alpha_1\right)-A+C\overline{T}\right), & y<\eta,\\ \dfrac{1}{B+C}\left(Qs(y)\left(1-\alpha_2\right)-A+C\overline{T}\right), & y>\eta,\end{cases}$$

$$=\begin{cases}26.85-34.13\mathrm{y}^2, & y<0.948,\\ 1.00-19.07y^2, & y>0.948.\end{cases}$$

plug in the numbers

If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°C, and *use linear approximation* to determine the latitude where the new temperature is the permafrost temperature.



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$$=\begin{cases}\boxed{26.85-34.13\mathrm{y}^2}, & y<0.948,\\ 1.00-19.07y^2, & y>0.948.\end{cases}$$

derivative

derivative

permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=\boxed{-68.26y_p}=-59.70$$

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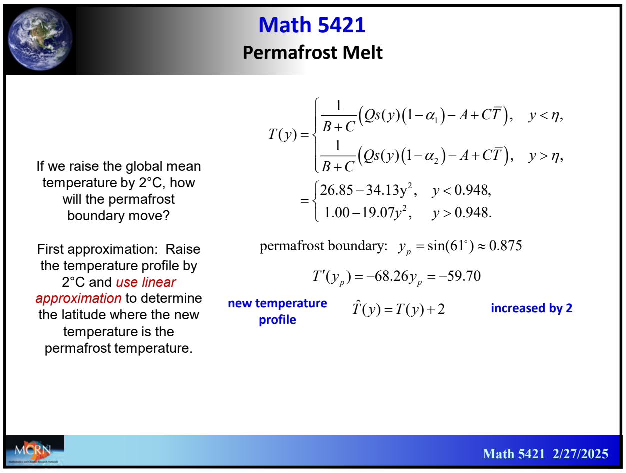

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permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

new temperature profile $\hat{T}(y)=T(y)+2$ **increased by 2**

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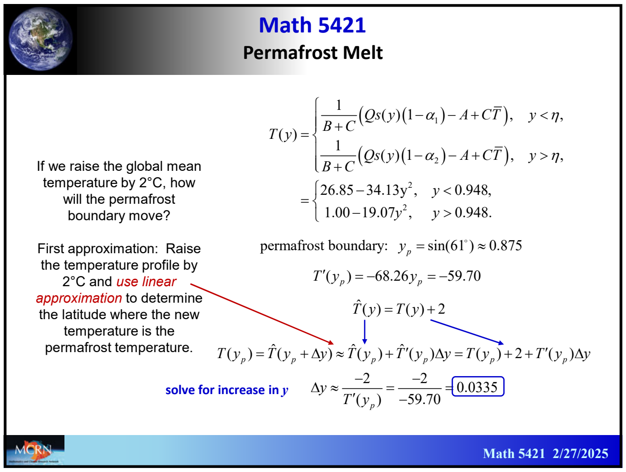

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permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

$$\hat{T}(y)=T(y)+2$$

$$T(y_p)=\hat{T}(y_p+\Delta y)\approx\hat{T}(y_p)+\hat{T}'(y_p)\Delta y=T(y_p)+2+T'(y_p)\Delta y$$

solve for increase in *y* $\Delta y\approx\dfrac{-2}{T'(y_p)}=\dfrac{-2}{-59.70}=\boxed{0.0335}$

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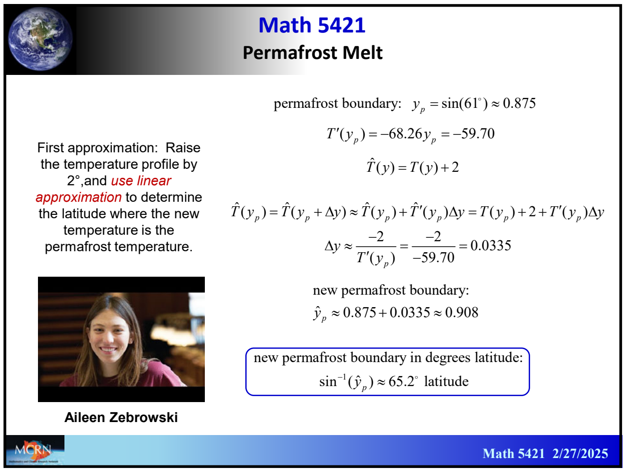

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First approximation: Raise the temperature profile by 2°,and *use linear approximation* to determine the latitude where the new temperature is the permafrost temperature.

Aileen Zebrowski

permafrost boundary: $y_p=\sin(61^\circ)\approx 0.875$

$$T'(y_p)=-68.26y_p=-59.70$$

$$\hat{T}(y)=T(y)+2$$

$$\hat{T}(y_p)=\hat{T}(y_p+\Delta y)\approx\hat{T}(y_p)+\hat{T}'(y_p)\Delta y=T(y_p)+2+T'(y_p)\Delta y$$

$$\Delta y\approx\frac{-2}{T'(y_p)}=\frac{-2}{-59.70}=0.0335$$

new permafrost boundary:

$\hat{y}_p\approx 0.875+0.0335\approx 0.908$

new permafrost boundary in degrees latitude:

$\sin^{-1}(\hat{y}_p)\approx 65.2^\circ$ latitude

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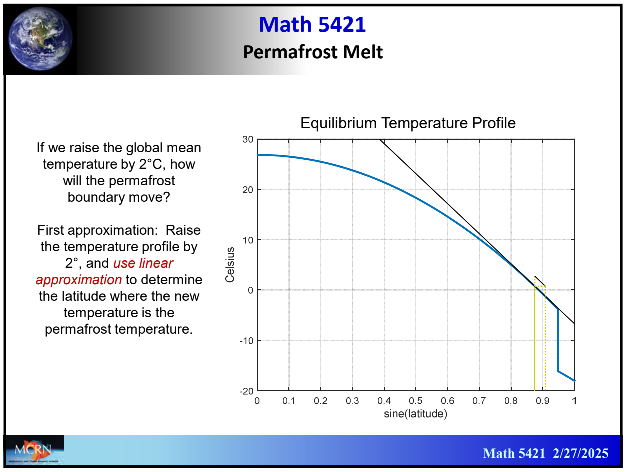

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If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

First approximation: Raise the temperature profile by 2°, and *use linear approximation* to determine the latitude where the new temperature is the permafrost temperature.

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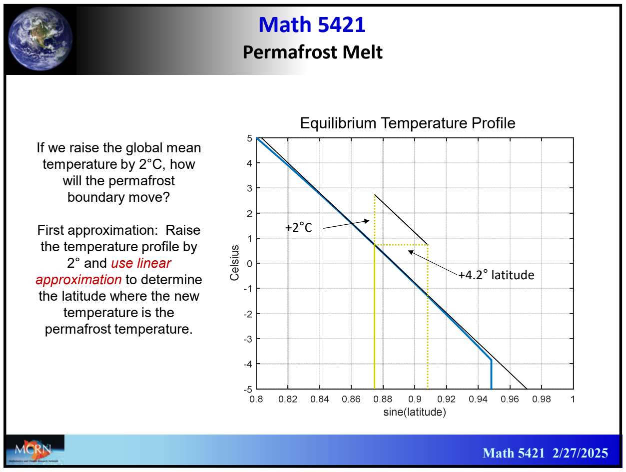

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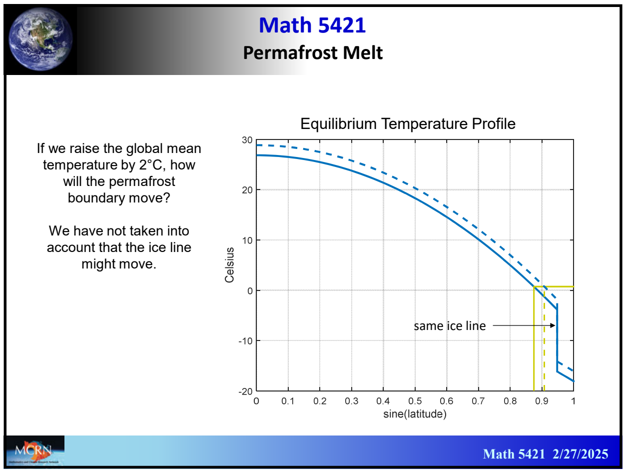

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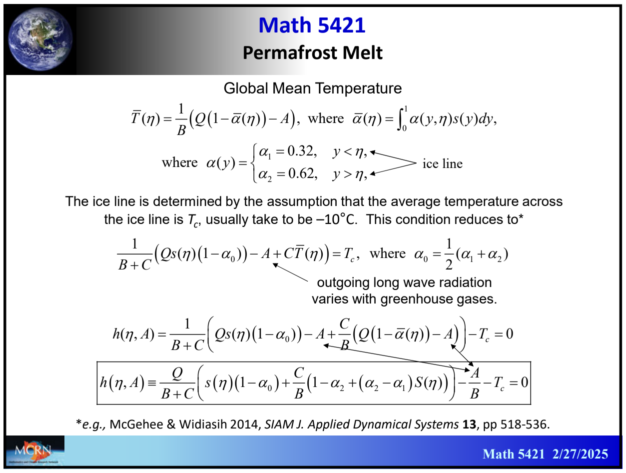
If we raise the global mean temperature by 2°C, how will the permafrost boundary move?

We have not taken into account that the ice line might move.



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Permafrost Melt

Global Mean Temperature

$$\overline{T}(\eta)=\frac{1}{B}\big(Q\big(1-\overline{\alpha}(\eta)\big)-A\big),\ \text{where}\ \overline{\alpha}(\eta)=\int_0^1\alpha(y,\eta)s(y)dy,$$

$$\text{where}\ \ \alpha(y)=\begin{cases}\alpha_1=0.32, & y<\eta,\\ \alpha_2=0.62, & y>\eta,\end{cases}$$

ice line

The ice line is determined by the assumption that the average temperature across the ice line is T_c, usually take to be –10°C. This condition reduces to*

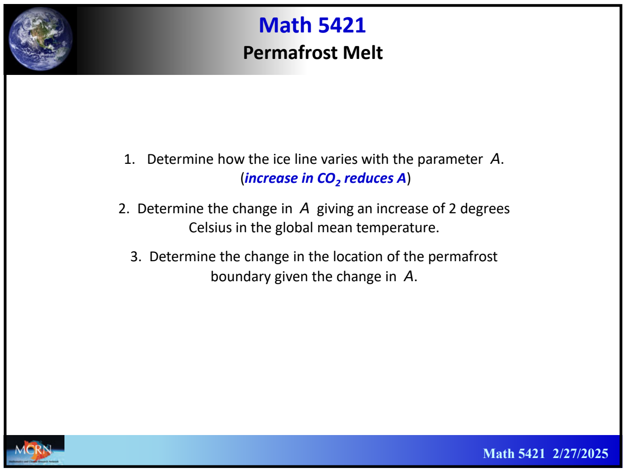
$$\frac{1}{B+C}\big(Qs(\eta)\big(1-\alpha_0\big)\big)-A+C\overline{T}(\eta)\big)=T_c,\ \ \text{where}\ \ \alpha_0=\frac{1}{2}(\alpha_1+\alpha_2)$$

outgoing long wave radiation varies with greenhouse gases.

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\big(1-\alpha_0\big)\big)-A+\frac{C}{B}\big(Q\big(1-\overline{\alpha}(\eta)\big)-A\big)\right)-T_c=0$$

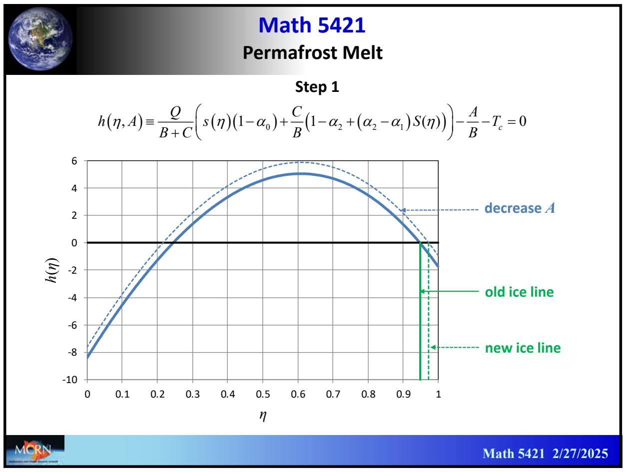
$$\boxed{h(\eta,A)\equiv\frac{Q}{B+C}\left(s(\eta)\big(1-\alpha_0\big)+\frac{C}{B}\big(1-\alpha_2+\big(\alpha_2-\alpha_1\big)S(\eta)\big)\right)-\frac{A}{B}-T_c=0}$$

e.g.*, McGehee & Widiasih 2014, *SIAM J. Applied Dynamical Systems* **13, pp 518-536.



Math 5421
Permafrost Melt

1. Determine how the ice line varies with the parameter A. (***increase in CO$_2$ reduces A***)
2. Determine the change in A giving an increase of 2 degrees Celsius in the global mean temperature.
3. Determine the change in the location of the permafrost boundary given the change in A.



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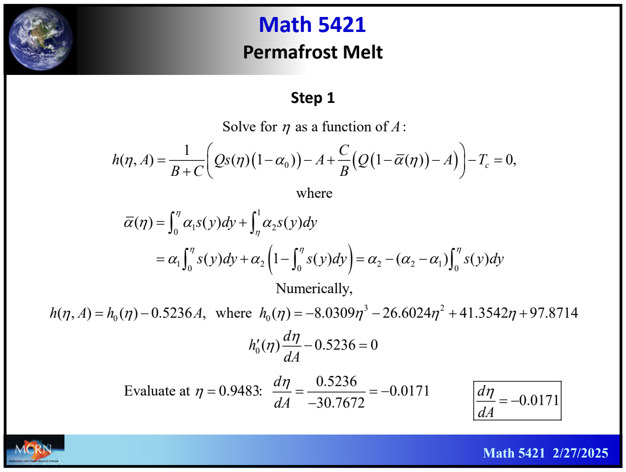
Permafrost Melt

Step 1

$$h(\eta,A)\equiv\frac{Q}{B+C}\left(s(\eta)\big(1-\alpha_0\big)+\frac{C}{B}\big(1-\alpha_2+\big(\alpha_2-\alpha_1\big)S(\eta)\big)\right)-\frac{A}{B}-T_c=0$$



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Permafrost Melt

Step 1

Solve for η as a function of A:

$$h(\eta,A)=\frac{1}{B+C}\left(Qs(\eta)\big(1-\alpha_0\big)\big)-A+\frac{C}{B}\big(Q\big(1-\overline{\alpha}(\eta)\big)-A\big)\right)-T_c=0,$$

where

$$\overline{\alpha}(\eta)=\int_0^\eta\alpha_1s(y)dy+\int_\eta^1\alpha_2s(y)dy$$

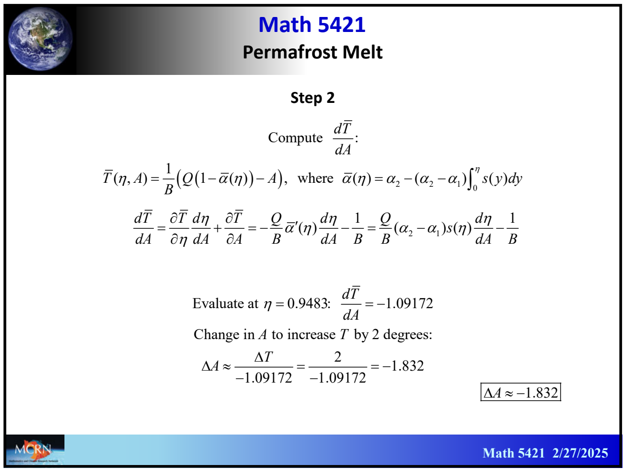
$$=\alpha_1\int_0^\eta s(y)dy+\alpha_2\left(1-\int_0^\eta s(y)dy\right)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

Numerically,

$$h(\eta,A)=h_0(\eta)-0.5236A,\ \ \text{where}\ \ h_0(\eta)=-8.0309\eta^3-26.6024\eta^2+41.3542\eta+97.8714$$

$$h_0'(\eta)\frac{d\eta}{dA}-0.5236=0$$

Evaluate at $\eta=0.9483$: $\dfrac{d\eta}{dA}=\dfrac{0.5236}{-30.7672}=-0.0171$ $\boxed{\dfrac{d\eta}{dA}=-0.0171}$



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Permafrost Melt

Step 2

Compute $\dfrac{d\overline{T}}{dA}$:

$$\overline{T}(\eta,A)=\frac{1}{B}\big(Q\big(1-\overline{\alpha}(\eta)\big)-A\big),\ \ \text{where}\ \ \overline{\alpha}(\eta)=\alpha_2-(\alpha_2-\alpha_1)\int_0^\eta s(y)dy$$

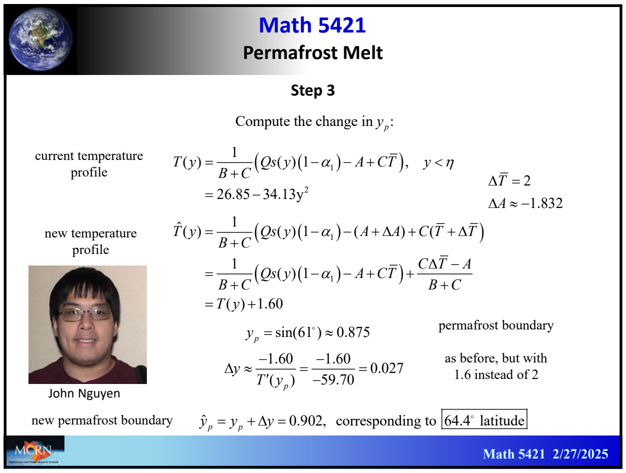
$$\frac{d\overline{T}}{dA}=\frac{\partial\overline{T}}{\partial\eta}\frac{d\eta}{dA}+\frac{\partial\overline{T}}{\partial A}=-\frac{Q}{B}\overline{\alpha}'(\eta)\frac{d\eta}{dA}-\frac{1}{B}=\frac{Q}{B}(\alpha_2-\alpha_1)s(\eta)\frac{d\eta}{dA}-\frac{1}{B}$$

Evaluate at $\eta=0.9483$: $\dfrac{d\overline{T}}{dA}=-1.09172$

Change in A to increase T by 2 degrees:

$$\Delta A\approx\frac{\Delta T}{-1.09172}=\frac{2}{-1.09172}=-1.832$$

$$\boxed{\Delta A\approx-1.832}$$



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Permafrost Melt

Step 3

Compute the change in y_p:

current temperature profile

$$T(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha_1)-A+C\bar{T}\right),\quad y<\eta$$
$$=26.85-34.13y^2$$

$$\Delta\bar{T}=2$$
$$\Delta A\approx-1.832$$

new temperature profile

$$\hat{T}(y)=\frac{1}{B+C}\left(Qs(y)(1-\alpha_1)-(A+\Delta A)+C(\bar{T}+\Delta\bar{T}\right)$$
$$=\frac{1}{B+C}\left(Qs(y)(1-\alpha_1)-A+C\bar{T}\right)+\frac{C\Delta\bar{T}-A}{B+C}$$
$$=T(y)+1.60$$

$$y_p=\sin(61^\circ)\approx 0.875$$

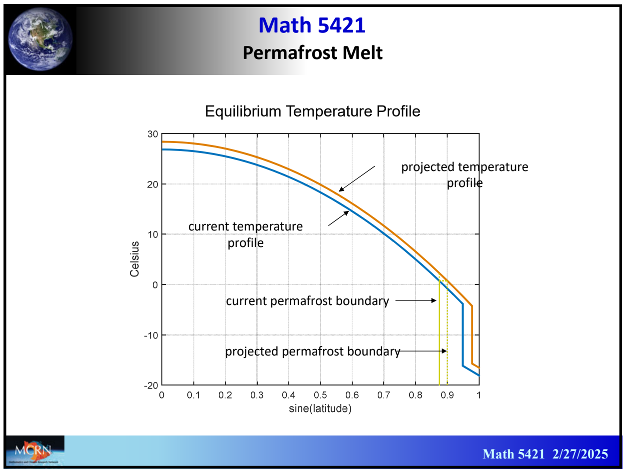
permafrost boundary

$$\Delta y\approx\frac{-1.60}{T'(y_p)}=\frac{-1.60}{-59.70}=0.027$$

as before, but with 1.6 instead of 2

John Nguyen

new permafrost boundary $\hat{y}_p=y_p+\Delta y=0.902,$ corresponding to $\boxed{64.4^\circ\text{ latitude}}$



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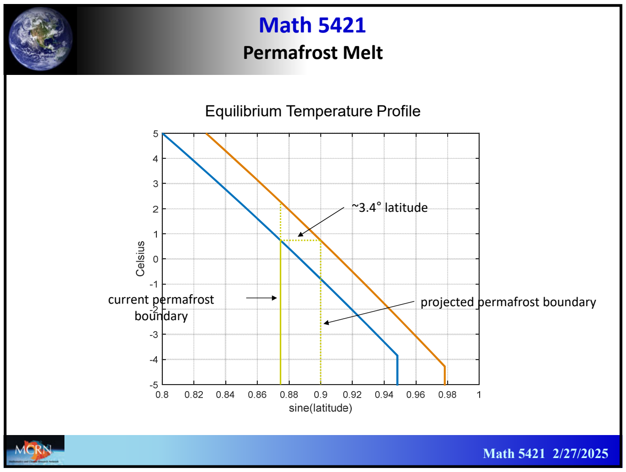
Equilibrium Temperature Profile



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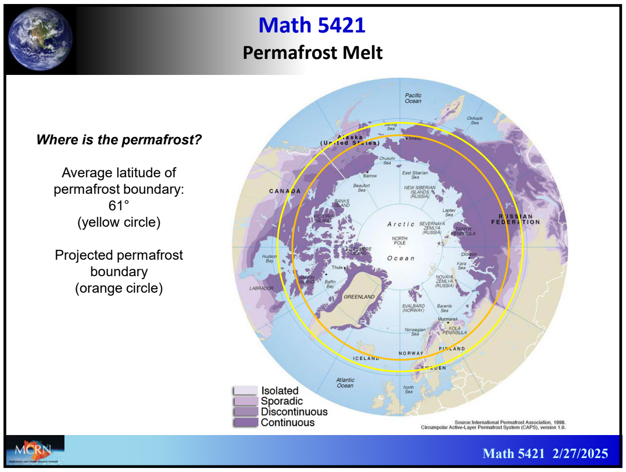
Equilibrium Temperature Profile



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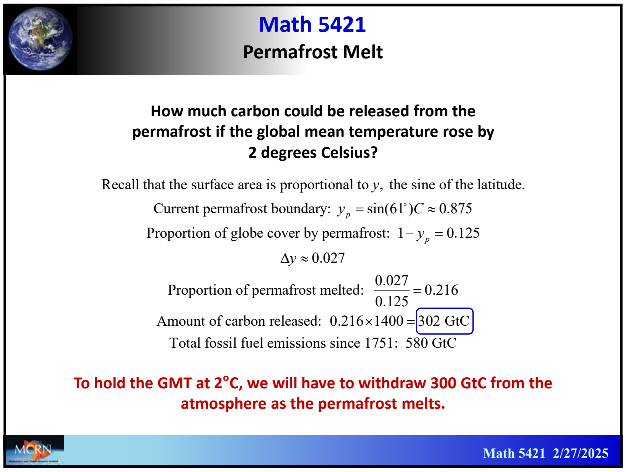
Where is the permafrost?

Average latitude of permafrost boundary:
61°
(yellow circle)

Projected permafrost boundary
(orange circle)

Source: International Permafrost Association, 1998. Circumpolar Active-Layer Permafrost System (CAPS), version 1.0.



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Permafrost Melt

How much carbon could be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

Recall that the surface area is proportional to y, the sine of the latitude.

Current permafrost boundary: $y_p=\sin(61^\circ)C\approx 0.875$

Proportion of globe cover by permafrost: $1-y_p=0.125$

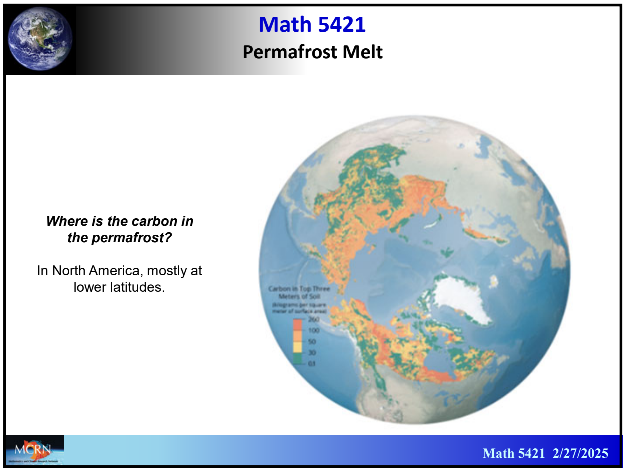
$\Delta y\approx 0.027$

Proportion of permafrost melted: $\frac{0.027}{0.125}=0.216$

Amount of carbon released: $0.216\times 1400=\boxed{302\text{ GtC}}$

Total fossil fuel emissions since 1751: 580 GtC

To hold the GMT at 2°C, we will have to withdraw 300 GtC from the atmosphere as the permafrost melts.



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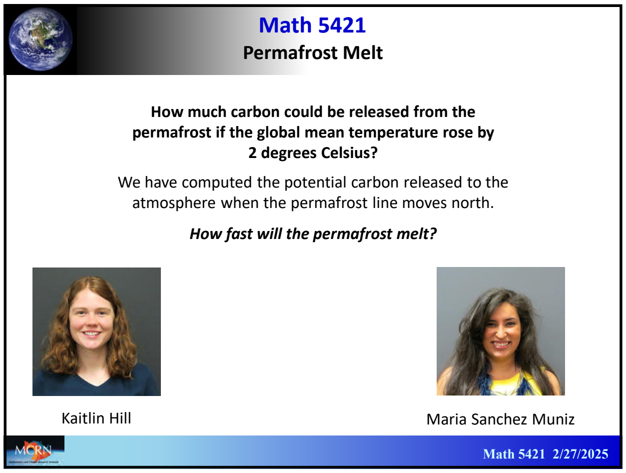
Where is the carbon in the permafrost?

In North America, mostly at lower latitudes.



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Permafrost Melt

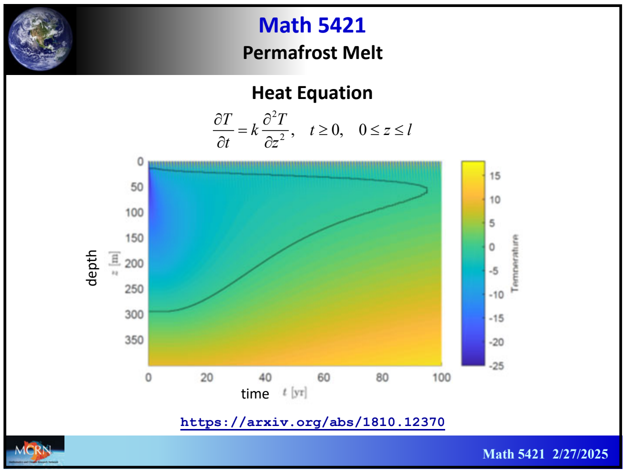
How much carbon could be released from the permafrost if the global mean temperature rose by 2 degrees Celsius?

We have computed the potential carbon released to the atmosphere when the permafrost line moves north.

How fast will the permafrost melt?

Kaitlin Hill

Maria Sanchez Muniz



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Heat Equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2}, \quad t \ge 0, \quad 0 \le z \le l$$

https://arxiv.org/abs/1810.12370

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Siberian Permafrost Sinkhole

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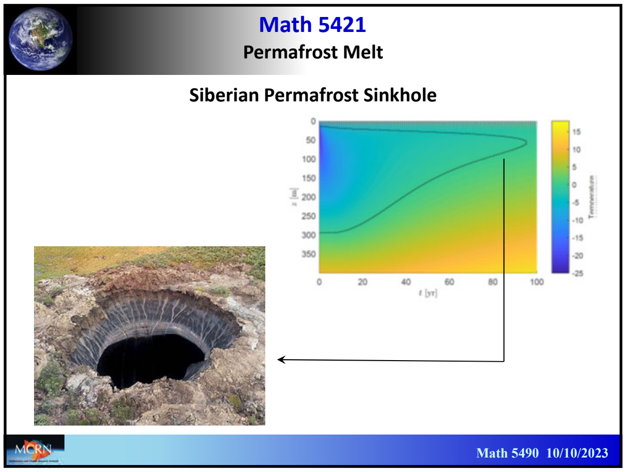
Siberian Permafrost Sinkhole



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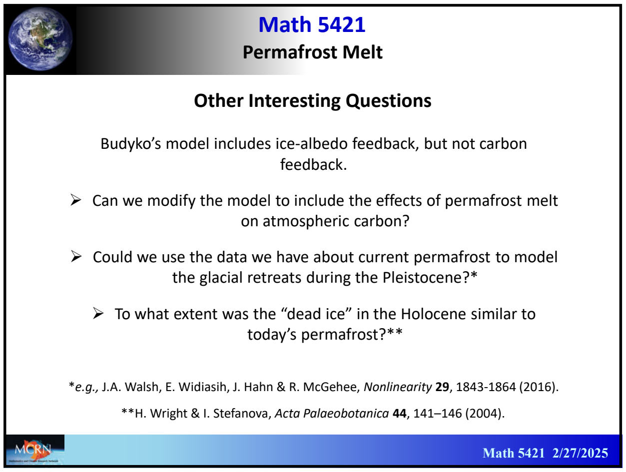
Siberian Permafrost Sinkhole



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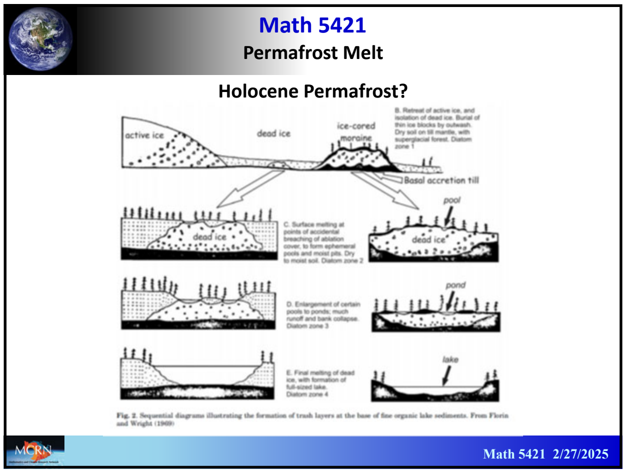
Other Interesting Questions

Budyko's model includes ice-albedo feedback, but not carbon feedback.

- Can we modify the model to include the effects of permafrost melt on atmospheric carbon?
- Could we use the data we have about current permafrost to model the glacial retreats during the Pleistocene?*
- To what extent was the "dead ice" in the Holocene similar to today's permafrost?**

*e.g., J.A. Walsh, E. Widiasih, J. Hahn & R. McGehee, *Nonlinearity* **29**, 1843-1864 (2016).

H. Wright & I. Stefanova, *Acta Palaeobotanica* **44, 141–146 (2004).



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Holocene Permafrost?

Fig. 2. Sequential diagrams illustrating the formation of trash layers at the base of fine organic lake sediments. From Florin and Wright (1969)

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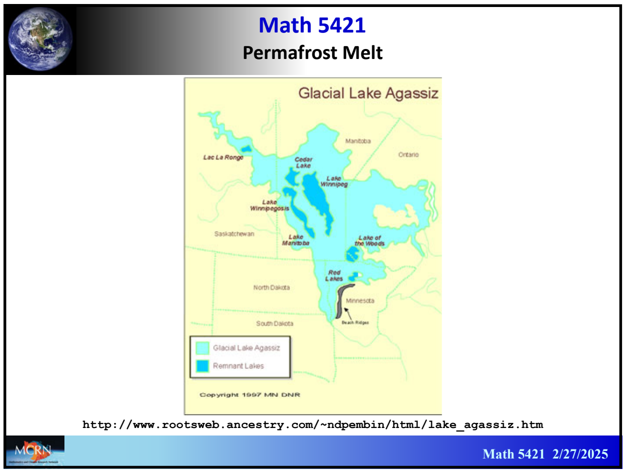

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http://www.rootsweb.ancestry.com/~ndpembin/html/lake_agassiz.htm

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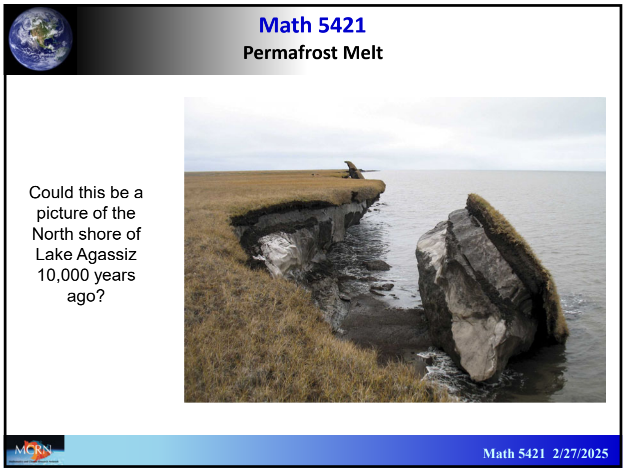

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Could this be a picture of the North shore of Lake Agassiz 10,000 years ago?

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